NEW SMARANDACHE SEQUENCES: THE FAMILY OF METALLIC MEANS

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ABSTRACT

The family of Metallic Means comprises every quadratic irrational number that is the positive solution of algebraic equations of the types

$$x^2 - nx - 1 = 0 \qquad \text{and} \qquad x^2 - x - n = 0,$$

where n is a natural number. The most prominent member of this family is the Golden Mean, then it comes the Silver Mean, the Bronze Mean, the Nickel Mean, the Copper Mean, etc. All of them are closely related to quasi-periodic dynamics, being therefore important clues in the study of the onset to chaos. However, they also constitute the basis of musical and architectural proportions. Through the analysis of their common mathematical properties, it becomes evident that they interconnect different human fiels of knowledge, in the sense defined by Florentin Smarandache ("Paradoxist Mathematics").

Keywords: continued fractions, quadratic irrationals, Fibonacci sequences, Smarandache sequences, hyperbolic map.

1. INTRODUCTION

Let us introduce a new family of positive quadratic irrational numbers. The family is called the **"Metallic Means Family"** **(MMF)** . Its members have, among other common characteristics, the one of carrying the name of a metal (see [1], [2]) . E.g., the most distinguished member is the well known "Golden Mean". Then, we have the Silver Mean, the Bronze Mean, the Copper Mean, the Nickel Mean and many others.

The Golden Mean has been widely utilized by a great quantity of ancient cultures as basis of proportions to compose music, to make sculptures and paintings or construct temples and palaces (in Reference [3], see the first chapter dedicated to this subject). With respect to the many relatives of the Golden Mean, a great part of them have been used by physicists in different researchs, in trying to systematize the behavior of non linear dynamical systems that suffer the transition from periodicity to quasi-periodicity. Notwithstanding, there other instances of using these relatives in quite different fields:
Jay Kappraff [4] appealed to the Silver Mean to describe and explain the roman system of proportions, making use of a mathematical property of this Mean that is, as we are going to prove, common to all the members of this curious family.

Being irrational numbers the members of the MMF, in the applications to different scientific disciplines, they have to be approximated by ratios of integer numbers and the analysis of the relation between the MMF and the approximant ratios is one of the goals of this paper. A direct consequence of this study will be the possibility of interconnecting quite distinct (sometimes opposite) human fields of knowledge, in the sense defined by Florentin Smarandache ("Paradoxist Mathematics").

2. CONTINUED FRACTIONS EXPANSIONS

Every real number x admits a continued fraction expansion, that is, an expression of the type

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

that is written $x = [a_0, a_1, a_2, \ldots]$. The first coefficient can be zero (in such a case the real number is between 0 and 1) but the rest of the coefficients are positive integers. This continued fraction expansion is <u>finite</u> if and only if x is a rational number (that is, a number of the form p/q with q different from zero and p,q natural numbers without common factors). For example,

$$\frac{18}{7} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}} = [2,1,1,3].$$

If x is an irrational number, the expansion is infinite and if we take a finite number of terms like

$$\sigma_k = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_k}}}$$

we get a sequence of *"rational approximants"* to the number x such that they converge to x when $k \to \infty$.

Some irrational numbers, like π and e have approximants that converge very quickly. In particular, the number $\pi = [3, 7, 15, 1, 292, \ldots]$ converges so quickly that the third rational approximant $\sigma_3 = \frac{335}{113} = 3{,}1415929\ldots$ has six exact decimals!

Amazingly, this result was already known by Tsu Chung Chi in China, 5th century!. Instead, the base of the napierian logarithms, the number $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 2, 2, 8, 1, ...]$ converges more slowly at the beginning, due to the pressence of many 'ones' in its expansion. Comparatively, the quadratic irrationals converge much slower.

Similarly to the periodic decimal expansions, the *"periodic"* continued fractions are denoted with a line over the period and if the continued fraction expansion is of the form $x = [\overline{a_0, a_1, ..., a_n}]$, we say that the continued fraction is *"purely periodic"*. In this context, the french mathematician Joseph Louis Lagrange (1736-1813) proved that *a real number is a quadratic irrational if and only if its continued fraction expansion is periodic (not necessarily purely periodic).* This result was improved by Evariste Galois (1811-1832) in the following form: *The continued fraction of an irrational number x is purely periodic if and only if x > 1 and it is a root of a second degree equation with integer coefficients, the other root being between* -1 *and* 0.

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They are all positive quadratic irrationals.

In fact, if we take the quadratic equation

$$x^2 - n\,x - 1 = 0 \tag{2.1}$$

where n is a natural number and solve it, we find that the positive solutions of this equation are of the form

$$x = \frac{n + \sqrt{n^2 + 4}}{2}$$

For $n = 1$, the result is the well known Golden Mean $\phi = \dfrac{1 + \sqrt{5}}{2} = 1.618....$ To find the continued fraction expansions of this quadratic irrationals, simply we take equation (2.1) and divide it by x (different from zero):

$$x = n + \frac{1}{x}.$$

Then, we replace the x of the second member iteratively by $n + 1/x$. In this way, we get, after N iterations:

$$x = n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{\cdots + \cfrac{1}{n + \cfrac{1}{x}}}}}.$$

If $N \to \infty$, we have

$$x = n + \cfrac{1}{n + \cfrac{1}{n + \cdots}} = [\overline{n}],$$

a purely periodic continued fraction expansion.

Obviously, the Golden Mean has the most simple continued fraction expansion

$$\phi = [\,\overline{1}\,].$$

For $n = 2$, we have the Silver Mean $\sigma_{Ag} = 1 + \sqrt{2}$, which continued fraction expansion is

$$\sigma_{Ag} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}} = [\,\overline{2}\,].$$

For $n = 3$, the result is the Bronze Mean

$$\sigma_{Br} = \frac{3 + \sqrt{13}}{2} = [\,\overline{3}\,].$$

Summarizing, solving quadratic equations of the form

$$x^2 - nx - 1 = 0$$

with n natural, we obtain as positive solutions, the members of the MMF, which continued fraction expansion is purely periodic

$$x = [\,\overline{n}\,].$$

Instead, if we solve quadratic equations of the form

$$x^2 - x - n = 0, \tag{2.2}$$

with n natural, we obtain members of the MMF which continued fraction expansion is periodic, not necessarily purely periodic, e.g.

$$[m, \overline{n_1, n_2, \ldots, n_n}].$$

This last subset of Metallic Means has curious mathematical properties, with reference to the frequence of apparition of the natural numbers, as well as to the length of the period or the presence of *"stable cycles"* (see Reference [1] for more details).

Obviously, of all these Metallic Means, the one that converges more slowly is the Golden Mean, since all the denominators are the smallest possible -- ones. This fact allows us to state the following

The Golden Mean ϕ is the most irrational of all irrational numbers.

Note: In the restant posible cases of quadratic equations with integer coefficients, we find the following results, looking for positive solutions

a) $x^2 + n\,x - 1 = 0$. Same solutions as for equation (2.1), but only their decimal part.
b) $x^2 + n\,x + 1 = 0$. There are no positive solutions.
c) $x^2 - n\,x + 1 = 0$. The positive solutions have periodic continued fraction expansions.
d) $x^2 + x - n = 0$. The positive solutions have periodic continued fraction expansions.
e) $x^2 + x + n = 0$. There are no positive solutions.
f) $x^2 - x + n = 0$. There are no positive solutions.

3. FIBONACCI SEQUENCES

The Fibonacci sequence is a sequence of natural numbers formed by taking each number equal to the sum of the two precedent terms. For this reason, this type of sequences is called a *"secondary Fibonacci sequence"*, to distinguish them from the ternary Fibonacci sequences, in which each term is a linear combination of the three precedent terms.

Beginning with $F(0) = 1$; $F(1) = 1$, we have the following secondary Fibonacci sequence

(3.1) $$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

where

(3.2) $$F(n \div 1) = F(n) \div F(n - 1).$$

Secondary Fibonacci sequences can be generalized, originating what is known as *"generalized secondary Fibonacci sequences"* GSFS, like

$$a,\ b,\ pb - qa,\ p\,(pb - qa) - qb,\ \ldots$$

that satisfy relations of the type

(3.3) $$\boxed{G(n+1) = p\,G(n) - q\,G(n-1)}$$

with p and q natural numbers.

From equation (3.3), we get

$$\frac{G(n+1)}{G(n)} = p + q\frac{G(n-1)}{G(n)} = p + \frac{q}{\dfrac{G(n)}{G(n-1)}}.$$

Taking limits in both members of this equation and assuming that $\lim\limits_{n\to\infty}\dfrac{G(n+1)}{G(n)}$ exists and is equal to a real number x -- fact that will be proved in next theorem--, we have

$$x = p + \frac{q}{x}$$

or $x^2 - px - q = 0$, which positive solution is

$$x = \frac{p + \sqrt{p^2 + 4q}}{2}.$$

This means that

(3.4) $$\boxed{\lim_{n\to\infty}\frac{G(n+1)}{G(n)} = \frac{p + \sqrt{p^2+4q}}{2}}$$

Now, let us prove the existence of this limit:

<u>Theorem</u>

Given a generalized secondary Fibonacci sequence (GSFS)

$$a,\ b,\ pb + qa,\ p(pb + qa) + qb,\ \ldots$$

such that

$$G(n+1) = p\,G(n) + q\,G(n-1)$$

with p, q natural numbers, then there exists $\lim\limits_{n\to\infty}\dfrac{G(n+1)}{G(n)}$ and is a real positive number σ.

