

Non-Solvable Equation Systems with Graphs Embedded in \mathbb{R}^n

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Abstract: Different from the homogenous systems, a Smarandache system is a contradictory system in which an axiom behaves in at least two different ways within the same system, i.e., validated and invalidated, or only invalidated but in multiple distinct ways. Such systems widely exist in the world. In this report, we discuss such a kind of Smarandache system, i.e., non-solvable equation systems, such as those of non-solvable algebraic equations, non-solvable ordinary differential equations and non-solvable partial differential equations by topological graphs, classify these systems and characterize their global behaviors, particularly, the sum-stability and prod-stability of such equations. Applications of such systems to other sciences, such as those of controlling of infectious diseases, interaction fields and flows in network are also included in this report.

Key Words: Non-solvable equation, Smarandache system, topological graphs, vertex-edge labeled graph, G-solution, sum-stability, prod-stability.

AMS(2010): 05C15, 34A30, 34A34, 37C75, 70F10, 92B05

§1. Introduction

Consider two systems of linear equations following:

$$(LES_4^N) \begin{cases} x + y = 1 \\ x + y = -1 \\ x - y = -1 \\ x - y = 1 \end{cases} \quad (LES_4^S) \begin{cases} x = y \\ x + y = 2 \\ x = 1 \\ y = 1 \end{cases}$$

Clearly, (LES_4^N) is non-solvable because $x + y = -1$ is contradictory to $x + y = 1$, and so that for equations $x - y = -1$ and $x - y = 1$. Thus there are no solutions x_0, y_0 hold with all equations in this system. But (LES_4^S) is solvable clearly with a solution $x = 1$ and $y = 1$.

It should be noted that each equation in systems (LES_4^N) and (LES_4^S) is a straight line in Euclidean space \mathbb{R}^2 , such as those shown in Fig.1.

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space \mathbb{R}^n and

$$(ES_e) \begin{cases} f_1^e(x_1, x_2, \dots, x_n) = 0 \\ f_2^e(x_1, x_2, \dots, x_n) = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ f_{n-1}^e(x_1, x_2, \dots, x_n) = 0 \end{cases}$$

be a system of equations for determining an edge $e \in E(G)$ in \mathbb{R}^n . Then the system

$$\left. \begin{array}{l} f_1^e(x_1, x_2, \dots, x_n) = 0 \\ f_2^e(x_1, x_2, \dots, x_n) = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ f_{n-1}^e(x_1, x_2, \dots, x_n) = 0 \end{array} \right\} \forall e \in E(G)$$

is a non-solvable system of equations. Generally, let \mathcal{G} be a geometrical figure consisting of m parts $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_m$, where \mathcal{G}_i is determined by a system of algebraic equations

$$\begin{cases} f_1^{[i]}(x_1, x_2, \dots, x_n) = 0 \\ f_2^{[i]}(x_1, x_2, \dots, x_n) = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ f_{n-1}^{[i]}(x_1, x_2, \dots, x_n) = 0 \end{cases}$$

Similarly, we get a non-solvable system

$$\left. \begin{array}{l} f_1^{[i]}(x_1, x_2, \dots, x_n) = 0 \\ f_2^{[i]}(x_1, x_2, \dots, x_n) = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ f_{n-1}^{[i]}(x_1, x_2, \dots, x_n) = 0 \end{array} \right\} 1 \leq i \leq m.$$

Thus we obtain the following result.

Proposition 1.2 *Any geometrical figure \mathcal{G} consisting of m parts, each of which is determined by a system of algebraic equations in $\mathbb{R}^n, n \geq 2$ posses an algebraic representation by system of equations, solvable or not in \mathbb{R}^n .*

For example, let G be a planar graph with vertices v_1, v_2, v_3, v_4 and edges $v_1v_2, v_1v_3, v_2v_3, v_3v_4, v_4v_1$, shown in Fig.2.

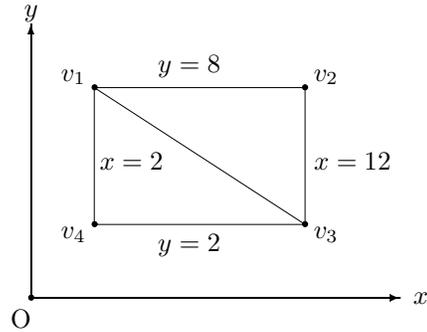


Fig.2

Then we get a non-solvable system of linear equations

$$\begin{cases} x = 2 \\ y = 8 \\ x = 12 \\ y = 2 \\ 3x + 5y = 46. \end{cases}$$

More results on non-solvable linear systems of equations can be found in [9]. Terminologies and notations in this paper are standard. For those not mentioned in this paper, we follow [12] and [15] for partial or ordinary differential equations. [5-7], [13-14] for algebra, topology and Smarandache systems, and [1] for mechanics.

