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On Multi-Metric Spaces

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Abstract A Smarandache multi-space is a union of n spaces $A_1, A_2, \cdots, A_n$ with some additional conditions holding. Combining Smarandache multi-spaces with classical metric spaces, the conception of multi-metric spaces is introduced. Some characteristics of multi-metric spaces are obtained and the Banach's fixed-point theorem is generalized in this paper.

Keywords metric; multi-space; multi-metric space; Banach theorem.

§1. Introduction

The notion of multi-spaces is introduced by Smarandache in [6] under his idea of hybrid mathematics: *combining different fields into a unifying field* ([7]), which is defined as follows.

Definition 1.1 *For any integer $i, 1 \leq i \leq n$ let A_i be a set with ensemble of law L_i, and the intersection of k sets $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$ of them constrains the law $I(A_{i_1}, A_{i_2}, \cdots, A_{i_k})$. Then the union of A_i, $1 \leq i \leq n$*

$$\widetilde{A} = \bigcup_{i=1}^{n} A_i$$

is called a multi-space.

As we known, a set M associative a function $\rho : M \times M \rightarrow R^+ = \{x \mid x \in R, x \geq 0\}$ is called a *metric space* if for $\forall x, y, z \in M$, the following conditions for the metric function ρ hold:

(1) (*definiteness*) $\rho(x, y) = 0$ if and only if $x = y$;
(*ii*) (*symmetry*) $\rho(x, y) = \rho(y, x)$;
(*iii*) (*triangle inequality*) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$.

By combining Smarandache multi-spaces with classical metric spaces, a new kind of spaces called multi-metric spaces is found, which is defined in the following.

Definition 1.2 *A multi-metric space is a union $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ such that each M_i is a space with metric ρ_i for $\forall i, 1 \leq i \leq m$.*

When we say *a multi-metric space* $\widetilde{M} = \bigcup_{i=1}^{m} M_i$, it means that a multi-metric space with metrics $\rho_1, \rho_2, \cdots, \rho_m$ such that (M_i, ρ_i) is a metric space for any integer $i, 1 \leq i \leq m$. For a multi-metric space $\widetilde{M} = \bigcup_{i=1}^{m} M_i$, $x \in \widetilde{M}$ and a positive number R, a *R-disk* $B(x, R)$ in $\widetilde{M}$ is defined by

$$B(x,R) = \{\ y \mid \text{there exists an integer } k, 1 \le k \le m \text{ such that } \rho_k(y,x) \ < R, y \in \widetilde{M}\}$$

The main purpose of this paper is to find some characteristics of multi-metric spaces. For terminology and notations not defined here can be seen in $[1]-[2], [4]$ for terminologies in the metric space and in $[3], [5]-[9]$ for multi-spaces and logics.

§2. Characteristics of multi-metric spaces

For metrics on spaces, we have the following result.

Theorem 2.1. *Let $\rho_1, \rho_2, \cdots, \rho_m$ be m metrics on a space M and F a function on $\mathbf{E}^m$ such that the following conditions hold:*

(i) $F(x_1, x_2, \cdots, x_m) \ge F(y_1, y_2, \cdots, y_m)$ if for $\forall i, 1 \le i \le m$, $x_i \ge y_i$;

(ii) $F(x_1, x_2, \cdots, x_m) = 0$ only if $x_1 = x_2 = \cdots = x_m = 0$;

(iii) For two m-tuples $(x_1, x_2, \cdots, x_m)$ and $(y_1, y_2, \cdots, y_m)$,

$$F(x_1, x_2, \cdots, x_m) + F(y_1, y_2, \cdots, y_m) \ge F(x_1 + y_1, x_2 + y_2, \cdots, x_m + y_m).$$

Then $F(\rho_1, \rho_2, \cdots, \rho_m)$ is also a metric on M.

Proof. We only need to prove that $F(\rho_1, \rho_2, \cdots, \rho_m)$ satisfies the metric conditions for $\forall x, y, z \in M$.