<u>Proof</u>: To find the nth term of the GSFS, let us put

$$G(n+1) = p\,G(n) + qH(n)$$

$$H(n+1) = G(n)$$

and

$$\overline{G(n)} = \begin{bmatrix} G(n) \\ H(n) \end{bmatrix}; A = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}.$$

Then it is easy to prove that

$$\overline{G(n+1)} = A.\overline{G(n)}.$$

Let us assume that $G(0) = G(1) = 1$ for simplicity. If $\overline{G(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then $\overline{G(n+1)} = A^n.\overline{G(1)}$ and the problem is reduced to the finding of the nth power of the matrix A. We know that the eigenvalues of A are

$$\sigma = \frac{p + \sqrt{p^2 + 4q}}{2};\ \sigma' = \frac{p - \sqrt{p^2 + 4q}}{2}.$$

To diagonalize A so as to transform it in $A_d = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma' \end{bmatrix}$, we shall use the change of base matrix $P = \begin{bmatrix} \sigma & \sigma' \\ 1 & 1 \end{bmatrix}$. The nth power of A is calculated applying the similarity transformation

$$A^n = P.A_d^n.P^{-1} = \frac{1}{\sigma - \sigma'} \begin{bmatrix} \sigma^{n+1} - \sigma'^{(n+1)} & \sigma\sigma'(\sigma'^n - \sigma^n) \\ \sigma^n - \sigma'^n & \sigma\sigma'(\sigma'^{(n-1)} - \sigma^{n-1}) \end{bmatrix}.$$

and the nth term of the GSFS

$$1, 1, p + q, p(p + q) + q, \ldots$$

is given by the following expression

$$G(n+1) = \frac{\sigma^{n+2} - \sigma'^{(n+2)}}{\sigma - \sigma'}.$$

Replacing $\sigma - \sigma' = \sqrt{p^2 + 4q}$; $\sigma' = -\frac{q}{\sigma}$ we have

$$\lim_{n\to\infty}\frac{G(n+1)}{G(n)}=\lim_{n\to\infty}\frac{\sigma^{n+1}+\left(\frac{q}{\sigma}\right)^{n+1}}{\sigma^{n}+\left(\frac{q}{\sigma}\right)^{n}}=\sigma$$

and the proof is completed.

Note: if instead of choosing $G(0) = G(1) = 1$ we begin the GSFS with two arbitrary values a and b, it is easy to prove that the result is the same. Indeed, given the GSFS

$$a,\ b,\ pb \div qa,\ p\,(pb \div qa) + qb,\ \dots$$

we have to evaluate the quotient

$$\frac{G(n+1)}{G(n)}=\frac{pbG(n)+qaG(n-1)}{pbG(n-1)+qaG(n-2)}=\frac{pb\dfrac{G(n)}{G(n-1)}+qa}{pb+\dfrac{qa}{\dfrac{G(n-1)}{G(n-2)}}}\xrightarrow[n\to\infty]{}\sigma.$$

Let us put $G(0) = G(1) = 1$ and consider different possibilities for the coefficients of (3.4). Then, if $p = q = 1$, we have the Golden Mean

$$x=\frac{1+\sqrt{5}}{2}=\phi=[\overline{1}].$$

If $p = 2$ and $q = 1$, the sequence has the form

(3.5) $$1, 1, 3, 7, 17, 41, 99, 140, \dots$$

where

(3.6) $$G(n \div 1) = 2\,G(n) \div G(n - 1),$$

and from (3.4) we get the Silver Mean

$$\sigma_{Ag}=\lim_{n\to\infty}\frac{G(n+1)}{G(n)}=[\overline{2}\,].$$

Analogously, if $p = 3$ and $q = 1$, the sequence is

(3.7) $$1, 1, 4, 13, 43, 142, 469, \dots$$

where

(3.8) $$G(n+1) = 3\,G(n) + G(n-1),$$

and we get the Bronze Mean

$$\sigma_{Br} = \lim_{n\to\infty} \frac{G(n+1)}{G(n)} = \frac{3+\sqrt{13}}{2} = [\bar{3}],$$

If $p = 1$ and $q = 2$, the sequence is

(3.9) $$1, 1, 3, 5, 11, 21, 43, 85, \ldots$$

where

$$G(n+1) = G(n) + 2\,G(n-1)$$

and we get the Copper Mean

$$\sigma_{Cu} = 2 = [\,2, \bar{0}\,].$$

If $p = 1$ and $q = 3$, the sequence is

(3.10) $$1, 1, 4, 7, 19, 40, 97, \ldots$$

where

$$G(n+1) = G(n) + 3\,G(n-1)$$

and we get the Nickel Mean

$$\sigma_{Ni} = \lim_{n\to\infty} \frac{G(n+1)}{G(n)} = \frac{1+\sqrt{13}}{2} = [2, \bar{3}].$$

Summarizing our results, we may affirm

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All of them are obtained as limits of ratios of two consecutive terms of generalized secondary Fibonacci sequences.

4. ADDITIVE PROPERTIES

Let us form now the sequence of ratios of consecutive terms of the sequence (3.1)

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \dots \tag{4.1}$$

Obviously, this sequence converges directly to the Golden Mean ϕ. This sequence is very useful as a good approximation: indeed the term u(11) = 233/144 = 1.6180 with four exact decimals!

If we take now a geometric progression of ratio ϕ such as

$$\dots, \frac{1}{\phi^2}, \frac{1}{\phi}, 1, \phi, \phi^2, \phi^3, \dots$$

we can easily verify that this geometric progression is also a GSFS. In fact

$$\frac{1}{\phi^2} + \frac{1}{\phi} = \frac{1+\phi}{\phi^2} = 1.$$

The same happens for the Silver Mean σ_{Ag}, starting from the sequence

$$\frac{1}{1}, \frac{3}{1}, \frac{7}{3}, \frac{17}{7}, \frac{41}{17}, \frac{99}{41}, \frac{140}{99}, \dots, \tag{4.2}$$

that converges to σ_{Ag}. The sequence

$$\dots \frac{1}{\sigma_{Ag}^2}, \frac{1}{\sigma_{Ag}}, 1, \sigma_{Ag}, \sigma_{Ag}^2, \sigma_{Ag}^3, \dots$$

is a geometric progression of ratio σ_{Ag} that satisfies condition (3.6). Indeed

$$\frac{1}{\sigma_{Ag}} + 2 = \sigma_{Ag}; 1 + 2\sigma_{Ag} = \sigma_{Ag}^2; \sigma_{Ag} + 2\sigma_{Ag}^2 = \sigma_{Ag}^3; \dots$$

Similarly, it is easy to prove that the sequence of ratios

$$\frac{1}{1}, \frac{4}{1}, \frac{13}{4}, \frac{43}{13}, \frac{142}{43}, \frac{469}{142}, \dots, \tag{4.3}$$

converges to the Bronze Mean $\sigma_B = \dfrac{3+\sqrt{13}}{2} = [\bar{3}]$ and the sequence

$$\cdots \frac{1}{\sigma_B^2}, \frac{1}{\sigma_B}, 1, \sigma_B, \sigma_B^2, \sigma_B^3, \cdots$$

is a geometric progression of ratio σ_B that satisfies condition (3.6). This is due to the fact that

$$\frac{1}{\sigma_B} + 3 = \sigma_B;\ 1 + 3\sigma_B = \sigma_B^2;\ \sigma_B + 3\sigma_B^2 = \sigma_B^3;\ \cdots$$

Similarly for all GSFS. These numerical sequences (4.1), (4.2), (4.3), and so on, are new Smarandache sequences that have to be empirically used as approximations to the values of the members of the MMF. Furthermore, the sequences formed by taking these members as ratios enjoy the following unique mathematical property:

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They are the only positive quadratic irrational numbers that originate GSFS (with additive properties) which are, simultaneously, geometric progressions.

This curious property of satisfying both arithmetic additive and geometric properties, bestow all the members of the MMF with interesting characteristics to become basis of different systems of geometric proportions in Design.

