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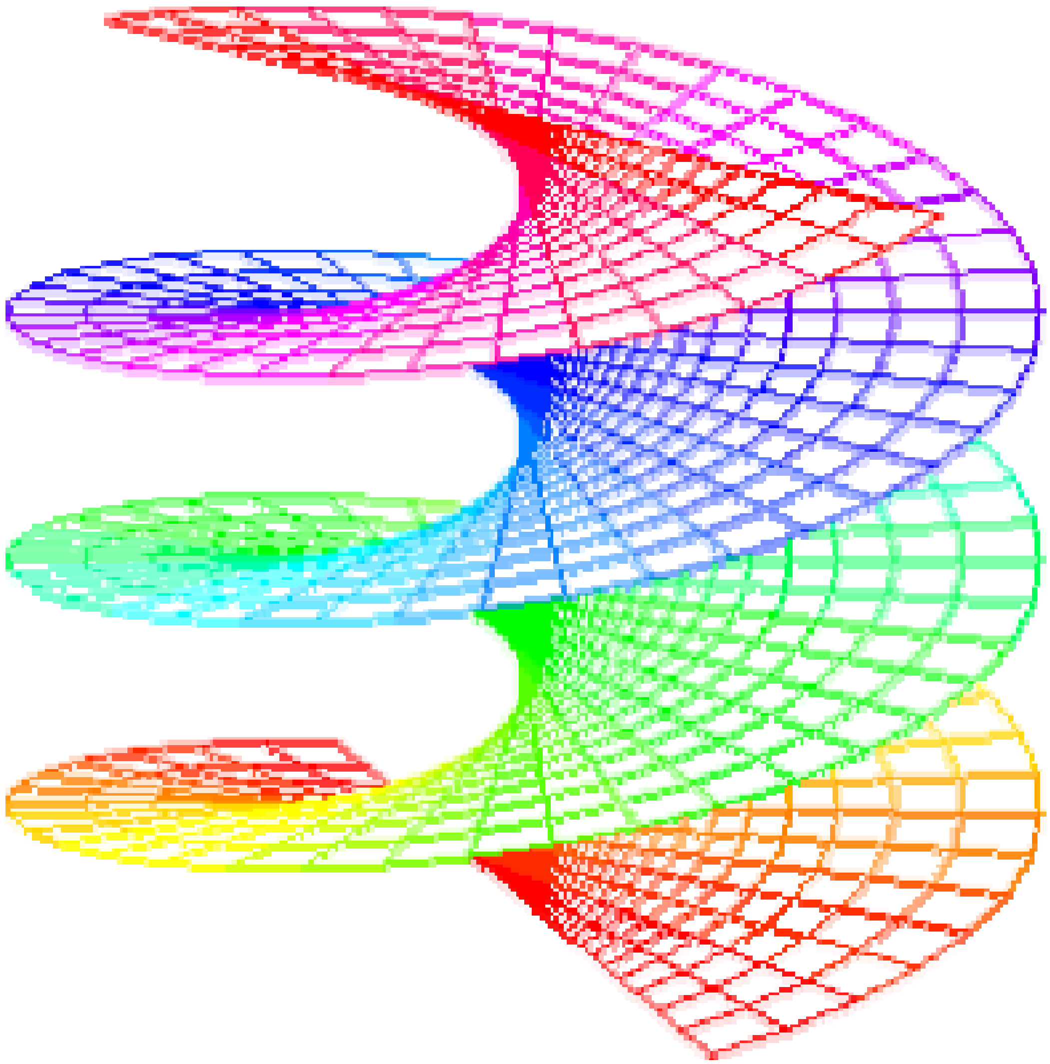

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Contents

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Smarandache inversion sequence

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Abstract We study the Smarandache inversion sequence which is a new concept, related sequences, conjectures, properties, and problems. This study was conducted by using (Maple 8)–a computer Algebra System.

Keywords Smarandache inversion, Smarandache reverse sequence.

Introduction

In [1], C.Ashbacher, studied the Smarandache reverse sequence:

$$1, 21, 321, 4321, 54321, 654321, 7654321, 87654321, 987654321, 10987654321, 1110987654321, \quad (1)$$

and he checked the first 35 elements and no prime were found. I will study sequence (1), from different point of view than C. Ashbacher. The importance of this sequence is to consider the place value of digits for example the number 1110987654321, to be considered with its digits like this : 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, and so on. (This consideration is the soul of this study because our aim is to study all relations like this (without loss of generality): $11 > 10 > 9 > 8 > 7 > 6 > 5 > 4 > 3 > 2 > 1$).

Definition. The value of the Smarandache Inversions (SI) of a positive integers, is the number of the relations $i > j$ (i and j are the digits of the positive integer that we concern with it), where i always in the left of j as the case of all numbers in (1). I will study the following cases of above equation.

Examples. The number 1234 has no inversions ((SI) $= 0$, or zero inversion), also the number 1, while the number 4321 has 6 inversions, because $4 > 3 > 2 > 1$, $3 > 2 > 1$, and $2 > 1$. The number 1110987654321 has 55 inversions, and 1342 has two inversions. So our interest will be of the numbers in Smarandache reverse sequence i.e. (1), because it has mathematical patterns and interesting properties.

Theorem. The values of SI of (1), is given by the following formula:

$$SI(n) = \frac{n(n-1)}{2}, \quad (2)$$

n is the number of inversions.

Proof. For $n = 1$, $SI(1) = \frac{1(1-1)}{2} = 0$, this is clearly true.
Now suppose that $SI(k) = \frac{k(k-1)}{2}$ is true, then $SI(k+1) = \frac{k+1(k+1-1)}{2} = \frac{k(k+1)}{2}$, thus the assertion is true for $n = k+1$, if it is true for $n = k$.

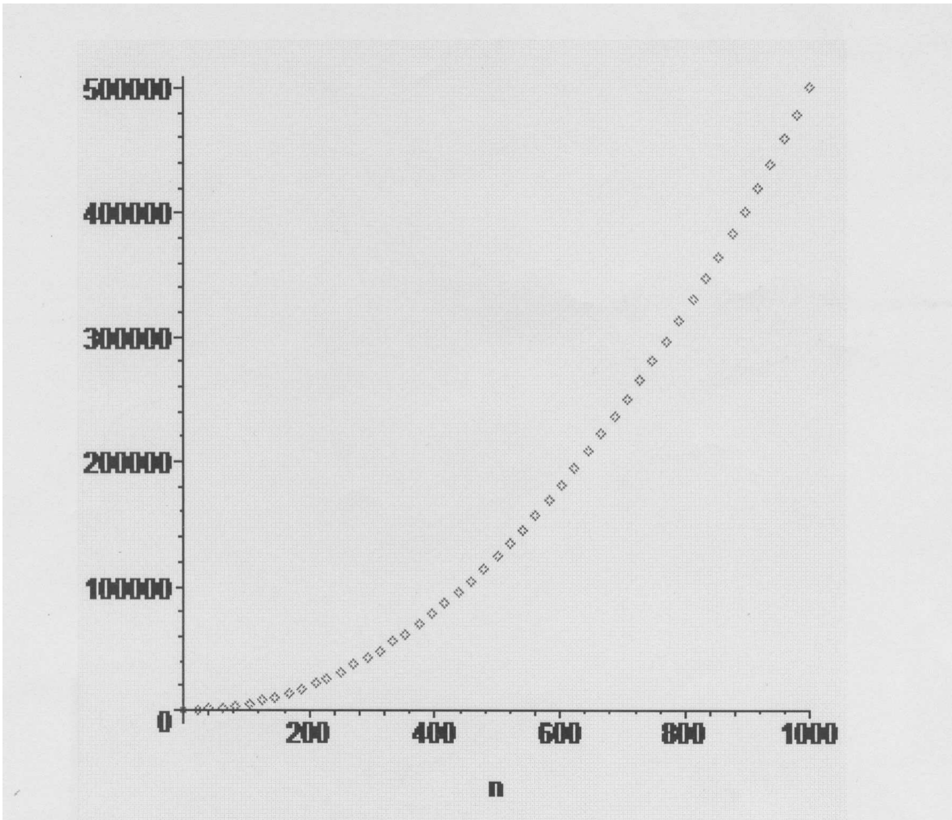

Figure 1: Plot of function $SI(n) = \frac{n(n-1)}{2}$

From the above figure, we can see although n is small, $SI(n) = \frac{n(n-1)}{2}$ it has big values. For example, if $n = 1000$, then $SI(1000) = 499500$.

Using Maple 8 programming language [2], verifying the first 100 terms of $SI(n)$:

$$SI(1) = 0, \quad SI(2) = 1, \quad SI(3) = 3, \quad SI(4) = 6,$$

$$SI(5) = 10, \quad SI(6) = 15, \quad SI(7) = 21, \quad SI(8) = 28,$$

$$SI(7) = 21, \quad SI(8) = 28, \quad SI(9) = 36, \quad SI(10) = 45,$$

$$SI(11) = 55, \quad SI(12) = 66, \quad SI(13) = 78, \quad SI(14) = 91,$$

$$SI(15) = 105, \quad SI(16) = 120, \quad SI(17) = 136,$$

$$SI(18) = 153, \quad SI(19) = 171, \quad SI(20) = 190,$$

$$SI(21) = 210, \quad SI(22) = 231, \quad SI(23) = 253,$$

$$SI(24) = 276, \quad SI(25) = 300, \quad SI(26) = 325,$$
$$SI(27) = 351, \quad SI(28) = 378, \quad SI(29) = 406,$$
$$SI(30) = 435, \quad SI(31) = 465, \quad SI(32) = 496,$$
$$SI(33) = 528, \quad SI(34) = 561, \quad SI(35) = 595,$$
$$SI(36) = 630, \quad SI(37) = 666, \quad SI(38) = 703,$$
$$SI(39) = 741, \quad SI(40) = 780, \quad SI(41) = 820,$$
$$SI(42) = 861, \quad SI(43) = 903, \quad SI(44) = 946,$$
$$SI(45) = 990, \quad SI(46) = 1035, \quad SI(47) = 1081,$$
$$SI(48) = 1128, \quad SI(49) = 1176, \quad SI(50) = 1225,$$
$$SI(51) = 1275, \quad SI(52) = 1326, \quad SI(53) = 1378,$$
$$SI(54) = 1431, \quad SI(55) = 1485, \quad SI(56) = 1540,$$
$$SI(57) = 1596, \quad SI(58) = 1653, \quad SI(59) = 1711,$$
$$SI(60) = 1770, \quad SI(61) = 1830, \quad SI(62) = 1891,$$
$$SI(63) = 1953, \quad SI(64) = 2016, \quad SI(65) = 2080,$$
$$SI(66) = 2145, \quad SI(67) = 2211, \quad SI(68) = 2278,$$
$$SI(69) = 2346, \quad SI(70) = 2415, \quad SI(71) = 2485,$$
$$SI(72) = 2556, \quad SI(73) = 2628, \quad SI(74) = 2701,$$
$$SI(75) = 2775, \quad SI(76) = 2850, \quad SI(77) = 2926,$$
$$SI(78) = 3003, \quad SI(79) = 3081, \quad SI(80) = 3160,$$
$$SI(81) = 3240, \quad SI(82) = 3321, \quad SI(83) = 3403,$$
$$SI(84) = 3486, \quad SI(85) = 3570, \quad SI(86) = 3655,$$
$$SI(87) = 3741, \quad SI(88) = 3828, \quad SI(89) = 3916,$$
$$SI(90) = 4005, \quad SI(91) = 4095, \quad SI(92) = 4186,$$
$$SI(93) = 4278, \quad SI(94) = 4371, \quad SI(95) = 4465,$$
$$SI(96) = 4560, \quad SI(97) = 4656, \quad SI(98) = 4753,$$
$$SI(99) = 4851, \quad SI(100) = 4950.$$

Summation of $SI(n) = \frac{n(n-1)}{2}$, we have

$$\sum_{i=1}^{n} \frac{i(i-1)}{2} = \frac{n(n^2-1)}{6}. \tag{3}$$

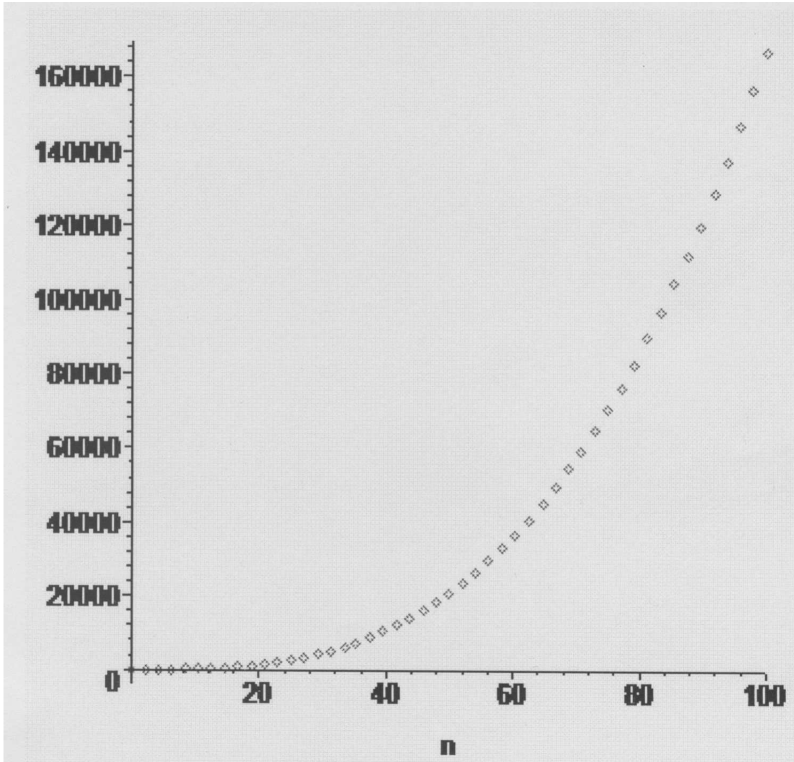

Figure 2: Plot of summation of $\sum_{i=1}^{n} \frac{i(i-1)}{2} = \frac{n(n^2-1)}{6}$

Proof. For $n = 1$, the assertion of (3) is that

$$\sum_{i=1}^{n} \frac{i(i-1)}{2} = 0 = \frac{1(1^2-1)}{6},$$

and this is clearly true.

Now suppose that

$$\sum_{i=1}^{k} \frac{i(i-1)}{2} = \frac{k(k^2-1)}{6},$$

then adding $\frac{k(k-1)}{2}$ to both sides of this equation, we obtain

$$\sum_{i=1}^{k+1} \frac{i(i-1)}{2} = \frac{k(k^2-1)}{6} + \frac{k(k-1)}{2} = \frac{k^3+3k^2+2k}{6} = \frac{k(k+1)(k+2)}{6}.$$

Thus the assertion is true for $n = k + 1$ if it is true for $n = k$.

Using Maple 8 programming language, verifying the first 73 terms of $\sum_{i=1}^{n} \frac{i(i-1)}{2} = \frac{n(n^2-1)}{6}$:

$$\sum SI(1) = 0, \qquad \sum SI(2) = 1, \qquad \sum SI(3) = 4,$$

$$\sum SI(4) = 10, \qquad \sum SI(5) = 20, \qquad \sum SI(6) = 35,$$
$$\sum SI(7) = 56, \qquad \sum SI(8) = 84, \qquad \sum SI(9) = 120,$$
$$\sum SI(10) = 165, \qquad \sum SI(11) = 220, \qquad \sum SI(12) = 286,$$
$$\sum SI(13) = 364, \qquad \sum SI(14) = 455, \qquad \sum SI(15) = 560,$$
$$\sum SI(16) = 680, \qquad \sum SI(17) = 816, \qquad \sum SI(18) = 969,$$
$$\sum SI(19) = 1140, \qquad \sum SI(20) = 1330, \qquad \sum SI(21) = 1540,$$
$$\sum SI(22) = 1771, \qquad \sum SI(23) = 2024, \qquad \sum SI(24) = 2300,$$
$$\sum SI(25) = 2600, \qquad \sum SI(26) = 2925, \qquad \sum SI(27) = 3276,$$
$$\sum SI(28) = 3654, \qquad \sum SI(29) = 4060, \qquad \sum SI(30) = 4495,$$
$$\sum SI(31) = 4960, \qquad \sum SI(32) = 5456, \qquad \sum SI(33) = 5984,$$
$$\sum SI(34) = 6545, \qquad \sum SI(35) = 7140, \qquad \sum SI(36) = 7770,$$
$$\sum SI(37) = 8436, \qquad \sum SI(38) = 9139, \qquad \sum SI(39) = 9880,$$
$$\sum SI(40) = 10660, \qquad \sum SI(41) = 11480,$$
$$\sum SI(42) = 12341, \qquad \sum SI(43) = 13244,$$
$$\sum SI(44) = 14190, \qquad \sum SI(45) = 15180,$$
$$\sum SI(46) = 16215, \qquad \sum SI(47) = 17296,$$
$$\sum SI(48) = 18424, \qquad \sum SI(49) = 19600,$$
$$\sum SI(50) = 20825, \qquad \sum SI(51) = 22100,$$
$$\sum SI(52) = 23426, \qquad \sum SI(53) = 24804,$$
$$\sum SI(54) = 26235, \qquad \sum SI(55) = 27720,$$
$$\sum SI(56) = 29260, \qquad \sum SI(57) = 30856,$$
$$\sum SI(58) = 32509, \qquad \sum SI(59) = 34220,$$
$$\sum SI(60) = 35990, \qquad \sum SI(61) = 37820,$$
$$\sum SI(62) = 39711, \qquad \sum SI(63) = 41664,$$
$$\sum SI(64) = 43680, \qquad \sum SI(65) = 45760,$$
$$\sum SI(66) = 47905, \qquad \sum SI(67) = 50116,$$
$$\sum SI(68) = 52394, \qquad \sum SI(69) = 54740,$$

$$\sum SI(70) = 57155, \qquad \sum SI(71) = 59640,$$
$$\sum SI(72) = 62196, \qquad \sum SI(73) = 64824.$$

Properties of $SI(n) = \dfrac{n(n-1)}{2}$:

1).

$$SI(n) + SI(n-1) = (n-1)^2. \tag{4}$$

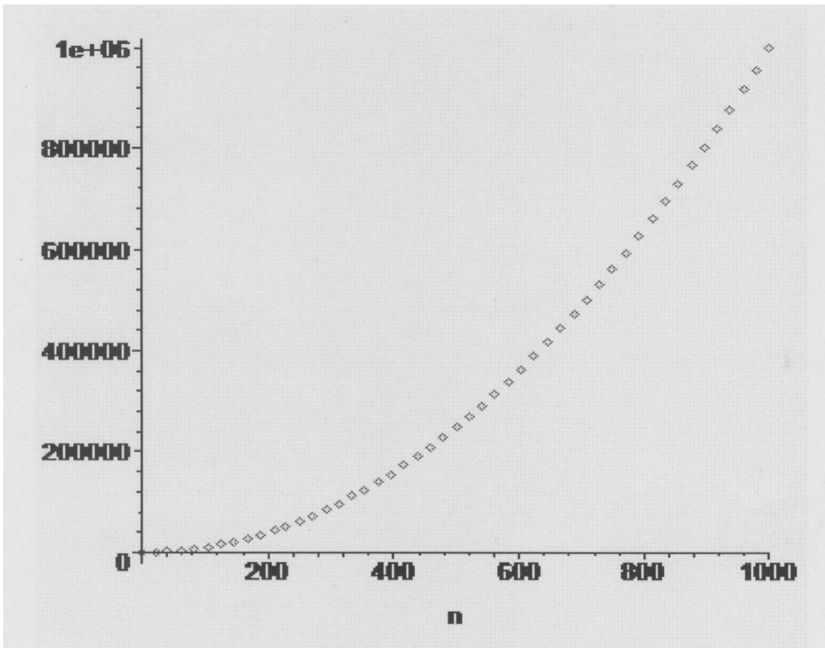

Fig 3: Plot of function $SI(n) + SI(n-1) = (n-1)^2$

Proof.

$$\begin{aligned} SI(n) + SI(n-1) &= \frac{n(n-1)}{2} + \frac{(n-1)(n-1-1)}{2} \\ &= \frac{n(n-1) + (n-1)(n-2)}{2} \\ &= \frac{(n-1)(2n-2)}{2} \\ &= (n-1)^2. \end{aligned}$$

Using Maple 8 programming language, verifying the first 40 terms of $SI(n) + SI(n-1) = (n-1)^2$:

$$SI(1) + SI(0) = [SI(0)]^2, \qquad SI(2) + SI(1) = [SI(1)]^2,$$
$$SI(3) + SI(2) = [SI(2)]^2, \qquad SI(4) + SI(3) = [SI(3)]^2,$$
$$SI(5) + SI(4) = [SI(4)]^2, \qquad SI(6) + SI(5) = [SI(5)]^2,$$

$$SI(7)+SI(6)=[SI(0)]^2, \quad SI(8)+SI(7)=[SI(7)]^2,$$
$$SI(9)+SI(8)=[SI(0)]^2, \quad SI(10)+SI(9)=[SI(9)]^2,$$
$$SI(11)+SI(10)=[SI(10)]^2, \quad SI(12)+SI(11)=[SI(11)]^2,$$
$$SI(13)+SI(12)=[SI(12)]^2, \quad SI(14)+SI(13)=[SI(13)]^2,$$
$$SI(15)+SI(14)=[SI(14)]^2, \quad SI(16)+SI(15)=[SI(15)]^2,$$
$$SI(17)+SI(16)=[SI(16)]^2, \quad SI(18)+SI(17)=[SI(17)]^2,$$
$$SI(19)+SI(18)=[SI(18)]^2, \quad SI(20)+SI(19)=[SI(19)]^2,$$
$$SI(21)+SI(20)=[SI(20)]^2, \quad SI(22)+SI(19)=[SI(21)]^2,$$
$$SI(23)+SI(22)=[SI(22)]^2, \quad SI(24)+SI(23)=[SI(23)]^2,$$
$$SI(25)+SI(24)=[SI(24)]^2, \quad SI(26)+SI(25)=[SI(25)]^2,$$
$$SI(27)+SI(26)=[SI(26)]^2, \quad SI(28)+SI(27)=[SI(27)]^2,$$
$$SI(29)+SI(28)=[SI(28)]^2, \quad SI(30)+SI(29)=[SI(29)]^2,$$
$$SI(31)+SI(30)=[SI(30)]^2, \quad SI(32)+SI(31)=[SI(31)]^2,$$
$$SI(33)+SI(32)=[SI(32)]^2, \quad SI(34)+SI(33)=[SI(33)]^2,$$
$$SI(35)+SI(34)=[SI(34)]^2, \quad SI(36)+SI(35)=[SI(35)]^2,$$
$$SI(37)+SI(36)=[SI(36)]^2, \quad SI(38)+SI(37)=[SI(37)]^2,$$
$$SI(39)+SI(38)=[SI(38)]^2, \quad SI(40)+SI(39)=[SI(39)]^2.$$

From the above values we can notes the following important conjecture.
Conjecture. There are other values on n such that
$SI2(4)+SI2(5)=SI(6)$, (three consecutive positive number)
$SI2(7)+SI2(9)=SI(11)$, (three odd consecutive positive number)
$SI2(6)+SI2(13)=SI(14)$, $\cdots$ etc.
2).

$$SI(n)^2 - SI(n-1)^2 = (n-1)^3. \tag{5}$$

Proof.

$$\begin{aligned} SI(n)^2 - SI(n-1)^2 &= \left[\frac{n(n-1)}{2}\right]^2 - \left[\frac{(n-1)(n-1-1)}{2}\right]^2 \\ &= \left[\frac{n(n-1)}{2}\right]^2 - \left[\frac{(n-1)(n-2)}{2}\right]^2 \\ &= \frac{(n-1)^2(4n-4)}{4} \\ &= (n-1)^3. \end{aligned}$$