By (ii), $F(\rho_1(x,y), \rho_2(x,y), \cdots, \rho_m(x,y)) = 0$ only if for any integer i, $\rho_i(x,y) = 0$. Since ρ_i is a metric on M, we know that $x = y$.

For any integer $i, 1 \le i \le m$, since ρ_i is a metric on M, we know that $\rho_i(x,y) = \rho_i(y,x)$. Whence,

$$F(\rho_1(x,y), \rho_2(x,y), \cdots, \rho_m(x,y)) = F(\rho_1(y,x), \rho_2(y,x), \cdots, \rho_m(y,x)).$$

Now by (i) and (iii), we get that

$$\begin{aligned}
&F(\rho_1(x,y), \rho_2(x,y), \cdots, \rho_m(x,y)) + F(\rho_1(y,z), \rho_2(y,z), \cdots, \rho_m(y,z)) \\
&\ge F(\rho_1(x,y) + \rho_1(y,z), \rho_2(x,y) + \rho_2(y,z), \cdots, \rho_m(x,y) + \rho_m(y,z)) \\
&\ge F(\rho_1(x,z), \rho_2(x,z), \cdots, \rho_m(x,z)).
\end{aligned}$$

Therefore, $F(\rho_1, \rho_2, \cdots, \rho_m)$ is a metric on M.

Corollary 2.1. If $\rho_1, \rho_2, \cdots, \rho_m$ are m metrics on a space M, then $\rho_1 + \rho_2 + \cdots + \rho_m$ and $\frac{\rho_1}{1+\rho_1} + \frac{\rho_2}{1+\rho_2} + \cdots + \frac{\rho_m}{1+\rho_m}$ are also metrics on M.

A sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ is said *convergence to a point* $x, x \in \widetilde{M}$ if for any positive number $\epsilon > 0$, there exist numbers N and $i, 1 \le i \le m$ such that if $n \ge N$ then

$$\rho_i(x_n, x) \ < \ \epsilon.$$

If $\{x_n\}$ convergence to a point $x, x \in \widetilde{M}$, we denote it by $\lim_n x_n = x$.

We have a characteristic for convergent sequences in a multi-metric space.

Theorem 2.2. *A sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ is convergent if and only if there exist integers N and $k, 1 \leq k \leq m$ such that the subsequence $\{x_n | n \geq N\}$ is a convergent sequence in (M_k, ρ_k).*

Proof. If there exist integers N and $k, 1 \leq k \leq m$, such that $\{x_n | n \geq N\}$ is a convergent subsequence in (M_k, ρ_k), then for any positive number $\epsilon > 0$, by definition there exists a positive integer P and a point $x, x \in M_k$ such that

$$\rho_k(x_n, x) \ < \ \epsilon$$

if $n \geq \max\{N,\ P\}$.

Now if $\{x_n\}$ is a convergent sequence in the multi-space $\widetilde{M}$, by definition for any positive number $\epsilon > 0$, there exist a point $x, x \in \widetilde{M}$ and natural numbers $N(\epsilon)$ and $k, 1 \leq k \leq m$ such that if $n \geq N(\epsilon)$, then

$$\rho_k(x_n, x) \ < \ \epsilon,$$

that is, $\{x_n | n \geq N(\epsilon)\} \subset M_k$ and $\{x_n | n \geq N(\epsilon)\}$ is a convergent sequence in (M_k, ρ_k).