5. PROPORTIONS SYSTEMS

The golden Mean $\phi = \frac{1+\sqrt{5}}{2}$, is indissolubly linked to pentagonal symmetry. Indeed, if we take a regular pentagon of unitary edge, like the one depicted in Fig. 5.1, it is easy to prove that its diagonal is equal to ϕ. Considering the geometric similarity of the two isosceles triangles ADC and ABF we have

$$\frac{AD}{DC} = \frac{DC}{AD - FD}$$

Being $DC = FD = 1$ and calling $x = AD$, we obtain the quadratic equation $x\,(x - 1) = 1$ or $x^2 - x - 1 = 0$, that is equation (2.1) with $n = 1$ and positive solution $x = \phi$. It is not difficult to prove besides the following *"golden relations"* in the regular pentagon

$$GB = \phi - 1 = \frac{1}{\phi} = 0{,}618...$$

$$GI = FG = 1 - \frac{1}{\phi}$$

$$FG = \frac{1}{\phi^2} = 0{,}382...$$

$$JG = \frac{1}{\phi^3} = 0{,}236...$$

These *"golden relations"* determine, for example, the proportions of the ancient mask of Hermes (Medusa), shown in Fig. 5.2. It is a wonderful Roman marble after Greek original, 1st century BC. , pertaining to the artistic collection of the Glyptothek, Munich, Germany.

Innumerable are the references to the apparition of the Golden Mean ϕ in the proportion systems adopted by antique civilizations in their constructions, as well as its presence in the human body proportions and in Botany. Among the many authors that have dedicated their researchs to this subject, we have to mention Matila Ghyka [5], [6] and [7], H. E. Huntley [8] and Theodore Andrea Cook, whose book [9], published in 1979, is a reprint of the original published by Constable, London, England, as early as 1914.

Instead, the Silver Mean is linked to octogonal symmetry, as it is shown in Fig. 5.3. *"Silver relations"* have been found in many examples, coming from quite different fields of human knowledge. In particular, the mathematician Jay Kappraff [4], at the conference *Nexus'96: Relations between Architecture and Mathematics*, that took place in Fucecchio (province of Florence) in June 1996, carried out a carefully analysis of the three architectonic proportion systems presented by P. H. Scholfield in his excellent book [10]. These three proportion systems are the following

1) the system of musical proportions used during the Italian Renaissance, developed by Leon Battista Alberti [11];
2) the Modulor created by the twentieth-century architect Le Corbusier [12] and
3) the Roman proportion system.

The musical system was based on rational proportions inherent in the musical scale. Although it succeeded in creating harmonic relationships in which key proportions were repeated in a design, this system did not have the additive properties necessary for a successful proportion system. Notwithstanding, the very well known contemporary Modulor that is based on the Golden Mean ϕ , and the ancient Roman proportion system, based on the Silver Mean, both conform to the relationships inherent in the system of musical proportions, with the great advantage of having additive properties.

Unlike the Renaissance system, which used a static sequence of commensurable reatios to proportion the length, width and height of rooms, Le Corbusier's system developed a scale of lengths based on the irrational number ϕ , through a GSFS and geometric sequence:

$$\ldots, \frac{a}{\phi^2}, \frac{a}{\phi}, a, a\phi, a\phi^2, a\phi^3, \ldots$$

for some convenient unit a, directly determined by ergonomic reasons. In general, the ratios involved in this system are incommensurable and Le Corbusier, in his designs, used an integer GSFS approximation, that is a Smarandache sequence. More details about this proportion system may be consulted in References [13] and [14].

Now, we are going to consider in detail the third proportion system. With this purpose, let us consider a couple of sequences

$$\begin{array}{lccccccccccc} & 1 & & 3 & & 7 & & 17 & & 41 \ldots & \\ (5.1) & 1 & 2 & & 5 & & 12 & & 29 & & 70 \ldots \end{array}$$

such that

$$(5.2) \qquad A(n+2) = 2\,A(n+1) + A(n).$$

These sequences satisfy three additive fundamental properties: in addition to relation (5.2) they obey the following numerical relations

$$\begin{gathered} 7 = 2.3 + 1;\ 17 = 2.7 + 3;\ \ldots \\ 5 = 2.2 + 1;\ 12 = 5.2 + 1;\ \ldots \end{gathered}$$

and

$$\begin{gathered} 2 + 5 = 7;\ 5 + 12 = 17;\ 12 + 29 = 41;\ \ldots \\ 2 + 3 = 5;\ 5 + 7 = 12;\ 12 + 17 = 29;\ 29 + 41 = 70;\ \ldots \end{gathered}$$

Furthermore,the ratios of diagonally adjacent terms of the sequences (5.1) are aproximants to $\sqrt{2}$

$$(5.3) \qquad \frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \cdots \to \sqrt{2}.$$

But since the sum of any couple of numbers of the upper sequence, is not represented in this system, we may expand it adding a third sequence obtained by duplicating the terms of the lower sequence

$$\begin{array}{lccccccccccc} & & 2 & & 4 & & 10 & & 24 & & 58 \ldots \\ (5.4) & 1 & & 3 & & 7 & & 17 & & 41 \ldots & \\ & 1 & 2 & & 5 & & 12 & & 29 & & 70 \ldots \end{array}$$

Finally, the Roman architectonic system utilizes the following incommensurable schema based on the Silver Mean, which is equivalent to the commensurable system (5.4)

$$
\begin{array}{ccccccccccc}
 & & & 2\sqrt{2} & & 2\sqrt{2}\,\sigma_{Ag} & & 2\sqrt{2}\,\sigma_{Ag}^{2} & & 2\sqrt{2}\,\sigma_{Ag}^{3} & \cdots \\
 & & 2 & & 2\sigma_{Ag} & & 2\sigma_{Ag}^{2} & & 2\sigma_{Ag}^{3} & & \cdots \\
 & \sqrt{2} & & \sqrt{2}\,\sigma_{Ag} & & \sqrt{2}\,\sigma_{Ag}^{2} & & \sqrt{2}\,\sigma_{Ag}^{3} & & \cdots & \\
1 & & \sigma_{Ag} & & \sigma_{Ag}^{2} & & \sigma_{Ag}^{3} & & \cdots & &
\end{array}
\tag{5.5}
$$

This system holds all the additive relations of sequences (5.4), as it is easy to prove. Donald and Carol Watts [15], a couple of american architects, have carefully studied the ruins of the Garden Houses at Ostia, the city-port of the Roman Empire and they found that all these houses have been designed using theoretically the proportion system (5.5) and practically, its integer approximation (5.4). These are not the only examples of the antiquity where the Silver Mean is present, since the italian-american architect Kim Williams has found similar results while surveying:
1) the pavement of the baptistery of San Giovanni, Florence, Italy [16],
2) Verrocchio's Tombslab for Cosimo de' Medici, patriarch of the wealthiest of Florentine families [17] and
3) the famous Medici Chapel in Florence, Italy, built by Michaelangelo [18].

6. FRACTAL STRUCTURES OF ST. GEORGE

Alan St. George is a British retired architect, living in Portugal and dedicated to the creation of mathematical sculptures. In december 1995 he presented at Lisboa his exposition *"La forma del número"* [19]. His originals are fabricated with acrylic or metallic plates and they can be reproduced by computerized graphics. The generation of these original structures is based on the **fractal** principle of adding to each one of the five platonic solids -- tetrahedra, cube or hexahedra, octahedra, dodecahedra, icosahedra -- reduced versions of the same solid. In such a way, adding in each iteration auto-similar versions of the original structure, the result are fractal variations of regular solids.

For example, to convert a cube in a fractal octahedra, we begin with a cube which faces are divided in nine equal squares, as indicated in Fig. 6.1. Then, we bild a cross with six smaller cubes, which faces are of the size of the above mentioned squares. Five of these cubes are located in form of a *"greek cross"* and the sixth is put over the central cube, forming a sort of stepping pyramid. The construction goes on sticking one of such units over each face of the original cube. Then, each of the faces of the resulting structure is subdivided in nine even smaller squares, over which we stick more reduced copies of the stepping pyramid.

It is also possible to fractalize an octahedra and obtain a tetrahedra or a cube, like the mathematician Ian Stewart suggested in an interesting paper [20]. And why not? It would also be feasible to apply this fractalization process to semi-regular solids, a task that has not been focussed yet ...

Another variant of St. George consists in constructing three-dimensional spirals, starting also from the five platonic solids. In particular, let us consider the icosahedra of pentagonal symmetry (Fig. 6.2), which main characteristics we detail in what follows

Faces: 20 Vertices: 12 Edges: 30

Edge length: 1

Distance from the polyhedra centre to the face centre: $\phi^2 / 2\sqrt{3} = 0,7558...$

Distance from the polyhedra centre to the edge mid-point: $\phi / 2 = 0,8090...$

Distance from the polyhedra centre to a vertex: $\sqrt[4]{5}\sqrt{\phi}/2 = 0,9511...$

Volumen: $5\phi^2 / 6 = 2,1817...$

Starting with an icosahedra, it is possible to construct the so called *"icosahedrical spiral"*, following a path that passes through the twelve triangular edges of the icosahedra, visiting each vertex once and only once (Fig. 6.3). The construction is fulfilled by means of a sequence of "legs" ,which correspond to the twelve edges of the icosahedra. Each leg is connected to the previous one and is parallel to an edge. But the successive legs have different lengths: each of them has $\phi^{1/12} = 1,040916...$ times the length of its antecessor. The answer to the question: why this strange figure?, is that after having added twelve edges to a given one, the last edge is parallel to the original, having increased its length in $(\phi^{1/12})^{12} = \phi$.