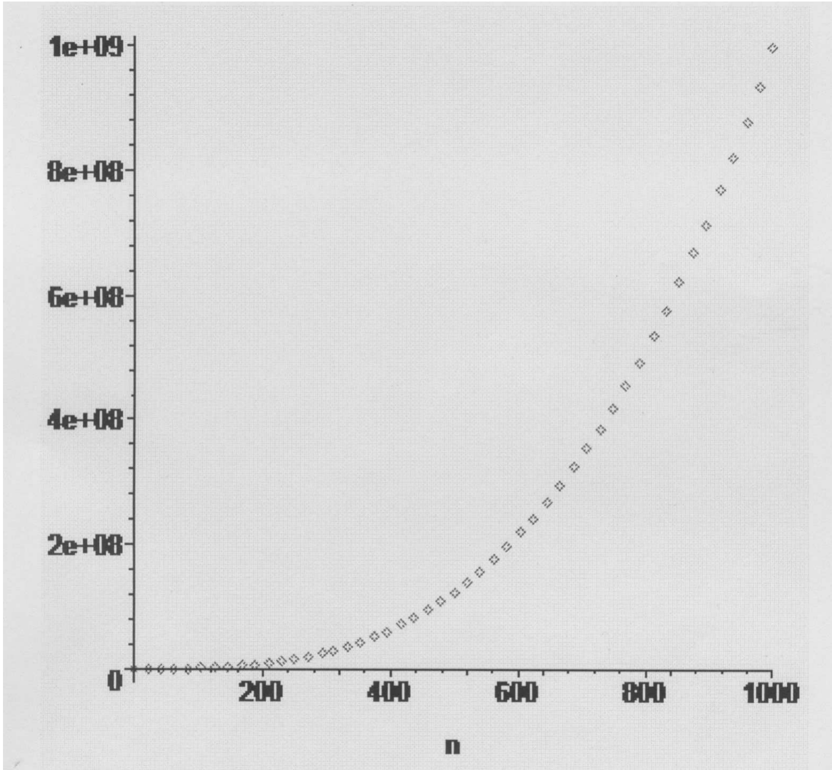

Fig 4: Plot of function $SI(n)^2 - SI(n-1)^2 = (n-1)^3$

Using Maple 8 programming language, verifying the first 28 terms of $SI(n)^2 - SI(n-1)^2 = SI(n-1)^3$:

$$SI(1)^2 - SI(0)^2 = [SI(0)]^3, \quad SI(2)^2 - SI(1)^2 = [SI(1)]^3,$$

$$SI(3)^2 - SI(2)^2 = [SI(2)]^3, \quad SI(4)^2 - SI(3)^2 = [SI(3)]^3,$$

$$SI(5)^2 - SI(4)^2 = [SI(4)]^3, \quad SI(6)^2 - SI(5)^2 = [SI(5)]^3,$$

$$SI(7)^2 - SI(6)^2 = [SI(0)]^3, \quad SI(8)^2 - SI(7)^2 = [SI(7)]^3,$$

$$SI(9)^2 - SI(8)^2 = [SI(0)]^3, \quad SI(10)^2 - SI(9)^2 = [SI(9)]^3,$$

$$SI(11)^2 - SI(10)^2 = [SI(10)]^3, \quad SI(12)^2 - SI(11)^2 = [SI(11)]^3,$$

$$SI(15)^2 - SI(14)^2 = [SI(14)]^3, \quad SI(16)^2 - SI(15)^2 = [SI(15)]^3,$$

$$SI(17)^2 - SI(16)^2 = [SI(16)]^3, \quad SI(18)^2 - SI(17)^2 = [SI(17)]^3,$$

$$SI(19)^2 - SI(18)^2 = [SI(18)]^3, \quad SI(20)^2 - SI(19)^2 = [SI(19)]^3,$$

$$SI(21)^2 - SI(20)^2 = [SI(20)]^3, \quad SI(22)^2 - SI(19)^2 = [SI(21)]^3,$$

$$SI(23)^2 - SI(22)^2 = [SI(22)]^3, \quad SI(24)^2 - SI(23)^2 = [SI(23)]^3,$$

$$SI(25)^2 - SI(24)^2 = [SI(24)]^3, \quad SI(26)^2 - SI(25)^2 = [SI(25)]^3,$$

$$SI(27)^2 - SI(26)^2 = [SI(26)]^3, \quad SI(28)^2 - SI(27)^2 = [SI(27)]^3.$$

Adding (4) and (5) , and only with slight modifications , we could have:
3).

$$(n^2-1)^2+(n^2-1)^3=[n(n-1)(n+1)]^2. \tag{6}$$

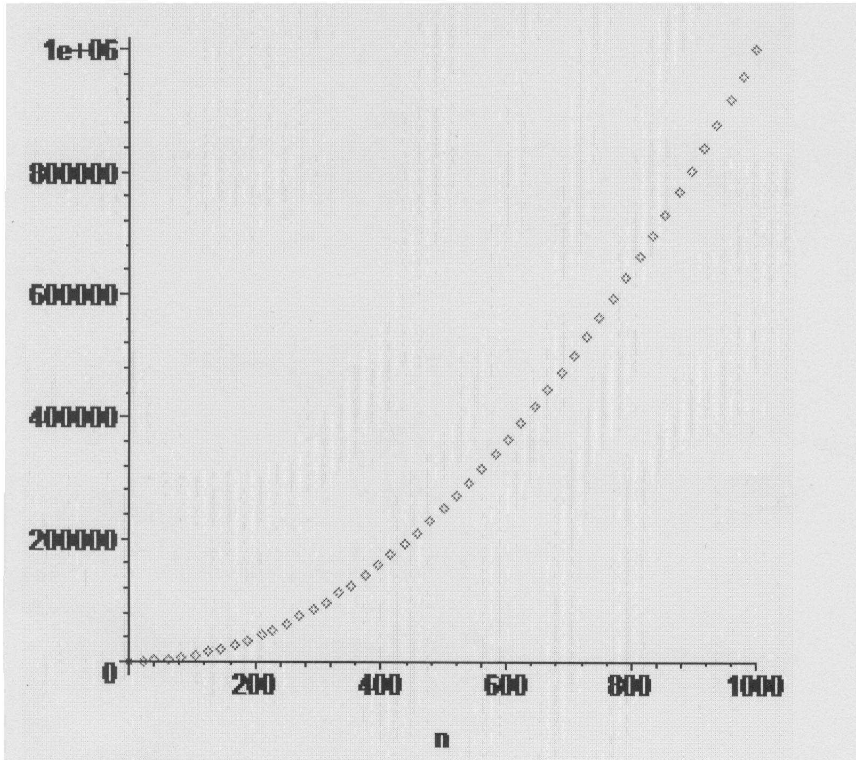

Fig 5: Plot of function $(n^2-1)^2+(n^2-1)^3=[n(n-1)(n+1)]^2$

By direct factorizations and calculations we can easily prove (6).
Using Maple 8 programming language, verifying the first 1680 terms of (6):

$$[0]^2+[0]^3=[0]^2, \qquad [3]^2+[3]^3=[6]^2,$$

$$[8]^2+[8]^3=[24]^2, \qquad [15]^2+[15]^3=[60]^2,$$

$$[24]^2+[24]^3=[120]^2, \qquad [35]^2+[35]^3=[210]^2,$$

$$[48]^2+[48]^3=[336]^2, \qquad [63]^2+[63]^3=[504]^2,$$

$$[80]^2+[80]^3=[720]^2, \qquad [99]^2+[99]^3=[990]^2,$$

$$[120]^2+[120]^3=[1320]^2, \qquad [143]^2+[143]^3=[1716]^2,$$

$$[168]^2+[168]^3=[2184]^2, \qquad [195]^2+[195]^3=[2730]^2,$$

$$[224]^2+[224]^3=[3360]^2, \qquad [255]^2+[255]^3=[4080]^2,$$

$$[288]^2+[288]^3=[4896]^2, \qquad [323]^2+[323]^3=[5814]^2,$$

$$[360]^2+[360]^3=[6840]^2, \qquad [399]^2+[399]^3=[7980]^2,$$

$$[440]^2 + [440]^3 = [9240]^2, \quad [483]^2 + [483]^3 = [10626]^2,$$
$$[528]^2 + [528]^3 = [12144]^2, \quad [575]^2 + [575]^3 = [13800]^2,$$
$$[624]^2 + [624]^3 = [15600]^2, \quad [675]^2 + [675]^3 = [17550]^2,$$
$$[728]^2 + [728]^3 = [19656]^2, \quad [783]^2 + [783]^3 = [21924]^2,$$
$$[840]^2 + [840]^3 = [24360]^2, \quad [899]^2 + [899]^3 = [26970]^2,$$
$$[960]^2 + [960]^3 = [29760]^2, \quad [1023]^2 + [1023]^3 = [32736]^2,$$
$$[1088]^2 + [1088]^3 = [35904]^2, \quad [1155]^2 + [1155]^3 = [39270]^2,$$
$$[1224]^2 + [1224]^3 = [42840]^2, \quad [1295]^2 + [1295]^3 = [46620]^2,$$
$$[1368]^2 + [1368]^3 = [50616]^2, \quad [1443]^2 + [1443]^3 = [54834]^2,$$
$$[1520]^2 + [1520]^3 = [59280]^2, \quad [1599]^2 + [1599]^3 = [63960]^2,$$
$$[1680]^2 + [1680]^3 = [68880]^2.$$

4). Subtracting (4) from (5), and only with slight modifications, we could have:

$$(n^2 + 1)^3 - (n^2 + 1)^2 = n^2(n^2 + 1)^2. \tag{7}$$

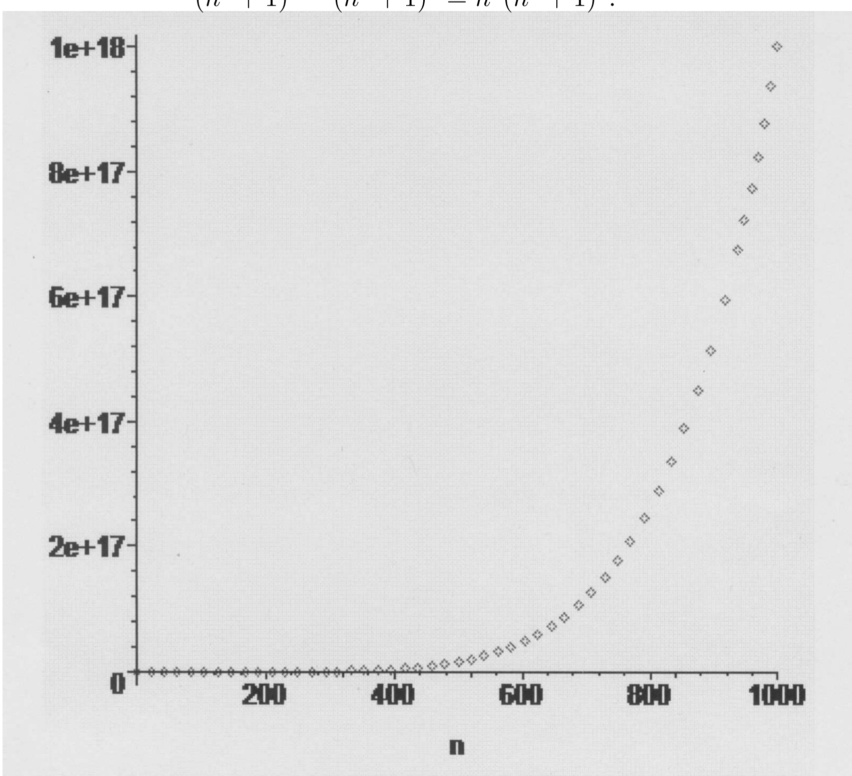

Fig 6: Plot of function $(n^2 + 1)^3 - (n^2 + 1)^2 = n^2(n^2 + 1)^2$

By direct factorizations and calculations we can easily prove (7).

Using Maple 8 programming language, verifying the first 2026 terms of (7):

$$[82]^3 - [82]^2 = [738]^2, \quad [101]^3 - [101]^2 = [1010]^2,$$

$$[122]^3 - [122]^2 = [1342]^2, \quad [145]^3 - [145]^2 = [1740]^2,$$

$$[170]^3 - [170]^2 = [2210]^2, \quad [197]^3 - [197]^2 = [2758]^2,$$

$$[226]^3 - [226]^2 = [3390]^2, \quad [257]^3 - [257]^2 = [4112]^2,$$

$$[290]^3 - [290]^2 = [4930]^2, \quad [325]^3 - [325]^2 = [5850]^2,$$

$$[362]^3 - [362]^2 = [6878]^2, \quad [401]^3 - [401]^2 = [8020]^2,$$

$$[442]^3 - [442]^2 = [9282]^2, \quad [485]^3 - [485]^2 = [10670]^2,$$

$$[530]^3 - [530]^2 = [12190]^2, \quad [577]^3 - [577]^2 = [13848]^2,$$

$$[626]^3 - [626]^2 = [15650]^2, \quad [677]^3 - [677]^2 = [17602]^2,$$

$$[730]^3 - [730]^2 = [19710]^2, \quad [785]^3 - [785]^2 = [21980]^2,$$

$$[842]^3 - [842]^2 = [24418]^2, \quad [901]^3 - [901]^2 = [27030]^2,$$

$$[962]^3 - [962]^2 = [29822]^2, \quad [1025]^3 - [1025]^2 = [32800]^2,$$

$$[1090]^3 - [1090]^2 = [35970]^2, \quad [1157]^3 - [1157]^2 = [39338]^2,$$

$$[1226]^3 - [1226]^2 = [42910]^2, \quad [1297]^3 - [1297]^2 = [46692]^2,$$

$$[1370]^3 - [1370]^2 = [50690]^2, \quad [1445]^3 - [1445]^2 = [54910]^2,$$

$$[1522]^3 - [1522]^2 = [59358]^2, \quad [1601]^3 - [1601]^2 = [64040]^2,$$

$$[1682]^3 - [1682]^2 = [68962]^2, \quad [1765]^3 - [1765]^2 = [74130]^2,$$

$$[1850]^3 - [1850]^2 = [79550]^2, \quad [1937]^3 - [1937]^2 = [85228]^2,$$

$$[2026]^3 - [2026]^2 = [91170]^2.$$

5).

$$SI(n) - SI(n-1) = n - 1. \tag{8}$$

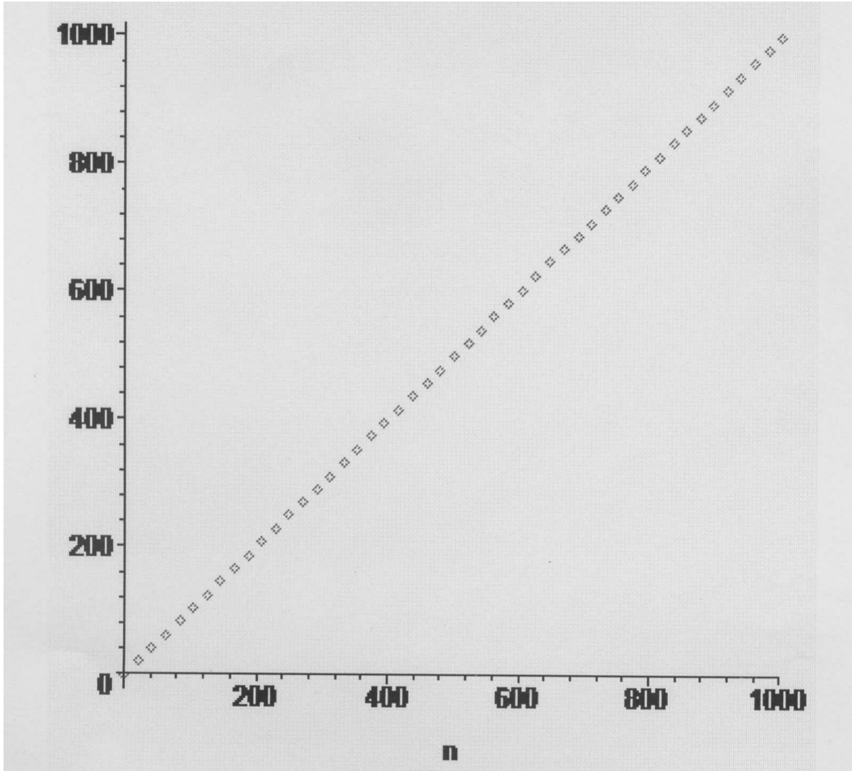

Fig 7: Plot of function $SI(n) - SI(n-1) = n - 1$

Proof.

$$\begin{aligned} SI(n) - SI(n-1) &= \frac{n(n-1)}{2} - \frac{(n-1)(n-1-1)}{2} \\ &= \frac{n(n-1) - (n-1)(n-2)}{2} \\ &= \frac{2n-2}{2} \\ &= n-1. \end{aligned}$$

Using Maple 8 programming language, verifying the first 80 terms of $SI(n) - SI(n-1) = n - 1$:

$$SI(1) - SI(0) = 0, \qquad SI(2) - SI(1) = 1,$$

$$SI(3) - SI(2) = 2, \qquad SI(4) - SI(3) = 3,$$

$$SI(5) - SI(4) = 4, \qquad SI(6) - SI(5) = 5,$$

$$SI(7) - SI(6) = 6, \qquad SI(8) - SI(7) = 7,$$

$$SI(9) - SI(8) = 8, \qquad SI(10) - SI(9) = 9,$$

$$SI(11) - SI(10) = 10, \qquad SI(12) - SI(11) = 11,$$

$$SI(13) - SI(12) = 12, \qquad SI(14) - SI(13) = 13,$$

$$SI(15) - SI(14) = 14, \qquad SI(16) - SI(15) = 15,$$
$$SI(17) - SI(16) = 16, \qquad SI(18) - SI(17) = 17,$$
$$SI(19) - SI(18) = 18, \qquad SI(20) - SI(19) = 19,$$
$$SI(21) - SI(20) = 20, \qquad SI(22) - SI(21) = 21,$$
$$SI(23) - SI(22) = 22, \qquad SI(24) - SI(23) = 23,$$
$$SI(25) - SI(24) = 24, \qquad SI(26) - SI(25) = 25,$$
$$SI(27) - SI(26) = 26, \qquad SI(28) - SI(27) = 27,$$
$$SI(29) - SI(28) = 28, \qquad SI(30) - SI(29) = 29,$$
$$SI(31) - SI(30) = 30, \qquad SI(32) - SI(31) = 31,$$
$$SI(33) - SI(32) = 32, \qquad SI(34) - SI(33) = 33,$$
$$SI(35) - SI(34) = 34, \qquad SI(36) - SI(35) = 35,$$
$$SI(37) - SI(36) = 36, \qquad SI(38) - SI(37) = 37,$$
$$SI(39) - SI(38) = 38, \qquad SI(40) - SI(39) = 39,$$
$$SI(41) - SI(40) = 40, \qquad SI(42) - SI(41) = 41,$$
$$SI(43) - SI(42) = 42, \qquad SI(44) - SI(43) = 43,$$
$$SI(45) - SI(44) = 44, \qquad SI(46) - SI(45) = 45,$$
$$SI(47) - SI(46) = 46, \qquad SI(48) - SI(47) = 47,$$
$$SI(49) - SI(48) = 48, \qquad SI(50) - SI(49) = 49,$$
$$SI(51) - SI(50) = 50, \qquad SI(52) - SI(51) = 51,$$
$$SI(53) - SI(52) = 52, \qquad SI(54) - SI(53) = 53,$$
$$SI(55) - SI(54) = 54, \qquad SI(56) - SI(55) = 55,$$
$$SI(57) - SI(56) = 56, \qquad SI(58) - SI(57) = 57,$$
$$SI(59) - SI(58) = 58, \qquad SI(60) - SI(59) = 59,$$
$$SI(61) - SI(60) = 60, \qquad SI(62) - SI(61) = 61,$$
$$SI(63) - SI(62) = 62, \qquad SI(64) - SI(63) = 63,$$
$$SI(65) - SI(64) = 64, \qquad SI(66) - SI(65) = 65,$$
$$SI(67) - SI(66) = 68, \qquad SI(68) - SI(67) = 67,$$
$$SI(69) - SI(68) = 68, \qquad SI(70) - SI(69) = 69,$$
$$SI(71) - SI(70) = 70, \qquad SI(72) - SI(71) = 71,$$
$$SI(73) - SI(72) = 72, \qquad SI(74) - SI(73) = 73,$$
$$SI(75) - SI(74) = 74, \qquad SI(76) - SI(75) = 75,$$

$$SI(77) - SI(76) = 76, \quad SI(78) - SI(77) = 77,$$
$$SI(79) - SI(78) = 78, \quad SI(80) - SI(79) = 79.$$

6).

$$SI(n+1)SI(n-1) + SI(n) = SI(n)^2.$$

Proof.

$$\begin{aligned} SI(n+1)SI(n-1) + SI(n) &= \frac{n(n+1)}{2} \cdot \frac{(n-2)(n-1)}{2} + \frac{n(n-1)}{2} \\ &= \left[\frac{n(n-1)}{2}\right]^2. \end{aligned}$$

7).

$$SI(n)^2 + SI(n-1)^2 = k^2. \tag{9}$$

In this case I find the following two solutions:
i) $SI(8)^2 + SI(7)^2 = (35)^2$, i.e. $(28)^2 + (21)^2 = (35)^2$,
ii) $SI(42)^2 + SI(41)^2 = (1189)^2$, i.e.$(861)^2 + (820)^2 = (1189)^2$.

8). General SI identities given by numbers:
$SI(0) + SI(1) + SI(2) = 1$,
$SI(1) + SI(2) + SI(3) = 2^2$,
$SI(6) + SI(7) + SI(8) = 2^6$,
$SI(15) + SI(16) + SI(17) = 19^2$,
$SI(64) + SI(65) + SI(66) = 79^2$,
$SI(153) + SI(154) + SI(155) = 2^4 \cdot 47^2$,
$SI(0) + SI(1) + SI(2) + SI(3) = 2^2$,
$SI(6) + SI(7) + SI(8) + SI(9) = 2^2 \cdot 5^2$,
$SI(40) + SI(41) + SI(42) + SI(43) = 2^2 \cdot 29^2$,
$SI(238) + SI(239) + SI(240) + SI(241) = 2^2 \cdot 13^2$,
$SI(19) + SI(20) + SI(21) + SI(22) + SI(23) + SI(24) = 11^3$,
$SI(4) + SI(5) + SI(6) + SI(7) + SI(8) + SI(9) + SI(10) + SI(11) = 2^3 \cdot 3^3$,
$SI(1) - SI(0) + SI(2) = 1$,
$SI(7) - SI(6) + SI(8) = 2^3 \cdot 5^3$,
$SI(1) + SI(0) + SI(2) = 1$,
$SI(5) + SI(4) + SI(6) = 19^2$.

References

[1] Ashbacher.C, Plucking from the tree of Smarandache Sequences and Functions, http://www.gallup.unm.edu/~Smarandache.

[2] M.B. Monagan and others, Maple 7, Programming Guide, Waterloo Maple Inc.

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Global attractivity of a recursive sequence

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Abstract In this paper we investigate the boundedness, the periodic character and the global attractivity of the recursive sequence

$$x_{n+1} = \frac{a + bx_{n-1}}{A - x_n}, \quad n = 0, 1, \cdots,$$

where $a \geq 0$, A, $b > 0$ are real numbers, and the initial conditions x_{-1}, x_0 are arbitrary real numbers. We show that the positive equilibrium of the equation is a global attractor with a basin that depends on certain conditions posed on the coefficients.

Keywords Difference equation, boundedness, global attractivity, global asymptotic stability.

§1. Introduction

Our goal in this paper is to investigate the boundedness, periodic character and global attractivity of all positive solutions of the rational recursive sequence

$$x_{n+1} = \frac{a + bx_{n-1}}{A - x_n}, \quad n = 0, 1, \cdots, \tag{1}$$

where $a \geq 0, A, b > 0$ are real numbers, and the initial conditions x_{-1}, x_0 are arbitrary real numbers.

In [1], C. H. Gibbons et.al. investigated the global behavior of the rational recursive sequence

$$x_{n+1} = \frac{a + bx_{n-1}}{A + x_n}, \quad n = 0, 1, \cdots,$$

where $a, A, b > 0$, and the initial conditions x_{-1}, x_0 are arbitrary positive real numbers. In [2], He, Li and Yan investigated the global behavior of the rational recursive sequence

$$x_{n+1} = \frac{a - bx_{n-1}}{A + x_n}, \quad n = 0, 1, \cdots,$$

where $a \geq 0, A, b > 0$ are real numbers, and the initial conditions x_{-1}, x_0 are arbitrary positive real numbers. For the global behavior of solutions of some related equations, see [3–6]. Other related results reffer to [7–14].

