International J.Math. Combin. Vol.2 (2010), 102-111

Euler-Savary Formula for the Lorentzian Planar Homothetic Motions

M.A. Gungor, A.Z. Pirdal and M. Tosun

(Department of Mathematics of Faculty of Arts and Science of Sakarya University, Sakarya, Turkey)

Email: agungor@sakarya.edu.tr, apirdal@sakarya.edu.tr, tosun@sakarya.edu.tr

Abstract: One-parameter planar homothetic motion of 3-lorentzian planes, two are moving and one is fixed, have been considered in ref. [19]. In this paper we have given the canonical relative systems of a plane with respect to other planes so that the plane has a curve on it, which is spacelike or timelike under homothetic motion. Therefore, Euler-Savary formula giving the relation between curvatures of the trajectory curves drawn on the points on moving L and fixed plane L' is expressed separately for the cases whether the curves are spacelike or timelike. As a result it has been found that Euler-Savary formula stays the same whether these curves are spacelike or timelike. We have also found that if homothetic scala h is equal to 1 then the Euler-Savary formula becomes an equation which exactly the same is given by ref. [6].

Key Words: Homothetic Motion, Euler-Savary Formula, Lorentz Plane, kinematics, Smarandache Geometry.

AMS(2000): 53A04, 53A17, 53B50.

§1. Introduction

We know that the angular velocity vector has an important role in kinematics of two rigid bodies, especially one Rolling on another, [15] and [16]. To investigate to geometry of the motion of a line or a point in the motion of plane is important in the study of planar kinematics or planar mechanisms or in physics. Mathematicians and physicists have interpreted rigid body motions in various ways. K. Nomizu [16] has studied the 1-parameter motions of orientable surface M on tangent space along the pole curves using parallel vector fields at the contact points and he gave some characterizations of the angular velocity vector of rolling without sliding. H.H. Hacısalihoğlu showed some properties of 1-parameter homothetic motions in Euclidean space [8]. The geometry of such a motion of a point or a line has a number of applications in geometric modeling and model-based manufacturing of the mechanical products or in the design of robotic motions. These are specifically used to generate geometric models of shell-type objects and thick surfaces, [4,7,17].

[1]Received June 7, 2010. Accepted June 30, 2010.

As a model of spacetimes in physics, various geometries such as those of Euclid, Riemannian and Finsler geometries are established by mathematicians.

A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(1969), i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways, [11, 18].

In the Euclidean geometry, also called parabolic geometry, the fifth Euclidean postulate that there is only one parallel to a given line passing through an exterior point, is kept or validated. While in the Riemannian geometry, called elliptic geometry, the fifth Euclidean postulate is also invalidated as follows: there is no parallel to a given line passing through an exterior point [11].

Thus, as a particular case, Euclidean, Lobachevsky-Bolyai-Gauss, and Riemannian geometries may be united altogether, in the same space, by some Smarandache geometries. These last geometries can be partially Euclidean and partially Non-Euclidean. Howard Iseri [10] constructed a model for this particular Smarandache geometry, where the Euclidean fifth postulate is replaced by different statements within the same space, i.e. one parallel, no parallel, infinitely many parallels but all lines passing through the given point, all lines passing through the given point are parallel. Linfan Mao [12,13] showed that Smarandache geometries are generalizations of Pseudo-Manifold Geometries, which in their turn are generalizations of Finsler Geometry, and which in its turn is a generalization of Riemann Geometry.

The Euler-Savary theorem is a well-known theorem and studied systematically in two and three dimensional Euclidean space E^2 and E^3 by [2,3,14]. This theorem is used in serious fields of study in engineering and mathematics. For each mechanism type a simple graphical procedure is outlined to determine the circles of inflections and cusps, which are useful to compute the curvature of any point of the mobile plane through the Euler-Savary equation. By taking Lorentzian plane L^2 instead of Euclidean plane E^2, Ergin [5] has introduced 1-parameter planar motion in Lorentzian plane. Furthermore he gave the relation between the velocities, accelerations and pole curves of these motions. In the L^2 Lorentz plane Euler-Savary formula is given in references, [1], [6] and [9].