Obviously, the choice of the Golden Mean ϕ in the construction of the icosahedrical spiral of St. George, obeys to mathematical as well as purely aesthetic reasons. In any case, it is impossible to deny the underlying mathematical reality inherent to a pentagonal symmetry so directly related to the Golden Mean ...

7. INFLATIONARY SYSTEM

We may consider that the terms of the different GSFS that define the Metallic Means family, can be ordered in generations in such a way that each generation "inherits" a property from his antecessor. This type of inheritance is completely normal in iterative processes and frequently, produces auto-similar structures that are the base of fractal configurations [20]. Let us denote such processes as *"inflationary"*, using an usual noun in Economy.

Let us consider two types of building blocks A and B that are distributed according to the inflation schema

$$S_{p+1} = S_{p-1}{}^{m}\, S_{p}{}^{n}$$

where m and n are integers; $p \geq 2$. $S_L{}^m$ represents m adjacent repetitions of the stack S_L.

It is easily proved that the Golden Mean ϕ is generated by the recurrence relation

It is easily proved that the Golden Mean ϕ is generated by the recurrence relation

$$S_{p+1} = S_{p-1}\, S_p\,,$$

that is,

$$S_1 = \{A\}; S_2 = \{BA\}; S_3 = \{ABA\}; S_4 = \{BAABA\};\ldots$$

in which each term is the "sum" of its two immediate antecessors.

The Silver Mean, instead, is generated by the recurrence relation

$$S_{p+1} = S_{p-1}\, S_p^{\,2}\,,$$

$$S_1 = \{A\}; S_2 = \{BA\}; S_3 = \{ABABA\}; S_4 = \{BAABABAABABA\};\ldots$$

such that each term of the chain is formed by writing contiguously two replicas of the precedent term and adding its antecessor to the left of the replicas.

In the case of the Bronze Mean, the relation is

$$S_{p+1} = S_{p-1}\, S_p^{\,3}\,,$$

$$S_1 = \{A\}; S_2 = \{BA\}; S_3 = \{ABABABA\}; S_4 = \{BAABABABAABABABAABABABA\};\ldots$$

For the Copper Mean, we have the relation

$$S_{p+1} = S_{p-1}^{\,2}\, S_p$$

$$S_1 = \{B\}; S_2 = \{A\}; S_3 = \{BBA\}; S_4 = \{AABBA\};\ldots$$

And for the Nickel Mean

$$S_{p+1} = S_{p-1}^{\,3}\, S_p$$

$$S_1 = \{B\}; S_2 = \{A\}; S_3 = \{BBBA\}; S_4 = \{AAABBBA\};\ldots$$

Finally, we may assert

PROPERTY Nr. 4 OF THE METALLIC MEANS FAMILY

All the members of this family are obtained through an "inflationary schema" that produces a binary chain originated by two primitive blocks A and B that are distributed according to the inflation schema

$$S_{p+1} = S_{p-1}^{\ m}\ S_p^{\ n}$$

where m and n are integers and $p \geq 2$.

8. THE HYPERBOLIC MAP

In analyzing dynamical systems -- that is, physical systems which behavior changes with time -- it is crucial to detect periodic orbits. This periodic behavior, as well as the transition to quasi-periodic orbits, is mathematically studied considering irrational values of some characteristic parameter and, in such a case, as the important fact is the *"irrationality"* of such a value, the integer part is omitted and only the decimal part of the number is taken into account. More precisely, the main subject is restricted to the analysis of **maps** (transformations) of the unitary interval (0,1) in itself.

Returning to the continued fraction expansion, there is another possibility of expressing the continued fraction expansion of a positive real number $\alpha < 1$. Let us put $x_1 = \frac{1}{\alpha}$ and apply the iterative process described by the following relation

(8.1) $$x_n = \frac{1}{mant\ x_{n-1}}$$

where *mant* x means *"mantissa of* x*"* and is the rest of the number x when it is taken modulo 1, that is, when one substracts as many times 1 as possible.

E.g. $mant\ \pi = 0{,}1416...$; $mant\ \phi = 0.618...$

Then we may state that the continued fraction expansion of the number α is

$$[\lfloor x_0 \rfloor, \lfloor x_1 \rfloor, \ldots]$$

where $\lfloor x_i \rfloor$, the so called *"floor function"* by Manfred Schroeder [21], is the biggest integer not greater than x_i.

Notice that:

$$mant\ \phi = 1/\phi$$

or

$$mant\ \phi = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}} = [0,1,1,\ldots] = [0,\bar{1}].$$

The iterative process (8.1) is called the "hyperbolic *map*" [22]. This map is very simple to execute if the number x is given as a continued fraction expansion:

In each iteration move all the terms of the expansion $x = [a_0, a_1, a_2, \ldots]$ *one place to the left and leave out the first coefficient of the expansion.*

In Fig. 8.1a we show the iteration of the hyperbolic map, starting from the number $x = \pi$ and in Fig. 8.1b the ordered sequence of 200 points is depicted. The same procedure have been applied to the hyperbolic map starting from the number e (see Figs. 8.2a y 8.2b) . It is highly interesting to compare in both cases the graphics 8.1a and 8.1b as well as 8.2a and 8.2b: notice how the 200 points of the hyperbolic map ordered themselves when in reality, they are following a completely chaotic[1] [24] trajectory!

Obviously, being the continued fraction expansion of the Golden Mean a purely periodic expansion, it is a **"fixed point"** or an **"equilibrium value"** of the hyperbolic map, through all the iterations. That means that if the initial value is $A(0) = a$, then $A(k) = a$ is a constant solution to the iterated dynamical system, for all values of k.

The same happens with all the members of the family that have a purely periodic continued fraction expansion. In the restant cases, where the continued fraction expansion is only periodic, we have also fixed points of the hyperbolic map, since leaving aside the first iteration, then the obtained value is invariant.

In fact, we have depicted in Fig. 8.3 the hyperbolic map starting from the Golden Mean ϕ and in Fig. 8.4 the hyperbolic map starting from all the others Metallic Means we have already considered. As is easily seen, they appear as fixed points of the hyperbolic map. We have taken 50 digits and 1,000 iterations.

In conclusion, we may assert

[1] "Chaotic" is a process with respect to its dynamics, that is, when it is not possible to adventure any prognosis about its future evolution, since very similar initial conditions produce behaviors of the system that differ enormously among them.

PROPERTY NR. 5 OF THE METALLIC MEANS FAMILY

Since the continued fraction expansions of the Golden, Silver and Bronze Means are of the form $[\overline{1}]$, $[\overline{2}]$, $[\overline{3}]$, respectively, these numbers are "fixed points" of the hyperbolic map. For the restant members of this family, that possess periodic continued fraction expansions of the form $[\alpha, \overline{n}]$, being all the terms (with the exception of the first) equal to n, we have also fixed points of the hyperbolic map.

NOTE: Of course, the number of members of the MMF that satisfies Properties 1, 2, 3, 4 and 5, is infinite, since we could add to the above mentioned irrational numbers, all the irrational numbers which continued fraction expansion is purely periodic of period 1, such as for example

$$[\overline{4}]=1+2\phi\,;\,[\overline{5}]=\frac{5+\sqrt{29}}{2}\,;[\overline{6}]=3+\sqrt{10}\,;[\overline{7}]=\frac{7+\sqrt{53}}{2};\,[\overline{8}]=4+\sqrt{17};\ldots$$

as well as all the possible combinations of continued fraction expansions of the form $[n, \overline{p}]$, with n natural and p an uneven number:

$$[2,\overline{3}]=\frac{1+\sqrt{13}}{2};[3,\overline{5}]=\frac{1+\sqrt{29}}{2};[4,\overline{7}]=\frac{1+\sqrt{53}}{2}.$$

The rest of the members of the family are integer numbers with continued fraction expansions $[n,\overline{0}]$ or else numbers with continued fraction expansions that include **"stable cycles"** obeying certain regularity rules that will be published elsewhere. Some of them are

$$\frac{1+\sqrt{21}}{2}=[2,\overline{1,3}];\frac{1+\sqrt{33}}{2}=[3,\overline{2,1,2,5}];\frac{1+\sqrt{73}}{2}=[4,\overline{1,3,2,1,1,2,3,1,7}].$$

9. QUASI-CRYSTALS: FORBIDDEN SYMMETRIES

Among the many problems in Physics, Chemistry, Biology and Ecology where the members of the MMF appear, one of the most striking is the structure of a quasi-crystal. The most symmetric, regular and periodic of all real entities, are the *"crystals"*. At the opposite end of the scale, we have the disordered or amorphous substances, like the *"glasses"*.