§2. Smarandache Systems with Labeled Topological Graphs

A non-solvable system of algebraic equations is in fact a contradictory system in classical meaning of mathematics. As we have shown, such systems extensively exist in mathematics and possess real meaning even if in classical mathematics. This fact enables one to introduce the conception of Smarandache system following.

Definition 2.1([5-7]) *A rule \mathcal{R} in a mathematical system $(\Sigma; \mathcal{R})$ is said to be Smarandachely denied if it behaves in at least two different ways within the same set Σ , i.e., validated and invalidated, or only invalidated but in multiple distinct ways.*

A Smarandache system $(\Sigma; \mathcal{R})$ is a mathematical system which has at least one Smarandachely denied rule \mathcal{R} .

Without loss of generality, let $(\Sigma_1; \mathcal{R}_1)$, $(\Sigma_2; \mathcal{R}_2)$, \dots , $(\Sigma_m; \mathcal{R}_m)$ be mathematical systems, where \mathcal{R}_i is a rule on Σ_i for integers $1 \leq i \leq m$. If for two integers i, j , $1 \leq i, j \leq m$, $\Sigma_i \neq \Sigma_j$ or $\Sigma_i = \Sigma_j$ but $\mathcal{R}_i \neq \mathcal{R}_j$, then they are said to be *different*, otherwise, *identical*. If we can list all systems of a Smarandache system $(\Sigma; \mathcal{R})$, then we get a *Smarandache multi-space* defined following.

Definition 2.2([5-7],[11]) Let $(\Sigma_1; \mathcal{R}_1), (\Sigma_2; \mathcal{R}_2), \dots, (\Sigma_m; \mathcal{R}_m)$ be $m \geq 2$ mathematical spaces, different two by two. A Smarandache multi-space $\tilde{\Sigma}$ is a union $\bigcup_{i=1}^m \Sigma_i$ with rules $\tilde{\mathcal{R}} = \bigcup_{i=1}^m \mathcal{R}_i$ on $\tilde{\Sigma}$, denoted by $(\tilde{\Sigma}; \tilde{\mathcal{R}})$.

The conception of Smarandache multi-space reflects the notion of the whole $\tilde{\Sigma}$ is consisting of its parts $(\Sigma_i; \mathcal{R}_i), i \geq 1$ for a thing in philosophy. The laterality of human beings implies that one can only determines lateral feature of a thing in general. Such a typical example is the proverb of blind men with an elephant.



Fig. 3

In this proverb, there are 6 blind men were be asked to determine what an elephant looked like by feeling different parts of the elephant's body. The man touched the elephant's leg, tail, trunk, ear, belly or tusk claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe, respectively. They then entered into an endless argument and each of them insisted his view right. *All of you are right!* A wise man explains to them: *Why are you telling it differently is because each one of you touched the different part of the elephant. So, actually the elephant has all those features what you all said, i.e., a Smarandache multi-space consisting of these 6 parts.*

Usually, a man is blind for an unknowing thing and takes himself side as the dominant factor. That makes him knowing only the lateral features of a thing, not the whole. That is also the reason why one used to harmonious, not contradictory systems in classical mathematics. But the world is filled with contradictions. Being a wise man knowing the world, we need to find the whole, not just the parts. Thus the Smarandache multi-space is important for sciences.

Notice that a Smarandache multi-space $(\tilde{\Sigma}; \tilde{\mathcal{R}})$ naturally inherits a combinatorial structure, i.e., a vertex-edge labeled topological graph defined following.

Definition 2.3([5-7]) Let $(\tilde{\Sigma}; \tilde{\mathcal{R}})$ be a Smarandache multi-space with $\tilde{\Sigma} = \bigcup_{i=1}^m \Sigma_i$ and $\tilde{\mathcal{R}} = \bigcup_{i=1}^m \mathcal{R}_i$. Then a inherited graph $G[\tilde{\Sigma}, \tilde{\mathcal{R}}]$ of $(\tilde{\Sigma}; \tilde{\mathcal{R}})$ is a labeled topological graph defined by

$$V(G[\tilde{\Sigma}, \tilde{\mathcal{R}}]) = \{\Sigma_1, \Sigma_2, \dots, \Sigma_m\},$$

$$E(G[\tilde{\Sigma}, \tilde{\mathcal{R}}]) = \{(\Sigma_i, \Sigma_j) \mid \Sigma_i \cap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\}$$

with an edge labeling

$$l^E : (\Sigma_i, \Sigma_j) \in E \left(G \left[\tilde{S}, \tilde{R} \right] \right) \rightarrow l^E(\Sigma_i, \Sigma_j) = \varpi \left(\Sigma_i \cap \Sigma_j \right),$$

where ϖ is a characteristic on $\Sigma_i \cap \Sigma_j$ such that $\Sigma_i \cap \Sigma_j$ is isomorphic to $\Sigma_k \cap \Sigma_l$ if and only if $\varpi(\Sigma_i \cap \Sigma_j) = \varpi(\Sigma_k \cap \Sigma_l)$ for integers $1 \leq i, j, k, l \leq m$.

