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On the mean value of the Smarandache LCM function $SL(n)$

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Abstract For any positive integer n, let $SL(n)$ denotes the least positive integer k such that $L(k) \equiv 0 \pmod n$, where $L(k)$ denotes the Least Common Multiple of all integers from 1 to k. The main purpose of this paper is to study the properties of the Smarandache LCM function $SL(n)$, and give an asymptotic formula for its mean value.

Keywords Smarandache LCM function, mean value, asymptotic formula.

§1. Introduction

For any positive integer n, we define $SL(n)$ as the least positive integer k such that $L(k) \equiv 0$ (mod n). That is, $SL(n) = \min\{k : n \mid [1, 2, 3, \cdots, k]\}$. For example, $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 4$, $\cdots$, $SL(997) = 997$, $SL(998) = 499$, $SL(999) = 37$, $SL(1000) = 15$, $\cdots$. The arithmetical function $SL(n)$ is called the Smarandache LCM function. About its elementary properties, there are some people studied it, and obtained many interesting results. For example, in reference [1], Murthy showed that if n is a prime, then $SL(n) = S(n) = n$. Simultaneously, he proposed the following problem,

$$SL(n) = S(n), \quad S(n) \neq n? \tag{1}$$

Maohua Le [2] solved this problem completely, and proved that every positive integer n satisfying (1) can be expressed as

$$n = 12 \quad \text{or} \quad n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where p_1, p_2, $\cdots$, p_r, p are distinct primes, and α_1, α_2, $\cdots$, α_r are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1, 2, \cdots, r$.

But about the deeply arithmetical properties of $SL(n)$, it seems that none had studied it before, at least we have not seen such a result at present. It is clear that the value of $SL(n)$ is not regular, but we found that the mean value of $SL(n)$ has good value distribution properties. In this paper, we want to show this point. That is, we shall prove the following conclusion:

Theorem. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} SL(n) = Ax^2 + c_1 \frac{x^2}{\ln x} + c_2 \frac{x^2}{\ln^2 x} + \cdots + c_k \frac{x^2}{\ln^k x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where $A = \frac{1}{2}\sum_{p}\frac{1}{p^2-1}$, k is any fixed positive integer, and c_1, c_2, $\cdots$, c_k are calculable constants.

§2. Proof of the theorem

In this section, we shall complete the proof of Theorem. First we need the following two simple lemmas.

Lemma 1. Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}$ be the factorization of n into prime power, then

$$SL(n) = \max\left(p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots\ p_t^{\alpha_t}\right).$$

Proof. This Lemma can be deduce by the definition of $SL(n)$, see reference [1].

Lemma 2. For any arithmetical function $a(n)$, let $A(x) = \sum_{n\le x} a(n)$, where $A(x) = 0$ if $x < 1$. Assume $f(x)$ has a continuous derivative on the interval $[y,\ x]$, where $0 < y < x$. Then we have

$$\sum_{y<n\le x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt.$$

Proof. This is the famous Abel's identity, its proof see Theorem 4.2 of [2].

Now we complete the proof of our Theorem. We will discuss it in two cases. Let p be the greatest prime divisor of n,

(a) if $n = p\cdot l$, $p > l$, then use Lemma 1 we obtain $SL(n) = p$.

(b) If $n = p\cdot l$, $p \le l$, then we will discuss it in three cases.

(i) If n is complete power of prime, that is $n = p^\alpha$, $\alpha \ge 2$, then $SL(n) \le n$, but the number of this kind n is not exceed $\sqrt{n}$. Thus $\sum_{\substack{n\le x\\ n=p^\alpha\\ \alpha\ge 2}} SL(n) = O\left(x^{\frac{3}{2}}\right)$.

(ii) If n is not complete power of prime, that is $n = l\cdot p^\alpha$, and $l < p^\alpha$, $l \le \sqrt{n}$, then $SL(n) = p^\alpha$. Thus