To distinguish between a crystal and a glass let us consider that a real crystal can be modellized putting an atom or a molecule at all the vertices of a regular triangular, cuadrangular or hexagonal lattice, lattices that have symmetries of order 3, 4 and 6 (Fig. 9.1). In such a way, the problem of matter structure is reduced to one of pure geometry. This was the state of the art until 1984, when Schechtman et al. [25], [26], registering diffraction schema of electrons in an alloy of Aluminium and Manganese quickly cooled, found in cutting with planes forming determined angles, pentagonal symmetries of order 5, wholly impossible in a crystal since it is, obviously, impossible to tessellate the plane with regular pentagons.

These configurations with pentagonal symmetry, that possess a quasi-periodic spatial structure, were called *"quasi-crystals"*. And they are really a new solid state of matter!

What is extremely interesting is the fact that the projections were taken cutting with a plane which slope with respect to the ground was equal to the Golden Mean ϕ.

Starting with this discovery, there appeared another quasi-crystals with other forbidden symmetries. E.g. the Silver Mean $\sigma_{Ag} = 1 + \sqrt{2} = [\ \overline{2}\ ]$, generates a quasi-crystal with a forbidden symmetry of order 8 (see [27], [28]), while $[\ \overline{4}\ ] = \phi^3$ appears in another forbidden symmetry, of order 12 (see [29]). Both symmetries, have been empirically detected.

In particular, Gumbs, Ali et al., in various highly interesting papers [30], [31], [32], [33] and [34] studied electronic, optical, acoustic and super-conducting properties of quasi-periodic layered systems. For that purpose, they constructed geometric one-dimensional models of a new type of quasi-crystals devised taking as basis GSFS. They were interested in these quasi-crystals because of their important physical applications, i.e. the problem of light transmission through a multi-layered medium. Among their most remarkable experimental results, they found fundamental differences in the behavior of Metallic Means which continued fraction expansion is purely periodic (the Golden Mean, the Silver Mean and the Bronze Mean) and the Metallic Means with only periodic continued fraction expansions (the Copper Mean and the Nickel Mean):

1) In studying the electronic properties of a GSFS lattice, it was found that the trace maps of the Golden, Silver and Bronze Mean lattices are volume-preserving (non-dissipative) while those of the Copper and Nickel Mean lattices are volume-non-preserving (dissipative).

2) In investigating the magnetic excitation spectra of a Nickel-Molybdene GSFS lattice, it was found that only in the case of purely periodic continued fraction expansions, the whole spectrum is self-similar. In the case of periodic continued fraction expansions, only some parts of the whole spectrum are self-similar.

3) In considering quasi-periodic quantum Ising models in which the exchange interaction follows a GSFS, it was proved that in the case of dissipative maps (Copper and Nickel Mean lattices), the spectral properties are directly determined by the attractor of the map. And that the Copper and Nickel Mean lattices can be classified as between quasi-periodic and random, with the Nickel Mean more random than the Copper Mean.

10. CANTOR SPECTRA IN CRITICAL STATES

In 1919, the brilliant mathematician Félix Hausdorff published a fundamental paper on the concept of *"dimensión"* of a set. This paper opened the possibility of constructing sets with non integer topological dimension! The topological dimension corresponds to the common meaning of the word "dimension" and is an integer: it is zero for a point, one for a straight line, two for a certain portion of the plane and three for any body in space. But evidently, the curves, surfaces and volumes may be so complex as to make it necessary to differentiate among them, taking into account how quickly the length, the surface or the volumen vary with respect to measure scales each time smaller. This notion established the base to define the **"fractal dimension"**, introduced by the polish mathematician Benoit B. Mandelbrot [35], [36].

Mandelbrot defined a **"fractal"** as a set with a Hausdorff dimension greater or equal to its topological dimension. It can be stated that the concept of dimension he used was a simplification of Hausdorff dimension.

The notion of self-similarity is strictly related with the intuitive concept of dimension. A segment may be divided into N equal sub-segments, each of which is in a relation $\varepsilon = 1/N$ with the original segment (Fig. 10.1). Analogously, in dividing a square into N equal sub-squares, obviously self-similar, we have a relation $\varepsilon = 1/N^{1/2}$ with the complete figure; this ratio is $\varepsilon = 1/N^{1/3}$ in the case of a cube and $\varepsilon = 1/N^{D}$ for a D-dimensional object. Then

$$\varepsilon^{D} = 1/N.$$

Taking logarithms in both members, we get

$$D \ln \varepsilon = -\ln N,$$

from where we get the fractal dimension D:

$$\boxed{D = \frac{\ln N}{\ln(1/\varepsilon)}} \qquad (10.1)$$

We shall apply this formula to calculate the fractal dimension of the famous *"Cantor ternary set"*, that is the most ancient known fractal. It was introduced by the german mathematician Georg Cantor (1845-1918), who is considered one of the founders of set theory. To construct this set, let us begin with a given segment that is divided into three equal parts (Fig. 10.2) and leaving aside the middle third. Then the left and right thirds are again divided in three equal parts and the middle third is left aside. The process is repeated until after many iterations, we get discrete points that form the so called *"Cantor powder"*. If we take the initial length equal to unity, after three iterations, we shall have $2^3 = 8$ segments, each of them of length $3^{-3} = 1/27$. After n iterations there will be 2^n segments, each of length 3^{-n}. The total length of the restant segments is equal to $(2/3)^n$, a quantity that tends evidently to zero when n tends to infinity. This implies that the fractal dimension of the Cantor ternary set is

$$D = \frac{\ln N}{\ln(1/\varepsilon)} = \frac{\ln 2^n}{\ln(1/3^{-n})} = 0{,}6309...$$

This value is an irrational number, being nearer from one than from zero, and this is, in a certain sense, a measure of its irregularity.

M. S. El Naschie has carefully analyzed the relations existent among the Hausdorff dimension of Cantor sets of higher order and the Golden Mean and the Silver Mean [37], [38]. In particular, in Reference [39], he proved five important theorems, three of them main theorems (Bijection Theorem, Theorem of the Golden Mean and Generalized Fibonacci Theorem) and two auxiliary theorems (Silver Mean Theorem and Arithmetic Mean Theorem). These theorems are related to the notion of KAM instability[2] and the global chaos in hamiltonian (that conserve the energy) physical systems.

Indeed, certain members of the MMF play a very important rol in relation to the stability of some orbits in the n-dimensional phase space. For example, it is a very well known fact that orbits with a *"winding number"* equal to the Golden Mean are the most stable -- the winding number measures the mean displacement of a certain angle at each iteration of a discrete dynamical system. Furthermore, the connection between the hyperbolic map and more general dynamical systems, is closely related to *period duplication* and the *Golden Mean route to chaos*. The empirical finding of period duplication in a certain physical phenomenon, as well as the existence of certain irrational ratios that produce the onset to chaos when this ratio is equal to the Golden Mean, are very well known in modern References (see References [3] and [21]).

The forbidden symmetries we have already encountered in analyziung quasi-crystals, like the symmetries of order eight and twelve, may also be generated by Cantor multiplicative sets of higher order, together with the Golden Mean [40].

[2] Kolmogorov (1954), Arnold (1963) and Moser (1967), proved what is today known as KAM theorem. This theorem states that the motion in the phase space of Classical Mechanics is neither completely regular nor completely irregular, but that the sort path depends sensibly from the initial conditions..

Comparing the terms of the secondary Fibonacci sequence (3.1), with the ternary Fibonacci sequence, defined by the relation

(10.2) $$B_{n+1} = B_{n-2} + B_{n-1} + B_n ,$$

like it is indicated in the following table:

n	1	2	3	4	5	6	7	8	9
F_n	1	1	2	3	5	8	13	21	34
B_n	1	1	2	4	7	13	24	44	81

it is easy to verify that for the first sequence, F_n and n are equal only when $n = 5$, while for the second one, B_n and n are equal only when $n = 4$. These type of states is normally used to modellize some forms of *ergodic*[3] behavior of physical systems and they can be considered as **"ergodic-type states"**. The connections of this research with statistical mechanics, classic as well as quantum mechanics, as is proved by El Naschie [41], determine the existence of two types of quasi-ergodic Cantor sets:

a) an even set of four dimensions, that describes the behavior of classical particles and *bosons*[4];
b) an odd set of five dimensions, related with *fermions*[5] and with the pentagonal symmetry of quasi-crystals.

11. TIME IRREVERSIBILITY

Ilia Prigogine is, without any doubt, one of the most important scientists of this century. He awarded the Nobel Prize in Chemistry and nowadays, he is the leader of a brilliant research group at the Free University in Brussel, Belgium. The fundamental question of time irreversibility and its consequences in science philosophy, has been one of his main preoccupations.

The basic laws in Physics, from newtonian Mechanics to the generalized relativity theory of Einstein, as well as the present theories for the elementary particles, satisfy all the hypothesis of time reversibility.