For example, let $S_1 = \{a, b, c\}$, $S_2 = \{c, d, e\}$, $S_3 = \{a, c, e\}$ and $S_4 = \{d, e, f\}$. Then the multi-space $\tilde{S} = \bigcup_{i=1}^4 S_i = \{a, b, c, d, e, f\}$ with its labeled topological graph $G[\tilde{S}]$ is shown in Fig.4.

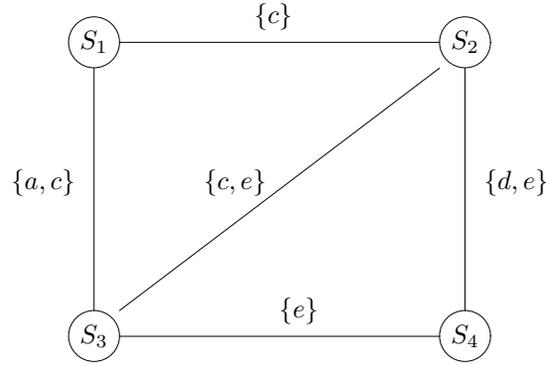


Fig.4

The labeled topological graph $G \left[\tilde{\Sigma}, \tilde{R} \right]$ reflects the notion that there exist linkages between things in philosophy. In fact, each edge $(\Sigma_i, \Sigma_j) \in E \left(G \left[\tilde{\Sigma}, \tilde{R} \right] \right)$ is such a linkage with coupling $\varpi(\Sigma_i \cap \Sigma_j)$. For example, let $a = \{\text{tusk}\}$, $b = \{\text{nose}\}$, $c_1, c_2 = \{\text{ear}\}$, $d = \{\text{head}\}$, $e = \{\text{neck}\}$, $f = \{\text{belly}\}$, $g_1, g_2, g_3, g_4 = \{\text{leg}\}$, $h = \{\text{tail}\}$ for an elephant \mathcal{C} . Then its labeled topological graph is shown in Fig.5,

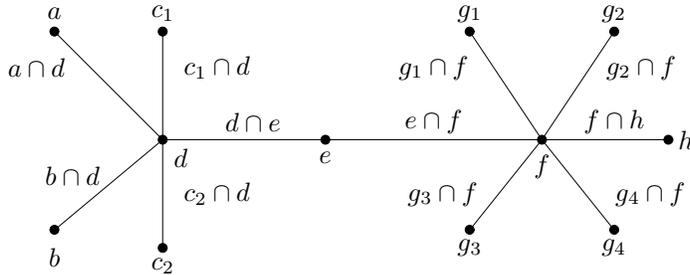


Fig.5

which implies that one can characterizes the geometrical behavior of an elephant combinatorially.

§3. Non-Solvable Systems of Ordinary Differential Equations

3.1 Linear Ordinary Differential Equations

For integers $m, n \geq 1$, let

$$\dot{X} = F_1(X), \dot{X} = F_2(X), \dots, \dot{X} = F_m(X) \tag{DES_m^1}$$

be a differential equation system with continuous $F_i : \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that $F_i(\bar{0}) = \bar{0}$, particularly, let

$$\dot{X} = A_1X, \dots, \dot{X} = A_kX, \dots, \dot{X} = A_mX \tag{LDES_m^1}$$

be a linear ordinary differential equation system of first order with

$$A_k = \begin{bmatrix} a_{11}^{[k]} & a_{12}^{[k]} & \cdots & a_{1n}^{[k]} \\ a_{21}^{[k]} & a_{22}^{[k]} & \cdots & a_{2n}^{[k]} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}^{[k]} & a_{n2}^{[k]} & \cdots & a_{nn}^{[k]} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}$$

where each $a_{ij}^{[k]}$ is a real number for integers $0 \leq k \leq m, 1 \leq i, j \leq n$.

Definition 3.1 *An ordinary differential equation system (DES_m^1) or $(LDES_m^1)$ are called non-solvable if there are no function $X(t)$ hold with (DES_m^1) or $(LDES_m^1)$ unless the constants.*

As we known, the general solution of the i th differential equation in $(LDES_m^1)$ is a linear space spanned by the elements in the solution basis

$$\mathcal{B}_i = \{ \bar{\beta}_k(t)e^{\alpha_k t} \mid 1 \leq k \leq n \}$$

for integers $1 \leq i \leq m$, where

$$\alpha_i = \begin{cases} \lambda_1, & \text{if } 1 \leq i \leq k_1; \\ \lambda_2, & \text{if } k_1 + 1 \leq i \leq k_2; \\ \cdots & \cdots \cdots \cdots \cdots \cdots \cdots; \\ \lambda_s, & \text{if } k_1 + k_2 + \cdots + k_{s-1} + 1 \leq i \leq n, \end{cases}$$

λ_i is the k_i -fold zero of the characteristic equation

$$\det(A - \lambda I_{n \times n}) = |A - \lambda I_{n \times n}| = 0$$

with $k_1 + k_2 + \cdots + k_s = n$ and $\bar{\beta}_i(t)$ is an n -dimensional vector consisting of polynomials in t with degree $\leq k_i - 1$.