$$\begin{aligned}
\sum_{\substack{n\le x\\ n=l\cdot p^\alpha\\ l<p^\alpha\\ 2\le\alpha<\frac{\ln x}{\ln 2}}} SL(n) &= \sum_{2\le\alpha<\frac{\ln x}{\ln 2}}\ \sum_{l\le\sqrt{x}}\ \sum_{p^\alpha\le\frac{x}{l}} p^\alpha \\
&= \sum_{2\le\alpha<\frac{\ln x}{\ln 2}}\ \sum_{l\le\sqrt{x}}\ \sum_{p\le\left(\frac{x}{l}\right)^{\frac{1}{\alpha}}} p^\alpha \\
&= \sum_{2\le\alpha<\frac{\ln x}{\ln 2}}\left(\sum_{l\le\sqrt{x}}\frac{\left(\frac{x}{l}\right)^{\frac{1}{\alpha}}}{\ln\left(\frac{x}{l}\right)^{\frac{1}{\alpha}}}\cdot\left(\frac{x}{l}\right)^{\frac{1}{\alpha}\cdot\alpha} + O\left(\frac{x^{1+\frac{1}{\alpha}}}{\ln^2 x}\right)\right) \\
&= \sum_{2\le\alpha<\frac{\ln x}{\ln 2}}\left(\sum_{l\le\sqrt{x}}\frac{\alpha x^{\frac{1+\alpha}{\alpha}}}{l^{\frac{1+\alpha}{\alpha}}\ln\frac{x}{l}} + O\left(\frac{x^{\frac{1+\alpha}{\alpha}}}{\ln^2 x}\right)\right).
\end{aligned}$$

Because $\alpha \geq 2$, so $\frac{\alpha+1}{\alpha} \leq \frac{3}{2}$. We obtain

$$\sum_{\substack{n\leq x\\ n=l\cdot p^{\alpha}\\ l<p^{\alpha}\\ 2\leq\alpha<\frac{\ln x}{\ln 2}}} SL(n) = O\left(x^{\frac{3}{2}}\ln x\right).$$

(iii) If n is not complete power of prime, that is $n = l\cdot p^{\alpha}$, and $l > p^{\alpha}$, $\alpha \geq 1$, $p^{\alpha} \leq \sqrt{n}$, then $SL(n) = l$. Thus

$$\begin{aligned}\sum_{\substack{n\leq x\\ n=l\cdot p^{\alpha}\\ l>p^{\alpha}\\ 1\leq\alpha<\frac{\ln x}{\ln 2}}} SL(n) &= \sum_{1\leq\alpha<\frac{\ln x}{\ln 2}}\sum_{p^{\alpha}\leq\sqrt{x}}\sum_{l\leq\frac{x}{p^{\alpha}}} l\\
&= \sum_{1\leq\alpha<\frac{\ln x}{\ln 2}}\sum_{p\leq x^{\frac{1}{2\alpha}}}\left(\frac{1}{2}\left(\frac{x}{p^{\alpha}}\right)^2 + O\left(\frac{x}{p^{\alpha}}\right)\right)\\
&= \sum_{1\leq\alpha<\frac{\ln x}{\ln 2}}\left(\frac{x^2}{2}\sum_{p\leq x^{\frac{1}{2\alpha}}}\frac{1}{p^{2\alpha}} + O\left(\sum_{p\leq x^{\frac{1}{2\alpha}}}\frac{x}{p^{\alpha}}\right)\right),\end{aligned}$$

where

$$\begin{aligned}\frac{x^2}{2}\sum_{1\leq\alpha<\frac{\ln x}{\ln 2}}\sum_{p\leq x^{\frac{1}{2\alpha}}}\frac{1}{p^{2\alpha}} &= \frac{x^2}{2}\sum_{1\leq\alpha<\frac{\ln x}{\ln 2}}\left(\sum_{p}\frac{1}{p^{2\alpha}} - \sum_{p>x^{\frac{1}{2\alpha}}}\frac{1}{p^{2\alpha}}\right)\\
&= \frac{x^2}{2}\sum_{p}\frac{1}{p^2}\frac{1-\frac{1}{p^{2[\frac{\ln x}{\ln 2}]}}}{1-\frac{1}{p^2}} - \frac{x^2}{2}\sum_{1\leq\alpha<\frac{\ln x}{\ln 2}}\sum_{p>x^{\frac{1}{2\alpha}}}\frac{1}{p^{2\alpha}}\\
&= \frac{x^2}{2}\sum_{p}\frac{1}{p^2-1} + O\left(\sum_{1\leq\alpha<\frac{\ln x}{\ln 2}} x^2\cdot x^{-\frac{2\alpha-1}{2\alpha}}\right)\\
&= 7Ax^2 + O\left(\sum_{1\leq\alpha<\frac{\ln x}{\ln 2}} x^{\frac{2\alpha+1}{2\alpha}}\right).\end{aligned}$$

Using the same method, we deduce that

$$O\left(\sum_{1\leq\alpha<\frac{\ln x}{\ln 2}}\sum_{p\leq x^{\frac{1}{2\alpha}}}\frac{x}{p^{\alpha}}\right) = O(x).$$