Here, we recall some results which will be useful in the sequel.

Let I be some interval of real numbers and let $f \in C^1[I \times I, I]$. Let $\overline{x} \in I$ be an equilibrium point of the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \cdots, \tag{2}$$

that is $\overline{x} = f(\overline{x}, \overline{x})$. Let

$$s = \frac{\partial f}{\partial u}(\overline{x}, \overline{x}) \quad \text{and} \quad t = \frac{\partial f}{\partial v}(\overline{x}, \overline{x}),$$

denote the partial derivatives of $f(u, v)$ evaluated at an equilibrium $\overline{x}$ of Eq.(2). Then the equation

$$y_{n+1} = sy_n + ty_{n-1}, \ n = 0, 1, \cdots, \tag{3}$$

is called the linearized equation associated with Eq.(2) about the equilibrium point $\overline{x}$.

An interval $J \in I$ is called an invariant interval of Eq.(2), if

$$x_{-1}, x_0 \in J \Rightarrow x_n \in J, \quad \text{for all } n \geq 1.$$

That is, every solution of Eq.(2) with initial conditions in J remains in J.

Definition 1.1. **[9]** The difference equation (2) is said to be permanent, if there exist numbers P and Q with $0 < P \leqslant Q < \infty$ such that for any initial conditions x_{-1}, x_0 there exists a positive integer N which depends on the initial conditions such that

$$P \leqslant x_n \leqslant Q \quad \text{for } n \geq N.$$

Theorem A. **[5]** Linearized stability.

(a) If both roots of the quadratic equation

$$\lambda^2 - s\lambda - t = 0 \tag{4}$$

lie in the open unit disk $\mid \lambda \mid< 1$, then the equilibrium $\overline{x}$ of Eq.(2) is locally asymptotically stable.

(b) If at least one of the roots of Eq.(4) has absolute value greater than one, then the equilibrium $\overline{x}$ of Eq.(2) is unstable.

(c) A necessary and sufficient condition for both roots of Eq.(4) to lie in the open unit disk $\mid \lambda \mid< 1$, is

$$|s| < 1 - t < 2.$$

In this case the locally asymptotically stable equilibrium $\overline{x}$ is also called a sink.

(d) A necessary and sufficient condition for both roots of Eq.(4) to have absolute value greater than one is

$$|t| > 1 \text{ and } |s| < |1 - t|.$$

In this case $\overline{x}$ is called a repeller.

(e) A necessary and sufficient condition for both roots of Eq.(4) to have absolute value greater than one and for the other to have less than one is

$$s^2 + 4t > 0 \text{ and } |s| > |1 - t|.$$

In this case the unstable equilibrium $\overline{x}$ is called a saddle point.

Theorem B. [9] Consider the difference equation

$$x_{n+1} = g(x_n, \cdots, x_{n-k}),\ n = 0, 1, \cdots, \tag{5}$$

where $k \in \{1, 2, \cdots\}$, $g \in C[(0,\infty)^{k+1}, (0,\infty)]$ is increasing in each of its arguments and the initial conditions $x_{-k}, \cdots, x_0$ are positive. Assume that Eq.(5) has a unique positive equilibrium $\overline{x}$ and that the function h defined by

$$h(x) = g(x, \cdots, x), \quad x \in (0,\infty)$$

satisfies

$$(h(x) - x)(x - \overline{x}) < 0 \quad \text{for } x \neq \overline{x}.$$

Then $\overline{x}$ is a global attractor of all positive solutions of Eq.(5).

§2. Period Two Solutions and Linearized Stability

Consider the difference equation (1) with

$$a \geq 0 \text{ and } A, b > 0, \tag{6}$$

we have the following result for its period two solutions.

Theorem 2.1. The Eq.(1) has a prime period two solution if and only if $A = b$.

Proof. Let

$$\cdots,\ \phi,\ \psi,\ \phi,\ \psi,\ \cdots,$$

be a prime period two solution of the Eq.(1). Then we have

$$\phi = \frac{a + b\phi}{A - \psi} \quad \text{and} \quad \psi = \frac{a + b\psi}{A - \phi}.$$

Hence

$$(\phi - \psi)(A - b) = 0,$$

and from which it follows that $A = b$. On the other hand, if $A = b$, then the period two solutions of Eq.(1) must be of the form:

$$\cdots,\ u,\ -\frac{a}{u},\ u,\ -\frac{a}{u},\ \cdots,$$

where $u \in R\backslash\{0\}$ is arbitrary real number. The proof is complete.

The equilibria of Eq.(1) are the solutions of the quadratic equation

$$\overline{x}^2 - (A - b)\overline{x} + a = 0, \tag{7}$$

From Eq.(7) we see that under condition (6), if $a = (A-b)^2/4$, then Eq.(1) has a unique positive equilibrium $\overline{x}_1 = (A - b)/2$; if

$$0 < a < (A - b)^2/4 \ \text{ and } \ A > b > 0, \tag{8}$$

then Eq.(1) has two positive equilibria

$$\overline{x}_2 = \frac{A-b+\sqrt{(A-b)^2-4a}}{2} \quad \text{and} \quad \overline{x}_3 = \frac{A-b-\sqrt{(A-b)^2-4a}}{2}.$$

The linearized equation of Eq.(1) about $\overline{x}_i (i = 1, 2, 3)$ is

$$y_{n+1} - \frac{\overline{x}_i}{A-\overline{x}_i} y_n - \frac{b}{A-\overline{x}_i} y_{n-1} = 0, \quad i = 1, 2, 3, \quad n = 0, 1, \cdots .$$

The following results are consequence of Theorem A by straight forword computations.

Lemma 2.1. Assume that (8) holds, and let $f(u, v) = \dfrac{a+bv}{A-u}$. Then the following statements are true:

(a) The positive equilibrium $\overline{x}_3$ of Eq.(1) is locally asymptotic stable (in the sequel, we will denote $\overline{x}_3$ as $\overline{x}$) .

(b) The positive equilibrium $\overline{x}_2$ of Eq.(1) is a saddle point.

(c) $0 < \overline{x} < \overline{x}_2 < A$.

(d) $f(x, x)$ is a strictly increasing function in $(-\infty, +\infty)$.

(e) Let $u, v \in (-\infty, A) \times [-a/b, +\infty)$, then $f(u, v)$ is a strictly increasing function in u and in v.

§3. The Case $a > 0$

In this section, we will study the global attractivity of all positive solutions of Eq.(1). We show that the positive equilibrium $\overline{x}$ of Eq.(1) is a global attractor with a basin that depends on certain conditions posed on the coefficients.

The following theorem shows that Eq.(1) is permanent under certain conditions.

Theorem 3.1. Assume that (8) holds. Let $\{x_n\}$ be a solution of Eq.(1), if

$$(x_{-1}, x_0) \in [-a/b, \overline{x}_2]^2,$$

then

$$0 < \frac{a}{A} \leqslant x_n \leqslant \overline{x}_2 \quad \text{for } n \geq 3.$$

Proof. By part (e) of Lemma 2.1, we have

$$0 = f(x_0, -\frac{a}{b}) \leqslant x_1 = f(x_0, x_{-1}) \leqslant f(\overline{x}_2, \overline{x}_2) = \overline{x}_2,$$

and

$$0 = f(x_1, -\frac{a}{b}) \leqslant x_2 = f(x_1, x_0) \leqslant f(\overline{x}_2, \overline{x}_2) = \overline{x}_2.$$

Furthermore, we have

$$0 < \frac{a}{A} = f(0, 0) \leqslant x_3 = f(x_2, x_1) \leqslant f(\overline{x}_2, \overline{x}_2) = \overline{x}_2,$$

and

$$0 < \frac{a}{A} = f(0, 0) \leqslant x_4 = f(x_3, x_2) \leqslant f(\overline{x}_2, \overline{x}_2) = \overline{x}_2.$$

Hence, the result follows by induction. The proof is complete.

By Theorem 3.1 and definition of the invariant interval, we know that the interval $[0, \overline{x}_2]$ is an invariant interval of Eq.(1).

Theorem 3.2. Assume that (8) holds. Then the positive equilibrium $\overline{x}$ of Eq.(1) is a global attractor with a basin

$$S = (0, \overline{x}_2)^2.$$

Proof. By part (e) of Lemma 2.1, for any $u, v \in (0, \overline{x}_2)$, we have

$$0 < f(u, v) = \frac{a + bv}{A - u} < \frac{a + b\overline{x}_2}{A - \overline{x}_2} = \overline{x}_2.$$

Hence, $f \in C[(0, \overline{x}_2)^2, (0, \overline{x}_2)]$. Let the function g be defined as

$$g(x) = f(x, x), \quad x \in (0, \overline{x}_2).$$

Then we have

$$\begin{aligned} & (g(x) - x)(x - \overline{x}) \\ = & \left(\frac{a + bx}{A - x} - x\right)(x - \overline{x}) = \frac{1}{A - x}\left(x^2 - (A - b)x + a\right)(x - \overline{x}) \\ = & \frac{(x - \overline{x})}{A - x}\left(x - \frac{(A - b) + \sqrt{(A - b)^2 - 4a}}{2}\right)\left(x - \frac{(A - b) - \sqrt{(A - b)^2 - 4a}}{2}\right) \\ = & \frac{1}{A - x}(x - \overline{x}_2)(x - \overline{x})^2 < 0, \quad \text{for } x \neq \overline{x}. \end{aligned}$$

By Theorem B, $\overline{x}$ is a global attractor of all positive solutions of Eq.(1) with the initial conditions $(x_{-1}, x_0) \in S$. That is, let $\{x_n\}$ be a solution of Eq.(1) with initial conditions $(x_{-1}, x_0) \in S$, then we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

The proof is complete.

Theorem 3.3. Assume that (8) holds. Then the positive equilibrium $\overline{x}$ of Eq.(1) is a global attractor with a basin

$$S = \left[-\frac{a}{b}, \overline{x}_2\right]^2 \backslash \{(\overline{x}_2, \overline{x}_2)\}.$$

Proof. Let $\{x_n\}$ be a solution of Eq.(1) with initial conditions $(x_{-1}, x_0) \in S$. Then, by Theorem 3.2, we have $x_n \in (0, \overline{x}_2)$, for $n \geq 1$. Hence, Theorem 3.2 implies that

$$\lim_{n \to \infty} x_n = \overline{x}.$$

The proof is complete.

§4. The Case $a = 0$

In this section, we study the asymptotic stability for the difference equation

$$x_{n+1} = \frac{bx_{n-1}}{A - x_n}, \quad n = 0, 1, \cdots, \tag{9}$$

where $b, A \in (0, \infty)$, and the initial conditons x_{-1}, x_0 are arbitrary real numbers.

By putting $x_n = by_n$, Eq.(9) yields

$$y_{n+1} = \frac{y_{n-1}}{C - y_n}, \quad n = 0, 1, \cdots, \tag{10}$$

where $C = A/b > 0$, Eq.(10) has two equilibria $\overline{y}_1 = 0$, $\overline{y}_2 = C - 1$. The linearized equation of the Eq.(10) about the equilibria $\overline{y}_i,\ i = 1, 2$, is

$$z_{n+1} - \frac{\overline{y}_i}{C - \overline{y}_i} z_n - \frac{1}{C - \overline{y}_i} z_{n-1} = 0,\ i = 1, 2,\ n = 0, 1, \cdots,$$

For $\overline{y}_2 = C - 1$, by Theorem A we can see that it is a saddle point.

For $\overline{y}_1 = 0$, we have

$$z_{n+1} - \frac{1}{C} z_{n-1} = 0, \quad n = 0, 1, \cdots, \tag{11}$$

the characteristic equation of Eq.(11) is

$$\lambda^2 - \frac{1}{C} = 0.$$

Hence, by Theorem A, we have

(i) If $A > b$, then $\overline{y}_1$ is locally asymptotically stable.

(ii) If $A < b$, then $\overline{y}_1$ is a repeller.

(iii) If $A = b$, then linearized stability analysis fails.

In the sequel, we discuss the global attractivity of the zero equilibrium of Eq.(10). So we assume that $A > b$, namely, $C > 1$.

Lemma 4.1. Assume that $(y_{-1}, y_0) \in [-C+1, C-1]^2$. Then any solution $\{y_n\}$ of Eq.(10) satisfies $y_n \in [-C+1, C-1]$ for $n \geq 1$.

Proof. Since $(y_{-1}, y_0) \in [-C+1, C-1]^2$, then we have

$$-C + 1 \leqslant \frac{-C+1}{C - (-C+1)} \leqslant y_1 = \frac{y_{-1}}{C - y_0} \leqslant \frac{C-1}{C - (C-1)} = C - 1$$

and

$$-C + 1 \leqslant \frac{-C+1}{C - (-C+1)} \leqslant y_2 = \frac{y_0}{C - y_1} \leqslant \frac{C-1}{C - (C-1)} = C - 1.$$

The result follows by induction, and the proof is complete.

By Lemma 4.1 and the definition of the invariant interval, we know that the interval $[-C+1, C-1]$ is an invariant interval of Eq.(10). Also Lemma 4.1 implies that the following result is true.

Theorem 4.1 The equilibrium $\overline{y}_1 = 0$ of Eq.(10) is a global attractor with a basin

$$S = (-C+1, C-1)^2.$$

References

[1] C. H. Gibbons, M. R. S. Kulenovic, G. Ladas, On the recursive sequence $y_{n+1} = ((\alpha + \beta y_{n-1})/(\gamma + y_n))$, Math. Sci. Res. Hot-Line, **4**(2000), No.2, 1-11.

[2] W. S. He, W. T. Li and X. X. Yan, Global attractivity in a nonlinear difference equation, International Journal of Applied Mathematics, **11**(2002), No.3, 283-293.

[3] X. X. Yan and W. T. Li, Global Attractivity in the recursive sequence $x_{n+1} = (a - bx_n)/(A - x_{n-1})$, Appl. Math. Comput., **138**(2003), 415-423.

[4] X. X. Yan, W. T. Li and H. R. Sun, Global attractivity in a higher order nonlinear difference equation, Appl. Math. E-Notes, **2**(2002), 51-58.

[5] K. C. Cunningham, M. R. S. Kulenovic, G. Ladas and S. V. Valicenti, On the recursive sequence $x_n = (\alpha + \beta x_n)/(Bx_n + Cx_{n-1})$, Nonlinear Analysis TMA, **47**(2001), 4603-4614.

[6] M. T. Aboutaleb, M. A. EI-Sayed and A. E. Hamza, Stability of the recursive sequence $x_{n+1} = (\alpha - \beta x_n)/(\gamma + x_{n-1})$, J. Math. Anal. Appl., **261**(2001), 126-133.

[7] W. T. Li and H. R. Sun, Global attractivity in a rational recursive, Dynamic Systems and Applications, **11**(2002), No.3, 339-345.

[8] M. R. S. Kulenovic, G. Ladas and N. R. Prokup, On The recursive sequence $x_{n+1} = (\alpha x_n + \beta x_{n-1})/(1 + x_n)$, J. Differ. Equation Appl., **5**(2000), 563-576.

[9] V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Application, Kluwer Academic Publishers, Dordrecht, 1993.

[10] R. DeVault, W. Kosmala, G. Ladas and S. W. Schultz, Global behavior of $y_{n+1} = (p + y_{n-k})/(qy_n + y_{n-k})$, Nonlinear Analysis TMA, **47**(2001), 4743-4751.

[11] C. H. Gibbons, M. R. S. Kulenovic, G. Ladas and H. D. Voulov, On the trichotomy character of $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + x_n)$, J. Difference Equations and applications, **8**(2002), No.1, 75-92.

[12] W. T. Li, H. R. Sun and X. X. Yan, The asymtotic behavior of a higher order delay nonlinear difference equations, Indian J. Pure Appl. Math., **34**(2003), No.10, 1431-1441.

[13] X. X. Yan and W. T. Li, Global attractivity in a rational recursive sequence, Appl. Math. Comput., **145**(2003), No.1, 1-12.

[14] X. X. Yan and W. T. Li, Global attractivity for a class of higher order nonlinear differential equations, Appl. Math. Comput., **149**(2004), No.2, 533-546.

[15] W. S. He and W. T. Li, Global attractivity of a higher order nonlinear difference equation, International Journal of Mathematics Sciences, **2**(2003), No.2, 251-259.

[16] W. S. He, W. T. Li and X. X. Yan, Global attractivity of the difference equation $x_{n+1} = \alpha + (x_{n-k}/x_n)$, Appl. Math. Comput., **151**(2004), No.3, 879-885.

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Smarandache reversed slightly excessive numbers

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In Charles Ashbacher's paper, "On Numbers that are Pseudo-Smarandache and Smarandache Perfect" [1] he discussess the operation of summing the divisors of a number after a function has been applied to those divisors. In this note we consider the process of applying the reverse function to the divisors of a number and then summing them.

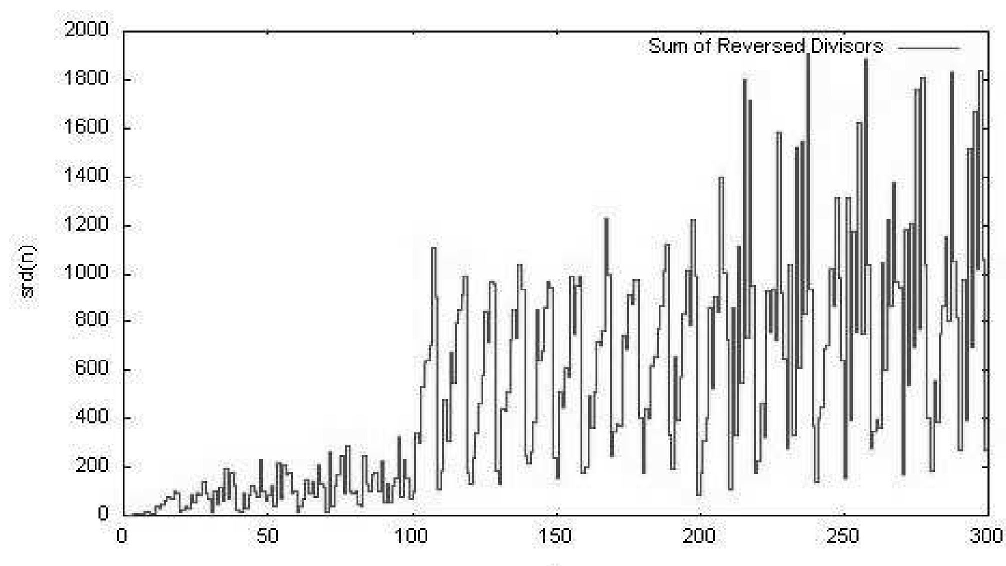
Let $srd(n)$ be the Smarandache sum of reversed divisors of n. This function produces the following sequence:

$$n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, \cdots,$$

$$srd(n) = 1, 3, 4, 7, 6, 12, 8, 15, 13, 9, 12, 37, 32, 51, \cdots.$$

For example, $srd(14) = 51$, because the divisors of 14 are 1, 2, 7, 14, then reversing and summing gives $1 + 2 + 7 + 41 = 51$.

Here is a graph of the first 300 values:



Joseph L. Pe investigated some aspects of this function in his paper, “The Picture-Perfect Numbers” [2]. The purpose of this note is to provide some results on a new $srd(n)$ problem.

Background

Pythagoras and his followers were the first to notice they could not find integers equal to one more than the sum of their divisors [3]. That is, they could not find solutions to $\sigma(n) = 2n+1$. And they referred to these numbers–still hoping at least one might exist–as “slightly excessive numbers.” To this day a solution has not been found, and the problem remains open, even though a couple of things are known about “slightly excessives” – if one exists it is larger than 1035; and it has more than 7 distinct prime factors [4].

We shall alter this “slightly excessive” problem in two ways and search for solutions. Our modification will consist of (1) summing all the divisors of n, not just the proper divisors; (2) reversing the divisors before summing them–i.e., we'll use the $srd(n)$ function.

Solutions to $srd(n) = n + 1$

It is easy to see that palprimes will always be solutions: The only divisors of a prime are 1and itself, and because a palindrome's largest nonproper divisor is already the same as its reversal, the sum of a palprime's divisors will always be equal to $n + 1$.

Interestingly, a computer search reveals that there are also non-palprime solutions to $srd(n) = n + 1$. Namely: 965, 8150, 12966911 satisfy the equation, with no more found up to $2 * 107$. Example: The divisors of 965 are 1, 5, 193, 965; and reversing and summing produces $1 + 5 + 391 + 569 = 966$. Also note that there are no obvious patterns in the factorizations of these solutions:

$$965 = 5 * 193,$$

$$8150 = 2 * 5 * 5 * 163,$$

$$12966911 = 19 * 251 * 2719.$$

Open questions. Are there any more non-palprime solutions to $srd(n) = n + 1$? If so, do the solutions have any properties in common? Are there infinitely many solutions?

References

[1] Charles Ashbacher, On Numbers That Are Pseudo-Smarandache and Smarandache Perfect, Smarandache Notions Journal, **14** (2004), pp. 40-41.

[2] Joseph L. Pe, Picture-Perfect Numbers and Other Digit-Reversal Diversions, http://jlpe.tripod.com/ppn/ppnpaper.htm.

[3] S. Singh, Fermat's Enigma, The Epic Quest to Solve the World's Greatest Mathematical Problem, Walker and Co. NY, 1997, pp. 13.

[4] Eric W. Weisstein, Quasiperfect Number, From MathWorld–A Wolfram Web Resource, http://mathworld.wolfram.com/QuasiperfectNumber.html.

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A new structure of super $\mathcal{R}^*$-unipotent semigroups[1]

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Abstract A super $\mathcal{R}^*$-unipotent semigroup is a super abundant semigroup S in which every $\mathcal{R}^*$-class of S contains a unique idempotent and whose idempotents form a subsemigroup of S. This kinds of semigroups have been investigated in [1], [3] and [5]. The aim of this paper is to introduce the concept of the generalized left Δ-product of semigroups and to establish a new construction of super $\mathcal{R}^*$-unipotent semigroups, namely the generalized left Δ-product structure of a super $\mathcal{R}^*$-unipotent semigroup.

Keywords Generalized left Δ-product, super $\mathcal{R}^*$-unipotent semigroups, left regular bands, cancellative monoids

§1. Introduction

On a semigroup S the relation $\mathcal{L}^*$ is defined by the rule that $a\mathcal{L}^*b$ if and only if the elements a and b of S are $\mathcal{L}$-related on some over semigroup of S. Dually the relation $\mathcal{R}^*$ can be defined. The relation $\mathcal{H}^*$ is the intersection of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$. It was noted in [2] that $\mathcal{L} \subseteq \mathcal{L}^*$ on any semigroup S. In particular, $\mathcal{L} = \mathcal{L}^*$ on a regular semigroup S. It is easy to see that the relation $\mathcal{L}^*$ is a generalization of the usual Green's relation $\mathcal{L}$ on a semigroup S. A semigroup S is called abundant if every $\mathcal{L}^*$-class and $\mathcal{R}^*$-class of S contains an idempotent. A semigroup S is called super abundant if every $\mathcal{H}^*$-class of S contains an idempotent (see [2]). According to [1], a super $\mathcal{R}^*$-unipotent semigroup is a super abundant semigroup S if the set of all idempotents of S forms a subsemigroup and every $\mathcal{R}^*$-class of S contains a unique idempotent. Clearly, a super $\mathcal{R}^*$-unipotent semigroup is indeed a generalization of a left C–semigroup in the class of abundant semigroups. This class of abundant semigroups was first introduced by El-Qallali in [1]. Later, Guo, Guo and Shum studied this kinds of semigroups from another view point and they called such semigroups the left C–a semigroups in [5]. It was shown in [5] that such a semigroup can be constructed by a semi-spined product of semigroups. In this paper, we will introduce the concept of generalized left Δ-product of semigroups and establish a new construction of a super $\mathcal{R}^*$-unipotent semigroup by using the generalized left Δ-product of semigroups.

Terminologies and notations not mentioned in this paper should be referred to [1],[3] and [7].

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§2. Preliminaries and generalized left Δ-products

We first recall some basic results known which are useful for our construction of a super $\mathcal{R}^*$-unipotent semigroup.