In this case, we can simplify the labeled topological graph $G[\widetilde{\Sigma}, \widetilde{R}]$ replaced each \sum_i by the solution basis \mathcal{B}_i and $\sum_i \cap \sum_j$ by $\mathcal{B}_i \cap \mathcal{B}_j$ if $\mathcal{B}_i \cap \mathcal{B}_j \neq \emptyset$ for integers $1 \leq i, j \leq m$, called the *basis graph* of $(LDES_m^1)$, denoted by $G[LDES_m^1]$. For example, let $m = 4$ and $\mathcal{B}_1^0 = \{e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t}\}$, $\mathcal{B}_2^0 = \{e^{\lambda_3 t}, e^{\lambda_4 t}, e^{\lambda_5 t}\}$, $\mathcal{B}_3^0 = \{e^{\lambda_1 t}, e^{\lambda_3 t}, e^{\lambda_5 t}\}$ and $\mathcal{B}_4^0 = \{e^{\lambda_4 t}, e^{\lambda_5 t}, e^{\lambda_6 t}\}$, where $\lambda_i, 1 \leq i \leq 6$ are real numbers different two by two. Then $G[LDES_m^1]$ is shown in Fig.6.

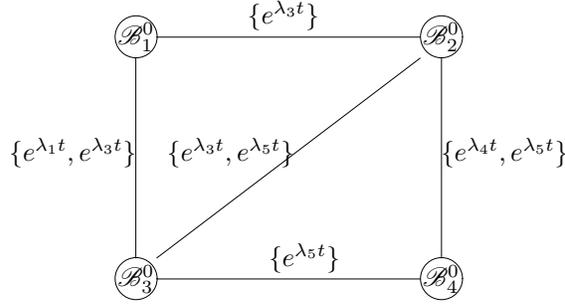


Fig.6

We get the following results.

Theorem 3.2([10]) *Every linear homogeneous differential equation system $(LDES_m^1)$ uniquely determines a basis graph $G[LDES_m^1]$ inherited in $(LDES_m^1)$. Conversely, every basis graph G uniquely determines a homogeneous differential equation system $(LDES_m^1)$ such that $G[LDES_m^1] \simeq G$.*

Such a basis graph $G[LDES_m^1]$ is called the G -solution of $(LDES_m^1)$. Theorem 3.2 implies that

Theorem 3.3([10]) *Every linear homogeneous differential equation system $(LDES_m^1)$ has a unique G -solution, and for every basis graph H , there is a unique linear homogeneous differential equation system $(LDES_m^1)$ with G -solution H .*

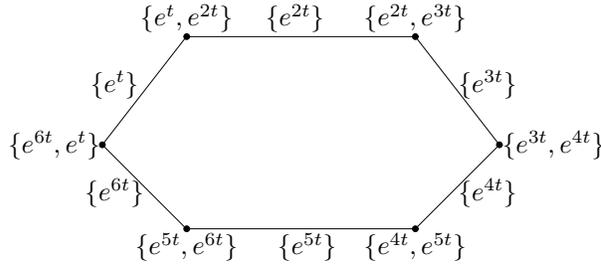


Fig.7 A basis graph

Example 3.4 Let (LDE_m^n) be the following linear homogeneous differential equation system

$$\begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 & (1) \\ \ddot{x} - 5\dot{x} + 6x = 0 & (2) \\ \ddot{x} - 7\dot{x} + 12x = 0 & (3) \\ \ddot{x} - 9\dot{x} + 20x = 0 & (4) \\ \ddot{x} - 11\dot{x} + 30x = 0 & (5) \\ \ddot{x} - 7\dot{x} + 6x = 0 & (6) \end{cases}$$

where $\ddot{x} = \frac{d^2x}{dt^2}$ and $\dot{x} = \frac{dx}{dt}$. Then the solution basis of equations (1) – (6) are respectively $\{e^t, e^{2t}\}$, $\{e^{2t}, e^{3t}\}$, $\{e^{3t}, e^{4t}\}$, $\{e^{4t}, e^{5t}\}$, $\{e^{5t}, e^{6t}\}$, $\{e^{6t}, e^t\}$ and its basis graph is shown in Fig.7.