Theorem 2.3. *Let $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ be a multi-metric space. For two sequences $\{x_n\}, \{y_n\}$ in $\widetilde{M}$, if $\lim_n x_n = x_0$, $\lim_n y_n = y_0$ and there is an integer p such that $x_0, y_0 \in M_p$, then $\lim_n \rho_p(x_n, y_n) = \rho_p(x_0, y_0)$.*

Proof. According to Theorem 2.2, there exist integers N_1 and N_2 such that if $n \geq max\{N_1, N_2\}$, then $x_n, y_n \in M_p$. Whence, we have that

$$\rho_p(x_n, y_n) \leq \rho_p(x_n, x_0) + \rho_p(x_0, y_0) + \rho_p(y_n, y_0)$$

and

$$\rho_p(x_0, y_0) \leq \rho_p(x_n, x_0) + \rho_p(x_n, y_n) + \rho_p(y_n, y_0).$$

Therefore,

$$|\rho_p(x_n, y_n) - \rho_p(x_0, y_0)| \ \leq \ \rho_p(x_n, x_0) + \rho_p(y_n, y_0).$$

For any positive number $\epsilon > 0$, since $\lim_n x_n = x_0$ and $\lim_n y_n = y_0$, there exist numbers $N_1(\epsilon), N_1(\epsilon) \geq N_1$ and $N_2(\epsilon), N_2(\epsilon) \geq N_2$ such that $\rho_p(x_n, x_0) \leq \frac{\epsilon}{2}$ if $n \geq N_1(\epsilon)$ and $\rho_p(y_n, y_0) \leq \frac{\epsilon}{2}$ if $n \geq N_2(\epsilon)$. Whence, if $n \geq max\{N_1(\epsilon), N_2(\epsilon)\}$, then

$$|\rho_p(x_n, y_n) - \rho_p(x_0, y_0)| \ < \ \epsilon.$$

Whether a convergent sequence can has more than one limit point? The following result answers this question.

Theorem 2.4. *If $\{x_n\}$ is a convergent sequence in a multi-metric space $\widetilde{M} = \bigcup_{i=1}^{m} M_i$, then $\{x_n\}$ has only one limit point.*

Proof. According to Theorem 2.2, there exist integers N and $i, 1 \leq i \leq m$ such that $x_n \in M_i$ if $n \geq N$. Now if

$$\lim_n x_n = x_1 \text{ and } \lim_n x_n = x_2,$$

and $n \geq N$, by definition,

$$0 \leq \rho_i(x_1, x_2) \leq \rho_i(x_n, x_1) + \rho_i(x_n, x_2).$$

Whence, we get that $\rho_i(x_1, x_2) = 0$. Therefore, $x_1 = x_2$.

Theorem 2.5. *Any convergent sequence in a multi-metric space is a bounded points set.*

Proof. According to Theorem 2.4, we obtain this result immediately.

A sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ is called a *Cauchy sequence* if for any positive number $\epsilon > 0$, there exist integers $N(\epsilon)$ and $s, 1 \leq s \leq m$ such that for any integers $m, n \geq N(\epsilon)$, $\rho_s(x_m, x_n) < \epsilon$.

Theorem 2.6. *A Cauchy sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ is convergent if and only if for $\forall k, 1 \leq k \leq m$, $|\{x_n\} \bigcap M_k|$ is finite or infinite but $\{x_n\} \bigcap M_k$ is convergent in (M_k, ρ_k).*

Proof. The necessity of these conditions is by Theorem 2.2.

Now we prove the sufficiency. By definition, there exist integers $s, 1 \leq s \leq m$ and N_1 such that $x_n \in M_s$ if $n \geq N_1$. Whence, if $|\{x_n\} \bigcap M_k|$ is infinite and $\lim\limits_n \{x_n\} \bigcap M_k = x$, then there must be $k = s$. Denoted by $\{x_n\} \bigcap M_k = \{x_{k1}, x_{k2}, \cdots, x_{kn}, \cdots\}$.

For any positive number $\epsilon > 0$, there exists an integer $N_2, N_2 \geq N_1$ such that $\rho_k(x_m, x_n) < \frac{\epsilon}{2}$ and $\rho_k(x_{kn}, x) < \frac{\epsilon}{2}$ if $m, n \geq N_2$. According to Theorem 4.7, we get that

$$\rho_k(x_n, x) \leq \rho_k(x_n, x_{kn}) + \rho_k(x_{kn}, x) < \epsilon$$

if $n \geq N_2$. Whence, $\lim\limits_n x_n = x$.