Lemma 2.1. [2] Let S be a semigroup and let $a, b \in S$. Then the following statements hold:

(1) $a\mathcal{L}^*b$ if and only if for all $x, y \in S^1, ax = ay \Leftrightarrow bx = by$.

(2) for any idempotent $e \in S, e\mathcal{L}^*a$ if and only if $ae = a$ and for all $x, y \in S^1, ax = ay \Rightarrow ex = ey$.

It is easy to see that on any semigroup S, $\mathcal{L} \subseteq \mathcal{L}^*$ holds and for any regular elements $a, b \in S$, we have $(a, b) \in \mathcal{L}^*$ if and only if $(a, b) \in \mathcal{L}$. The dual results for $\mathcal{R}^*$ also hold.

The following characterizations of super $\mathcal{R}^*$-unipotent semigroups are crucial in the studying of a super $\mathcal{R}^*$-unipotent semigroup.

Lemma 2.2. [3] The following conditions on a semigroup S are equivalent:

(i) S is a super $\mathcal{R}^*$-unipotent semigroup;

(ii) S is an abundant semigroup in which $\mathcal{R}^* = \mathcal{H}^*$;

(iii) S is a super abundant semigroup and $eS \subseteq Se$ for any $e \in S \cap E$;

(iv) S is a semilattice Y of the direct product S_α of a left zero band I_α and a cancellative monoid M_α and $H^*_a(S) = H^*_a(S_\alpha)$ for $a \in S_\alpha$ and $\alpha \in Y$.

We now introduce the concept of generalized left Δ-product of semigroups which is a modification of left Δ-product of semigroups given in [6].

Let Y be a semilattice and $M = [Y; M_\alpha, \theta_{\alpha,\beta}]$ be a strong semilattice of monoids M_α with structure homomorphism $\theta_{\alpha,\beta}$, and let $I = \bigcup_{\alpha\in Y} I_\alpha$ be a semilattice decomposition of left regular band I into left zero band I_α. For every $\alpha \in Y$ we use S_α to denote the direct product $I_\alpha \times M_\alpha$ of a left zero band I_α and a monoid M_α, and use $\mathcal{J}^*(\mathcal{I}_\alpha)$ to denote the left transformation semigroup on I_α.

Now for any $\alpha, \beta \in Y$ with $\alpha \geqslant \beta$ and $a \in S_\alpha$, define a mapping

$$\Phi_{\alpha,\beta} : S_\alpha \to \mathcal{J}^*(\mathcal{I}_\beta)$$

$$a \mapsto \varphi^a_{\alpha,\beta}$$

satisfying the following conditions:

(C1) If $(i, g) \in S_\alpha$, $j \in I_\alpha$, then $\varphi^{(i,g)}_{\alpha,\alpha} j = i$;

(C2) For any $\alpha, \beta \in Y$ and any $(i, g) \in S_\alpha$, $(j, f) \in S_\beta$,

(i) $\varphi^{(i,g)}_{\alpha,\alpha\beta}\varphi^{(j,f)}_{\beta,\alpha\beta}$ is a constant mapping on $I_{\alpha\beta}$, denote the constant value by $\langle\varphi^{(i,g)}_{\alpha,\alpha\beta}\varphi^{(j,f)}_{\beta,\alpha\beta}\rangle$;

(ii) if $\alpha, \beta, \delta \in Y$ with $\alpha\beta \geqslant \delta$ and $\langle\varphi^{(i,g)}_{\alpha,\alpha\beta}\varphi^{(j,f)}_{\beta,\alpha\beta}\rangle = k$, then $\varphi^{(k,gf)}_{\alpha\beta,\delta} = \varphi^{(i,g)}_{\alpha,\delta}\varphi^{(j,f)}_{\beta,\delta}$;

(iii) for any $\gamma \in Y$ and any $(k,u) \in S_\beta, (l,v) \in S_\gamma$,
$\varphi^{(i,g)}_{\alpha,\alpha\beta}\varphi^{(k,u)}_{\beta,\alpha\beta} = \varphi^{(i,g)}_{\alpha,\alpha\beta}\varphi^{(l,v)}_{\gamma,\alpha\gamma}$ implies $\varphi^{(i,1_\alpha)}_{\alpha,\alpha\beta}\varphi^{(k,u)}_{\beta,\alpha\beta} = \varphi^{(i,1_\alpha)}_{\alpha,\alpha\beta}\varphi^{(l,v)}_{\gamma,\alpha\gamma}$,
and also
$\varphi^{(k,u)}_{\beta,\alpha\beta}\varphi^{(i,g)}_{\alpha,\alpha\beta} = \varphi^{(l,v)}_{\gamma,\alpha\gamma}\varphi^{(i,g)}_{\alpha,\alpha\beta}$ implies $\varphi^{(k,u)}_{\beta,\alpha\beta}\varphi^{(i,1_\alpha)}_{\alpha,\alpha\beta} = \varphi^{(l,v)}_{\gamma,\alpha\gamma}\varphi^{(i,1_\alpha)}_{\alpha,\alpha\beta}$,
where 1_α is the identity of a monoid M_α.

Now form the set $S = \bigcup_{\alpha\in Y} S_\alpha$ and for any $(i,g) \in S_\alpha$, $(j,f) \in S_\beta$, define a multiplication "$\circ$" on S by the rule that

$$(i,g)\circ(j,f) = (\langle\varphi^{(i,g)}_{\alpha,\alpha\beta}\varphi^{(j,f)}_{\beta,\alpha\beta}\rangle, gf). \tag{1}$$

It can be easily verified that the multiplication "$\circ$" on S is associative and hence $(S,\circ)$ becomes a semigroup. We refer to this semigroup as the generalized left Δ-product of semigroups I and M with respect to structure mapping $\Phi_{\alpha,\beta}$, denote it by $S = I\Delta_{Y,\Phi}M$.

§3. Structure of super $\mathcal{R}^*$-unipotent semigroups

The aim of this section is to establish another construction of a super $\mathcal{R}^*$-unipotent semigroup S. We have the following result

Theorem 3.1. Let $M = [Y; M_\alpha, \theta_{\alpha,\beta}]$ be a strong semilattice of cancellative monoids M_α with structure homomorphism $\theta_{\alpha,\beta}$. Let $I = \bigcup_{\alpha\in Y} I_\alpha$ be a semilattice decomposition of left regular band I into left zero band I_α. Then the generalized left Δ-product $I\Delta_{Y,\Phi}M$ of I and M is a super $\mathcal{R}^*$-unipotent semigroup.

Conversely, every super $\mathcal{R}^*$-unipotent semigroup can be constructed in this way.

Proof. To prove the direct part of Theorem 3.1, we suppose that a semigroup S is a generalized left Δ-product $I\Delta_{Y,\Phi}M$ of a left regular band I and a strong semilattice M of cancellative monoids M_α. We will show that S is a super $\mathcal{R}^*$-unipotent semigroup by the following steps.

(1) We first show that $E(S) = \bigcup_{\alpha\in Y}\{(i,1_\alpha) \in I_\alpha \times M_\alpha | i \in I_\alpha\}$, where 1_α is the identity element of M_α. By using the multiplication given in (1) and the condition (C1), we immediately have

$$(i,1_\alpha)\circ(i,1_\alpha) = (\langle\varphi^{(i,1_\alpha)}_{\alpha,\alpha}\varphi^{(i,1_\alpha)}_{\alpha,\alpha}\rangle, 1_\alpha) = (i,1_\alpha).$$

On the other hand, if $(i,g)\circ(i,g) = (i,g)$, then we can see that g is the identity of the cancellative monoid M_α. Thus, the set $E(S)$ is the set of all idempotent in S.

(2) If $(i,g),(j,f) \in S_\alpha$, then we have

$$(i,g)\circ(j,f) = (\langle\varphi^{(i,g)}_{\alpha,\alpha}\varphi^{(j,f)}_{\alpha,\alpha}\rangle, gf) = (i,gf).$$

This shows that the restriction of the multiplication on S to S_α coincides with the previous multiplication on S_α. Also, it is easy to see that S itself is a semilattice Y of semigroups S_α.

To prove that S is a super $\mathcal{R}^*$-unipotent semigroup, by using Lemma 2.2 (iv), we only need to show that $H^*_a(S) = H^*_a(S_\alpha)$ for any $\alpha \in Y$ and any $a \in S_\alpha$. Suppose that $a\mathcal{L}^*(S_\alpha)b$

for any $a, b \in S_\alpha$. Then taking $a = (i,g) \in S_\alpha$ and $e = (i, 1_\alpha) \in S_\alpha$, by the definition of multiplication on S, we have $ae = (i,g)(i,1_\alpha) = (\langle \varphi_{\alpha,\alpha}^{(i,g)} \varphi_{\alpha,\alpha}^{(i,1_\alpha)} \rangle, g) = (i,g) = a$. If $ax = ay$ for any $x = (k,u) \in S_\beta, y = (l,v) \in S_\gamma$, then by the condition (C2)(iii), we have that $ex = ey$ and so $a\mathcal{L}^*(S)e$ by Lemma 2.1. If $b = (j,h) \in S_\alpha$, then there exists idempotent $f = (j, 1_\alpha) \in S_\alpha$ such that $b\mathcal{L}^*(S)f$. Clearly, $e\mathcal{L}f$. This leads to $a\mathcal{L}^* e\mathcal{L} f\mathcal{L}^* b$ and $a\mathcal{L}^*(S)b$. This shows that $L_a^*(S_\alpha) \subseteq L_a^*(S)$. Clearly, $L_a^*(S) \subseteq L_a^*(S_\alpha)$ and hence $L_a^*(S) = L_a^*(S_\alpha)$. Similarly, we have $R_a^*(S) = R_a^*(S_\alpha)$. Thus, we have that $H_a^*(S) = H_a^*(S_\alpha)$. By Lemma 2.2 (iv), we have shown that $S = I\Delta_{Y,\Phi}M$ is a super $\mathcal{R}^*$-unipotent semigroup.

Next we proceed to prove the converse part of Theorem 3.1. Suppose that S is any super $\mathcal{R}^*$-unipotent semigroup. In fact, by Lemma 2.2(iv), there exists a semilattice Y of semigroup $S_\alpha = I_\alpha \times M_\alpha$, where each I_α is a left zero band and each M_α is a cancellative monoid. Since M_α is a cancellative monoid, it is easy to check that $E(S) = \{(i, 1_\alpha) \in I_\alpha \times M_\alpha | i \in I_\alpha \ \&\ \alpha \in Y\}$. By Lemma 2.2(iii), we can deduce that $E(S)$ is a left regular band. Now we form the set $I = \bigcup_{\alpha \in Y} I_\alpha$ and $M = \bigcup_{\alpha \in Y} M_\alpha$.

In order to show that S is isomorphic to a generalized left Δ-product $I\Delta_{Y,\Phi}M$, we have to do by the following steps:

(1) We will point out that I forms a left regular band. For this purpose, it suffices to show that I is isomorphic to $E(S)$ which is the set of all idempotent of S. Hence, we consider the mapping $\eta : E(S) \to I$ by $(i, 1_\alpha) \mapsto i$ for any $i \in I_\alpha$.

It is easy to see that η is a bijection. If a multiplication on I given by $ji = k$ for $i \in I_\alpha, j \in I_\beta$ if and only if

$$(j, 1_\beta)(i, 1_\alpha) = (k, 1_{\beta\alpha}), \tag{2}$$

then the set $I = \bigcup_{\alpha \in Y} I_\alpha$ under the above multiplication forms a semigroup isomorphic to $E(S)$, that is, I is a left regular band.

(2) We can also claim that M is a strong semilattice of cancellative monoid M_α. To see this, we suppose that $(i,g) \in S_\alpha, (j, 1_\beta) \in S_\beta \bigcap E(S)$ for any $\alpha, \beta \in Y$ with $\alpha \geqslant \beta$. Let $(j,1_\beta)(i,g) = (k,f) \in S_\beta$. Since $(j,1_\beta)(i,g) = (j,1_\beta)[(j,1_\beta)(i,g)] = (j,1_\beta)(k,f) \in S_\beta$, it follows that

$$(j,1_\beta)(i,g) = (j,f) \in S_\beta.$$

Moreover, since E is a left regular band, we know that for any $l \in I_\alpha$

$$\begin{aligned} & (j,1_\beta)(l,g) = (j,1_\beta)(l,1_\alpha)(i,g) \\ = \quad & (j,1_\beta)[(l,1_\alpha)(j,1_\beta)](i,g) = (j,1_\beta)(i,g). \end{aligned}$$

This implies that the choice of f is independent of i. Consequently, define $\theta_{\alpha,\beta} : M_\alpha \to M_\beta$ by $g \mapsto g\theta_{\alpha,\beta}$ if and only if

$$(j,1_\beta)(i,g) = (j, g\theta_{\alpha,\beta}). \tag{3}$$

Clearly, the mapping $\theta_{\alpha,\beta}$ is well defined and $\theta_{\alpha,\alpha}$ is the identity mapping on M_α. For any $(i,g),(l,f) \in S_\alpha$, we have

$$(j,1_\beta)[(i,g)(l,f)] = (j,1_\beta)(i,gf) = (j,(gf)\theta_{\alpha,\beta}).$$

and

$$\begin{aligned} & [(j,1_\beta)(i,g)](l,f) \\ = \ & (j,g\theta_{\alpha,\beta})[(k,1_\beta)(l,f)] \\ = \ & (j,g\theta_{\alpha,\beta})(k,f\theta_{\alpha,\beta}) \\ = \ & (j,g\theta_{\alpha,\beta}f\theta_{\alpha,\beta}). \end{aligned}$$

Hence, $(gf)\theta_{\alpha,\beta} = g\theta_{\alpha,\beta}f\theta_{\alpha,\beta}$. This shows that $\theta_{\alpha,\beta}$ is a homomorphism of semigroup.

If $\alpha,\beta,\gamma \in Y$ with $\alpha \geqslant \beta \geqslant \gamma$ and $(i,g) \in S_\alpha$, then using (3), we have that

$$(k,1_\gamma)[(j,1_\beta)(i,g)] = (k,1_\gamma)(j,g\theta_{\alpha,\beta}) = (k,g\theta_{\alpha,\beta}\theta_{\beta,\gamma})$$

and

$$[(k,1_\gamma)(j,1_\beta)](i,g) = (k,1_\gamma)(i,g) = (k,g\theta_{\alpha,\gamma}).$$

This implies that $\theta_{\alpha,\beta}\theta_{\beta,\gamma} = \theta_{\alpha,\gamma}$. Thus $M = [Y; M_\alpha, \theta_{\alpha,\beta}]$ is a strong semilattice of M_α.

(3) We now consider how to obtain the mapping $\Phi_{\alpha,\beta}$ defined as in a generalized left Δ-product $I\Delta_{Y,\Phi}M$ of semigroups I and M. For this purpose, suppose that $\alpha,\beta \in Y$ with $\alpha \geqslant \beta$. Then for any $(i,g) \in S_\alpha = I_\alpha \times M_\alpha, (j,1_\beta) \in S_\beta \bigcap E(S)$, let $(i,g)(j,1_\beta) = (k,h) \in S_\beta$ for some $k \in I_\beta, h \in M_\beta$. By applying (2), we deduce that

$$\begin{aligned} & (i,g)(j,1_\beta) = (k,1_\beta)(i,g)(j,1_\beta) \\ = \ & (k,1_\beta)(j,1_\beta)(i,g)(j,1_\beta) \\ = \ & (k,1_\beta)(k,g\theta_{\alpha,\beta})(j,1_\beta) \\ = \ & (k,g\theta_{\alpha,\beta}) \in S_\beta. \end{aligned}$$

Consequently, we have

$$(i,g)(j,1_\beta) = (k,g\theta_{\alpha,\beta}) \in S_\beta. \tag{4}$$

From this, we can define a mapping $\Phi_{\alpha,\beta} : S_\alpha \longrightarrow \mathcal{J}^*(\mathcal{I}_\beta)$ given by $(i,g) \mapsto \varphi_{\alpha,\beta}^{(i,g)}$ such that

$$(i,g)(j,1_\beta) = (\varphi_{\alpha,\beta}^{(i,g)}j, g\theta_{\alpha,\beta}). \tag{5}$$

We will see that the conditions (C1) and (C2) in the generalized left Δ-product $I\Delta_{Y,\Phi}M$ of semigroups I and M are satisfied by the mapping $\Phi_{\alpha,\beta}$.

(i) It is a routine matter to verify that $\Phi_{\alpha,\beta}$ satisfies the condition (C1).

(ii) To show that $\Phi_{\alpha,\beta}$ satisfies the condition (C2)(i), we let $(i,g) \in S_\alpha, (j,f) \in S_\beta$ for any $\alpha,\beta \in Y$. Clearly, there exists $(\bar{k},\bar{g}) \in S_{\alpha\beta}$ such that $(i,g)(j,f) = (\bar{k},\bar{g})$. Hence, by applying

(5), for any $(k, 1_{\alpha\beta}) \in S_{\alpha\beta} \bigcap E(S)$, we have

$$\begin{aligned}
& (i,g)(j,f) \\
= \quad & (i,g)(j,f)(k,1_{\alpha\beta}) \\
= \quad & (i,g)[(j,f)(k,1_{\alpha\beta})] \\
= \quad & (i,g)(\varphi^{(j,f)}_{\beta,\alpha\beta}k, f\theta_{\beta,\alpha\beta}) \\
= \quad & (i,g)(\varphi^{(j,f)}_{\beta,\alpha\beta}k, 1_{\alpha\beta})(\varphi^{(j,f)}_{\beta,\alpha\beta}k, f\theta_{\beta,\alpha\beta}) \\
= \quad & (\varphi^{(i,g)}_{\alpha,\alpha\beta}\varphi^{(j,f)}_{\beta,\alpha\beta}k, g\theta_{\alpha,\alpha\beta})(\varphi^{(j,f)}_{\beta,\alpha\beta}k, f\theta_{\beta,\alpha\beta}) \\
= \quad & (\varphi^{(i,g)}_{\alpha,\alpha\beta}\varphi^{(j,f)}_{\beta,\alpha\beta}k, g\theta_{\alpha,\alpha\beta}f\theta_{\beta,\alpha\beta}).
\end{aligned}$$

Hence, $\bar{k} = \varphi^{(i,g)}_{\alpha,\alpha\beta}\varphi^{(j,f)}_{\beta,\alpha\beta}k$. This implies that $\varphi^{(i,g)}_{\alpha,\alpha\beta}\varphi^{(j,f)}_{\beta,\alpha\beta}$ is a constant value mapping on $I_{\alpha\beta}$. Consequently, condition (C2) (i) is satisfied.

By using (5) and by the associativity of semigroup, we may show that $\Phi_{\alpha,\beta}$ satisfies condition (C2) (ii).

To see that $\Phi_{\alpha,\beta}$ satisfies condition (C2) (iii), notice that S is a super abundant semigroup. Thus, for any $a \in S$ there exist a unique idempotent e such that $a\mathcal{L}^*e$ and $a\mathcal{R}^*e$. By applying Lemma 2.1 (2), we can easily check that $\Phi_{\alpha,\beta}$ satisfies the condition (C2) (iii).

(4) It remains to show that the super $\mathcal{R}^*$-unipotent semigroup S is isomorphic to $I\Delta_{Y,\Phi}M$. To do this, it suffices to show that the multiplication on S is the same as the multiplication on $I\Delta_{Y,\Phi}M$.

Suppose that $(i,g) \in S_\alpha, (j,f) \in S_\beta$ and $\alpha, \beta \in Y$. It is easy to see that $(i,g)(j,f) \in S_{\alpha\beta}$. Hence, for any $(k, 1_{\alpha\beta}) \in S_{\alpha\beta} \bigcap E(S)$, by using (5), we have

$$\begin{aligned}
& (i,g)(j,f) \\
= \quad & (i,g)(j,f)(k,1_{\alpha\beta}) \\
= \quad & (i,g)[(j,f)(k,1_{\alpha\beta})] \\
= \quad & (i,g)(\varphi^{(j,f)}_{\beta,\alpha\beta}k, f\theta_{\beta,\alpha\beta}) \\
= \quad & (i,g)(\varphi^{(j,f)}_{\beta,\alpha\beta}k, 1_{\alpha\beta})(\varphi^{(j,f)}_{\beta,\alpha\beta}k, f\theta_{\beta,\alpha\beta}) \\
= \quad & (\varphi^{(i,g)}_{\alpha,\alpha\beta}\varphi^{(j,f)}_{\beta,\alpha\beta}k, g\theta_{\alpha,\alpha\beta})(\varphi^{(j,f)}_{\beta,\alpha\beta}k, f\theta_{\beta,\alpha\beta}) \\
= \quad & (\varphi^{(i,g)}_{\alpha,\alpha\beta}\varphi^{(j,f)}_{\beta,\alpha\beta}k, g\theta_{\alpha,\alpha\beta}f\theta_{\beta,\alpha\beta}) \\
= \quad & (\langle\varphi^{(i,g)}_{\alpha,\alpha\beta}\varphi^{(j,f)}_{\beta,\alpha\beta}\rangle, gf) \\
= \quad & (i,g) \circ (j,f).
\end{aligned}$$

This shows that the multiplication on S coincides with the multiplication on the generalized left Δ-product $I\Delta_{Y,\Phi}M$. Hence, $S \simeq I\Delta_{Y,\Phi}M$. The proof is completed.

References

[1] El-Qallali A., $\mathcal{L}^*$-unipotent semigroups, J. Pure and Applied Algebra, **62** (1989), 19-33.

[2] Fountain J.B., Abunant semigroups, Proc. London Math. Soc., **44**(1982), No.3, 103-129.

[3] Ren X.M., Shum K.P. and Guo Y.Q., On super $\mathcal{R}^*$-unipotent semigroup, SEA. Bull. Math., **22**(1998), 199-208.

[4] Shum K.P. and Ren X.M., The structure of right C-rpp semigroups, Semigroup Forum, **68**(2004), 280-292.

[5] Guo X.J., Guo Y.Q. and Shum K.P., Semi-spined structure of left C-a semigroups, Semigroups, Springer-Verlag, Singapore, 1998, 157-166.

[6] Guo Y.Q., Ren X.M. and Shum K.P., Another structure of left C-semigroups, Advance in Mathematics(in Chinese), **24**(1995), No.1, 39-43.

[7] Howie J.H., An introduction to semigroup theory, Academic Press, London, 1976.

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An equation involving the Smarandache-type function

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Abstract The main purpose of this paper is using the elementary methods to study the solutions of an equation involving the Smarandache-type multiplicative function, and give its all positive integer solutions.

Keywords Smarandache-type function, equation, solutions

§1. Introduction and Results

For any positive integer n, the famous Smarandache ceil function $S_k(n)$ is defined by

$$S_k(n) = \min\{m \in N : n \mid m^k\}.$$

For example, if $k = 3$, we have the sequence $\{S_3(n)\}$ $(n = 1, 2, 3, \cdots)$ as following: $S_3(1) = 1$, $S_3(2) = 2$, $S_3(3) = 3$, $S_3(4) = 2$, $S_3(5) = 5$, $S_3(6) = 6$, $S_3(7) = 7$, $S_3(8) = 2$, $\cdots$. This arithmetic function is a multiplicative function, and has many interesting properties, so it had been studied by many people. For example, Li Jie [1] studied the asymptotic properties of this function, and obtained an interesting asymptotic formula. That is, for any positive integer k, then:

$$\Omega(S_k(n!)) = \frac{n}{k}(\ln\ln n + C) + O\left(\frac{n}{\ln n}\right),$$

where C is a computable constant.

Similarly, many scholars studied another Smarandache-type function $C_m(n)$, which is defined as:

$$C_m(n) = \max\{x \in N : x^m \mid n\}.$$

About this function, Liu Huaning [2] proved that for any integer $m \geq 3$ and real number $x \geq 1$, we have

$$\sum_{n\leq x} C_m(n) = \frac{\zeta(m-1)}{\zeta(m)}x + O(x^{\frac{1}{2}+\varepsilon}).$$

Guo Jinbao [3] also studied the properties of $C_m(n)$, and proved the following conclusion: let $d(n)$ denotes the divisor function, then for any real number $x \geq 1$ and any fixed positive integer $m \geq 2$, we have

$$\sum_{n\leq x} d(C_m(n)) = \zeta(m)x + O(x^{\frac{1}{2}+\varepsilon}).$$

In this paper, we introduce another Smarandache-type function $D_m(n)$, which denotes the m-th power free part of n. That is, for any positive integer n and $m \geq 2$, we define

$$D_m(n) = \min\{n/d^m : d^m | n, d \in N\}.$$

For example, $D_3(8) = 1, D_3(24) = 3, D_2(12) = 3, \cdots$.