3.2 Combinatorial Characteristics of Linear Differential Equations

Definition 3.5 Let $(LDES_m^1)$, $(LDES_m^1)'$ be two linear homogeneous differential equation systems with G -solutions H , H' . They are called combinatorially equivalent if there is an isomorphism $\varphi : H \rightarrow H'$, thus there is an isomorphism $\varphi : H \rightarrow H'$ of graph and labelings θ , τ on H and H' respectively such that $\varphi\theta(x) = \tau\varphi(x)$ for $\forall x \in V(H) \cup E(H)$, denoted by $(LDES_m^1) \stackrel{\varphi}{\simeq} (LDES_m^1)'$.

We introduce the conception of *integral graph* for $(LDES_m^1)$ following.

Definition 3.6 Let G be a simple graph. A vertex-edge labeled graph $\theta : G \rightarrow \mathbb{Z}^+$ is called *integral* if $\theta(uv) \leq \min\{\theta(u), \theta(v)\}$ for $\forall uv \in E(G)$, denoted by G^{I_θ} .

Let $G_1^{I_\theta}$ and $G_2^{I_\tau}$ be two integral labeled graphs. They are called *identical* if $G_1 \stackrel{\varphi}{\simeq} G_2$ and $\theta(x) = \tau(\varphi(x))$ for any graph isomorphism φ and $\forall x \in V(G_1) \cup E(G_1)$, denoted by $G_1^{I_\theta} = G_2^{I_\tau}$.

For example, these labeled graphs shown in Fig.8 are all integral on $K_4 - e$, but $G_1^{I_\theta} = G_2^{I_\tau}$, $G_1^{I_\theta} \neq G_3^{I_\sigma}$.

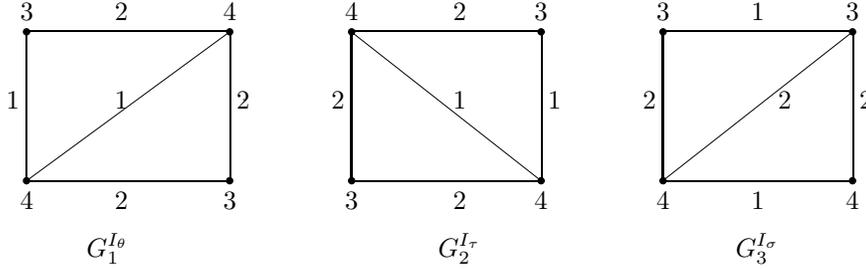


Fig.8

Then we get a combinatorial characteristic for combinatorially equivalent $(LDES_m^1)$ following.

Theorem 3.5([10]) Let $(LDES_m^1)$, $(LDES_m^1)'$ be two linear homogeneous differential equation systems with integral labeled graphs H , H' . Then $(LDES_m^1) \stackrel{\varphi}{\simeq} (LDES_m^1)'$ if and only if $H = H'$.

3.3 Non-Linear Ordinary Differential Equations

If some functions $F_i(X)$, $1 \leq i \leq m$ are non-linear in (DES_m^1) , we can linearize these non-linear equations $\dot{X} = F_i(X)$ at the point $\bar{0}$, i.e., if

$$F_i(X) = F_i'(\bar{0})X + R_i(X),$$

of partial differential equations of first order is non-solvable with initial values

$$\begin{cases} x_i|_{x_n=x_n^0} = x_i^0(s_1, s_2, \dots, s_{n-1}) \\ u|_{x_n=x_n^0} = u_0(s_1, s_2, \dots, s_{n-1}) \\ p_i|_{x_n=x_n^0} = p_i^0(s_1, s_2, \dots, s_{n-1}), \quad i = 1, 2, \dots, n \end{cases}$$

if and only if the system

$$F_k(x_1, x_2, \dots, x_n, u, p_1, p_2, \dots, p_n) = 0, \quad 1 \leq k \leq m$$

is algebraically contradictory, in this case, there must be an integer k_0 , $1 \leq k_0 \leq m$ such that

$$F_{k_0}(x_1^0, x_2^0, \dots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \dots, p_n^0) \neq 0$$

or it is differentially contradictory itself, i.e., there is an integer j_0 , $1 \leq j_0 \leq n-1$ such that

$$\frac{\partial u_0}{\partial s_{j_0}} - \sum_{i=0}^{n-1} p_i^0 \frac{\partial x_i^0}{\partial s_{j_0}} \neq 0.$$

Particularly, we get conclusions following by Theorem 4.2.

