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Value distribution of the F.Smarandache LCM function[1]

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Abstract For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of $1,\ 2,\ \cdots,\ k$. The main purpose of this paper is using the elementary methods to study the value distribution properties of the function $SL(n)$, and give a sharper value distribution theorem.

Keywords F.Smarandache LCM function, value distribution, asymptotic formula.

§1. Introduction and results

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of $1,\ 2,\ \cdots,\ k$. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $\cdots$. About the elementary properties of $SL(n)$, some authors had studied it, and obtained some interesting results, see reference [3] and [4]. For example, Murthy [3] showed that if n be a prime, then $SL(n) = S(n)$, where $S(n)$ denotes the Smarandache function, i.e., $S(n) = \min\{m:\ n|m!,\ m \in N\}$. Simultaneously, Murthy [3] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [4] completely solved this problem, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \quad \texttt{or} \quad n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1,\ p_2,\ \cdots,\ p_r,\ p$ are distinct primes, and $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1,\ 2,\ \cdots,\ r$.

Lv Zhongtian [6] studied the mean value properties of $SL(n)$, and proved that for any fixed positive integer k and any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} SL(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

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where c_i $(i = 2,\ 3, \cdots, k)$ are computable constants.

The main purpose of this paper is using the elementary methods to study the value distribution properties of $SL(n)$, and prove an interesting value distribution theorem. That is, we shall prove the following conclusion:

Theorem. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x}\left(SL(n)-P(n)\right)^2=\frac{2}{5}\cdot\zeta\left(\frac{5}{2}\right)\cdot\frac{x^{\frac{5}{2}}}{\ln x}+O\left(\frac{x^{\frac{5}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, and $P(n)$ denotes the largest prime divisor of n.

§2. Proof of the theorem

In this section, we shall prove our theorem directly. In fact for any positive integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of n into prime powers, then from [3] we know that

$$SL(n)=\max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_s^{\alpha_s}\}. \tag{2}$$

Now we consider the summation

$$\sum_{n\le x}\left(SL(n)-P(n)\right)^2. \tag{3}$$

We separate all integers n in the interval $[1,\ x]$ into four subsets A, B, C and D as follows:

A: $P(n)\ge\sqrt{n}$ and $n = m\cdot P(n)$, $m < P(n)$;

B: $n^{\frac{1}{3}} < P(n) \le \sqrt{n}$ and $n = m\cdot P^2(n)$, $m < n^{\frac{1}{3}}$;

C: $n^{\frac{1}{3}} < p_1 < P(n) \le \sqrt{n}$ and $n = m\cdot p_1\cdot P(n)$, where p_1 is a prime;

D: $P(n)\le n^{\frac{1}{3}}$.

It is clear that if $n\in A$, then from (2) we know that $SL(n) = P(n)$. Therefore,

$$\sum_{n\in A}\left(SL(n)-P(n)\right)^2=\sum_{n\in A}\left(P(n)-P(n)\right)^2=0. \tag{4}$$

Similarly, if $n\in C$, then we also have $SL(n) = P(n)$. So

$$\sum_{n\in C}\left(SL(n)-P(n)\right)^2=\sum_{n\in C}\left(P(n)-P(n)\right)^2=0. \tag{5}$$

Now we estimate the main terms in set B. Applying Abel's summation formula (see Theorem 4.2 of [5]) and the Prime Theorem (see Theorem 3.2 of [7])

$$\pi(x)=\sum_{p\le x}1=\frac{x}{\ln x}+O\left(\frac{x}{\ln^2 x}\right)$$

we have

$$
\begin{aligned}
&\sum_{n\in B}(SL(n)-P(n))^2=\sum_{\substack{mp^2\leq x\\ m<p}}\left(SL(mp^2)-P(mp^2)\right)^2\\
=&\sum_{m\leq x^{\frac{1}{3}}}\ \sum_{m<p\leq\sqrt{\frac{x}{m}}}\left(p^2-p\right)^2\\
=&\sum_{m\leq x^{\frac{1}{3}}}\left[\left(\frac{x}{m}\right)^2\cdot\pi\left(\sqrt{\frac{x}{m}}\right)-4\int_m^{\sqrt{\frac{x}{m}}}y^3\pi(y)dx+O\left(m^5+\frac{x^2}{m^2}\right)\right]\\
=&\sum_{m\leq x^{\frac{1}{3}}}\left(\frac{x^{\frac{5}{2}}}{5m^{\frac{5}{2}}\ln\sqrt{\frac{x}{m}}}+O\left(\frac{x^{\frac{5}{2}}}{m^{\frac{5}{2}}\ln^2\frac{x}{m}}\right)\right)\\
=&\ \frac{2}{5}\cdot\zeta\left(\frac{5}{2}\right)\cdot\frac{x^{\frac{5}{2}}}{\ln x}+O\left(\frac{x^{\frac{5}{2}}}{\ln^2 x}\right), \qquad (6)
\end{aligned}
$$

where $\zeta(s)$ is the Riemann zeta-function.

Finally, we estimate the error terms in set D. For any integer $n\in D$, let $SL(n)=p^{\alpha}$. If $\alpha=1$, then $SL(n)=p=P(n)$, so that $SL(n)-P(n)=0$. Therefore, we assume that $\alpha\geq 2$. This time note that $P(n)\leq n^{\frac{1}{3}}$, we have

$$
\begin{aligned}
&\sum_{n\in D}(SL(n)-P(n))^2\ll\sum_{n\in D}\left(SL^2(n)+P^2(n)\right)\\
\ll&\sum_{\substack{mp^{\alpha}\leq x\\ \alpha\geq 2,\ p<x^{\frac{1}{3}}}}p^{2\alpha}+\sum_{n\leq x}n^{\frac{2}{3}}\ll\sum_{\substack{p^{\alpha}\leq x\\ \alpha\geq 2,\ p\leq x^{\frac{1}{3}}}}p^{2\alpha}\sum_{m\leq\frac{x}{p^{\alpha}}}1+x^{\frac{5}{3}}\\
\ll&\ x\cdot\sum_{\substack{p^{\alpha}\leq x\\ \alpha\geq 2,\ p\leq x^{\frac{1}{3}}}}p^{\alpha}+x^{\frac{5}{3}}\ll x^2. \qquad (7)
\end{aligned}
$$

Combining (3), (4), (5), (6) and (7) we may immediately obtain the asymptotic formula

$$
\begin{aligned}
&\sum_{n\leq x}(SL(n)-P(n))^2=\sum_{n\in A}(SL(n)-P(n))^2+\sum_{n\in B}(SL(n)-P(n))^2\\
&\quad+\sum_{n\in C}(SL(n)-P(n))^2+\sum_{n\in D}(SL(n)-P(n))^2\\
=&\ \frac{2}{5}\cdot\zeta\left(\frac{5}{2}\right)\cdot\frac{x^{\frac{5}{2}}}{\ln x}+O\left(\frac{x^{\frac{5}{2}}}{\ln^2 x}\right).
\end{aligned}
$$

This completes the proof of Theorem.

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