Let L (moving), L' (fixed) be planes and the coordinate systems of these planes be $\{O; \vec{e}_1, \vec{e}_2(\text{timelike})\}$ and $\{O'; \vec{e}_1', \vec{e}_2'(\text{timelike})\}$, respectively. Therefore, one-parameter Lorentzian planar homothetic motion is defined by the transformation [19]

$$\vec{x}' = h\vec{x} - \vec{u}, \tag{1}$$

where h is homothetic scale, $\overrightarrow{OO'} = \vec{u}$, is vector combining the systems (fixed and moving) initial points and the vectors $\vec{X}$, $\vec{X}'$ show the position vectors of the point $X \in L$ with respect to moving and fixed systems, respectively. In the one-parameter Lorentzian planar homothetic motion the relation

$$\vec{V}_a = \vec{V}_f + h\vec{V}_r$$

holds where $\vec{V}_a$, $\vec{V}_f$ and $\vec{V}_r$ represent to absolute, sliding and relative velocity of the motion, respectively [19].

We have given the canonical relative systems of a plane with respect to others planes so that the plane has a curve on it which is spacelike or timelike under homothetic motions. Thus

Euler-Savary formula, which gives the relation between the curvatures of the trajectory curves drawn an the points of moving plane L and fixed plane L', is expressed separately for the cases whether the curves are spacelike or timelike. Finally it has been observed that Euler-Savary formula does not change whether these curves are spacelike or timelike and if homothetic scale is equal to 1 then the Euler-Savary formula takes the form in reference [6].

§2. Moving Coordinate Systems and Their Velocities

Let L_1, L be the moving planes and L' be the fixed plane. The perpendicular coordinate systems of the planes L_1, L and L' are $\{B;\vec{a}_1,\vec{a}_2\}$, $\{O;\vec{e}_1,\vec{e}_2\}$ and $\{O';\vec{e}'_1,\vec{e}'_2\}$, respectively. Suppose that θ and θ' are the rotation angles of one parameter Lorentzian homothetic motions of L_1 with respect to L and L', respectively. Therefore, in one parameter Lorentzian homothetic motions L_1/L and L_1/L' following relations are holds

$$\begin{aligned} \vec{a}_1 &= \cosh\theta\vec{e}_1 + \sinh\theta\vec{e}_2 \\ \vec{a}_2 &= \sinh\theta\vec{e}_1 + \cosh\theta\vec{e}_2 \end{aligned} \tag{2}$$

$$\overrightarrow{OB} = \vec{b} = b_1\vec{a}_1 + b_2\vec{a}_2 \tag{3}$$

and

$$\begin{aligned} \vec{a}_1 &= \cosh\theta'\vec{e}'_1 + \sinh\theta'\vec{e}'_2 \\ \vec{a}_2 &= \sinh\theta'\vec{e}'_1 + \cosh\theta'\vec{e}'_2 \end{aligned} \tag{4}$$

$$\overrightarrow{O'B} = \vec{b}' = b'_1\vec{a}_1 + b'_2\vec{a}_2 \tag{5}$$

respectively [19]. If we consider equations (2)-(3) and (4)-(5), then the differential equations for the motions L_1/L and L_1/L' are as follows, respectively [19]

$$\begin{aligned} &d\vec{a}_1 = d\theta\vec{a}_2, \qquad d\vec{a}_2 = d\theta\vec{a}_1 \\ &d\vec{b} = (db_1 + b_2 d\theta)\,\vec{a}_1 + (db_2 + b_1 d\theta)\,\vec{a}_2 \end{aligned} \tag{6}$$

and

$$\begin{aligned} &d'\vec{a}_1 = d\theta'\vec{a}_2, \qquad d'\vec{a}_2 = d\theta'\vec{a}_1 \\ &d'\vec{b}' = (db'_1 + b'_2 d\theta')\,\vec{a}_1 + (db'_2 + b'_1 d\theta')\,\vec{a}_2. \end{aligned} \tag{7}$$

If we use the following abbreviations

$$\begin{aligned} &d\theta = \lambda, && d\theta' = \lambda' \\ &db_1 + b_2 d\theta = \sigma_1, && db_2 + b_1 d\theta = \sigma_2 \\ &db'_1 + b'_2 d\theta' = \sigma'_1, && db'_2 + b'_1 d\theta' = \sigma'_2 \end{aligned} \tag{8}$$

then the differential equations for L_1/L and L_1/L' become

$$d\vec{a}_1 = \lambda\vec{a}_2, \qquad d\vec{a}_2 = \lambda\vec{a}_1, \qquad d\vec{b} = \sigma_1\vec{a}_1 + \sigma_2\vec{a}_2 \tag{9}$$

and

$$d'\vec{a}_1 = \lambda'\vec{a}_2, \qquad d'\vec{a}_2 = \lambda'\vec{a}_1, \qquad d'\vec{b} = \sigma_1'\vec{a}_1 + \sigma_2'\vec{a}_2 \tag{10}$$

respectively. Here the quantities σ_j, σ_j', λ and λ' are Pfaffian forms of one parameter Lorentzian homothetic motion [19].