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[3] In Dynamics, it is a very important problem to be able to describe the path of a particle in space. If the particle is limited to move inside a limited domain of space, it is essential to know if the path fills out all the space with an uniform distribution in a sufficiently long time. Such paths are called *"ergodic"* and to postulate their existence is a fundamental problem in classic Dynamics as well as in Quantum Mechanics.

[4] Bosons are elementary particles with a *"spin"* or angular momentum that is an integer multiple of Planck's constant. Photons and mesons are bosons.

[5] Fermions are elementary particles with a *"spin"* that is a half-integer multiple of Planck's constant. Electrons, protons and neutrons are fermions.

As Einstein stated: *"the distinction among past, present and future, is only an illusion"*. However, time seems to flow in one sense. How is it possible to reconcile the **fundamental statement** with the **empirical fact**?

In his recently appeared book [42], Prigogine considers this question and the finding of an answer obliges him to revise and restate all the Physics, starting from Epicur's dilemma for whom the problem of the intelligibility of nature is undetachable from men destiny.

Together with Prigogine and other scientists, El Naschie proposes a solution valid for classical Mechanics as well as for Quantum Mechanics [43]. The solution consists in the introduction of the notion of a *"cantorian"* (from Cantor) space-time, in which time behaves statistically and is completely undistinguishable from the restant three space coordinates. What is really remarkable of this Cantorian space-time is that applying all the probabilistic necessary laws, the values of Hausdorff dimensions are intrinsically linked to the Golden Mean ϕ and its successive powers, like $\phi^2 = [2,\overline{1}]$ and $\phi^3 = [\overline{4}]$ (see Reference [44])!

Obviously, Hausdorff dimension, being an intermediate measure between volume and dimension, plays in this new theory a preponderant rol as a linkage between dimension and information. We may as well conjecture a relation between the irrationality grade and the information content, since when the dimension is equal to the Golden Mean ϕ -- the most irrational of all irrational numbers -- the information content is the largest possible.

12. CONCLUSIONS

We have already verified how the MMF is closely related to the transition from a periodic dynamics to a quasi-periodic dynamics, as well as to the onset from order to chaos and with time irreversibility.

But simultaneously, since the beginning of humanity, there have been philosophical, natural and aesthetic considerations that have had primacy in the establishment of proportions based on some members of this family. They appeared more or less explicitly in the sacred art of Egypt, India, China and Islam and other ancient civilizations. They have dominated greek art and architecture, they persisted concealed in the monuments of the Gothic Middle Ages and re-emerged openly to be celebrated in the Renaissance.

Summarizing, we can state that wherever there is an intensification of function or a particular beauty and harmony of form, there at least the two first members of the MMF, e.g. the Golden Mean and the Silver Mean, will be found. If the restant members of this family are also involucred in these considerations, future researchs will give the answer.

Such a wide range of applications where the members of the MMF are present, opens the road to new inter-disciplinary investigations that undoubtedly will clear up the existent relations between Art and Technique, building a bridge linking the rational scientific approach and the aesthetic emotion. And perhaps this new perspective could help us in giving Technology, from which we depend each time more and more for our survival, a more human character.

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FIGURE CAPTIONS

Fig. 5.1: Golden relations in a pentagon or unity side.

Fig. 5.2: Golden divisions in an ancient mask of Hermes.

Fig. 5.3: The Silver Mean in a regular octagon.

Fig. 6.1: Fractalization of a cube.

Fig. 6.2: Icosahedron.

Fig. 6.3: Icosahedrical spiral.

Fig. 8.1a: Hyperbolic map starting from π.

Fig. 8.1b: Ordered sequence of 200 points for the hyperbolic map of Fig. 8.1a.

Fig. 8..2a: Hyperbolic map starting from e.

Fig. 8.2b: Ordered sequence of 200 points for the hyperbolic map of Fig. 8.2a.

Fig. 8.3: Hyperbolic map starting from the Golden Mean.

Fig. 8.4: Hyperbolic map starting from the other Metallic Means.

Fig. 9.1: Regular tilings for tessellating the plane.

Fig. 9.2: Cantor "powder" set.