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ denotes the prime powers decomposition of n, then we have:

$$D_m(n) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}, \qquad \alpha_i \leq m-1, \qquad i = 1, 2, \cdots, s.$$

The properties of this function has been studied by many authors, for example, Liu Yanni [4] obtained an interesting asymptotic formula for it. That is, let p be a prime, k be any fixed positive integer. Then for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} e_p(D_m(n)) = \left(\frac{p^m - p}{(p^m-1)(p-1)} - \frac{m-1}{p^m - 1} \right) x + O(x^{\frac{1}{2}+\varepsilon}),$$

where ε denotes any fixed positive number.

In reference [5], Li Zhanhu has also studied the asymptotic properties of $D_m(n)$, and proved the following conclusion:

$$\sum_{\substack{n \leq x \\ n \in \mathcal{A}_k}} D_m(n) = \frac{1}{\zeta(k)} \frac{m}{m+1} x^{\frac{m+1}{m}} + O(x),$$

where $x \geq 1$ is a real number, $\zeta(k)$ is the Riemann $zeta$-function, and $\mathcal{A}_k$ denotes the set of all k-power free numbers.

On the other hand, Le Maohua [6] studied the solutions of an equation involving the Smarandache-type multiplicative function $SSC(n)$, where $SSC(n)$ denotes the Smarandache square complementary function of n. And he obtained all solutions of the equation

$$SSC(n)^r + SSC(n)^{r-1} + \cdots + SSC(n) = n, \qquad r > 1$$

as follows:

(i) $(n, r) = (363, 5)$;

(ii) $(n, r) = (ab^2, 2)$, where a and b are co-prime positive integers satisfying $a > 1, b > 1, a = b^2 - 1$ and a is a square free number.

The main purpose of this paper is using the elementary methods to determine the solutions of an equation involving the Smarandache-type function $D_m(n)$, and give its all positive integer solutions. That is, we shall prove the following:

Theorem. For any fixed positive integer $m \geq 2$ and nonnegative integer t, the equation

$$D_m^{t+r}(n) + D_m^{t+r-1}(n) + \cdots + D_m^{t+1}(n) = n, \qquad r > 1 \tag{1}$$

has positive integer solutions (n, r), and all its positive integer solutions (n, r) are given by the following three cases:

(i) $(n, r) = (b^m, b^m)$, where b is any positive integer;

(ii) $(n, r) = (7^{t+1} \cdot 400, 4)$, $(3^{t+1} \cdot 121, 5)$, $(18^{t+1} \cdot 343, 3)$, where $t \equiv 0(\mod m)$;

(iii) $(n, 2) = (a^{t+1}(a+1), 2)$, where a is a m-power free number, and $t \equiv 0(\mod m)$.

Especially, if we take $t = 0$ on the above, then we can get the following:

Corollary. For any fixed positive integer $m \geq 2$, the equation

$$D_m^r(n) + D_m^{r-1}(n) + \cdots + D_m(n) = n, \qquad r > 1 \tag{2}$$

has following positive integer solutions:

(i) $(n, r) = (b^m, b^m)$, where b is any positive integer;

(ii) $(n, r) = (2800, 4), (363, 5), (6174, 3)$;

(iii) $(n, 2) = (a(a+1), 2)$, where a is a m-power free number.

§2. Proof of the theorem

The proof of our Theorem depends on the following Lemma.

Lemma. The equation

$$\frac{x^r - 1}{x - 1} = y^q, \qquad x > 1, y > 1, r > 2, q > 1$$

has only the following three positive integer solutions $(x, y, r, q) = (7, 20, 4, 2), (3, 11, 5, 2), (18, 7, 3, 3)$.(See [7])

Now we use this Lemma to complete the proof of our Theorem. Let $n = u^m v$, where v is a m-power free number. According to the definition of $D_m(n)$, we have $D_m(n) = v$. Then from the equation (1) we have

$$v^{t+r} + v^{t+r-1} + \cdots + v^{t+1} = u^m v$$

or

$$v^t(v^{r-1} + v^{r-2} + \cdots + 1) = u^m, \qquad r > 1. \tag{3}$$

If $v = 1$, we have $r = u^m$, let $b = u$, then $(n, r) = (b^m, b^m)$.

If $v \neq 1$, when $t = 0$, the equation (3) becomes

$$\frac{v^r - 1}{v - 1} = u^m, \qquad r > 1. \tag{4}$$

By Lemma, if $r > 2$, we can immediately get $(v, u, r, m) = (7, 20, 4, 2)$, $(3, 11, 5, 2)$, $(18, 7, 3, 3)$.

So $(n, r) = (2800, 4)$, $(363, 5)$, $(6174, 3)$.

If $r = 2$, from (4) we have $v + 1 = u^m$. Let $a = v$, we can get $(n, 2) = (a(a+1), 2)$, where a is a m-power free number.

Therefore the corollary is proved.

When $t > 0$, we may let $u = u_1 u_2$, where u_1, u_2 are positive integers.

In the equation (3), because $(v, v^{r+1} + \cdots + 1) = 1$, so we can obtain

$$v^t = u_1^m, \tag{5}$$

$$v^{r-1} + \cdots + 1 = u_2^m. \tag{6}$$

In expression (6), by Lemma, if $r > 2$, we have $(v, u_2, r, m) = (7, 20, 4, 2)$, $(3, 11, 5, 2)$, $(18, 7, 3, 3)$.

So $(n, r) = (7^{t+1} \cdot 400, 4)$, $(3^{t+1} \cdot 121, 5)$, $(18^{t+1} \cdot 343, 3)$.

If $r = 2$, we have

$$v^t = u_1^m, \qquad v + 1 = u_2^m.$$

Let $a = v$, we can get $(n, 2) = (a^{t+1}(a + 1), 2)$, where a is a m-power free number.

In expression (5), because u_1 is an integer, and v is a m-power free number. Therefore, $t \equiv 0(\mod m)$.

This completes the proof of Theorem.

References

[1] Li Jie, An Asymptotic Formula on Smarandache Ceil Function, Research On Smarandache Problems In Number Theory (Collected papers), 2004, 103-106.

[2] Liu Huaning, Mean Value on two Smarandache-type Multiplicative Functions, Research On Smarandache Problems In Number Theory (Collected papers), 2004, 69-72.

[3] Guo Jinbao and He Yanfeng, Several Asymptotic Formulae On A New Arithmetical Function, Research On Smarandache Problems In Number Theory (Collected papers), 2004, 115-118.

[4] Liu Yanni, Mean Value of A New Arithmetic Function, Scientia Magna Journal, **1**(2005), No.1, 187-189.

[5] Li Zhanhu, On the Integer Part of the M-th Root and the k-th Power Free Number, Research On Smarandache Problems In Number Theory, **2**(2005), 41-43.

[6] Le Maohua, Some Problems Concerning the Smarandache Square Complementary Function(IV), Smarandache Notions Journal, **14**(2004), 335-337.

[7] W.Ljunggren, Noen Setninger om ubestemte likninger av formen $(x^n - 1)/(x - 1) = y^q$, Norsk.Mat.Tidsskr, **25**(1943), 17-20.

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An equation involving the function $\delta_k(n)$

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Abstract In this paper, we use the elementary methods to study the number of the solutions of an equation involving the function $\delta_k(n)$, and give its all positive integer solutions.

Keywords Equation, positive integer solution, arithmetical function

§1. Introduction and Results

Let k be a fixed positive integer, for any positive integer n, we define the arithmetical function $\delta_k(n)$ as following:

$$\delta_k(n) = \max\{d : d \mid n, (d, k) = 1\}.$$

For example, $\delta_2(1) = 1$, $\delta_2(2) = 1$, $\delta_2(3) = 1$, $\delta_2(4) = 1$, $\delta_3(6) = 2, \cdots$. About the elementary properties of this function, many scholars have studied it, and got some useful results. For example, Xu Zhefeng [1] studied the divisibility of $\delta_k(n)$ by $\varphi(n)$, and proved that $\varphi(n) \mid \delta_k(n)$ if and only if $n = 2^\alpha 3^\beta$, where $\alpha > 0$, $\beta \geq 0$, $\alpha, \beta \in N$. In [2], Liu Yanni and Gao Peng studied the mean value properties of $\delta_k(b_m(n))$, and obtained an interesting mean value formula for it. That is, they obtained the following conclusion:

Let k and m are two fixed positive integers. Then for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x} \delta_k(b_m(n)) = \frac{x^2}{2}\frac{\zeta(2m)}{\zeta(m)} \prod_{p|k} \frac{p^m + 1}{p^{m-1}(p+1)} + O(x^{\frac{3}{2}+\epsilon}),$$

where ϵ denotes any fixed positive number, $\zeta(s)$ is the Riemann zeta-function, and $\prod_{p|k}$ denotes the product over all different prime divisors of k.

In this paper, we will use the elementary methods to study the existence of the solutions of the equation

$$\sum_{i=1}^{n} \delta_k(i) = \delta_k\left(\frac{n(n+1)}{2}\right),$$

and get some interesting results. It is clearly that $n = 1$ is a solution of this equation. But except $n = 1$, are there any other solutions? In this paper, we solved this problem, and obtained all its positive solutions. That is, we shall prove the following:

Theorem 1. If $k = 2^\alpha$ ($\alpha = 1,\ 2,\ 3,\ \cdots$), then the equation has only one solution $n = 1$.

Theorem 2. If $k = p^\alpha$ (p is an odd prime, $\alpha = 1,\ 2,\ 3,\ \cdots$), then the equation has $p-2$ positive integer solutions, they are $n = 1,\ 2,\ 3, \cdots,\ p-2$.

§2. Some useful lemmas

To complete the proofs of the theorems, we need the following lemmas.

Lemma 1. If $n = 4m$ or $n = 4m+3$ ($m = 1,\ 2,\ \cdots$), then we have

$$\sum_{i=1}^{n} \delta_2(i) > \delta_2\left(\frac{n(n+1)}{2}\right).$$

Proof. From the definition of the arithmetic function $\delta_k(n)$, we have the following:

(1) If $n = 4m$, then

$$\sum_{i=1}^{4m} \delta_2(i) \geq \sum_{\substack{l\leq 4m\\ 2\nmid l}} l + \sum_{\substack{l\leq 4m\\ 2\mid l}} 1 = \frac{2m((4m-1)+1)}{2} + 2m = 4m^2 + 2m,$$

but

$$\delta_2\left(\frac{4m(4m+1)}{2}\right) \leq m(4m+1) = 4m^2 + m < 4m^2 + 2m,$$

therefore

$$\sum_{i=1}^{4m} \delta_2(i) > \delta_2\left(\frac{4m(4m+1)}{2}\right).$$

(2) If $n = 4m+3$, then

$$\sum_{i=1}^{4m+3} \delta_2(i) \geq \sum_{\substack{l\leq 4m+3\\ 2\nmid l}} l + \sum_{\substack{l\leq 4m+3\\ 2\mid l}} 1 = \frac{(2m+1)((4m+3)+1)}{2} + 2m + 1 = 4m^2 + 8m + 3,$$

but

$$\delta_2\left(\frac{(4m+3)(4m+3+1)}{2}\right) \leq (4m+3)(m+1) = 4m^2 + 7m + 3 < 4m^2 + 8m + 3,$$

therefore

$$\sum_{i=1}^{4m+3} \delta_2(i) > \delta_2\left(\frac{(4m+3)(4m+3+1)}{2}\right).$$

Combining (1) and (2), the proof of Lemma 1 is completed.

Lemma 2. For $k = p$ (an odd prime), we have

$$\sum_{i=1}^{n} \delta_k(i) > \delta_k\left(\frac{n(n+1)}{2}\right),$$

if $n = tp$ or $n = tp - 1$ ($t = 1,\ 2,\ \cdots$).

Proof. (3) If $n = tp$, then we have

$$\sum_{i=1}^{tp} \delta_k(i) > \sum_{\substack{l\leq tp\\ (l,p)=1}} l = \sum_{l\leq tp} l - \sum_{\substack{l\leq tp\\ p\mid l}} l = \frac{tp(tp+1)}{2} - \frac{tp(t+1)}{2} = \frac{t^2p^2 - t^2p}{2},$$

but
$$\delta_k\left(\frac{tp(tp+1)}{2}\right) \le \frac{t(tp+1)}{2} < \frac{t^2p^2-t^2p}{2}.$$

(4) If $n = tp-1$, then we have
$$\sum_{i=1}^{tp-1}\delta_k(i) > \sum_{\substack{l\le tp-1\\(l,p)=1}} l = \sum_{l\le tp-1} l - \sum_{\substack{l\le tp-1\\p\mid l}} l = \frac{tp(tp-1)}{2} - \frac{tp(t-1)}{2} = \frac{t^2p^2-t^2p}{2},$$
but
$$\delta_k\left(\frac{tp(tp-1)}{2}\right) \le \frac{t(tp-1)}{2} < \frac{t^2p^2-t^2p}{2}.$$

Therefore, Lemma 2 follows from (3) and (4).

§3. Proof of the theorems

In this section, we will use the elementary methods to complete the proof of the theorems. Note that for any prime p and positive integer n, we have $(n,p) = (n,p^\alpha)(\alpha = 1,\ 2,\ \cdots)$, then $\delta_p(n) = \delta_{p^\alpha}(n)$, so the equation
$$\sum_{i=1}^{n}\delta_{p^\alpha}(i) = \delta_{p^\alpha}\left(\frac{n(n+1)}{2}\right)$$
is equivalent to
$$\sum_{i=1}^{n}\delta_{p}(i) = \delta_{p}\left(\frac{n(n+1)}{2}\right).$$
Therefore, we just need to prove the case $k = p$. Now we prove Theorem 1. First, we separate all positive integers into two cases:

(i) If $n \le 3$, then from the definition of the function $\delta_k(n)$ we have
$$\delta_2(1) = 1, \quad \delta_2(2) = 1, \quad \delta_2(3) = 1.$$

It is clear that $n = 1$ is a solution of this equation.

(ii) If $n > 3$, then for any positive integer m $(m = 1,\ 2,\ \cdots)$, we have the following:

(a) If $n = 4m+1$ or $n = 4m+2$, it is obviously true that
$$\delta_2\left(\frac{n(n+1)}{2}\right) = \frac{n(n+1)}{2}.$$

Note that in the sum $\sum\limits_{i=1}^{n}\delta_2(i)$, there are at least one term such that $\delta_2(i) < i$ (for example: $\delta_2(4m) \le m < 4m$). Therefore, we have
$$\sum_{i=1}^{n}\delta_2(i) < \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

That is,

$$\sum_{i=1}^{n} \delta_2(i) < \delta_2\left(\frac{n(n+1)}{2}\right).$$

(b) If $n = 4m$ or $n = 4m + 3$, then from Lemma 1 we have

$$\sum_{i=1}^{n} \delta_2(i) > \delta_2\left(\frac{n(n+1)}{2}\right).$$

That is to say, the equation has no solution in this case.

From the former discussing we know that the equation has only one positive solution $n = 1$. This completes the proof of Theorem 1.

Now we come to prove Theorem 2. Using the same methods of proving Theorem 1 we have the following:

(iii) If $n \le p - 2$, then for $1 \le i \le n \le p - 2$, we have $(i, p) = 1$, $\delta_k(i) = i$. Therefore,

$$\sum_{i=1}^{n} \delta_k(i) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Next we prove that $\delta_k\left(\frac{n(n+1)}{2}\right) = \frac{n(n+1)}{2}$.

In fact, because $(n, n+1) = 1$ and $n \le p - 2$, so $(n, p) = 1$. When $n \le p - 2$, we have $n+1 \le p-1$. If $n+1 = p-1$, then $(\frac{n+1}{2}, p) = 1$; If $n+1 < p-1$, then $(n+1, p) = 1$. So, if $n \le p - 2$, then we have $(\frac{n(n+1)}{2}, p) = 1$, $\delta_k\left(\frac{n(n+1)}{2}\right) = \frac{n(n+1)}{2}$.

Therefore, for every positive integer $n \le p - 2$, we have

$$\sum_{i=1}^{n} \delta_k(i) = \delta_k\left(\frac{n(n+1)}{2}\right).$$

That is, $n = 1,\ 2,\ \cdots,\ p - 2$ are the positive integer solutions of the equation.

(iv) If $n > p - 2$, then for any positive integer $t(t = 1,\ 2,\ \cdots)$, we have the following:

(c) If $n = tp + r$ $(1 \le r \le p - 2$ is a positive integer), then we have

$$\delta_k\left(\frac{n(n+1)}{2}\right) = \frac{n(n+1)}{2}.$$

Note that in the sum $\sum_{i=1}^{n} \delta_k(i)$, there are at least one term satisfying $\delta_k(i) < i$ (for example: $\delta_k(tp) \le t < tp$), so we have

$$\sum_{i=1}^{n} \delta_k(i) < \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Therefore,

$$\sum_{i=1}^{n} \delta_k(i) < \delta_k\left(\frac{n(n+1)}{2}\right).$$

(d) If $n = tp$ or $n = tp - 1$, then by Lemma 2 we have

$$\sum_{i=1}^{n} \delta_k(i) > \delta_k\left(\frac{n(n+1)}{2}\right).$$

Combining (c) and (d) we know that the equation has no solution if $n > p-2$.

Now, by (iii) and (iv), we know that the equation has $p-2$ positive integer solutions, they are $n = 1, 2, \cdots, p-2$. This completes the proof of Theorem 2.

Open Problem. If $k = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_s^{\alpha_s}$ ($\alpha_i = 1, 2, 3, \cdots, i = 1, 2, \cdots, s$), let $p = \min\{p_1, p_2, \cdots, p_s\}$, then it is obviously true that $n = 1, 2, 3, \cdots, p-2$ are the positive integer solutions of the equation, if p is an odd prime; It has one positive integer solution $n = 1$, if $p = 2$. Whether there exists any other solutions for the equation is an open problem.

References

[1] Zhefeng Xu, On the function $\varphi(n)$ and $\delta_k(n)$, Research on Smarandache Problems in Number Theory, Hexis, 2005, 9-11.

[2] Liu Yanni and Gao Peng, On the m-power free part of an integer, Scientia Magna, **1.1**(2005), 203-206.

[3] Tian Qing, A discussion on a number theoretic function, Scientia Magna, **2.1**(2006), 38-41.

[4] Tom M. Apostol, Introduction to analytical number theory, Spring-Verlag, New York, 1976.

[5] Chengdong Pan and Chengbiao Pan, The elementary number theory, Beijing University Press, Beijing, 2003.

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Smarandache fantastic ideals of Smarandache BCI-algebras

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Abstract The notion of Smarandache fantastic ideals is introduced, examples are given, and related properties are investigated. Relations among Q-Smarandache fresh ideals, Q-Smarandache clean ideals and Q-Smarandache fantastic ideals are given. A characterization of a Q-Smarandache fantastic ideal is provided. The extension property for Q-Smarandache fantastic ideals is established.

Keywords Smarandache BCI-algebra, Smarandache fresh ideal, Smarandache clean ideal, Smarandache fantastic ideal

1. Introduction

Generally, in any human field, a Smarandache structure on a set A means a weak structure W on A such that there exists a proper subset B of A which is embedded with a strong structure S. In [5], W.B.Vasantha Kandasamy studied the concept of Smarandache groupoids, sub-groupoids, ideal of groupoids, semi-normal subgroupoids, Smarandache Bol groupoids and strong Bol groupoids and obtained many interesting results about them. Smarandache semigroups are very important for the study of congruences, and it was studied by R.Padilla [4]. It will be very interesting to study the Smarandache structure in BCK/BCI-algebras. In [1], Y.B.Jun discussed the Smarandache structure in BCI-algebras. He introduced the notion of Smarandache (positive implicative, commutative, implicative) BCI-algebras, Smarandache subalgebras and Smarandache ideals, and investigated some related properties. Also, he studied Smarandache ideal structures in Smarandache BCI-algebras. He introduced the notion of Smarandache fresh ideals and Smarandache clean ideals in Smarandache BCI-algebras, and investigated its useful properties. He gave relations between Q-Smarandache fresh ideals and Q-Smarandache clean ideals, and established extension properties for Q-Smarandache fresh ideals and Q-Smarandache clean ideals (see [2]). In this paper, we introduce the notion of Q-Smarandache fantastic ideals, and investigate its properties. We give relations among Q-Smarandache fresh ideals, Q-Smarandache clean ideals and Q-Smarandache fantastic ideals. We also provide a characterization of a Q-Smarandache fantastic ideal. We finally establish the extension property for Q-Smarandache fantastic ideals.

2. Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI-algebra if it satisfies the following conditions:

(a1) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(a2) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(a3) $(\forall x \in X)\ (x * x = 0)$,

(a4) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(a5) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a BCK-algebra. We can define a partial order $\leq$ on X by $x \leq y \iff x * y = 0$. Every BCI-algebra X has the following properties:

(b1) $(\forall x \in X)\ (x * 0 = x)$.

(b2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$.

(b3) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$.

(b4) $(\forall x, y \in X)\ (x * (x * (x * y)) = x * y)$.

A Smrandache BCI-algebra [1] is defined to be a BCI-algebra X in which there exists a proper subset Q of X such that

(s1) $0 \in Q$ and $|Q| \geq 2$,

(s2) Q is a BCK-algebra under the operation of X.

3. Smarandache Fantastic Ideals

In what follows, let X and Q denote a Smarandache BCI-algebra and a BCK-algebra which is properly contained in X, respectively.

Definition 3.1. [1] A nonempty subset I of X is called a Smarandache ideal of X related to Q (or briefly, Q-Smarandache ideal of X) if it satisfies:

(c1) $0 \in I$,

(c2) $(\forall x \in Q)\ (\forall y \in I)\ (x * y \in I \Rightarrow x \in I)$.

If I is a ideal of X related to every BCK-algebra contained in X, we simply say that I is a Smarandache ideal of X.

Definition 3.2. [2] A nonempty subset I of X is called a Smarandache fresh ideal of X related to Q (or briefly, Q-Smarandache fresh ideal of X) if it satisfies the condition (c1) and (c3) $(\forall x, y, z \in Q)$ $((x * y) * z \in I,\ y * z \in I \ \Rightarrow\ x * z \in I)$.

Lemma 3.3. [2] If I is a Q-Smarandache fresh ideal of X, then

(i) $(\forall x, y \in Q)$ $((x * y) * y \in I \ \Rightarrow\ x * y \in I)$.

(ii) $(\forall x, y, z \in Q)$ $((x * y) * z \in I \ \Rightarrow\ (x * z) * (y * z) \in I)$.

Definition 3.4. [2] A nonempty subset I of X is called a Smarandache clean ideal of X related to Q (or briefly, Q-Smarandache clean ideal of X) if it satisfies the condition (c1) and

(c4) $(\forall x, y \in Q)$ $(\forall z \in I)$ $((x * (y * x)) * z \in I \ \Rightarrow\ x \in I)$.

Lemma 3.5. [2] Every Q-Smarandache clean ideal is a Q-Smarandache fresh ideal.

Lemma 3.6. Let I be a Q-Smarandache ideal of X. Then I is a Q-Smarandache clean ideal of $X \iff I$ satisfies the following condition:

$$(\forall x, y \in Q)\,(x * (y * x) \in I \ \Rightarrow\ x \in I). \tag{1}$$

Proof. Suppose that I satisfies the condition (1) and suppose that $(x * (y * x)) * z \in I$ for all $x, y \in Q$ and $z \in I$. Then $x * (y * x) \in I$ by (c2), and so $x \in I$ by (1). Conversely assume that I is a Q-Smarandache clean ideal of X and let $x, y \in Q$ be such that $x * (y * x) \in I$. Since $0 \in I$, it follows from (b1) that $(x * (y * x)) * 0 = x * (y * x) \in I$ so from (c4) that $x \in I$. This completes the proof.

Definition 3.7. A nonempty subset I of X is called a Smarandache fantastic ideal of X related to Q (or briefly, Q-Smarandache fantastic ideal of X) if it satisfies the condition (c1) and

(c5) $(\forall x, y \in Q)$ $(\forall z \in I)$ $((x * y) * z \in I \ \Rightarrow\ x * (y * (y * x)) \in I)$.