Because $\alpha \geq 1$, so $\frac{2\alpha+1}{2\alpha} \leq \frac{3}{2}$. Thus

$$\sum_{\substack{n\leq x\\ n=l\cdot p^{\alpha}\\ l>p^{\alpha}\\ 1\leq\alpha<\frac{\ln x}{\ln 2}}} SL(n) = Ax^2 + O\left(\sum_{1\leq\alpha<\frac{\ln x}{\ln 2}} x^{\frac{3}{2}}\right) = Ax^2 + O\left(x^{\frac{3}{2}}\ln x\right),$$

where $A=\dfrac{1}{2}\sum\limits_{p}\dfrac{1}{p^2-1}$.

Combining the above two cases, we may immediately get following equation:

$$\begin{aligned}\sum_{n\leq x} SL(n) &= \sum_{\substack{n\leq x\\ p>l\\ p>\sqrt{n}}} p+\sum_{\substack{n\leq x\\ p\leq l}} SL(n) \\ &= \sum_{l\leq\sqrt{x}}\sum_{l<p\leq\frac{x}{l}} p+Ax^2+O\left(x^{\frac{3}{2}}\ln x\right).\end{aligned} \tag{2}$$

Let $a(n)$ denote the characteristic function of the prime. That is,

$$a(n)=\begin{cases}1, & \text{if } n \text{ is prime;}\\ 0, & \text{otherwise.}\end{cases}$$

Then we have $A(x)=\sum\limits_{1<n\leq x} a(n)=\sum\limits_{p\leq x} 1=\pi(x)$.

Thus by Lemma 2

$$\sum_{l<p\leq\frac{x}{l}} p=\sum_{l<n\leq\frac{x}{l}} a(n)n=\pi\left(\frac{x}{l}\right)\frac{x}{l}+\pi(l)l-\int_l^{\frac{x}{l}}\pi(t)dt,$$

where $\pi(x)=\frac{x}{\ln x}+A_1\frac{x}{\ln^2 x}+A_2\frac{x}{\ln^3 x}+\ldots+A_n\frac{x}{\ln^n x}+O\left(\frac{x}{\ln^{n+1}x}\right)$, A_i are calculable constants, $i=1,2,\ldots,n$.

Then

$$\begin{aligned}\sum_{l\leq\sqrt{x}}\sum_{l<p\leq\frac{x}{l}} p &= \sum_{l\leq\sqrt{x}}\left(\pi\left(\frac{x}{l}\right)\frac{x}{l}+\pi(l)l-\int_l^{\frac{x}{l}}\pi(t)dt\right) \\ &= \sum_{l\leq\sqrt{x}}\left(\frac{x^2}{l^2\ln\frac{x}{l}}+\sum_{i=1}^{n}\frac{A_i x^2}{l^2\ln^i\frac{x}{l}}+O\left(\frac{x^2}{\ln^{n+1}\frac{x}{l}}\right)-\int_l^{\frac{x}{l}}\pi(t)dt\right),\end{aligned} \tag{3}$$

where

$$\int_l^{\frac{x}{l}}\pi(t)dt=\int_l^{\frac{x}{l}}\frac{t}{\ln t}dt+\sum_{i=1}^{n}\int_l^{\frac{x}{l}}\frac{A_i t}{\ln^{i+1}t}dt+O\left(\int_l^{\frac{x}{l}}\frac{t}{\ln^{n+1}x}dt\right).$$

Integration by parts give us

$$\int_l^{\frac{x}{l}}\frac{t}{\ln t}dt=\frac{x^2}{2l^2\ln\frac{x}{l}}+\frac{x^2}{4l^2\ln^2\frac{x}{l}}+\frac{x^2}{4l^2\ln^3\frac{x}{l}}+\ldots+O\left(\frac{x^2}{\ln^{n+1}x}\right),$$

$$\int_l^{\frac{x}{l}}\frac{t}{\ln^2 t}dt=\frac{x^2}{2l^2\ln^2\frac{x}{l}}+\frac{x^2}{2l^2\ln^3\frac{x}{l}}+\frac{3x^2}{4l^2\ln^4\frac{x}{l}}+\ldots+O\left(\frac{x^2}{\ln^{n+1}x}\right),$$

$$\cdots$$

$$\int_l^{\frac{x}{l}}\frac{t}{\ln^n t}dt=\frac{x^2}{2l^2\ln^n\frac{x}{l}}+\frac{nx^2}{4l^2\ln^{n+1}\frac{x}{l}}+\frac{n(n+1)x^2}{8l^2\ln^{n+2}\frac{x}{l}}+\ldots+O\left(\frac{x^2}{\ln^{2n}x}\right).$$