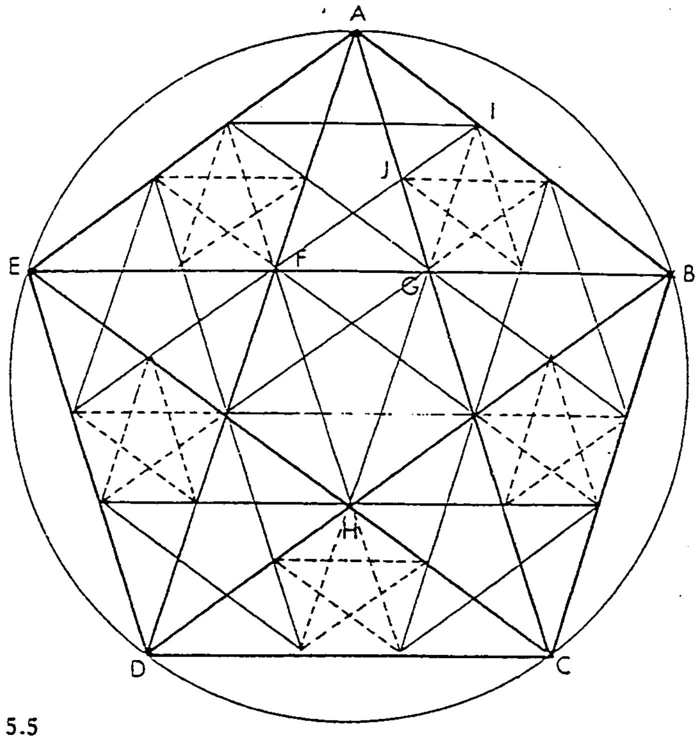


Fig. 5.1

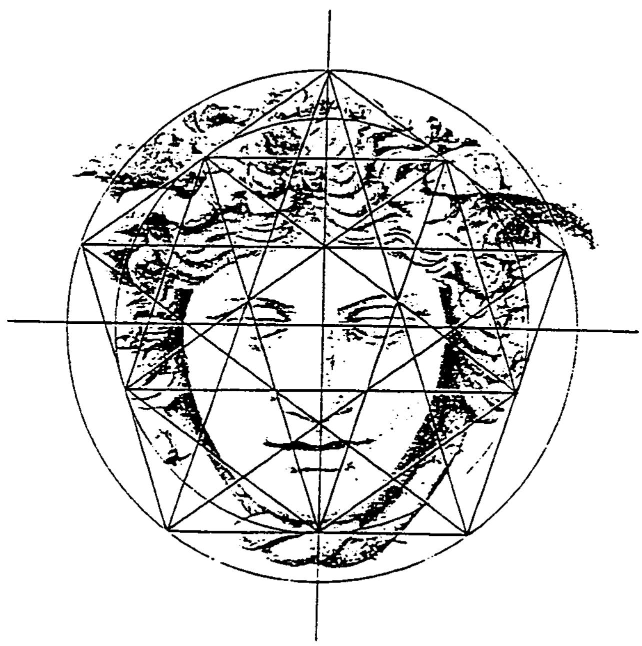


Fig. 5.2

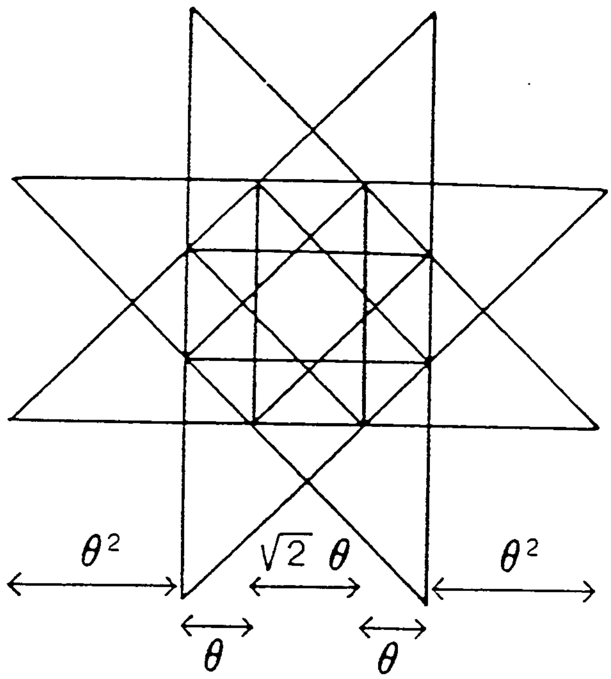


Fig. 5.3

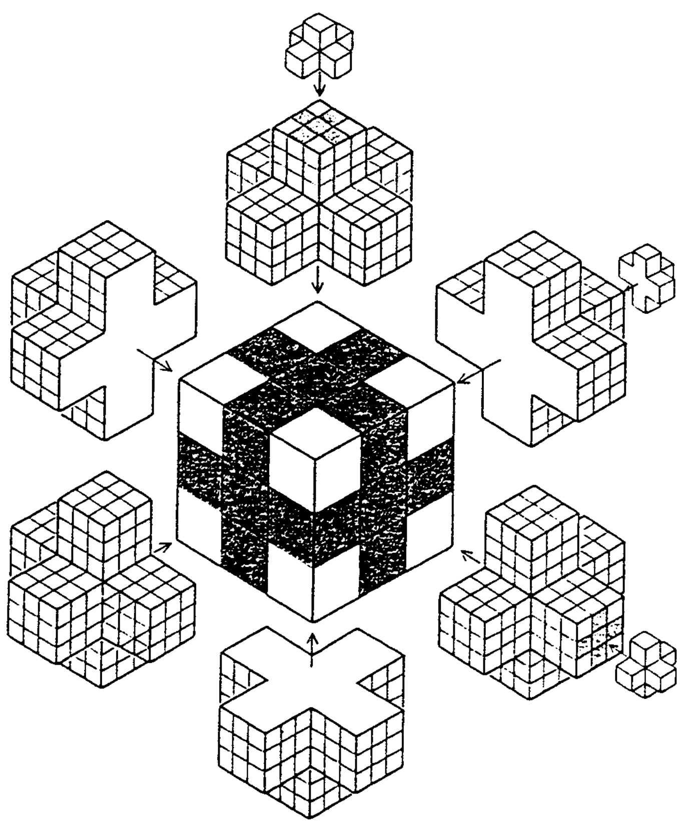


Fig. 6.1

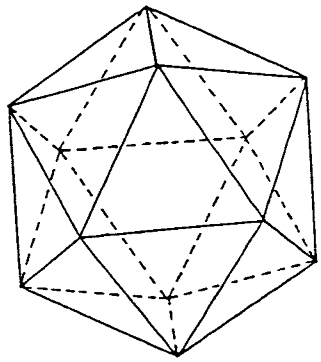


Fig. 6.2

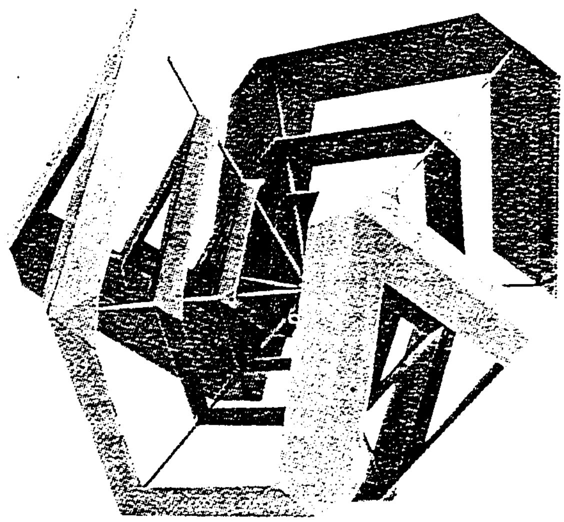


Fig. 6.3

```
> x:=evalf(Pi);
> R:=[seq(H(),i=1..1000)];
> plot(R,0..1,0..1,style=POINT);
```

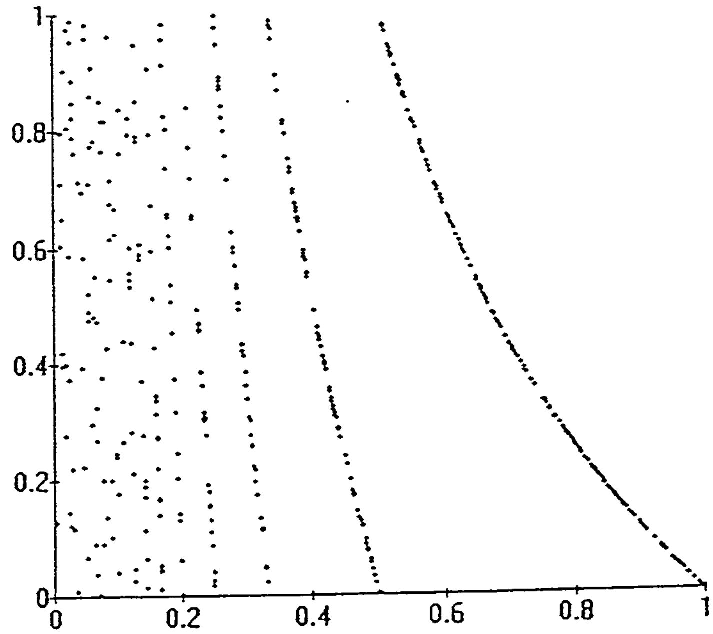


Fig. 8.1a

```
>
> R2:=[seq(H(),i=1..200)]:
> plot(R2,0..1,0..1,style=LINE);
```

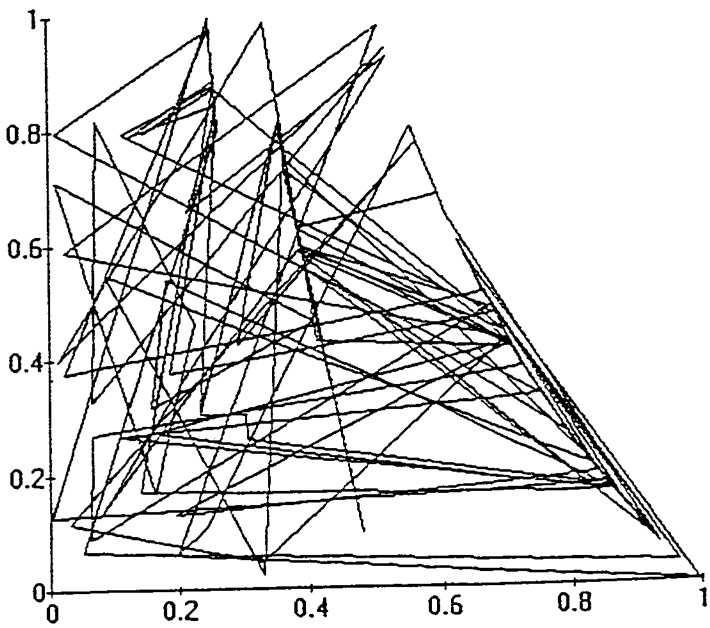


Fig. 8.1b

```
> x:=evalf(E);
```

$$x := 2.7182818284590452353602874713526624977572470937$$

```
> R:=[seq(H(),i=1..1000)]:
> plot(R,0..1,0..1,style=POINT);
```

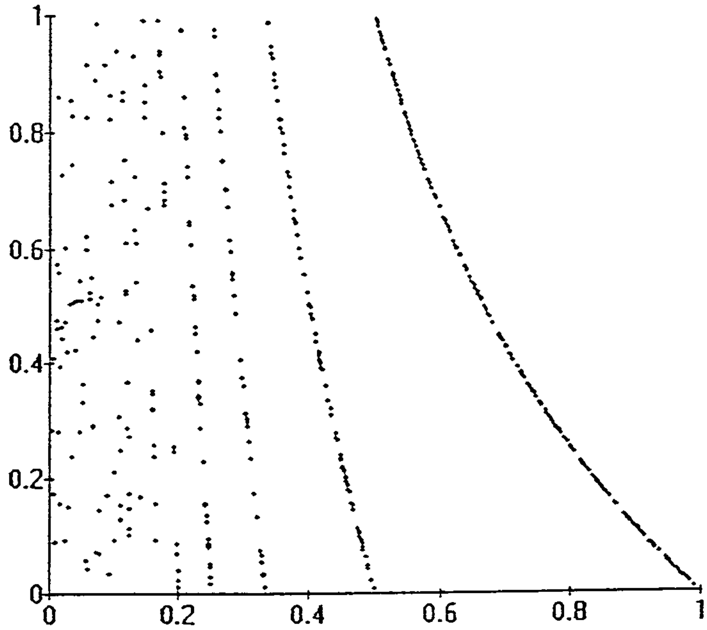


Fig. 8.2a

```
> R1:=[seq(H(),i=1..200)]:
> plot(R1,0..1,0..1,style=LINE);
```

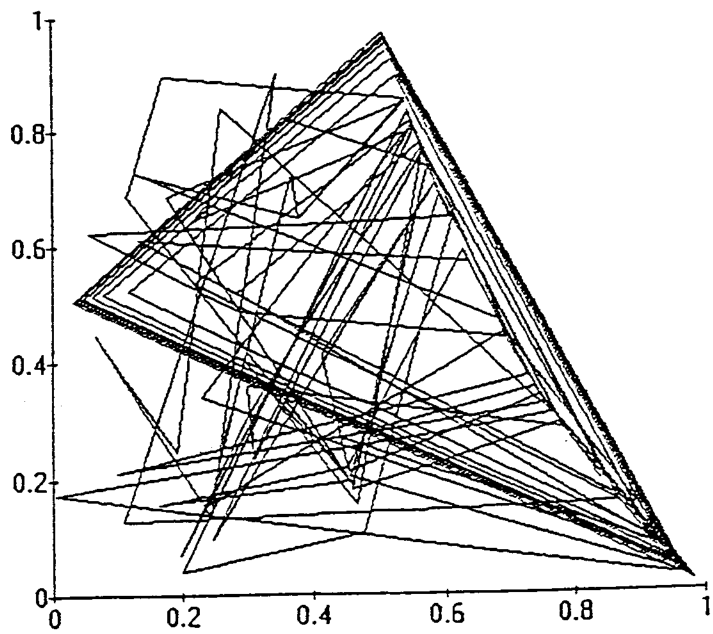


Fig. 8.2b

```
>
> Digits:=50;
> x:=evalf((1+sqrt(5))/2);
> F1:=array(1..1000);
> H1:=proc();
> if i=1 then F1[i]:=frac(x) else F1[i]:=frac(1/frac(F1[i-1]))
> fi;
> end;
> R1:=[seq(H1(),i=1..50)];
> R3:=[R1[2..49]];
> with(plots):
> RP:=plot(R1,0..1,0..1,style=POINT):
> RP3:=plot(R3,0..1,0..1,style=POINT):
> display({RP,RP3});
```