Example 3.8. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	1	0	1	5
2	2	2	0	2	0	5
3	3	1	3	0	3	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Table 3.1

Then $(X; *, 0)$ is a Smarandache BCI-algebra. Note that $Q = \{0, 1, 2, 3, 4\}$ is a BCK-algebra which is properly contained in X. It is easily checked that subsets $I_1 = \{0, 2\}$ and $I_2 = \{0, 2, 4\}$ are Q-Smarandache fantastic ideals of X, but not Q-Smarandache fresh ideals. A subset $I_3 = \{0, 1, 3\}$ is a Q-Smarandache fresh ideal, but not a Q-Smarandache fantastic ideal since $(2 * 4) * 3 = 0 \in I_3$ and $2 * (4 * (4 * 2)) = 2 \notin I_3$.

The example above suggests that a Q-Smarandache fantastic ideal need not be a Q-Smarandache fresh ideal, and a Q-Smarandache fresh ideal may not be a Q-Smarandache fantastic ideal.

Theorem 3.9. Let Q_1 and Q_2 be BCK-algebras which are properly contained in X such that $Q_1 \subset Q_2$. Then every Q_2-Smarandache fantastic ideal is a Q_1-Smarandache fantastic ideal of X.

Proof. Straightforward.

The converse of Theorem 3.9 is not true in general as seen in the following example.

Example 3.10. Consider the Smarandache BCI-algebra X described in Example 3.8. Note that $Q_1 := \{0, 2, 4\}$ and $Q_2 := \{0, 1, 2, 3, 4\}$ are BCK-algebras which are properly contained in X and $Q_1 \subset Q_2$. Then $I := \{0, 1, 3\}$ is a Q_1-Smarandache fantastic ideal, but not a Q_2-Smarandache fantastic ideal of X.

Theorem 3.11. Every Q-Smarandache fantastic ideal is a Q-Smarandache ideal.

Proof. Let I be a Q-Smarandache fantastic ideal of X and assume that $x * z \in I$ for all $x \in Q$ and $z \in I$. Using (b1), we get $(x*0)*z = x*z \in I$. Since $x \in Q$ and Q is a BCK-algebra, it follows from (a5), (b1) and (c5) that $x = x * (0 * (0 * x)) \in I$ so that I is a Q-Smarandache ideal of X.

As seen in Example 3.8, the converse of Theorem 3.11 is not true in general.

Theorem 3.12. Let I be a Q-Smarandache ideal of X. Then I is a Q-Smarandache fantastic ideal of X $\iff$ it satisfies the following implication:

$$(\forall x, y \in Q)\,(x * y \in I \Rightarrow x * (y * (y * x)) \in I). \tag{2}$$

Proof. Assume that I is a Q-Smarandache fantastic ideal of X and let $x, y \in Q$ be such that $x * y \in I$. Using (b1), we have $(x * y) * 0 = x * y \in I$ and $0 \in I$. It follows from (c5) that $x * (y * (y * x)) \in I$. Conversely suppose that I satisfies the condition (2). Assume that $(x * y) * z \in I$ for all $x, y \in Q$ and $z \in I$. Then $x * y \in I$ by (c2), and hence $x * (y * (y * x)) \in I$ by (2). This completes the proof.

Theorem 3.13. Let I be a nonempty subset of X. Then I is a Q-Smarandache clean ideal of X $\iff$ I is both a Q-Smarandache fresh ideal and a Q-Smarandache fantastic ideal of X.

Proof. Assume that I is a Q-Smarandache clean ideal of X. Then I is a Q-Smarandache fresh ideal of X (see Lemma 3.5). Suppose that $x * y \in I$ for all $x, y \in Q$. Since Q is a BCK-algebra, we have

$$(x * (y * (y * x))) * x = (x * x) * (y * (y * x)) = 0 * (y * (y * x)) = 0,$$

and so $(y * x) * (y * (x * (y * (y * x)))) = 0$, that is, $y * x \leq y * (x * (y * (y * x)))$. It follows from (b3), (b2) and (a1) that

$$\begin{aligned}
&(x * (y * (y * x))) * (y * (x * (y * (y * x)))) \\
&\leq (x * (y * (y * x))) * (y * x) \\
&= (x * (y * x)) * (y * (y * x)) \leq x * y,
\end{aligned}$$

that is, $((x*(y*(y*x)))*(y*(x*(y*(y*x)))))*(x*y) = 0 \in I$. Since $x*y \in I$, it follows from (c2) that $(x*(y*(y*x)))*(y*(x*(y*(y*x)))) \in I$, so from Lemma 3.6 that $x*(y*(y*x)) \in I$. Using Theorem 3.12, we know that I is a Q-Smarandache fantastic ideal of X.

Conversely, suppose that I is both a Q-Smarandache fresh ideal and a Q-Smarandache fantastic ideal of X. Let $x, y \in Q$ be such that $x * (y * x) \in I$. Since

$$((y * (y * x)) * (y * x)) * (x * (y * x)) = 0 \in I,$$

we get $(y * (y * x)) * (y * x) \in I$ by (c2). Since I is a Q-Smarandache fresh ideal, it follows from Lemma 3.3(i) that $y * (y * x) \in I$ so from (c2) that $x * y \in I$ since $(x * y) * (y * (y * x)) = 0 \in I$. Since I is a Q-Smarandache fantastic ideal, we obtain $x * (y * (y * x)) \in I$ by (2), and so $x \in I$ by (c2). Therefore I is a Q-Smarandache clean ideal of X by Lemma 3.6.

Theorem 3.14. (Extension Property) Let I and J be Q-Smarandache ideals of X and $I \subset J \subset Q$. If I is a Q-Smarandache fantastic ideal of X, then so is J.

Proof. Assume that $x * y \in J$ for all $x, y \in Q$. Since

$$(x * (x * y)) * y = (x * y) * (x * y) = 0 \in I,$$

it follows from (b2) and (2) that

$$(x * (y * (y * (x * (x * y))))) * (x * y) = (x * (x * y)) * (y * (y * (x * (x * y)))) \in I \subset J,$$

so from (c2) that $x * (y * (y * (x * (x * y)))) \in J$. Since $x, y \in Q$ and Q is a BCK-algebra, we get $(x * (y * (y * x))) * (x * (y * (y * (x * (x * y))))) = 0 \in J$, by using (a1) repeatedly. Since J is a Q-Smarandache ideal, we conclude that $x * (y * (y * x)) \in J$. Hence J is a Q-Smarandache fantastic ideal of X by Theorem 3.12.

References

[1] Y. B. Jun, Smarandache BCI-algebras, Sci. Math. Jpn., **62**(2005), No. 1, 137–142, **e2005**, 271–276.

[2] Y. B. Jun, Smarandache fresh and clean ideals of Smarandache BCI-algebras, Kyungpook Math. J., **46**(2006), 409–416.

[3] J. Meng and Y. B. Jun, BCK-algebras, Kyungmoonsa Co., Seoul, Korea, 1994.

[4] R. Padilla, Smarandache algebraic structures, Bull. Pure Appl. Sci., Delhi, **17E**(1998), No. 1, 119–121, http://www.gallup.unm.edu/∼Smarandache/alg-s-tx.txt.

[5] W. B. Vasantha Kandasamy, Smarandache groupoids, http://www.gallup.unm.edu/∼smarandache/Groupoids.pdf.

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On the universality of some Smarandache loops of Bol-Moufang type

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Abstract A Smarandache quasigroup(loop) is shown to be universal if all its f, g-principal isotopes are Smarandache f, g-principal isotopes. Also, weak Smarandache loops of Bol-Moufang type such as Smarandache: left(right) Bol, Moufang and extra loops are shown to be universal if all their f, g-principal isotopes are Smarandache f, g-principal isotopes. Conversely, it is shown that if these weak Smarandache loops of Bol-Moufang type are universal, then some autotopisms are true in the weak Smarandache sub-loops of the weak Smarandache loops of Bol-Moufang type relative to some Smarandache elements. Futhermore, a Smarandache left(right) inverse property loop in which all its f, g-principal isotopes are Smarandache f, g-principal isotopes is shown to be universal if and only if it is a Smarandache left(right) Bol loop in which all its f, g-principal isotopes are Smarandache f, g-principal isotopes. Also, it is established that a Smarandache inverse property loop in which all its f, g-principal isotopes are Smarandache f, g-principal isotopes is universal if and only if it is a Smarandache Moufang loop in which all its f, g-principal isotopes are Smarandache f, g-principal isotopes. Hence, some of the autotopisms earlier mentioned are found to be true in the Smarandache sub-loops of universal Smarandache: left(right) inverse property loops and inverse property loops.

Keywords Smarandache quasigroups, Smarandache loops, universality, f, g-principal isotopes

1. Introduction

W. B. Vasantha Kandasamy initiated the study of Smarandache loops (S-loop) in 2002. In her book [27], she defined a Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop called a Smarandache subloop (S-subloop). In [11], the present author defined a Smarandache quasigroup (S-quasigroup) to be a quasigroup with at least a non-trivial associative subquasigroup called a Smarandache subquasigroup (S-subquasigroup). Examples of Smarandache quasigroups are given in Muktibodh [21]. For more on quasigroups, loops and their properties, readers should check [24], [2],[4], [5], [8] and [27]. In her (W.B. Vasantha Kandasamy) first paper [28], she introduced Smarandache : left(right) alternative loops, Bol loops, Moufang loops, and Bruck loops. But in [10], the present author introduced Smarandache : inverse property loops (IPL), weak inverse property loops (WIPL), G-loops, conjugacy closed loops (CC-loop), central loops, extra

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loops, A-loops, K-loops, Bruck loops, Kikkawa loops, Burn loops and homogeneous loops. The isotopic invariance of types and varieties of quasigroups and loops described by one or more equivalent identities, especially those that fall in the class of Bol-Moufang type loops as first named by Fenyves [7] and [6] in the 1960s and later on in this 21^{st} century by Phillips and Vojtěchovský [25], [26] and [18] have been of interest to researchers in loop theory in the recent past. For example, loops such as Bol loops, Moufang loops, central loops and extra loops are the most popular loops of Bol-Moufang type whose isotopic invariance have been considered. Their identities relative to quasigroups and loops have also been investigated by Kunen [20] and [19]. A loop is said to be universal relative to a property $\mathcal{P}$ if it is isotopic invariant relative to $\mathcal{P}$, hence such a loop is called a universal $\mathcal{P}$ loop. This language is well used in [22]. The universality of most loops of Bol-Moufang types have been studied as summarised in [24]. Left(Right) Bol loops, Moufang loops, and extra loops have all been found to be isotopic invariant. But some types of central loops were shown to be universal in Jaíyéọlá [13] and [12] under some conditions. Some other types of loops such as A-loops, weak inverse property loops and cross inverse property loops (CIPL) have been found be universal under some neccessary and sufficient conditions in [3], [23] and [1] respectively. Recently, Michael Kinyon et. al. [16], [14], [15] solved the Belousov problem concerning the universality of F-quasigroups which has been open since 1967 by showing that all the isotopes of F-quasigroups are Moufang loops.

In this work, the universality of the Smarandache concept in loops is investigated. That is, will all isotopes of an S-loop be an S-loop? The answer to this could be 'yes' since every isotope of a group is a group (groups are G-loops). Also, the universality of weak Smarandache loops, such as Smarandache Bol loops (SBL), Smarandache Moufang loops (SML) and Smarandache extra loops (SEL) will also be investigated despite the fact that it could be expected to be true since Bol loops, Moufang loops and extra loops are universal. The universality of a Smarandache inverse property loop (SIPL) will also be considered.

2. Preliminaries

Definition 2.1. A loop is called a Smarandache left inverse property loop (SLIPL) if it has at least a non-trivial subloop with the LIP.

A loop is called a Smarandache right inverse property loop (SRIPL) if it has at least a non-trivial subloop with the RIP.

A loop is called a Smarandache inverse property loop (SIPL) if it has at least a non-trivial subloop with the IP.

A loop is called a Smarandache right Bol-loop (SRBL) if it has at least a non-trivial subloop that is a right Bol(RB)-loop.

A loop is called a Smarandache left Bol-loop (SLBL) if it has at least a non-trivial subloop that is a left Bol(LB)-loop.

A loop is called a Smarandache central-loop (SCL) if it has at least a non-trivial subloop that is a central-loop.

A loop is called a Smarandache extra-loop (SEL) if it has at least a non-trivial subloop that is a extra-loop.

A loop is called a Smarandache A-loop (SAL) if it has at least a non-trivial subloop that is a A-loop.

A loop is called a Smarandache Moufang-loop (SML) if it has at least a non-trivial subloop that is a Moufang-loop.

Definition 2.2. Let $(G, \oplus)$ and $(H, \otimes)$ be two distinct quasigroups. The triple (A, B, C) such that $A, B, C : (G, \oplus) \to (H, \otimes)$ are bijections is said to be an isotopism if and only if

$$xA \otimes yB = (x \oplus y)C \ \forall \ x, y \in G.$$

Thus, H is called an isotope of G and they are said to be isotopic. If $C = I$, then the triple is called a principal isotopism and $(H, \otimes) = (G, \otimes)$ is called a principal isotope of $(G, \oplus)$. If in addition, $A = R_g$, $B = L_f$, then the triple is called an f, g-principal isotopism, thus $(G, \otimes)$ is reffered to as the f, g-principal isotope of $(G, \oplus)$.

A subloop(subquasigroup) $(S, \otimes)$ of a loop(quasigroup) $(G, \otimes)$ is called a Smarandache f, g-principal isotope of the subloop(subquasigroup) $(S, \oplus)$ of a loop(quasigroup) $(G, \oplus)$ if for some $f, g \in S$,

$$xR_g \otimes yL_f = (x \oplus y) \ \forall \ x, y \in S.$$

On the other hand $(G, \otimes)$ is called a Smarandache f, g-principal isotope of $(G, \oplus)$ if for some $f, g \in S$,

$$xR_g \otimes yL_f = (x \oplus y) \ \forall \ x, y \in G,$$

where $(S, \oplus)$ is a S-subquasigroup(S-subloop) of $(G, \oplus)$. In these cases, f and g are called Smarandache elements(S-elements).

Theorem 2.1. [2] Let $(G, \oplus)$ and $(H, \otimes)$ be two distinct isotopic loops(quasigroups). There exists an f, g-principal isotope $(G, \circ)$ of $(G, \oplus)$ such that $(H, \otimes) \cong (G, \circ)$.

Corollary 2.1. Let $\mathcal{P}$ be an isomorphic invariant property in loops(quasigroups). If $(G, \oplus)$ is a loop(quasigroup) with the property $\mathcal{P}$, then $(G, \oplus)$ is a universal loop(quasigroup) relative to the property $\mathcal{P}$ if and only if every f, g-principal isotope $(G, \circ)$ of $(G, \oplus)$ has the property $\mathcal{P}$.

Proof. If $(G, \oplus)$ is a universal loop relative to the property $\mathcal{P}$ then every distinct loop isotope $(H, \otimes)$ of $(G, \oplus)$ has the property $\mathcal{P}$. By Theorem 2.1, there exists a f, g-principal isotope $(G, \circ)$ of $(G, \oplus)$ such that $(H, \otimes) \cong (G, \circ)$. Hence, since $\mathcal{P}$ is an isomorphic invariant property, every $(G, \circ)$ has it.

Conversely, if every f, g-principal isotope $(G, \circ)$ of $(G, \oplus)$ has the property $\mathcal{P}$ and since by Theorem 2.1, for each distinct isotope $(H, \otimes)$ there exists a f, g-principal isotope $(G, \circ)$ of $(G, \oplus)$ such that $(H, \otimes) \cong (G, \circ)$, then all $(H, \otimes)$ has the property. Thus, $(G, \oplus)$ is a universal loop relative to the property $\mathcal{P}$.

Lemma 2.1. Let $(G, \oplus)$ be a loop(quasigroup) with a subloop(subquasigroup) $(S, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then $(S, \circ)$ is a subloop(subquasigroup) of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$.

Proof. If $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$, then for some $f, g \in S$,

$$xR_g \circ yL_f = (x \oplus y) \ \forall \ x, y \in S \Rightarrow x \circ y = xR_g^{-1} \oplus yL_f^{-1} \in S \ \forall \ x, y \in S$$

since $f, g \in S$. So, $(S, \circ)$ is a subgroupoid of $(G, \circ)$. $(S, \circ)$ is a subquasigroup follows from the fact that $(S, \oplus)$ is a subquasigroup. $f \oplus g$ is a two sided identity element in $(S, \circ)$. Thus, $(S, \circ)$ is a subloop of $(G, \circ)$.

3. Main Results

Universality of Smarandache Loops

Theorem 3.1. A Smarandache quasigroup is universal if all its f, g-principal isotopes are Smarandache f, g-principal isotopes.

Proof. Let $(G, \oplus)$ be a Smarandache quasigroup with a S-subquasigroup $(S, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 2.1, $(S, \circ)$ is a subquasigroup of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

It shall now be shown that

$$(x \circ y) \circ z = x \circ (y \circ z) \ \forall \ x, y, z \in S.$$

But in the quasigroup $(G, \oplus)$, xy will have preference over $x \oplus y \ \forall \ x, y \in G$.

$$\begin{aligned}(x \circ y) \circ z &= (xR_g^{-1} \oplus yL_f^{-1}) \circ z = (xg^{-1} \oplus f^{-1}y) \circ z = (xg^{-1} \oplus f^{-1}y)R_g^{-1} \oplus zL_f^{-1} \\ &= (xg^{-1} \oplus f^{-1}y)g^{-1} \oplus f^{-1}z = xg^{-1} \oplus f^{-1}yg^{-1} \oplus f^{-1}z.\end{aligned}$$

$$\begin{aligned}x \circ (y \circ z) &= x \circ (yR_g^{-1} \oplus zL_f^{-1}) = x \circ (yg^{-1} \oplus f^{-1}z) = xR_g^{-1} \oplus (yg^{-1} \oplus f^{-1}z)L_f^{-1} \\ &= xg^{-1} \oplus f^{-1}(yg^{-1} \oplus f^{-1}z) = xg^{-1} \oplus f^{-1}yg^{-1} \oplus f^{-1}z.\end{aligned}$$

Thus, $(S, \circ)$ is a S-subquasigroup of $(G, \circ)$, $(G, \circ)$ is a S-quasigroup. By Theorem 2.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ which will now be a S-subquasigroup in $(H, \otimes)$. So, $(H, \otimes)$ is a S-quasigroup. This conclusion can also be drawn straight from Corollary 2.1.

Theorem 3.2. A Smarandache loop is universal if all its f, g-principal isotopes are Smarandache f, g-principal isotopes. Conversely, if a Smarandache loop is universal then

$$(I, L_f R_g^{-1} R_{f^\rho} L_f^{-1}, R_g^{-1} R_{f^\rho})$$

is an autotopism of a S-subloop of the S-loop such that f and g are S-elements.

Proof. Every loop is a quasigroup. Hence, the first claim follows from Theorem 3.1. The proof of the converse is as follows. If a Smarandache loop $(G, \oplus)$ is universal then every isotope $(H, \otimes)$ is a S-loop i.e. there exists a S-subloop $(S, \otimes)$ in $(H, \otimes)$. Let $(G, \circ)$ be the f, g-principal isotope of $(G, \oplus)$, then by Corollary 2.1, $(G, \circ)$ is a S-loop with say a S-subloop $(S, \circ)$. So,

$$(x \circ y) \circ z = x \circ (y \circ z) \ \forall \ x, y, z \in S$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \;\forall\; x, y \in S.$$

$$(xR_g^{-1} \oplus yL_f^{-1})R_g^{-1} \oplus zL_f^{-1} = xR_g^{-1} \oplus (yR_g^{-1} \oplus zL_f^{-1})L_f^{-1}.$$

Replacing xR_g^{-1} by x', yL_f^{-1} by y' and taking $z = e$ in $(S, \oplus)$ we have:

$$(x' \oplus y')R_g^{-1}R_{f^\rho} = x' \oplus y'L_fR_g^{-1}R_{f^\rho}L_f^{-1} \Rightarrow (I, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho})$$

is an autotopism of a S-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Universality of Smarandache Bol, Moufang and Extra Loops

Theorem 3.3. A Smarandache right(left)Bol loop is universal if all its f, g-principal isotopes are Smarandache f, g-principal isotopes. Conversely, if a Smarandache right(left)Bol loop is universal then

$$\mathcal{T}_1 = (R_gR_{f^\rho}^{-1}, L_{g^\lambda}R_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho})\Big(\mathcal{T}_2 = (R_{f^\rho}L_f^{-1}L_{g^\lambda}R_g^{-1}, L_fL_{g^\lambda}^{-1}, L_f^{-1}L_{g^\lambda})\Big)$$

is an autotopism of an SRB(SLB)-subloop of the SRBL(SLBL) such that f and g are S-elements.

Proof. Let $(G, \oplus)$ be a SRBL(SLBL) with a S-RB(LB)-subloop $(S, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 2.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \;\forall\; x, y \in S.$$

It is already known from [24] that RB(LB) loops are universal, hence $(S, \circ)$ is a RB(LB) loop thus a S-RB(LB)-subloop of $(G, \circ)$. By Theorem 2.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ which will now be a S-RB(LB)-subloop in $(H, \otimes)$. So, $(H, \otimes)$ is a SRBL(SLBL). This conclusion can also be drawn straight from Corollary 2.1.

The proof of the converse is as follows. If a SRBL(SLBL) $(G, \oplus)$ is universal then every isotope $(H, \otimes)$ is a SRBL(SLBL) i.e there exists a S-RB(LB)-subloop $(S, \otimes)$ in $(H, \otimes)$. Let $(G, \circ)$ be the f, g-principal isotope of $(G, \oplus)$, then by Corollary 2.1, $(G, \circ)$ is a SRBL(SLBL) with say a SRB(SLB)-subloop $(S, \circ)$. So for a SRB-subloop $(S, \circ)$,

$$[(y \circ x) \circ z] \circ x = y \circ [(x \circ z) \circ x] \;\forall\; x, y, z \in S$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \;\forall\; x, y \in S.$$

Thus,

$$[(yR_g^{-1} \oplus xL_f^{-1})R_g^{-1} \oplus zL_f^{-1}]R_g^{-1} \oplus xL_f^{-1} = yR_g^{-1} \oplus [(xR_g^{-1} \oplus zL_f^{-1})R_g^{-1} \oplus xL_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$(y'R_{f^\rho}R_g^{-1} \oplus z')R_g^{-1}R_{f^\rho} = y' \oplus z'L_{g^\lambda}R_g^{-1}R_{f^\rho}L_f^{-1}.$$

Again, replace $y'R_{f^\rho}R_g^{-1}$ by y'' so that

$$(y''\oplus z')R_g^{-1}R_{f^\rho}=y''R_gR_{f^\rho}^{-1}\oplus z'L_{g^\lambda}R_g^{-1}R_{f^\rho}L_f^{-1}\Rightarrow(R_gR_{f^\rho}^{-1},L_{g^\lambda}R_g^{-1}R_{f^\rho}L_f^{-1},R_g^{-1}R_{f^\rho})$$

is an autotopism of a SRB-subloop $(S,\oplus)$ of the S-loop $(G,\oplus)$ such that f and g are S-elements. On the other hand, for a SLB-subloop $(S,\circ)$,

$$[x\circ(y\circ x)]\circ z=x\circ[y\circ(x\circ z)]\ \forall\ x,y,z\in S,$$

where

$$x\circ y=xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

Thus,

$$[xR_g^{-1}\oplus(yR_g^{-1}\oplus xL_f^{-1})L_f^{-1}]R_g^{-1}\oplus zL_f^{-1}=xR_g^{-1}\oplus[yR_g^{-1}\oplus(xR_g^{-1}\oplus zL_f^{-1})L_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x=e$ in $(S,\oplus)$ we have

$$y'R_{f^\rho}L_f^{-1}L_{g^\lambda}R_g^{-1}\oplus z'=(y'\oplus z'L_{g^\lambda}L_f^{-1})L_f^{-1}L_{g^\lambda}.$$

Again, replace $z'L_{g^\lambda}L_f^{-1}$ by z'' so that

$$y'R_{f^\rho}L_f^{-1}L_{g^\lambda}R_g^{-1}\oplus z''L_fL_{g^\lambda}^{-1}=(y'\oplus z'')L_f^{-1}L_{g^\lambda}\Rightarrow(R_{f^\rho}L_f^{-1}L_{g^\lambda}R_g^{-1},L_fL_{g^\lambda}^{-1},L_f^{-1}L_{g^\lambda})$$

is an autotopism of a SLB-subloop $(S,\oplus)$ of the S-loop $(G,\oplus)$ such that f and g are S-elements.