For the point X with the coordinates of x_1 and x_2 in plane L_1 we get

$$\begin{aligned} \overrightarrow{BX} &= x_1\vec{a}_1 + x_2\vec{a}_2 \\ \vec{x} &= (hx_1 + b_1)\,\vec{a}_1 + (hx_2 + b_2)\,\vec{a}_2 \\ \vec{x}' &= (hx_1 + b_1')\,\vec{a}_1 + (hx_2 + b_2')\,\vec{a}_2. \end{aligned} \tag{11}$$

Therefore one obtains

$$d\vec{x} = (dhx_1 + hdx_1 + \sigma_1 + hx_2\lambda)\,\vec{a}_1 + (dhx_2 + hdx_2 + \sigma_2 + hx_1\lambda)\,\vec{a}_2 \tag{12}$$

and

$$d'\vec{x} = (dhx_1 + hdx_1 + \sigma_1' + hx_2\lambda')\,\vec{a}_1 + (dhx_2 + hdx_2 + \sigma_2' + hx_1\lambda')\,\vec{a}_2, \tag{13}$$

where $\vec{V}_r = \frac{d\vec{x}}{dt}$ and $\vec{V}_a = \frac{d'\vec{x}}{dt}$ are called relative and absolute velocities of the point X, [19]. If $\vec{V}_r = 0$ (i.e. $d\vec{x} = 0$) and $\vec{V}_a = 0$ (i.e. $d'\vec{x} = 0$), then the point X is fixed in the Lorentzian planes L and L', respectively. Thus, from equations (12) and (13) the condition that the point X are fixed in L and L' are given by following equations

$$\begin{aligned} hdx_1 &= -dhx_1 - \sigma_1 - hx_2\lambda \\ hdx_2 &= -dhx_2 - \sigma_2 - hx_1\lambda \end{aligned} \tag{14}$$

and

$$\begin{aligned} hdx_1 &= -dhx_1 - \sigma_1' - hx_2\lambda' \\ hdx_2 &= -dhx_2 - \sigma_2' - hx_1\lambda' \end{aligned} \tag{15}$$

respectively. Substituting equation (14) into equation (13), sliding velocities $\vec{V}_f = \frac{d_f\vec{x}}{dt}$ of the point X becomes

$$d_f\vec{x} = [(\sigma_1' - \sigma_1) + hx_2\,(\lambda' - \lambda)]\,\vec{a}_1 + [(\sigma_2' - \sigma_2) + hx_1\,(\lambda' - \lambda)]\,\vec{a}_2. \tag{16}$$

Thus, for the pole point $P = (p_1, p_2)$ of the motion, we write [19]

$$x_1 = p_1 = -\frac{\sigma_2' - \sigma_2}{h\,(\lambda' - \lambda)}, \qquad x_2 = p_2 = -\frac{\sigma_1' - \sigma_1}{h\,(\lambda' - \lambda)}. \tag{17}$$

§3. Euler-Savary Formula For One Parameter Lorentzian Planar Homothetic Motions

Now, we consider spacelike and timelike pole curves of one parameter lorentzian planar homothetic motions and calculate Euler-Savary formula for both cases individually.

3.1 Canonical Relative System For Spacelike Pole Curves and Euler-Savary Formula

Now, let us choose the moving plane A represented by the coordinate system $\{B; \vec{a}_1, \vec{a}_2\}$ in such way to meet following conditions:

i) The origin of the system B and the instantaneous rotation pole P coincide with each other, i.e. $B = P$;

ii) The axis $\{B; \vec{a}_1\}$ is the pole tangent, that is, it coincides with the common tangent of spacelike pole curves (P) and (P'), (see Figure 1).