Corollary 4.3 *Let*

$$\begin{cases} F_1(x_1, x_2, \dots, x_n, u, p_1, p_2, \dots, p_n) = 0 \\ F_2(x_1, x_2, \dots, x_n, u, p_1, p_2, \dots, p_n) = 0 \end{cases}$$

be an algebraically contradictory system of partial differential equations of first order. Then there are no values x_i^0, u_0, p_i^0 , $1 \leq i \leq n$ such that

$$\begin{cases} F_1(x_1^0, x_2^0, \dots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \dots, p_n^0) = 0, \\ F_2(x_1^0, x_2^0, \dots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \dots, p_n^0) = 0. \end{cases}$$

Corollary 4.4 *A Cauchy problem (LPDES_m^C) of quasilinear partial differential equations with initial values $u|_{x_n=x_n^0} = u_0$ is non-solvable if and only if the system (LPDES_m) of partial differential equations is algebraically contradictory.*

Denoted by $\widehat{G}[PDES_m^C]$ such a graph $G[PDES_m^C]$ eradicated all labels. Particularly, replacing each label $S^{[i]}$ by $S_0^{[i]} = \{u_0^{[i]}\}$ and $S^{[i]} \cap S^{[j]}$ by $S_0^{[i]} \cap S_0^{[j]}$ for integers $1 \leq i, j \leq m$, we get a new labeled topological graph, denoted by $G_0[PDES_m^C]$. Clearly, $\widehat{G}[PDES_m^C] \simeq \widehat{G}_0[PDES_m^C]$.

Theorem 4.5 ([11]) *For any system (PDES_m^C) of partial differential equations of first order, $\widehat{G}[PDES_m^C]$ is simple. Conversely, for any simple graph G , there is a system (PDES_m^C) of partial differential equations of first order such that $\widehat{G}[PDES_m^C] \simeq G$.*

Particularly, if (PDES_m^C) is linear, we can immediately find its underlying graph following.

Corollary 4.6 *Let (LPDES_m) be a system of linear partial differential equations of first order with maximal contradictory classes $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_s$ on equations in (LPDES). Then $\widehat{G}[LPDES_m^C] \simeq K(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_s)$, i.e., an s -partite complete graph.*

Definition 4.7 Let $(PDES_m^C)$ be the Cauchy problem of a partial differential equation system of first order. Then the labeled topological graph $G[PDES_m^C]$ is called its topological graph solution, abbreviated to G -solution.

Combining this definition with that of Theorems 4.5, the following conclusion is holden immediately.

Theorem 4.8([11]) A Cauchy problem on system $(PDES_m)$ of partial differential equations of first order with initial values $x_i^{[k^0]}, u_0^{[k]}, p_i^{[k^0]}, 1 \leq i \leq n$ for the k th equation in $(PDES_m)$, $1 \leq k \leq m$ such that

$$\frac{\partial u_0^{[k]}}{\partial s_j} - \sum_{i=0}^n p_i^{[k^0]} \frac{\partial x_i^{[k^0]}}{\partial s_j} = 0$$

is uniquely G -solvable, i.e., $G[PDES_m^C]$ is uniquely determined.

§5. Global Stability of Non-Solvable Differential Equations

Definition 5.1 Let H be a spanning subgraph of $G[LDES_m^1]$ of systems $(LDES_m^1)$ with initial value $X_v(0)$. Then $G[LDES_m^1]$ is called sum-stable or asymptotically sum-stable on H if for all solutions $Y_v(t), v \in V(H)$ of the linear differential equations of $(LDES_m^1)$ with $|Y_v(0) - X_v(0)| < \delta_v$ exists for all $t \geq 0$,

$$\left| \sum_{v \in V(H)} Y_v(t) - \sum_{v \in V(H)} X_v(t) \right| < \varepsilon,$$

or furthermore,

$$\lim_{t \rightarrow 0} \left| \sum_{v \in V(H)} Y_v(t) - \sum_{v \in V(H)} X_v(t) \right| = 0.$$

Similarly, a system $(PDES_m^C)$ is sum-stable if for any number $\varepsilon > 0$ there exists $\delta_v > 0, v \in V(\widehat{G}[0])$ such that each $G(t)$ -solution with $|u_0^{[v]} - u_0^{[v]}| < \delta_v, \forall v \in V(\widehat{G}[0])$ exists for all $t \geq 0$ and with the inequality

$$\left| \sum_{v \in V(\widehat{G}[t])} u^{[v]} - \sum_{v \in V(\widehat{G}[0])} u^{[v]} \right| < \varepsilon$$

holds, denoted by $G[t] \overset{\Sigma}{\approx} G[0]$. Furthermore, if there exists a number $\beta_v > 0, v \in V(\widehat{G}[0])$ such that every $G'[t]$ -solution with $|u_0^{[v]} - u_0^{[v]}| < \beta_v, \forall v \in V(\widehat{G}[0])$ satisfies

$$\lim_{t \rightarrow \infty} \left| \sum_{v \in V(\widehat{G}[t])} u^{[v]} - \sum_{v \in V(\widehat{G}[0])} u^{[v]} \right| = 0,$$

then the $G[t]$ -solution is called asymptotically stable, denoted by $G[t] \overset{\Sigma}{\rightarrow} G[0]$.