Theorem 3.4. A Smarandache Moufang loop is universal if all its f,g-principal isotopes are Smarandache f,g-principal isotopes. Conversely, if a Smarandache Moufang loop is universal then

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1},L_fR_g^{-1}R_{f^\rho}L_f^{-1},L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}),$$
$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1},L_fR_g^{-1}R_{f^\rho}L_f^{-1},R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}),$$
$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1},L_fL_{g^\lambda}^{-1},L_f^{-1}L_{g^\lambda}),$$
$$(R_gR_{f^\rho}^{-1},L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1},R_g^{-1}R_{f^\rho}),$$
$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1},L_{g^\lambda}R_g^{-1}R_{f^\rho}L_{g^\lambda}^{-1},R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}),$$
$$(R_{f^\rho}L_f^{-1}L_{g^\lambda}R_{f^\rho}^{-1},L_fR_g^{-1}R_{f^\rho}L_f^{-1},L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}),$$

are autotopisms of a SM-subloop of the SML such that f and g are S-elements.

Proof. Let $(G,\oplus)$ be a SML with a SM-subloop $(S,\oplus)$. If $(G,\circ)$ is an arbitrary f,g-principal isotope of $(G,\oplus)$, then by Lemma 2.1, $(S,\circ)$ is a subloop of $(G,\circ)$ if $(S,\circ)$ is a Smarandache f,g-principal isotope of $(S,\oplus)$. Let us choose all $(S,\circ)$ in this manner. So,

$$x\circ y=xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

It is already known from [24] that Moufang loops are universal, hence $(S,\circ)$ is a Moufang loop thus a SM-subloop of $(G,\circ)$. By Theorem 2.1, for any isotope $(H,\otimes)$ of $(G,\oplus)$, there exists a $(G,\circ)$ such that $(H,\otimes)\cong(G,\circ)$. So we can now choose the isomorphic image of $(S,\circ)$ which will now be a SM-subloop in $(H, \otimes)$. So, $(H, \otimes)$ is a SML. This conclusion can also be drawn straight from Corollary 2.1.

The proof of the converse is as follows. If a SML $(G, \oplus)$ is universal then every isotope $(H, \otimes)$ is a SML i.e there exists a SM-subloop $(S, \otimes)$ in $(H, \otimes)$. Let $(G, \circ)$ be the f, g-principal isotope of $(G, \oplus)$, then by Corollary 2.1, $(G, \circ)$ is a SML with say a SM-subloop $(S, \circ)$. For a SM-subloop $(S, \circ)$,

$$(x \circ y) \circ (z \circ x) = [x \circ (y \circ z)] \circ x \ \forall \ x, y, z \in S,$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$(xR_g^{-1} \oplus yL_f^{-1})R_g^{-1} \oplus (zR_g^{-1} \oplus xL_f^{-1})L_f^{-1} = [xR_g^{-1} \oplus (yR_g^{-1} \oplus zL_f^{-1})L_f^{-1}]R_g^{-1} \oplus xL_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1} \oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1} = (y' \oplus z')L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho} \Rightarrow$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho})$$

is an autotopism of a SM-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Again, for a SM-subloop $(S, \circ)$,

$$(x \circ y) \circ (z \circ x) = x \circ [(y \circ z) \circ x] \ \forall \ x, y, z \in S,$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$(xR_g^{-1} \oplus yL_f^{-1})R_g^{-1} \oplus (zR_g^{-1} \oplus xL_f^{-1})L_f^{-1} = xR_g^{-1} \oplus [(yR_g^{-1} \oplus zL_f^{-1})R_g^{-1} \oplus xL_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1} \oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1} = (y' \oplus z')R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda} \Rightarrow$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda})$$

is an autotopism of a SM-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Also, if $(S, \circ)$ is a SM-subloop then,

$$[(x \circ y) \circ x] \circ z = x \circ [y \circ (x \circ z)] \ \forall \ x, y, z \in S,$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$[(xR_g^{-1} \oplus yL_f^{-1})R_g^{-1} \oplus xL_f^{-1}]R_g^{-1} \oplus zL_f^{-1} = xR_g^{-1} \oplus [yR_g^{-1} \oplus (xR_g^{-1} \oplus zL_f^{-1})L_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x=e$ in $(S,\oplus)$ we have

$$y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1}\oplus z'=(y'\oplus z'L_{g^\lambda}L_f^{-1})L_f^{-1}L_{g^\lambda}.$$

Again, replace $z'L_{g^\lambda}L_f^{-1}$ by z'' so that

$$\begin{aligned} y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1}\oplus z''L_fL_{g^\lambda}^{-1} &= (y'\oplus z'')L_f^{-1}L_{g^\lambda} \\ &\Rightarrow (R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1}, L_fL_{g^\lambda}^{-1}, L_f^{-1}L_{g^\lambda}) \end{aligned}$$

is an autotopism of a SM-subloop $(S,\oplus)$ of the S-loop $(G,\oplus)$ such that f and g are S-elements.

Furthermore, if $(S,\circ)$ is a SM-subloop then,

$$[(y\circ x)\circ z]\circ x=y\circ[x\circ(z\circ x)]\ \forall\ x,y,z\in S,$$

where

$$x\circ y=xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

Thus,

$$[(yR_g^{-1}\oplus xL_f^{-1})R_g^{-1}\oplus zL_f^{-1}]R_g^{-1}\oplus xL_f^{-1}=yR_g^{-1}\oplus[xR_g^{-1}\oplus(zR_g^{-1}\oplus xL_f^{-1})L_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x=e$ in $(S,\oplus)$ we have

$$(y'R_{f^\rho}R_g^{-1}\oplus z')R_g^{-1}R_{f^\rho}=y'\oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1}.$$

Again, replace $y'R_{f^\rho}R_g^{-1}$ by y'' so that

$$\begin{aligned} (y''\oplus z')R_g^{-1}R_{f^\rho} &= y''R_gR_{f^\rho}^{-1}\oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1} \\ &\Rightarrow (R_gR_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1}, R_g^{-1}R_{f^\rho}) \end{aligned}$$

is an autotopism of a SM-subloop $(S,\oplus)$ of the S-loop $(G,\oplus)$ such that f and g are S-elements.

Lastly, $(S,\oplus)$ is a SM-subloop if and only if $(S,\circ)$ is a SRB-subloop and a SLB-subloop. So by Theorem 3.3, $\mathcal{T}_1$ and $\mathcal{T}_2$ are autotopisms in $(S,\oplus)$, hence $\mathcal{T}_1\mathcal{T}_2$ and $\mathcal{T}_2\mathcal{T}_1$ are autotopisms in $(S,\oplus)$.

Theorem 3.5. A Smarandache extra loop is universal if all its f,g-principal isotopes are Smarandache f,g-principal isotopes. Conversely, if a Smarandache extra loop is universal then

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_{f^\rho}^{-1}R_gL_f^{-1}, L_f^{-1}L_{g^\lambda}R_{f^\rho}^{-1}R_g),$$

$$(R_gR_{f^\rho}^{-1}R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_{g^\lambda}L_f^{-1}, L_f^{-1}L_{g^\lambda}),$$

$$(R_{f^\rho}R_g^{-1}, L_fL_{g^\lambda}^{-1}L_fR_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho}),$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}),$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}),$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1}, L_fL_{g^\lambda}^{-1}, L_f^{-1}L_{g^\lambda}),$$

$$(R_gR_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1}, R_g^{-1}R_{f^\rho}),$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_{g^\lambda}R_g^{-1}R_{f^\rho}L_{g^\lambda}^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}),$$

$$(R_{f^\rho}L_f^{-1}L_{g^\lambda}R_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}),$$

are autotopisms of a SE-subloop of the SEL such that f and g are S-elements.

Proof. Let $(G, \oplus)$ be a SEL with a SE-subloop $(S, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 2.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

In [9] and [17] respectively, it was shown and stated that a loop is an extra loop if and only if it is a Moufang loop and a CC-loop. But since CC-loops are G-loops(they are isomorphic to all loop isotopes) then extra loops are universal, hence $(S, \circ)$ is an extra loop thus a SE-subloop of $(G, \circ)$. By Theorem 2.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ which will now be a SE-subloop in $(H, \otimes)$. So, $(H, \otimes)$ is a SEL. This conclusion can also be drawn straight from Corollary 2.1.

The proof of the converse is as follows. If a SEL $(G, \oplus)$ is universal then every isotope $(H, \otimes)$ is a SEL i.e there exists a SE-subloop $(S, \otimes)$ in $(H, \otimes)$. Let $(G, \circ)$ be the f, g-principal isotope of $(G, \oplus)$, then by Corollary 2.1, $(G, \circ)$ is a SEL with say a SE-subloop $(S, \circ)$. For a SE-subloop $(S, \circ)$,

$$[(x \circ y) \circ z] \circ x = x \circ [y \circ (z \circ x)] \ \forall \ x, y, z \in S,$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$[(xR_g^{-1} \oplus yL_f^{-1})R_g^{-1} \oplus zL_f^{-1}]R_g^{-1} \oplus xL_f^{-1} = xR_g^{-1} \oplus [yR_g^{-1} \oplus (zR_g^{-1} \oplus xL_f^{-1})L_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$(y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1} \oplus z')R_g^{-1}R_{f^\rho} = (y' \oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1})L_f^{-1}L_{g^\lambda}.$$

Again, replace $z'L_fR_g^{-1}R_{f^\rho}L_f^{-1}$ by z'' so that

$$y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1} \oplus z''L_fR_{f^\rho}^{-1}R_gL_f^{-1} = (y' \oplus z'')L_f^{-1}L_{g^\lambda}R_{f^\rho}^{-1}R_g \Rightarrow$$

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_{f^\rho}^{-1}R_gL_f^{-1}, L_f^{-1}L_{g^\lambda}R_{f^\rho}^{-1}R_g)$$

is an autotopism of a SE-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Again, for a SE-subloop $(S, \circ)$,

$$(x \circ y) \circ (x \circ z) = x \circ [(y \circ x) \circ z] \ \forall \ x, y, z \in S,$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

Thus,

$$(xR_g^{-1} \oplus yL_f^{-1})R_g^{-1} \oplus (xR_g^{-1} \oplus zL_f^{-1})L_f^{-1} = xR_g^{-1} \oplus [(yR_g^{-1} \oplus xL_f^{-1})R_g^{-1} \oplus zL_f^{-1}]L_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$y'R_gL_f^{-1}L_{g^\lambda}R_g^{-1} \oplus z'L_{g^\lambda}L_f^{-1} = (y'R_{f^\rho}R_g^{-1} \oplus z')L_f^{-1}L_{g^\lambda}.$$

Again, replace $y'R_{f^\rho}R_g^{-1}$ by y'' so that

$$\begin{aligned} y''R_gR_{f^\rho}^{-1}R_gL_f^{-1}L_{g^\lambda}R_g^{-1} \oplus z'L_{g^\lambda}L_f^{-1} &= (y'' \oplus z')L_f^{-1}L_{g^\lambda} \\ &\Rightarrow (R_gR_{f^\rho}^{-1}R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_{g^\lambda}L_f^{-1}, L_f^{-1}L_{g^\lambda}), \end{aligned}$$

is an autotopism of a SE-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Also, if $(S, \circ)$ is a SE-subloop then,

$$(y \circ x) \circ (z \circ x) = [y \circ (x \circ z)] \circ x \;\forall\; x, y, z \in S$$

where

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \;\forall\; x, y \in S.$$

Thus,

$$(yR_g^{-1} \oplus xL_f^{-1})R_g^{-1} \oplus (zR_g^{-1} \oplus xL_f^{-1})L_f^{-1} = [(yR_g^{-1} \oplus (xR_g^{-1} \oplus zL_f^{-1})L_f^{-1}]R_g^{-1} \oplus xL_f^{-1}.$$

Replacing yR_g^{-1} by y', zL_f^{-1} by z' and taking $x = e$ in $(S, \oplus)$ we have

$$y'R_{f^\rho}R_g^{-1} \oplus z'L_fR_g^{-1}R_{f^\rho}L_f^{-1} = (y' \oplus z'L_{g^\lambda}L_f^{-1})R_g^{-1}R_{f^\rho}.$$

Again, replace $z'L_{g^\lambda}L_f^{-1}$ by z'' so that

$$\begin{aligned} y'R_{f^\rho}R_g^{-1} \oplus z''L_fL_{g^\lambda}^{-1}L_fR_g^{-1}R_{f^\rho}L_f^{-1} &= (y' \oplus z')R_g^{-1}R_{f^\rho} \\ &\Rightarrow (R_{f^\rho}R_g^{-1}, L_fL_{g^\lambda}^{-1}L_fR_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho}) \end{aligned}$$

is an autotopism of a SE-subloop $(S, \oplus)$ of the S-loop $(G, \oplus)$ such that f and g are S-elements.

Lastly, $(S, \oplus)$ is a SE-subloop if and only if $(S, \circ)$ is a SM-subloop and a SCC-subloop. So by Theorem 3.4, the six remaining triples are autotopisms in $(S, \oplus)$.

Universality of Smarandache Inverse Property Loops

Theorem 3.6. A Smarandache left(right) inverse property loop in which all its f, g-principal isotopes are Smarandache f, g-principal isotopes is universal if and only if it is a Smarandache left(right) Bol loop in which all its f, g-principal isotopes are Smarandache f, g-principal isotopes.

Proof. Let $(G, \oplus)$ be a SLIPL with a SLIP-subloop $(S, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 2.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \;\forall\; x, y \in S.$$

$(G, \oplus)$ is a universal SLIPL if and only if every isotope $(H, \otimes)$ is a SLIPL. $(H, \otimes)$ is a SLIPL if and only if it has at least a SLIP-subloop $(S, \otimes)$. By Theorem 2.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ to be $(S, \otimes)$ which is already a SLIP-subloop in $(H, \otimes)$. So, $(S, \circ)$ is also a SLIP-subloop in $(G, \circ)$. As shown in [24], $(S, \oplus)$ and its f, g-isotope(Smarandache f, g-isotope) $(S, \circ)$ are SLIP-subloops if and only if $(S, \oplus)$ is a left Bol subloop(i.e a SLB-subloop). So, $(G, \oplus)$ is SLBL.

Conversely, if $(G, \oplus)$ is SLBL, then there exists a SLB-subloop $(S, \oplus)$ in $(G, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 2.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

By Theorem 2.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ to be $(S, \otimes)$ which is a SLB-subloop in $(H, \otimes)$ using the same reasoning in Theorem 3.3. So, $(S, \circ)$ is a SLB-subloop in $(G, \circ)$. Left Bol loops have the left inverse property(LIP), hence, $(S, \oplus)$ and $(S, \circ)$ are SLIP-subloops in $(G, \oplus)$ and $(G, \circ)$ respectively. Thence, $(G, \oplus)$ and $(G, \circ)$ are SLBLs. Therefore, $(G, \oplus)$ is a universal SLIPL.

The proof for a Smarandache right inverse property loop is similar and is as follows. Let $(G, \oplus)$ be a SRIPL with a SRIP-subloop $(S, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 2.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

$(G, \oplus)$ is a universal SRIPL if and only if every isotope $(H, \otimes)$ is a SRIPL. $(H, \otimes)$ is a SRIPL if and only if it has at least a SRIP-subloop $(S, \otimes)$. By Theorem 2.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ to be $(S, \otimes)$ which is already a SRIP-subloop in $(H, \otimes)$. So, $(S, \circ)$ is also a SRIP-subloop in $(G, \circ)$. As shown in [24], $(S, \oplus)$ and its f, g-isotope(Smarandache f, g-isotope) $(S, \circ)$ are SRIP-subloops if and only if $(S, \oplus)$ is a right Bol subloop(i.e a SRB-subloop). So, $(G, \oplus)$ is SRBL.

Conversely, if $(G, \oplus)$ is SRBL, then there exists a SRB-subloop $(S, \oplus)$ in $(G, \oplus)$. If $(G, \circ)$ is an arbitrary f, g-principal isotope of $(G, \oplus)$, then by Lemma 2.1, $(S, \circ)$ is a subloop of $(G, \circ)$ if $(S, \circ)$ is a Smarandache f, g-principal isotope of $(S, \oplus)$. Let us choose all $(S, \circ)$ in this manner. So,

$$x \circ y = xR_g^{-1} \oplus yL_f^{-1} \ \forall \ x, y \in S.$$

By Theorem 2.1, for any isotope $(H, \otimes)$ of $(G, \oplus)$, there exists a $(G, \circ)$ such that $(H, \otimes) \cong (G, \circ)$. So we can now choose the isomorphic image of $(S, \circ)$ to be $(S, \otimes)$ which is a SRB-subloop in $(H, \otimes)$ using the same reasoning in Theorem 3.3. So, $(S, \circ)$ is a SRB-subloop in $(G, \circ)$. Right Bol loops have the right inverse property(RIP), hence, $(S, \oplus)$ and $(S, \circ)$ are SRIP-subloops in $(G, \oplus)$ and $(G, \circ)$ respectively. Thence, $(G, \oplus)$ and $(G, \circ)$ are SRBLs. Therefore, $(G, \oplus)$ is a universal SRIPL.

Theorem 3.7. A Smarandache inverse property loop in which all its f,g-principal isotopes are Smarandache f,g-principal isotopes is universal if and only if it is a Smarandache Moufang loop in which all its f,g-principal isotopes are Smarandache f,g-principal isotopes.

Proof. Let $(G,\oplus)$ be a SIPL with a SIP-subloop $(S,\oplus)$. If $(G,\circ)$ is an arbitrary f,g-principal isotope of $(G,\oplus)$, then by Lemma 2.1, $(S,\circ)$ is a subloop of $(G,\circ)$ if $(S,\circ)$ is a Smarandache f,g-principal isotope of $(S,\oplus)$. Let us choose all $(S,\circ)$ in this manner. So,

$$x\circ y = xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

$(G,\oplus)$ is a universal SIPL if and only if every isotope $(H,\otimes)$ is a SIPL. $(H,\otimes)$ is a SIPL if and only if it has at least a SIP-subloop $(S,\otimes)$. By Theorem 2.1, for any isotope $(H,\otimes)$ of $(G,\oplus)$, there exists a $(G,\circ)$ such that $(H,\otimes)\cong(G,\circ)$. So we can now choose the isomorphic image of $(S,\circ)$ to be $(S,\otimes)$ which is already a SIP-subloop in $(H,\otimes)$. So, $(S,\circ)$ is also a SIP-subloop in $(G,\circ)$. As shown in [24], $(S,\oplus)$ and its f,g-isotope(Smarandache f,g-isotope) $(S,\circ)$ are SIP-subloops if and only if $(S,\oplus)$ is a Moufang subloop(i.e a SM-subloop). So, $(G,\oplus)$ is SML.

Conversely, if $(G,\oplus)$ is SML, then there exists a SM-subloop $(S,\oplus)$ in $(G,\oplus)$. If $(G,\circ)$ is an arbitrary f,g-principal isotope of $(G,\oplus)$, then by Lemma 2.1, $(S,\circ)$ is a subloop of $(G,\circ)$ if $(S,\circ)$ is a Smarandache f,g-principal isotope of $(S,\oplus)$. Let us choose all $(S,\circ)$ in this manner. So,

$$x\circ y = xR_g^{-1}\oplus yL_f^{-1}\ \forall\ x,y\in S.$$

By Theorem 2.1, for any isotope $(H,\otimes)$ of $(G,\oplus)$, there exists a $(G,\circ)$ such that $(H,\otimes)\cong(G,\circ)$. So we can now choose the isomorphic image of $(S,\circ)$ to be $(S,\otimes)$ which is a SM-subloop in $(H,\otimes)$ using the same reasoning in Theorem 3.3. So, $(S,\circ)$ is a SM-subloop in $(G,\circ)$. Moufang loops have the inverse property(IP), hence, $(S,\oplus)$ and $(S,\circ)$ are SIP-subloops in $(G,\oplus)$ and $(G,\circ)$ respectively. Thence, $(G,\oplus)$ and $(G,\circ)$ are SMLs. Therefore, $(G,\oplus)$ is a universal SIPL.

Corollary 3.1. If a Smarandache left(right) inverse property loop is universal then

$$(R_gR_{f^\rho}^{-1}, L_{g^\lambda}R_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho})\Big((R_{f^\rho}L_f^{-1}L_{g^\lambda}R_g^{-1}, L_fL_{g^\lambda}^{-1}, L_f^{-1}L_{g^\lambda})\Big)$$

is an autotopism of a SLIP(SRIP)-subloop of the SLIPL(SRIPL) such that f and g are S-elements.

Proof. This follows by Theorem 3.6 and Theorem 3.1.

Corollary 3.2. If a Smarandache inverse property loop is universal then

$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}),$$
$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}),$$
$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}R_g^{-1}, L_fL_{g^\lambda}^{-1}, L_f^{-1}L_{g^\lambda}),$$
$$(R_gR_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}L_f^{-1}, R_g^{-1}R_{f^\rho}),$$
$$(R_gL_f^{-1}L_{g^\lambda}R_g^{-1}, L_{g^\lambda}R_g^{-1}R_{f^\rho}L_{g^\lambda}^{-1}, R_g^{-1}R_{f^\rho}L_f^{-1}L_{g^\lambda}),$$
$$(R_{f^\rho}L_f^{-1}L_{g^\lambda}R_{f^\rho}^{-1}, L_fR_g^{-1}R_{f^\rho}L_f^{-1}, L_f^{-1}L_{g^\lambda}R_g^{-1}R_{f^\rho}),$$

are autotopisms of a SIP-subloop of the SIPL such that f and g are S-elements.

Proof. This follows from Theorem 3.7 and Theorem 3.4.

References

[1] R. Artzy, Crossed inverse and related loops, Trans. Amer. Math. Soc., **91**(1959), No. 3, 480-492.

[2] R. H. Bruck, A survey of binary systems, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1966, 185.

[3] R. H. Bruck and L. J. Paige, Loops whose inner mappings are automorphisms, The annuals of Mathematics, **63**(1956), No. 2, 308-323.

[4] O. Chein, H. O. Pflugfelder and J. D. H. Smith, It quasigroups and loops : Theory and applications, Heldermann Verlag, 1990, 568.

[5] J. Déne and A. D. Keedwell, Latin squares and their applications, The Academic press Lts, 1974, 549.

[6] F. Fenyves, Extra loops I, Publ. Math. Debrecen, **15** (1968), 235-238.

[7] F. Fenyves, Extra loops II, Publ. Math. Debrecen, **16**(1969), 187-192.

[8] E. G. Goodaire, E. Jespers and C. P. Milies, Alternative loop rings, NHMS(184), Elsevier, 1996, 387.

[9] E. G. Goodaire and D. A. Robinson, Some special conjugacy closed loops, Canad. Math. Bull. **33**(1990), 73-78.

[10] T. G. Jaíyéọlá, An holomorphic study of the Smarandache concept in loops, Scientia Magna, **2**(2006), No. 1, 1-8.

[11] T. G. Jaíyéọlá, Parastrophic invariance of Smarandache quasigroups, Scientia Magna, **2**(2006), No. 3, 48-53.

[12] T. G. Jaíyéọlá, An isotopic study of properties of central loops, M.Sc. Dissertation, University of Agriculture, Abeokuta, 2005.

[13] T. G. Jaíyéọlá and J. O. Adéníran, On isotopic characterization of central loops, communicated for publication.

[14] T. Kepka, M. K. Kinyon and J. D. Phillips, F-quasigroups and generalised modules, communicated for publication.

[15] T. Kepka, M. K. Kinyon and J. D. Phillips, F-quasigroups isotopic to groups, communicated for publication.

[16] T. Kepka, M. K. Kinyon and J. D. Phillips, The structure of F-quasigroups, communicated for publication.

[17] M. K. Kinyon, K. Kunen, The structure of extra loops, Quasigroups and Related Systems, **12**(2004), 39-60.