Combining above formulas, and take them in (3), we obtain

$$\begin{aligned}
\sum_{l\leq\sqrt{x}}\sum_{l<p\leq\frac{x}{l}} p &= \sum_{l\leq\sqrt{x}}\left(B_1\frac{x^2}{l^2\ln\frac{x}{l}}+B_2\frac{x^2}{l^2\ln^2\frac{x}{l}}+\ldots+B_n\frac{x^2}{l^2\ln^n\frac{x}{l}}+O\left(\frac{x^2}{\ln^{n+1}x}\right)\right) \qquad (4)\\
&= \sum_{l\leq\sqrt{x}}\frac{x^2}{l^2}\left[\frac{B_1}{\ln x}\left(\frac{1}{1-\frac{\ln l}{\ln x}}\right)+\frac{B_2}{\ln^2 x}\left(\frac{1}{1-\frac{\ln l}{\ln x}}\right)^2+\ldots+\right.\\
&\qquad \left.\frac{B_n}{\ln^n x}\left(\frac{1}{1-\frac{\ln l}{\ln x}}\right)^n+O\left(\frac{1}{\ln^{n+1}x}\left(\frac{1}{1-\frac{\ln l}{\ln x}}\right)^{n+1}\right)\right]\\
&= \sum_{l\leq\sqrt{x}}\frac{x^2}{l^2}\left[B_1\left(\frac{1}{\ln x}+\frac{\ln l}{\ln^2 x}+\frac{\ln^2 l}{\ln^3 x}+\cdots+O\left(\frac{1}{\ln^n x}\right)\right)\right.\\
&\qquad +B_2\left(\frac{1}{\ln x}+\frac{\ln l}{\ln^2 x}+\frac{\ln^2 l}{\ln^3 x}+\cdots+O\left(\frac{1}{\ln^n x}\right)\right)^2\\
&\qquad +B_3\left(\frac{1}{\ln x}+\frac{\ln l}{\ln^2 x}+\frac{\ln^2 l}{\ln^3 x}+\cdots+O\left(\frac{1}{\ln^n x}\right)\right)^3+\ldots+\\
&\qquad \left.B_k\left(\frac{1}{\ln x}+\frac{\ln l}{\ln^2 x}+\frac{\ln^2 l}{\ln^3 x}+\cdots+O\left(\frac{1}{\ln^n x}\right)\right)^k+O\left(\frac{1}{\ln^{k+1}x}\right)\right].
\end{aligned}$$

where B_i are constants, $i=1,2,\ldots,n$.

We know that $\sum_{l=1}^{\infty}\frac{\ln^k l}{l^2}=c$, thus

$$\sum_{l\leq\sqrt{x}}\frac{\ln^k l}{l^2}=\sum_{l=1}^{\infty}\frac{\ln^k l}{l^2}-\sum_{l>\sqrt{x}}\frac{\ln^k l}{l^2}=c-O\left(\frac{\ln^k x}{\sqrt{x}}\right).$$

In (4) every coefficient of $\frac{x^2}{\ln^k x}$ is calculable, where k is any fixed positive integer.

So we obtain that

$$\sum_{l\leq\sqrt{x}}\sum_{l<p\leq\frac{x}{l}} p = c_1\frac{x^2}{\ln x}+c_2\frac{x^2}{\ln^2 x}+\cdots+c_k\frac{x^2}{\ln^k x}+O\left(\frac{x^2}{\ln^{k+1}x}\right).$$

Combining this formula with (2), we obtain

$$\begin{aligned}
\sum_{n\leq x} SL(n) &= \sum_{l\leq\sqrt{x}}\sum_{l<p\leq\frac{x}{l}} p+Ax^2+O\left(x^{\frac{3}{2}}\ln x\right)\\
&= Ax^2+c_1\frac{x^2}{\ln x}+c_2\frac{x^2}{\ln^2 x}+\cdots+c_k\frac{x^2}{\ln^k x}+O\left(\frac{x^2}{\ln^{k+1}x}\right),
\end{aligned}$$

where $A=\frac{1}{2}\sum_{p}\frac{1}{p^2-1}$, k is any fixed positive integer, and c_1, c_2, $\cdots$, c_k are calculable constants.

This completes the proof of the theorem.

References

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