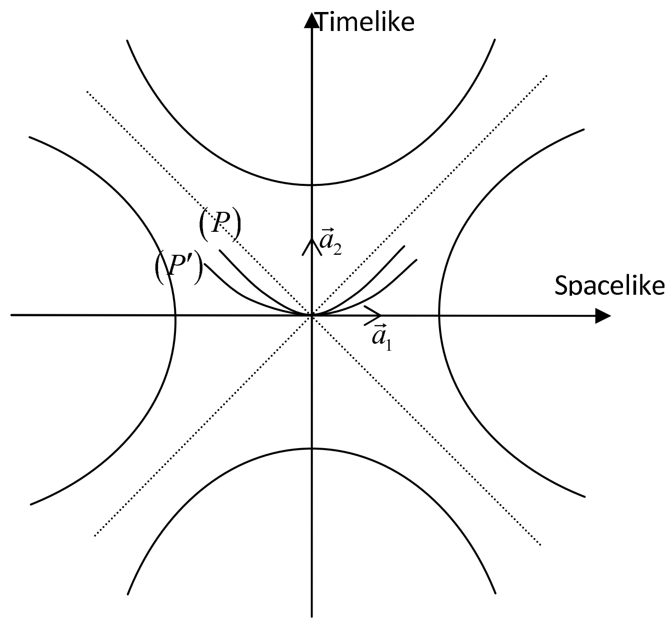

Figure 1. Spacelike Pole Curves (P) and (P')

If we consider the condition (i), then from equation (17) we reach that $\sigma_1 = \sigma_1'$ and $\sigma_2 = \sigma_2'$. Thus, from equation (9) and (10) we get

$$d\vec{b} = d\vec{p} = \sigma_1 \vec{a}_1 + \sigma_2 \vec{a}_2 = d'\vec{p} = d'\vec{b}.$$

Therefore, we have given the tangent of pole and constructed the rolling for the spacelike pole curves (P) and (P'). Considering the condition (ii) yields us that $\sigma_2 = \sigma_2' = 0$. If we choose $\sigma_1 = \sigma_1' = \sigma$ and consider equations (6) and (7), then we get the following equations for the differential equations related to the canonical relative system $\{P; \vec{a}_1, \vec{a}_2\}$ of the plane denoted by L_{1p},

$$d\vec{a}_1 = \lambda \vec{a}_2, \qquad d\vec{a}_2 = \lambda \vec{a}_1, \qquad d\vec{p} = \sigma \vec{a}_1 \tag{18}$$

and

$$d'\vec{a}_1 = \lambda' \vec{a}_2, \qquad d'\vec{a}_2 = \lambda' \vec{a}_1, \qquad d'\vec{p} = \sigma \vec{a}_1 \tag{19}$$

where $\sigma = ds$ is scalar arc element of the spacelike pole curves of (P) and (P') and λ is central cotangent angle, i.e. the angle between two neighboring tangents of (P). Therefore, the curvature of (P) at the point P is λ/σ. Similarly, taking λ' to be central cotangent angle, the curvature (P') at the point P becomes λ'/σ. Therefore, $r = \sigma/\lambda$ and $r' = \sigma/\lambda'$ are the curvature radii of spacelike pole curves (P) and (P'), respectively. Lorentzian plane L with respect to lorentz plane L' rotates about infinitesimal rotation angle $dv = \lambda' - \lambda$ at the time interval dt around the rotation pole P. Thus the rotational motions velocity of L with respect to L' becomes

$$\frac{\lambda' - \lambda}{dt} = \frac{dv}{dt} = \dot{v}\,. \tag{20}$$

Let us suppose that the direction of the unit tangent vector $\vec{a}_1$ is same as the direction of spacelike pole curves (P) and (P') (i.e., $ds/dt > 0$). In this case for the curvature radii (P) and (P'), $r > 0$ and $r' > 0$, respectively.