A multi-metric space $\widetilde{M}$ is said *completed* if every Cauchy sequence in this space is convergent. For a completed multi-metric space, we obtain two important results similar to the metric space theory in classical mathematics.

Theorem 2.7. *Let $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ be a completed multi-metric space. For a ϵ-disk sequence $\{B(\epsilon_n, x_n)\}$, where $\epsilon_n > 0$ for $n = 1, 2, 3, \cdots$, the following conditions hold:*

(i) $B(\epsilon_1, x_1) \supset B(\epsilon_2, x_2) \supset B(\epsilon_3, x_3) \supset \cdots \supset B(\epsilon_n, x_n) \supset \cdots$;

(ii) $\lim\limits_{n \to +\infty} \epsilon_n = 0$.

Then $\bigcap\limits_{n=1}^{+\infty} B(\epsilon_n, x_n)$ *only has one point.*

Proof. First, we prove that the sequence $\{x_n\}$ is a Cauchy sequence in $\widetilde{M}$. By the condition (i), we know that if $m \geq n$, then $x_m \in B(\epsilon_m, x_m) \subset B(\epsilon_n, x_n)$. Whence, for $\forall i, 1 \leq i \leq m$, $\rho_i(x_m, x_n) < \epsilon_n$ if $x_m, x_n \in M_i$.

For any positive number ϵ, since $\lim\limits_{n \to +\infty} \epsilon_n = 0$, there exists an integer $N(\epsilon)$ such that if $n \geq N(\epsilon)$, then $\epsilon_n < \epsilon$. Therefore, if $x_n \in M_l$, then $\lim x_m = x_n$. Whence, there exists an integer

N such that if $m \geq N$, then $x_m \in M_l$ by Theorem 2.2. Take integers $m, n \geq \max\{N, N(\epsilon)\}$. We know that

$$\rho_l(x_m, x_n) \ < \ \epsilon_n \ < \ \epsilon.$$

So $\{x_n\}$ is a Cauchy sequence.

By the assumption, $\widetilde{M}$ is completed. We know that the sequence $\{x_n\}$ is convergence to a point $x_0, x_0 \in \widetilde{M}$. By conditions (i) and (ii), we have that $\rho_l(x_0, x_n) \ < \epsilon_n$ if we take $m \to +\infty$. Whence, $x_0 \in \bigcap\limits_{n=1}^{+\infty} B(\epsilon_n, x_n)$.

Now if there a point $y \in \bigcap\limits_{n=1}^{+\infty} B(\epsilon_n, x_n)$, then there must be $y \in M_l$. We get that

$$0 \leq \rho_l(y, x_0) = \lim_n \rho_l(y, x_n) \leq \lim_{n \to +\infty} \epsilon_n = 0$$

by Theorem 2.3. Therefore, $\rho_l(y, x_0) = 0$. By definition of a metric on a space, we get that $y = x_0$.

Let $\widetilde{M}_1$ and $\widetilde{M}_2$ be two multi-metric spaces and $f : \widetilde{M}_1 \to \widetilde{M}_2$ a mapping, $x_0 \in \widetilde{M}_1, f(x_0) = y_0$. For $\forall \epsilon > 0$, if there exists a number δ such that for $forall x \in B(\delta, x_0)$, $f(x) = y \in B(\epsilon, y_0) \subset \widetilde{M}_2$, i.e.,

$$f(B(\delta, x_0)) \subset B(\epsilon, y_0),$$

then we say f is *continuous at point* x_0. If f is connected at every point of $\widetilde{M}_1$, then f is said a *continuous mapping* from $\widetilde{M}_1$ to $\widetilde{M}_2$.