We get results on the global stability for G -solutions of $(LDES_m^1)$ and $(PDES_m^C)$.

Theorem 5.2([10]) *A zero G -solution of linear homogenous differential equation systems ($LDES_m^1$) is asymptotically sum-stable on a spanning subgraph H of $G[LDES_m^1]$ if and only if $\text{Re}\alpha_v < 0$ for each $\bar{\beta}_v(t)e^{\alpha_v t} \in \mathcal{B}_v$ in ($LDES^1$) hold for $\forall v \in V(H)$.*

Example 5.3 Let a G -solution of ($LDES_m^1$) or (LDE_m^n) be the basis graph shown in Fig.4.1, where $v_1 = \{e^{-2t}, e^{-3t}, e^{3t}\}$, $v_2 = \{e^{-3t}, e^{-4t}\}$, $v_3 = \{e^{-4t}, e^{-5t}, e^{3t}\}$, $v_4 = \{e^{-5t}, e^{-6t}, e^{-8t}\}$, $v_5 = \{e^{-t}, e^{-6t}\}$, $v_6 = \{e^{-t}, e^{-2t}, e^{-8t}\}$. Then the zero G -solution is sum-stable on the triangle $v_4v_5v_6$, but it is not on the triangle $v_1v_2v_3$.

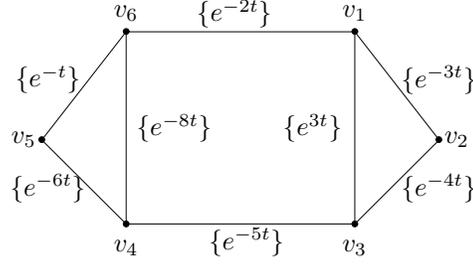


Fig.9

For partial differential equations, let the system ($PDES_m^C$) be

$$\left. \begin{aligned} \frac{\partial u}{\partial t} &= H_i(t, x_1, \dots, x_{n-1}, p_1, \dots, p_{n-1}) \\ u|_{t=t_0} &= u_0^{[i]}(x_1, x_2, \dots, x_{n-1}) \end{aligned} \right\} 1 \leq i \leq m \quad (APDES_m^C)$$

A point $X_0^{[i]} = (t_0, x_{10}^{[i]}, \dots, x_{(n-1)0}^{[i]})$ with $H_i(t_0, x_{10}^{[i]}, \dots, x_{(n-1)0}^{[i]}) = 0$ for $1 \leq i \leq m$ is called an *equilibrium point* of the i th equation in ($APDES_m$). Then we know that

Theorem 5.4([11]) *Let $X_0^{[i]}$ be an equilibrium point of the i th equation in ($APDES_m$) for each integer $1 \leq i \leq m$. If $\sum_{i=1}^m H_i(X) > 0$ and $\sum_{i=1}^m \frac{\partial H_i}{\partial t} \leq 0$ for $X \neq \sum_{i=1}^m X_0^{[i]}$, then the system ($APDES_m$) is sum-stability, i.e., $G[t] \stackrel{\Sigma}{\sim} G[0]$. Furthermore, if $\sum_{i=1}^m \frac{\partial H_i}{\partial t} < 0$ for $X \neq \sum_{i=1}^m X_0^{[i]}$, then $G[t] \stackrel{\Sigma}{\rightarrow} G[0]$.*

§6. Applications

6.1 Applications to Geometry

First, it is easily to shown that the G -solution of ($PDES_m^C$) is nothing but a differentiable manifold.

Theorem 6.1([11]) *Let the Cauchy problem be ($PDES_m^C$). Then every connected component of $\Gamma[PDES_m^C]$ is a differentiable n -manifold with atlas $\mathcal{A} = \{(U_v, \phi_v) | v \in V(\widehat{G}[0])\}$ underlying graph $\widehat{G}[0]$, where U_v is the n -dimensional graph $G[u^{[v]}] \simeq \mathbb{R}^n$ and ϕ_v the projection $\phi_v : ((x_1, x_2, \dots, x_n), u(x_1, x_2, \dots, x_n)) \rightarrow (x_1, x_2, \dots, x_n)$ for $\forall (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.*

Theorems 4.8 and 6.1 enables one to find the following result for vector fields on differentiable manifolds by non-solvable system ($PDES_m^C$).