Now we investigate the velocities of the point X which has the coordinates x_1 and x_2 with respect to canonical relative system. Considering equation (12) and (13) we find

$$d\vec{x} = (dhx_1 + hdx_1 + \sigma + hx_2\lambda)\,\vec{a}_1 + (dhx_2 + hdx_2 + hx_1\lambda)\,\vec{a}_2 \tag{21}$$

$$d'\vec{x} = (dhx_1 + hdx_1 + \sigma + hx_2\lambda')\,\vec{a}_1 + (dhx_2 + hdx_2 + hx_1\lambda')\,\vec{a}_2. \tag{22}$$

Thus, the condition that the point X to be fixed in the Lorentzian planes L and L' becomes

$$\begin{aligned} hdx_1 &= -dhx_1 - \sigma - hx_2\lambda \\ hdx_2 &= -dhx_2 - hx_1\lambda \end{aligned} \tag{23}$$

and

$$\begin{aligned} hdx_1 &= -dhx_1 - \sigma - hx_2\lambda' \\ hdx_2 &= -dhx_2 - hx_1\lambda'. \end{aligned} \tag{24}$$

Therefore, the sliding velocity $\vec{V}_f$ is written to be

$$d_f\vec{x} = h\,(x_2\vec{a}_1 + x_1\vec{a}_2)\,(\lambda' - \lambda)\,.$$

Any point X chosen at the moving Lorentzian plane L draws a trajectory at the fixed lorentz plane L' during one parameter Lorentzian planar homothetic motion L/L'. Now we search for the planar curvature center X' of this trajectory at the time t.

The points X and X' have coordinates (x_1, x_2) and (x_1', x_2') with respect to canonical relative system and stay on the trajectory normal of X at every time t with the instantaneous rotation pole P. Generally a curvature center of a planar curve with respect to the point of the plane stays on the normal with respect to the point of the curve. In addition to that, this curvature center can be thought to be the limit of the intersection's normal of two neighboring points on the curve (see Figure 2). Therefore the vectors

$$\begin{aligned} \overrightarrow{PX} &= x_1\vec{a}_1 + x_2\vec{a}_2 \\ \overrightarrow{PX'} &= x_1'\vec{a}_1 + x_2'\vec{a}_2 \end{aligned}$$

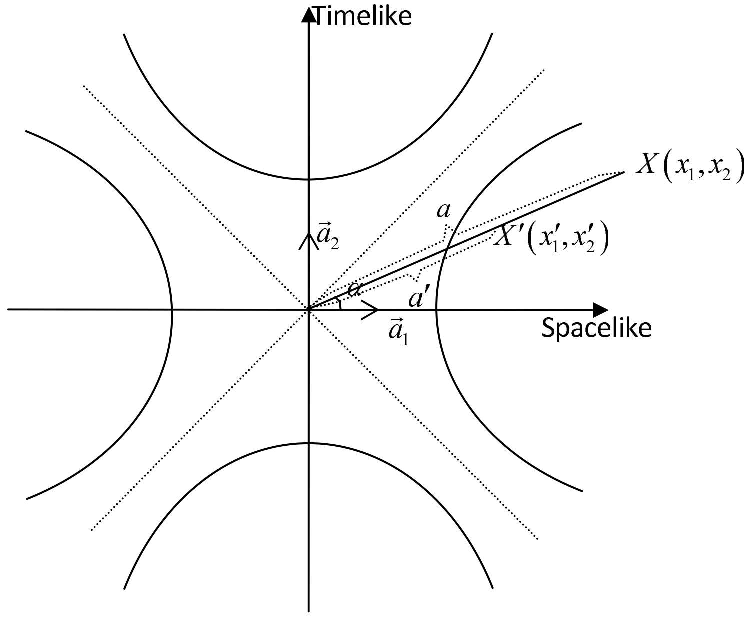

Figure 2. Spacelike vectors $\vec{PX}$ and $\vec{PX'}$

have same direction crossing the point P. Hence, the coordinates of the point X and X' satisfies the following equation:

$$x_1 x'_2 - x_2 x'_1 = 0. \tag{25}$$

Differentiation the last equation yields

$$dx_1 x'_2 + x_1 dx'_2 - dx'_1 x_2 - x'_1 dx_2 = 0. \tag{26}$$

The condition of being fixed of X in the Lorentzian plane L was given in equations (23). Moreover, the condition of being fixed of X' in the Lorentzian plane L' is

$$\begin{aligned} h dx'_1 &= -dh x'_1 - \sigma - h x'_2 \lambda' \\ h dx'_2 &= -dh x'_2 - h x'_1 \lambda'. \end{aligned} \tag{27}$$

Considering equation (26) with equations (23) and (27), we find

$$(x'_2 - x_2)\,\sigma + h\,(x_1 x'_1 - x_2 x'_2)\,(\lambda' - \lambda) = 0. \tag{28}$$