For a continuous mapping f from $\widetilde{M}_1$ to $\widetilde{M}_2$ and a convergent sequence $\{x_n\}$ in $\widetilde{M}_1$, $\lim\limits_n x_n = x_0$, we can prove that

$$\lim_n f(x_n) = f(x_0).$$

For a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ and a mapping $T : \widetilde{M} \to \widetilde{M}$, if there is a point $x^* \in \widetilde{M}$ such that $Tx^* = x^*$, then x^* is called a *fixed point* of T. Denoted by $^{\#}\Phi(T)$ the number of all fixed points of a mapping T in $\widetilde{M}$. If there are a constant $\alpha, 1 < \alpha < 1$ and integers $i, j, 1 \leq i, j \leq m$ such that for $\forall x, y \in M_i$, $Tx, Ty \in M_j$ and

$$\rho_j(Tx, Ty) \leq \alpha \rho_i(x, y),$$

then T is called a *contraction* on $\widetilde{M}$.

Theorem 2.8. *Let* $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ *be a completed multi-metric space and* T *a contraction on* $\widetilde{M}$. *Then*

$$1 \leq^{\#} \Phi(T) \leq m.$$

Proof. Choose arbitrary points $x_0, y_0 \in M_1$ and define recursively

$$x_{n+1} = Tx_n, \quad y_{n+1} = Tx_n$$

for $n = 1, 2, 3, \cdots$. By definition, we know that for any integer $n, n \geq 1$, there exists an integer $i, 1 \leq i \leq m$ such that $x_n, y_n \in M_i$. Whence, we inductively get that

$$0 \leq \rho_i(x_n, y_n) \leq \alpha^n \rho_1(x_0, y_0).$$

Notice that $0 < \alpha < 1$, we know that $\lim\limits_{n\to+\infty} \alpha^n = 0$. Therefore, there exists an integer i_0 such that

$$\rho_{i_0}(\lim_n x_n, \lim_n y_n) = 0.$$

Therefore, there exists an integer N_1 such that $x_n, y_n \in M_{i_0}$ if $n \geq N_1$. Now if $n \geq N_1$, we have that

$$\begin{aligned}
\rho_{i_0}(x_{n+1}, x_n) &= \rho_{i_0}(Tx_n, Tx_{n-1}) \\
&\leq \alpha\rho_{i_0}(x_n, x_{n-1}) = \alpha\rho_{i_0}(Tx_{n-1}, Tx_{n-2}) \\
&\leq \alpha^2\rho_{i_0}(x_{n-1}, x_{n-2}) \leq \cdots \leq \alpha^{n-N_1}\rho_{i_0}(x_{N_1+1}, x_{N_1}).
\end{aligned}$$

and generally, for $m \geq n \geq N_1$,

$$\begin{aligned}
\rho_{i_0}(x_m, x_n) &\leq \rho_{i_0}(x_n, x_{n+1}) + \rho_{i_0}(x_{n+1}, x_{n+2}) + \cdots + \rho_{i_0}(x_{n-1}, x_n) \\
&\leq (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n)\rho_{i_0}(x_{N_1+1}, x_{N_1}) \\
&\leq \frac{\alpha^n}{1-\alpha}\rho_{i_0}(x_{N_1+1}, x_{N_1}) \to 0 (m, n \to +\infty)
\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in $\widetilde{M}$. Similarly, we can prove $\{y_n\}$ is also a Cauchy sequence.

Because $\widetilde{M}$ is a completed multi-metric space, we have that

$$\lim_n x_n = \lim_n y_n = z^*.$$

We prove z^* is a fixed point of T in $\widetilde{M}$. In fact, by $\rho_{i_0}(\lim\limits_n x_n, \lim\limits_n y_n) = 0$, there exists an integer N such that

$$x_n, y_n, Tx_n, Ty_n \in M_{i_0}$$

if $n \geq N + 1$. Whence, we know that

$$\begin{aligned}
0 \leq \rho_{i_0}(z^*, Tz^*) &\leq \rho_{i_0}(z^*, x_n) + \rho_{i_0}(y_n, Tz^*) + \rho_{i_0}(x_n, y_n) \\
&\leq \rho_{i_0}(z^*, x_n) + \alpha\rho_{i_0}(y_{n-1}, z^*) + \rho_{i_0}(x_n, y_n).
\end{aligned}$$

Notice

$$\lim_{n\to+\infty} \rho_{i_0}(z^*, x_n) = \lim_{n\to+\infty} \rho_{i_0}(y_{n-1}, z^*) = \lim_{n\to+\infty} \rho_{i_0}(x_n, y_n) = 0.$$

We get that $\rho_{i_0}(z^*, Tz^*) = 0$, i.e., $Tz^* = z^*$.