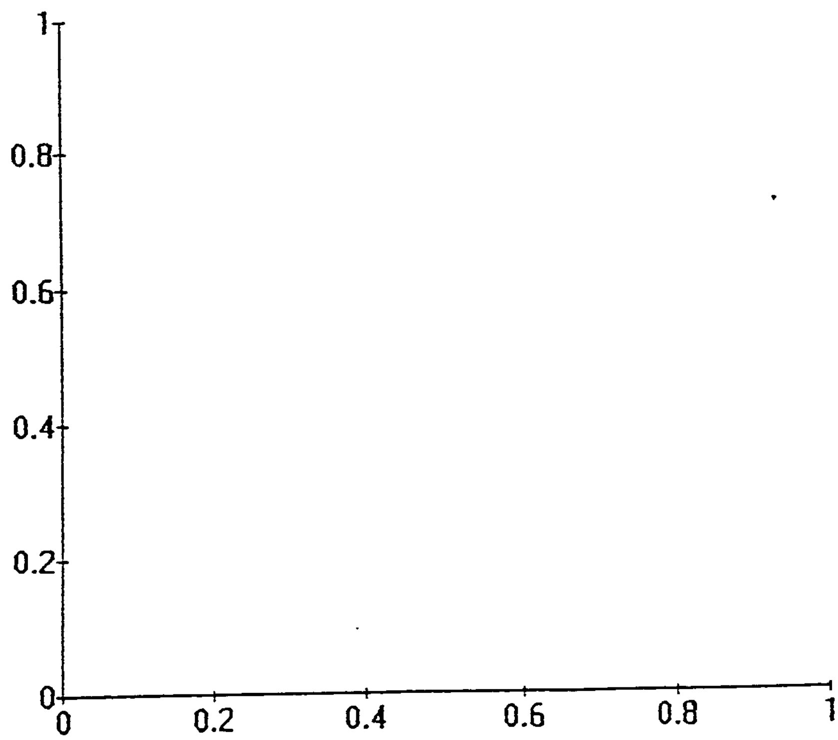


Fig. 8.3

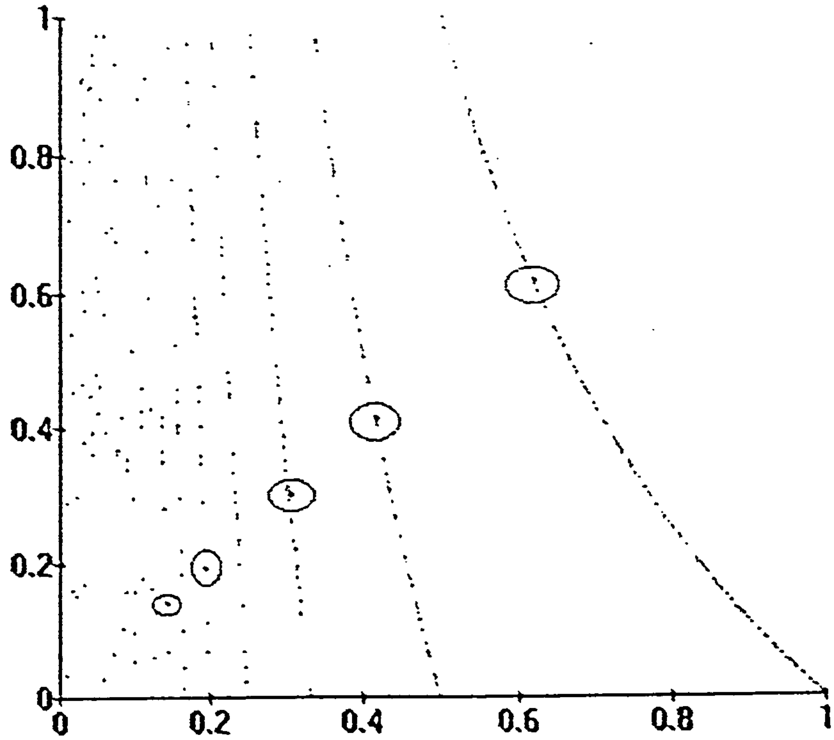


Fig. 8.4

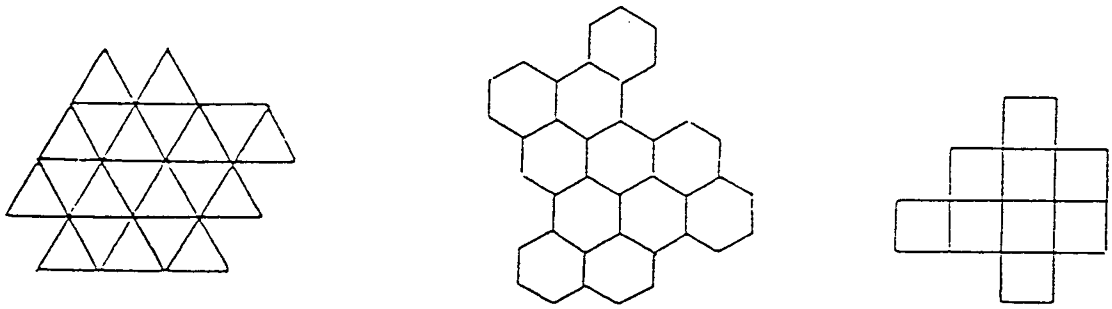
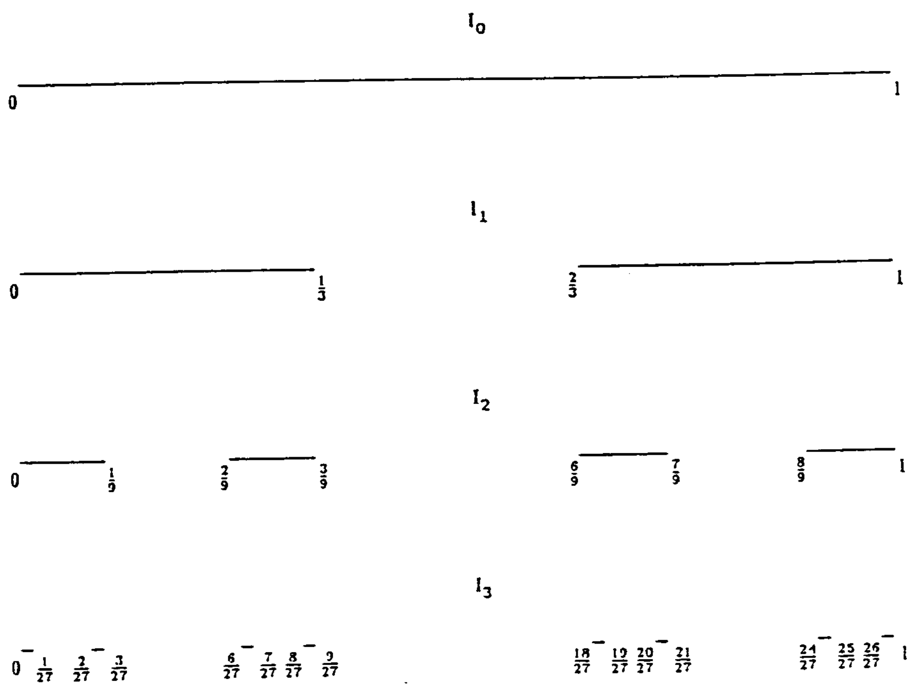


Fig. 9.1



I0
0
1
I1
0
1/3
2/3
1
I2
0
1/9
2/9
3/9
6/9
7/9
8/9
1
I3
0
1/27
2/27
3/27
6/27
7/27
8/27
9/27
18/27
19/27
20/27
21/27
24/27
25/27
26/27
1

Fig. 9.2