Taking the vectors $\overrightarrow{PX}$ and $\overrightarrow{PX'}$ to be spacelike vectors and switching to the polar coordinates, i.e.,

$$x_1 = a\cosh\alpha, \qquad x_2 = a\sinh\alpha$$

$$x'_1 = a'\cosh\alpha, \qquad x'_2 = a'\sinh\alpha$$

we find

$$\sigma\,(a' - a)\sinh\alpha + haa'\,(\lambda' - \lambda) = 0. \tag{29}$$

From equations (20) and (28) we obtain

$$\left(\frac{1}{a'} - \frac{1}{a}\right) \sinh\alpha = h\left(\frac{1}{r'} - \frac{1}{r}\right) = h\frac{dv}{ds}. \tag{30}$$

This last equation is called Euler-Savary formula for the lorentzian homothetic motion.

Therefore we can give the following theorem.

Theorem 1 *In the one parameter Lorentzian planar homothetic motion of moving Lorentz plane L with respect to fixed Lorentz plane L', any point X at the plane L draws a trajectory with the instantaneous curvature center X' in the plane L'. In reverse motion, any point X' at the plane L' draws a trajectory at the lorentz plane L, being the curvature center at the initial point X. The interrelation between the points X and X' is expressed in equation (30) which is Euler-Savary formula in the sense of Lorentz.*

3.2 Canonical Relative System For Timelike Pole Curves and Euler-Savary Formula

Let us choose the moving plane A represented by the coordinate system $\{B; \vec{a}_1, \vec{a}_2\}$ in such way to meet following conditions:

i) The origin of the system B and the instantaneous rotation pole P coincide with each other, i.e. $B = P$,

ii) The axis $\{B; \vec{a}_2\}$ is the pole tangent, that is, it coincides with the common tangent of timelike pole curves (P) and (P'), (see Figure 3.).

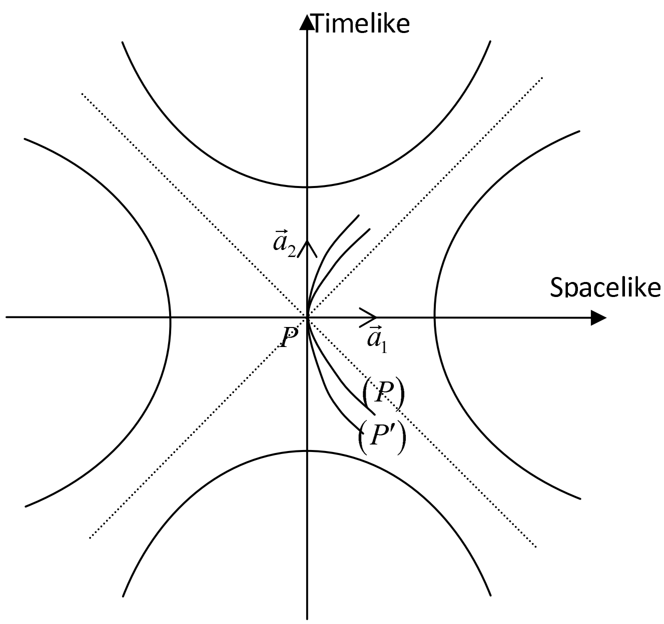

Figure 3. Timelike pole curves (P) and (P')

Thus, if the operations in III.1 section are performed considering the conditions i) and ii), the Euler-Savary formula for one-parameter lorentzian planar homothetic motion remains unchanged, that is, it is the same as in the equation (30), (see Figure 4.).