Theorem 6.2([11]) *For any integer $m \geq 1$, let $U_i, 1 \leq i \leq m$ be open sets in \mathbb{R}^n underlying a connected graph defined by*

$$V(G) = \{U_i | 1 \leq i \leq m\}, \quad E(G) = \{(U_i, U_j) | U_i \cap U_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

If X_i is a vector field on U_i for integers $1 \leq i \leq m$, then there always exists a differentiable manifold $M \subset \mathbb{R}^n$ with atlas $\mathcal{A} = \{(U_i, \phi_i) | 1 \leq i \leq m\}$ underlying graph G and a function $u_G \in \Omega^0(M)$ such that

$$X_i(u_G) = 0, \quad 1 \leq i \leq m.$$

More results on geometrical structure of manifold can be found in references [2-3] and [8].

6.2 Global Control of Infectious Diseases

Consider two cases of virus for infectious diseases:

Case 1 *There are m known virus $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_m$ with infected rate k_i , heal rate h_i for integers $1 \leq i \leq m$ and an person infected a virus \mathcal{V}_i will never infects other viruses \mathcal{V}_j for $j \neq i$.*

Case 2 *There are m varying $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_m$ from a virus \mathcal{V} with infected rate k_i , heal rate h_i for integers $1 \leq i \leq m$.*

We are easily to establish a non-solvable differential model for the spread of infectious viruses by applying the SIR model of one infectious disease following:

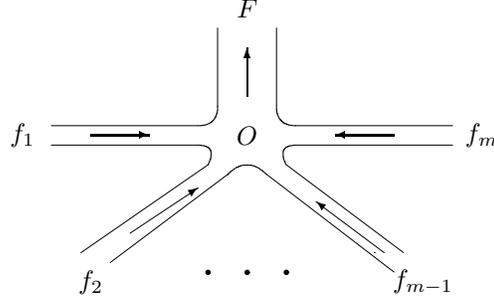
$$\begin{cases} \dot{S} = -k_1 SI \\ \dot{I} = k_1 SI - h_1 I \\ \dot{R} = h_1 I \end{cases} \quad \begin{cases} \dot{S} = -k_2 SI \\ \dot{I} = k_2 SI - h_2 I \\ \dot{R} = h_2 I \end{cases} \quad \cdots \quad \begin{cases} \dot{S} = -k_m SI \\ \dot{I} = k_m SI - h_m I \\ \dot{R} = h_m I \end{cases} \quad (DES_m^1)$$

and know the following result by Theorem 5.2 that

Conclusion 6.3([10]) *For m infectious viruses $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_m$ in an area with infected rate k_i , heal rate h_i for integers $1 \leq i \leq m$, then they decline to 0 finally if $0 < S < \sum_{i=1}^m h_i / \sum_{i=1}^m k_i$, i.e., these infectious viruses are globally controlled. Particularly, they are globally controlled if each of them is controlled in this area.*

6.3 Flows in Network

Let O be a node in N incident with m in-flows and 1 out-flow shown in Fig.10.

**Fig.10**

How can we characterize the behavior of flow F ? Denote the rate, density of flow f_i by $\rho^{[i]}$ for integers $1 \leq i \leq m$ and that of F by $\rho^{[F]}$, respectively. Then we know that

$$\frac{\partial \rho^{[i]}}{\partial t} + \phi_i(\rho^{[i]}) \frac{\partial \rho^{[i]}}{\partial x} = 0, \quad 1 \leq i \leq m.$$

We prescribe the initial value of $\rho^{[i]}$ by $\rho^{[i]}(x, t_0)$ at time t_0 . Replacing each $\rho^{[i]}$ by ρ in these flow equations of f_i , $1 \leq i \leq m$ enables one getting a non-solvable system ($PDES_m^C$) of partial differential equations following.

$$\left. \begin{array}{l} \frac{\partial \rho}{\partial t} + \phi_i(\rho) \frac{\partial \rho}{\partial x} = 0 \\ \rho|_{t=t_0} = \rho^{[i]}(x, t_0) \end{array} \right\} 1 \leq i \leq m.$$

Let $\rho_0^{[i]}$ be an equilibrium point of the i th equation, i.e., $\phi_i(\rho_0^{[i]}) \frac{\partial \rho_0^{[i]}}{\partial x} = 0$. Applying Theorem 5.4, if

$$\sum_{i=1}^m \phi_i(\rho) < 0 \quad \text{and} \quad \sum_{i=1}^m \phi_i(\rho) \left[\frac{\partial^2 \rho}{\partial t \partial x} - \phi'(\rho) \left(\frac{\partial \rho}{\partial x} \right)^2 \right] \geq 0$$

for $X \neq \sum_{k=1}^m \rho_0^{[k]}$, then we know that the flow F is stable and furthermore, if

$$\sum_{i=1}^m \phi_i(\rho) \left[\frac{\partial^2 \rho}{\partial t \partial x} - \phi'(\rho) \left(\frac{\partial \rho}{\partial x} \right)^2 \right] < 0$$

for $X \neq \sum_{k=1}^m \rho_0^{[k]}$, then it is also asymptotically stable.

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