For other chosen points $u_0, v_0 \in M_1$, we can also define recursively

$$u_{n+1} = Tu_n, \quad v_{n+1} = Tv_n$$

and get the limit points $\lim_n u_n = \lim_n v_n = w^* \in M_{i_0}, Tu^* \in M_{i_0}$. Since

$$\rho_{i_0}(z^*, u^*) = \rho_{i_0}(Tz^*, Tu^*) \leq \alpha\rho_{i_0}(z^*, u^*)$$

and $0 < \alpha < 1$, there must be $z^* = u^*$.

Similar consider the points in $M_i, 2 \leq i \leq m$, we get that

$$1 \leq^{\#} \Phi(T) \leq m.$$

Corollary 2.2.(Banach) *Let M be a metric space and T a contraction on M. Then T has just one fixed point.*

§3. Open problems for a multi-metric space

On a classical notion, only one metric maybe considered in a space to ensure the same on all the times and on all the situations. Essentially, this notion is based on an assumption that spaces are homogeneous. In fact, it is not true in general.

Multi-Metric spaces can be used to simplify or beautify geometrical figures and algebraic equations. One example is shown in Fig.1, in where the left elliptic curve is transformed to the right circle by changing the metric along x, y-axes and an elliptic equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

to equation

$$x^2 + y^2 = r^2$$

of a circle of radius r.

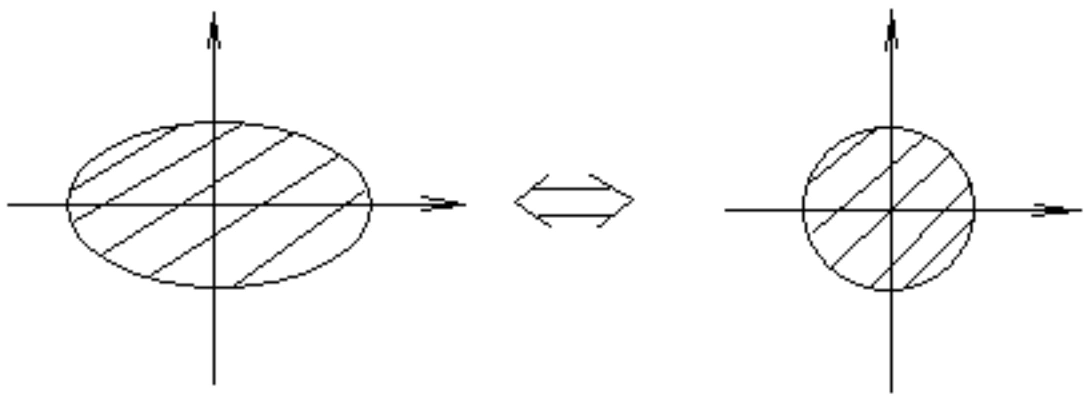

Fig.1

Generally, in a multi-metric space we can simplify a polynomial similar to the approach used in projective geometry. *Whether this approach can be contributed to mathematics with metrics?*

Problem 3.1 *Choose suitable metrics to simplify the equations of surfaces and curves in* $\mathbf{E}^3$.

Problem 3.2 *Choose suitable metrics to simplify the knot problem. Whether can it be used for classifying* 3*-dimensional manifolds?*

Problem 3.3 *Construct multi-metric spaces or non-linear spaces by Banach spaces. Simplify equations or problems to linear problems.*

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