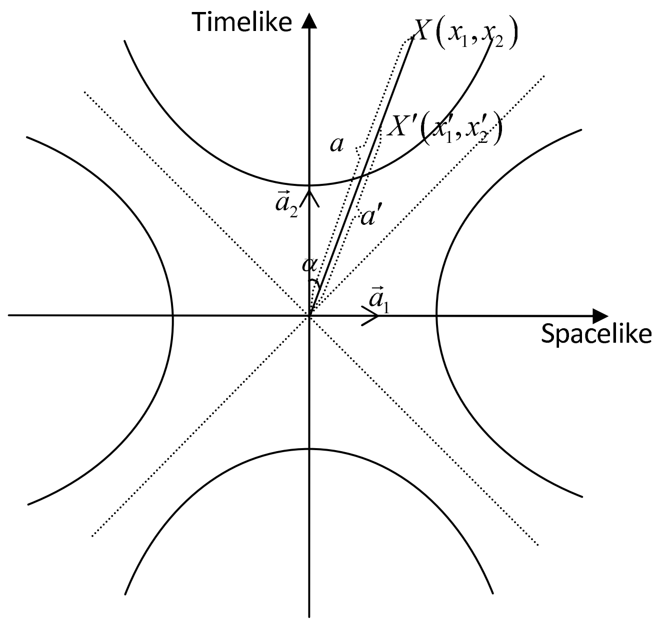

Figure 4. Timelike vectors $\vec{PX}$ and $\vec{PX'}$

Following Theorem 1 we reach the following corollaries:

Corollary 1 *In the one parameter Lorentzian homothetic motion L/L', whether the pole curves spacelike or timelike, the interrelation between the points X and X' is given by*

$$\left(\frac{1}{a'} - \frac{1}{a}\right)\sinh\alpha = h\left(\frac{1}{r'} - \frac{1}{r}\right)$$

which is Euler-Savary formula in the sense of Lorentz.

Corollary 2 *If $h \equiv 1$, then we reach the formula*

$$\left(\frac{1}{a'} - \frac{1}{a}\right)\sinh\alpha = \left(\frac{1}{r'} - \frac{1}{r}\right)$$

which is Euler-Savary formula in the Lorentzian plane given in references [1,6,9].

References

[1] I. Aytun, Euler-Savary formula for one-parameter Lorentzian plane motion and its lorentzian geometrical interpretation, M.Sc. Thesis, Celal Bayar University, 2002.

[2] O. Bottema, B. Roth, *Theoretical Kinematics*, North-Holland Series in Applied Mathematics and Mechanics, North-Holland, Amsterdam, 1979.

[3] R. R. Buckley, E. V. Whitfield, The Euler-Savary formula, *The Mathematical Gazette*, 33(306)(1949), 297-299.

[4] Y. J. Chen, B. Ravani, Offsets surface generation and contouring in computer aided design, *ASME J. Mech. Trans. and Autom. Des.* 109(1987), 348-352.

[5] A. A. Ergin, *The kinematic geometry on the lorentz plane*, Ph.D. Thesis, Ankara University, 1989.

[6] M. Ergüt, A. P. Aydın, N. Bildik, The geometry of the canonical system and one-parameter motions in 2-dimensional lorentzian space, *The Journal of Fırat University*, 3(1)(1988), 113-122.

[7] R. T. Farouki, The approximation of non-degenerate offset surface, *Comput. Aided Geom. Des.* 3(1986), 15-43.

[8] H. H. Hacısalihoğlu, On the rolling of one curve or surface upon another, *Proceedings of The Royal Irish Academy*, 71(2)(1971), 13-17.

[9] T. Ikawa, Euler-savary's formula on minkowski geometry, *Balkan Journal of Geometry and Its Applications*, 8(2)(2003), 31-36.

[10] H. Iseri, *Smarandache Manifolds*, Am. Res. Press, 2002.

[11] L. Kuciuk and M. Antholy, An Introduction to Smarandache Geometries, *Mathematics Magazine*, 12(2003), 1-12.

[12] L.F.Mao, *Smarandache Multi-Space Theory*, Hexis, Phoenix,American 2006.

[13] L.F.Mao, *Combinatorial Geometry with applications to Field Theory*, InfoQuest, USA, 2009.

[14] H. R. Müller, Kinematik, Sammlung Göschen, Walter de Gruyter, Berlin, 1963.

[15] H. R. Müller, Zur Bewegunssgeometrie In Raumen Höherer Dimension, *Mh. Math.* 70(1) (1966), 47-57.

[16] K. Nomizu, Kinematics and differential geometry of submanifolds, *Tohoku Math. J.* 30(4) (1978), 623-637.

[17] S. G. Papaioannou, D. Kiritsis, An application of bertrand curves and surface to CAD/CAM., *Comput.-Aided Des.* 17(8)(1985), 348-352.

[18] F.Smarandache, Mixed noneuclidean geometries, eprint: *arXiv: math/0010119, 10/2000.*

[19] A. Tutar, N. Kuruoğlu, M. Düldül, On the moving coordinate system and pole points on the Lorentzian plane, *International Journal of Applied Mathematics*, 7(4)(2001), 439-445.