[18] M. K. Kinyon, J. D. Phillips and P. Vojtěchovský, Loops of Bol-Moufang type with a subgroup of index two, Bull. Acad. Stinte. Rebub. Mold. Mat. **2**(2004), No. 45, 1-17.

[19] K. Kunen, Quasigroups, loops and associative laws, J. Alg. **185**(1996), 194-204.

[20] K. Kunen, Moufang quasigroups, J. Alg. **183**(1996), 231-234.

[21] A. S. Muktibodh, Smarandache quasigroups, Scientia Magna Journal, **2**(2006), No. 1, 13-19.

[22] P. T. Nagy and K. Strambach, Loops as invariant sections in groups, and their geometry, Canad. J. Math. **46**(1994), No. 5, 1027-1056.

[23] J. M. Osborn, Loops with the weak inverse property, Pac. J. Math. **10**(1961), 295-304.

[24] H. O. Pflugfelder, Quasigroups and loops : Introduction, Sigma series in Pure Math. 7, Heldermann Verlag, Berlin, 1990, 147.

[25] J. D. Phillips and P. Vojtěchovský, The varieties of loops of Bol-Moufang type, Algebra Universalis, **53**(2005), No. 3, 115-137.

[26] J. D. Phillips and P. Vojtěchovský, The varieties of quasigroups of Bol-Moufang type: An equational approach, J. Alg. **293**(2005), 17-33.

[27] W. B. Vasantha Kandasamy, Smarandache loops, Department of Mathematics, Indian Institute of Technology, Madras, India, 2002, 128.

[28] W. B. Vasantha Kandasamy, Smarandache Loops, Smarandache notions journal, **13** (2002), 252-258.

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Smarandache representation and its applications

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Here we for the first time define Smarandache representation of finite S-bisemigroup. We know every S-bisemigroup, $S = S_1 \cup S_2$ contains a bigroup $G = G_1 \cup G_2$. The Smarandache representation S-bisemigroups depends on the S-bigroup G which we choose. Thus this method widens the Smarandache representations. We first define the notion of Smarandache pseudo neutrosophic bisemigroup.

Definition 1. Let $S = S_1 \cup S_2$ be a neutrosophic bisemigroup. If S has only bigroup which is not a neutrosophic bigroup, then we all S a Smarandache pseudo neutrosophic bisemigroup(S-pseudo neutrosophic bisemigroup).

Example 1. Let $S = S_1 \cup S_2$ where $S_1 = Q(I) \times Q(I)$ and $S_2 = \{2 \times 2$ matrices with entries from $Q(I)\}$ both S_1 and S_2 under multiplication is a semigroup. Thus S is a neutrosophic bisemigroup. Take $G = G_1 \cup G_2$ where $G_1 = \{Q\backslash(0) \times Q\backslash(0)\}$ and $G_2 = \{$set of all 2×2 matrices A with entries from Q such that $|A| \neq 0\}$. G_1 and G_2 are groups under multiplication. So S is a pseudo Smarandache Neutrosophic bisemigroup.

Now we give the Smarandache representation of finite pseudo Smarandache neutrosophic bisemigroups.

Definition 2. Let $G = G_1 \cup G_2$ be a Smarandache neutrosophic bisemigroup and $V = V_1 \cup V_2$ be a bivector space. A Smarandache birepresentation of G on V is a mapping $S_\rho = S_\rho^1 \cup S_\rho^2$ from $H_1 \cup H_2$ ($H_1 \cup H_2$ is a subbigroup of G which is not a neutrosophic bigroup) to invertible linear bitransformation on $V = V_1 \cup V_2$ such that

$$S_{\rho_{xy}} = S^1_{\rho_{x_1 y_1}} \cup S^2_{\rho_{x_2 y_2}} = \left(S^1_{\rho_{x_1}} \circ S^1_{\rho_{y_1}}\right) \cup \left(S^2_{\rho_{x_2}} \circ S^2_{\rho_{y_2}}\right)$$

for all $x_1, y_1 \in H_1$ and for all $x_2, y_2 \in H_2$, $H_1 \cup H_2 \subset G_1 \cup G_2$. Here $\mathrm{S}_{\rho_x} = S^1_{\rho_{x_1}} \cup S^2_{\rho_{x_2}}$ to denote the invertible linear bitransformation on $V = V_1 \cup V_2$associated to $x = x_1 \cup x_2$ on $H = H_1 \cup H_2$, so that we may write

$$S_{\rho_x}(\nu) = S_{\rho_x}(\nu_1 \cup \nu_2) = S^1_{\rho_{x_1}}(\nu_1) \cup S^2_{\rho_{x_2}}(\nu_2)$$

for the image of the vector $\nu = \nu_1 \cup \nu_2$ in $V = V_1 \cup V_2$ under $S_{\rho_x} = S^1_{\rho_{x_1}} \cup S^2_{\rho_{x_2}}$. As a result, we have that $S_{\rho_e} = S^1_{\rho_{e_1}} \cup S^2_{\rho_{z_2}} = I^1 \cup I^2$ denotes the identity bitransformation on $V = V_1 \cup V_2$ and $S^{-1}_{\rho_x} = S^1_{\rho_{x_1^{-1}}} \cup S^2_{\rho_{x_2^{-1}}} = \left(S^1_{\rho_{x_1}}\right)^{-1} \cup \left(S^2_{\rho_{x_2}}\right)^{-1}$ for all $x = x_1 \cup x_2 \in H_1 \cup H_2 \subset G_1 \cup G_2 = G$.

In other words a birepresentation of $H = H_1 \cup H_2$ on $V = V_1 \cup V_2$ is a bihomomorphism from H into $GL(V)$ i.e. $(H_1$ into $GL(V_1)) \cup (H_2$ into $GL(V_2))$. The bidimension of $V = V_1 \cup V_2$ is called the bidegree of the representation.

Thus depending on the number of subbigroup of the S-neutrosophic bisemigroup we have several S-birepresentations of the finite S-neutrosophic bisemigroup.

Basic example of birepresentation would be Smarandache left regular birepresentation and Smarandache right regular birepresentation over a field K defined as follows.

We take $V_H = V_{H_1} \cup V_{H_2}$ to be a bivector space of bifunctions on $H_1 \cup H_2$ with values in K (where $H = H_1 \cup H_2$ is a subbigroup of the S-neutrosophic bisemigroup where H is not a neutrosophic bigroup). For Smarandache left regular birepresentation (S-left regular biregular representative) relative to $H = H_1 \cup H_2$ we define

$$SL_x = S^1 L^1_{x_1} \cup S^2 L^2_{x_2} = \left(S^1 \cup S^2\right)\left(L^1 \cup L_2\right)_{x_1 \cup x_2},$$

from $V_{H_1} \cup V_{H_2} \to V_{H_1} \cup V_{H_2}$ for each $x_1 \cup x_2 \in H = H_1 \cup H_2$ by for each $x = x_1 \cup x_2$ in $H = H_1 \cup H_2$ by $SL_x(f)(z) = f(x^{-1}z)$ for each function $f(z)$ in $V_H = V_{H_1} \cup V_{H_2}$ i.e. $S^1 L^1_{x_1} f_1(z_1) \cup S^2 L^2_{x_2} f_2(z_2) = f_1(x_1^{-1} z_1) \cup f_2(x_2^{-1} z_2)$.

For the Smarandache right regular birepresentation (S-right regular birepresentation) we define $SR_x = SR_{x_1 \cup x_2} : V_{H_1} \cup V_{H_2} \to V_{H_1} \cup V_{H_2}$; $H_1 \cup H_2 = H$ for each $x = x_1 \cup x_2 \in H_1 \cup H_2$ by $SR_x(f)(z) = f(zx)$.

$$S^1 R^1_{x_1} f_1(z_1) \cup S^2 R^2_{x_2} \left(f_2(z_2)\right) = f_1(z_1 x_1) \cup f_2(z_2 x_2),$$

for each function $f_1(z_1) \cup f_2(z_2) = f(z)$ in $V_H = V_{H_1} \cup V_{H_2}$.

Thus if $x = x_1 \cup x_2$ and $y = y_1 \cup y_2$ are elements $H_1 \cup H_2 \subset G_1 \cup G_2$.

Then

$$\begin{aligned}
(SL_x \circ SL_y)\ (f(z)) &= SL_x(SL_y)(f)(z) \\
&= (SL_y(f))x^{-1}z \\
&= f(y^{-1}x^{-1}z) \\
&= f_1 \left(y_1^{-1}x_1^{-1}z_1\right) \cup f_2\left(y_2^{-1}x_2^{-1}z_2\right) \\
&= f_1((x_1y_1)^{-1}z_1) \cup f_2((x_2y_2)^{-1}z_2) \\
&= S^1L^1_{x_1y_1}(f_1)(z_1) \cup S^2L^2_{x_2y_2}f_2(z_2) \\
&= \left[\left(S^1L^1_{x_1} \cup S^2L^2_{x_2}\right)\ \left(S^1L^1_{y_1} \cup S^2L^2_{y_2}\right)\right]\ f(z_1 \cup z_2) \\
&= \left[\left(S^1L^1_{x_1} \cup S^2L^2_{x_2}\right) \circ \left(S^1L^1_{y_1} \cup S^2L^2_{y_2}\right)\right]\ \left(f_1(z_1) \cup f_2(z_2)\right).
\end{aligned}$$

and

$$\begin{aligned}(SR_x \circ SR_y)(f)(z) &= \left(S^1R^1_{x_1} \cup S^2R^2_{x_2}\right) \circ \left(S^1R^1_{y_1} \cup S^2R^2_{y_2}\right)(f)(z)\\ &= SR_x(SR_y(f))(z)\\ &= \left(S^1R^1_{x_1} \cup S^2R^2_{x_2}\right) \circ \left(S^1R^1_{y_1} \cup S^2R^2_{y_2}\right)(f)(z)\\ &= \left(S^1R^1_{y_1}(f_1) \cup S^2R^2_{y_2}(f_2)\right)(z_1x_1 \cup z_2x_2)\\ &= f_1(z_1x_1y_1) \quad \cup f_2(z_2x_2y_2)\\ &= S^1R^1_{x_1y_1}f_1(z_1) \cup S^2R^2_{x_2y_2}f_2(z_2)\\ &= SR_{xy}(f)(z).\end{aligned}$$

Thus for a given S-neutrosophic bisemigroup we will have several V's associated with them i.e. bivector space functions on each $H_1 \cup H_2 \subset G_1 \cup G_2$, H a subbigroup of the S-neutrosophic bisemigroup with values from K. This study in this direction is innovative.

We have yet another Smarandache birepresentation which can be convenient is the following. For each $w = w_1 \cup w_2$ in $H = H_1 \cup H_2$, H bisubgroups of the S-neutrosophic bisemigroup $G = G_1 \cup G_2$.

Define a bifunction

$$\phi_w(z) = \phi^1_{w_1}(z_1) \cup \phi^2_{w_2}(z_2)$$

on $H_1 \cup H_2 = H$ by $\phi^1_{w_1}(z_1) \cup \phi^2_{w_2}(z_2) = 1 \cup 1$, where $w = w_1 \cup w_2 = z = z_1 \cup z_2$, $\phi^1_{w_1}(z_1) \cup \phi^2_{w_2}(z_2) = 0 \cup 0$ when $z \neq w$.

Thus the functions $\phi_w = \phi^1_{w_1} \cup \phi^2_{w_2}$ for $w = w_1 \cup w_2$ in $H = H_1 \cup H_2$ ($H \subset G$) form a basis for the space of bifunctions on each $H = H_1 \cup H_2$ contained in $G = G_1 \cup G_2$.

One can check that

$$SL_x\ (\phi_w) = (\phi_{xw}) \text{ i.e. } S^1L^1_{x_1}\ (\phi_{w_1}) \cup S^2L^2_{x_2}\ (\phi_{w_2}) = \phi^1_{x_1w_1} \cup \phi^2_{x_{12}w_2},$$

$$\mathrm{SR}_x\ (\phi_w) = \phi_{xw} \text{ i.e. } S^1R^1_{x_1}\ \left(\phi^1_{w_1}\right) \cup S^2R^2_{x_2}\ \left(\phi^2_{w_2}\right) = \phi^1_{x_1w_1} \cup \phi^2_{x_2w_2},$$

for all $x \in H_1 \cup H_2 \subset G$.

Observe that

$$SL_x \circ SR_y = SR_y \circ SL_x \text{ i.e. } \left(S^1L^1_{x_1} \cup S^2L^2_{x_2}\right) \circ \left(S^1L^1_{y_1} \cup S^2L^2_{y_2}\right)$$

$$\left(S^1L^1_{y_1} \cup S^2L^2_{y_2}\right) \circ \left(S^1L^1_{x_1} \cup S^2L^2_{x_2}\right),$$

for all $x = x_1 \cup x_2$ and $y = y_1 \cup y_2$ in $G = G_1 \cup G_2$.

More generally suppose we have a bihomomorphism from the bigroups $H = H_1 \cup H_2 \subset G = G_1 \cup G_2$ (G a S-neutrosophic bisemigroup) to the bigroup of permutations on a non empty finite biset. $E^1 \cup E^2$. That is suppose for each x_1 in $H_1 \subset G_1$ and x_2 in H_2, $H_2 \subset G_2$, x in $H_1 \cup H_2 \subset G_1 \cup G_2$ we have a bipermutation $\pi^1_{x_1} \cup \pi^1_{x_2}$ on $E_1 \cup E_2$ i.e. one to one mapping of E_1 on to E_1 and E_2 onto E_2 such that

$$\pi_x \circ \pi_y = \pi^1_{x_1} \circ \pi^1_{y_1} \cup \pi^2_{x_2} \circ \pi^2_{y_2}, \pi e = \pi^1_{e_1} \cup \pi^2_{e_2}$$

is the biidentity bimapping of $E_1 \cup E_2$ and $\pi_{x^{-1}} = \pi^1_{x_1^{-1}} \cup \pi^1_{x_2^{-1}}$ is the inverse mapping of $\pi_x = \pi^1_{x_1} \cup \pi^2_{x_2}$ on $E_1 \cup E_2$. Let $V_H = V^1_{H_1} \cup V^2_{H_2}$ be the bivector space of K-valued bifunctions on $E_1 \cup E_2$.

Then we get the Smarandache birepresentation of $H_1 \cup H_2$ on $V_{H_1} \cup V_{H_2}$ by associating to each $x = x_1 \cup x_2$ in $H_1 \cup H_2$ the linear bimapping

$$\pi_x = \pi^1_{x_1} \cup \pi^2_{x_2} : V_{H_1} \cup V_{H_2} \to V_{H_1} \cup V_{H_2},$$

defined by

π_x (f) (a) = f (π_x (a)) i.e. $(\pi_{x_1} \cup \pi_{x_2}) \left(f^1 \cup f^2\right) (a_1 \cup a_2) = \mathrm{f}^1 (\pi_{x_1}(a_1)) \cup f^2 (\pi_{x_2}(a_2))$

for every $f^1(a_1) \cup f^2(a_2) = f(a)$ in $V_{H_1} \cup V_{H_2}$.

This is called the Smarandache bipermutation birepresentation corresponding to the bihomomorphism $x \mapsto \pi x$ i.e. $x_1 \mapsto \pi_{x_1} \cup x_2 \mapsto \pi_{x_2}$ from $H = H_1 \cup H_2$ to permutations on $E = E_1 \cup E_2$.

It is indeed a Smarandache birepresentation for we have several E's and $V_H = V^1_{H_1} \cup V^2_{H_2}{}'s$ depending on the number of proper subsets $H = H_1 \cup H_2$ in $G_1 \cup G_2$ (G the S-bisemigroup) which are bigroups under the operations of $G = G_1 \cup G_2$ because for each $x = x_1 \cup x_2$ and $y = y_1 \cup y_2$ in $H = H_1 \cup H_2$ and each function $f(a) = f_1(a_1) \cup f_2(a_2)$ in $V_H = V^1_{H_1} \cup V^2_{H_2}$ we have

$$\begin{aligned}
(\pi_x \circ \pi_y)\ (f)\ (a) &= \left(\pi^1_{x_1} \cup \pi^2_{x_2}\right) \circ \left(\pi^1_{y_1} \cup \pi^2_{y_2}\right) (f_1 \cup f_2)\ (a_1 \cup a_2) \\
&= \left(\pi^1_{x_1} \circ \pi^1_{y_1}\right) (f_1)\ (a_1) \cup \left(\pi^2_{x_2} \circ \pi^2_{y_2}\right) (f_2)\ (a_2) \\
&= \pi^1_{x_1} \left(\pi^1_{y_1} (f_1)\ (a_1)\right) \cup \pi^2_{x_2} \left(\pi^2_{y_2} (f_2)\ (a_2)\right) \\
&= \pi^1_{y_1} (f_1) \left(\pi^1_{x_1} \left(I^1 (a_1)\right) \cup \pi^2_{y_2} (f_2) \left(\pi^2_{x_2} \left(I^2 (a_2)\right)\right.\right. \\
&= f_1 \left(\pi^1_{y_1} 1 \left(\pi^1_{x_1} (1 (a_1)\right) \cup f_2 \left(\pi^2_{y_2} 1 (\pi^2_{x_2} (1 (a_2)\right)\right. \\
&= f_1 \left(\pi^1_{(x_1 y_1)} 1 (a_1)\right) \cup f_2 \left(\pi^2_{(x_2 y_2)} 1 (a_2)\right).
\end{aligned}$$

Alternatively for each $b = b_1 \cup b_2 \in E_1 \cup E_2$ defined by

$$\psi_b (a) = \psi^1_{b_1}(a_1) \cup \psi^2_{b_2} (a_2)$$

be the function on $E_1 \cup E_2$ defined by $\psi_b(a) = 1$ i.e.,

$$\psi^1_{b_1}(a_1) \cup \psi^2_{b_2} (a_2) = 1 \cup 1.$$

When $a = b$ i.e. $a_1 \cup b_1 = a_2 \cup b_2$, $\psi_b(a) = 0$ when $a \neq b$, i.e. $\psi^1_{b_1} (a_1) \cup \psi^2_{b_2} (a_2) = 0 \cup 0$ when $a_1 \cup b_1 \neq a_2 \cup b_2$.

Then the collection of functions ψ_b for $b \in E_1 \cup E_2$ is a basis for $V_H = V^1_{H_1} \cup V^2_{H_2}$ and $\pi_x (\psi) = \psi_{\pi_x(b)}$ $\forall x$ *in* H and b in E i.e.

$$\pi_{x_1} (\psi^1) \cup \pi_{x_2} (\psi^2) = \psi^1_{\pi_{x_1(b_1)}} \cup \psi^2_{\pi_{x_2(b_2)}},$$

for $x = x_1 \cup x_2$ in $H = H_1 \cup H_2$ and $b_1 \cup b_2$ in $E_1 \cup E_2$. This is true for each proper subset $H = H_1 \cup H_2$ in the S-neutrosophic semigroup $G = G_1 \cup G_2$ and the bigroup $H = H_1 \cup H_2$ associated with the bipermutations of the non empty finite set $E = E_1 \cup E_2$.

Next we shall discuss about Smarandache isomorphic bigroup representation. To this end we consider two bivector spaces $V = V_1 \cup V_2$ and $W = W_1 \cup W_2$ defined over the same field K and that T is a linear biisomorphism from V on to W.

Assume $\rho H = \rho^1 H_1 \cup \rho^2 H_2$ and $\rho'_H = \rho'^1 H_1 \cup \rho'^2 H_2$ are Smarandache birepresentations of the subbigroup $H = H_1 \cup H_2$ in $G = G_1 \cup G_2$ (G a pseudo S-neutrosophic bisemigroup) on V and W respectively. To

$$(\rho H)_x = (\rho' H)_x \circ T \text{ for all } x = x_1 \cup x_2 \in H = H_1 \cup H_2,$$

i.e.

$$\begin{aligned}(T_1 \cup T_2) \circ \left(\rho^1 H_1 \cup \rho^2 H_2\right)_{x_1 \cup x_2} &= T_1 \left(\rho^1 H_1\right)_{x_1} \cup T_2 \left(\rho^2 H_2\right)_{x_2} \\ &= (\rho' H_1)^1_{x_1} \circ T_1 \cup (\rho' H_2)^2_{x_2} \circ T_2,\end{aligned}$$

then we say $T = T_1 \cup T_2$ determines a Smarandache bi-isomorphism between the birepresentation ρH and $\rho' H$. We may also say that ρH and $\rho' H$ are Smarandache biisomorphic S-bisemgroup birepresentations.

However it can be verified that Smarandache biisomorphic birepresentation have equal degree but the converse is not true in general.

Suppose $V = W$ be the bivector space of K-valued functions on $H = H_1 \cup H_2 \subset G_1 \cup G_2$ and define T on $V = W$ by

$T(f)(a) = f(a^{-1})$ i.e. $T_1(f_1)(a_1) \cup T_2(f_2)(a_2) = f_1\left(a_1^{-1}\right) \cup f_2\left(a_2^{-1}\right)$.

This is one to one linear bimapping from the space of K-valued bifunctions H_1 on to itself and

$$T \circ SR_x = SL_x \circ T,$$

i.e.

$$\left(T_1 \circ S^1 R^1_{x_1}\right) \cup \left(T_2 \circ S^2 R^2_{x_2}\right) = \left(S^1 L^1_{x_1} \circ T_1\right) \cup \left(S^2 L^2_{x_2} \circ T_2\right),$$

for all $x = x_1 \cup x_2$ in $H = H_1 \cup H_2$.

For if $f(a)$ is a bifunction on $G = G_1 \cup G_2$ then

$$\begin{aligned}(T \circ SR_x)(f)(a) &= T(SR_x(f))(a) \\ &= SR_x(f)(a^{-1}) \\ &= f(a^{-1}x) \\ &= T(f)(x^{-1}a) \\ &= SL_x(T(f))(a) \\ &= (SL_x \circ T)(f)(a).\end{aligned}$$

Therefore we see that S-left and S-right birepresentations of $H = H_1 \cup H_2$ are biisomorphic to each other.

Suppose now that $H = H_1 \cup H_2$ is a subbigroup of the S-bisemigroup G and $\rho H = \rho^1 H_1 \cup \rho^2 H_2$ is a birepresentation of $H = H_1 \cup H_2$ on the bivector space $V_H = V^1_{H_1} \cup V^2_{H_2}$over the field K and let $\nu_1, \ldots, \nu_n$ be a basis of $V_H = V^1_{H_1} \cup V^2_{H_2}$. For each $x = x_1 \cup x_2$ in $H = H_1 \cup H_2$

we can associate to $(\rho H)_x = \left(\rho^1 H_1\right)_{x_1} \cup \left(\rho^2 H_2\right)_{x_2}$ an invertible $n \times n$ bimatrix with entries in K using this basis we denote this bimatrix by $\left(M^1 H_1\right)_{x_1} \cup \left(M^2 H_2\right)_{x_2} = (MH)_x$ where $M = M_1 \cup M_2$.

The composition rule can be rewritten as

$$(MH)_{xy} = (MH)_x (MH)_y$$

$$\left(M^1 H_1\right)_{x_1 y_1} \cup \left(M^2 H_2\right)_{x_2 y_2}$$

$$\left[\left(M^1 H_1\right)_{x_1} \cup \left(M^2 H_2\right)_{x_2}\right] \left[\left(M^1 H_1\right)_{y_1} \cup \left(M^2 H_2\right)_{y_2}\right]$$

$$\left(M^1 H_1\right)_{x_1} \left(M^1 H_1\right)_{y_1} \cup \left(M^2 H_2\right)_{x_2} \left(M^2 H_2\right)_{y_2},$$

where the bimatrix product is used on the right side of the equation. We see depending on each $H = H_1 \cup H_2$ we can have different bimatrices $MH = M^1 H_1 \cup M_2 H_2$, and it need not in general be always a $n \times n$ bimatrices it can also be a $m \times m$ bimatrix $m \neq n$. A different choice of basis for $V = V_1 \cup V_2$ will lead to a different mapping $x \mapsto Nx$ *i.e.* $x_1 \cup x_2 \mapsto N^1_{x_1} \cup N^2_{x_2}$ from H to invertible $n \times n$ bimatrices.

However the two mappings

$$x \mapsto M_x = M^1_{x_1} \cup M^2_{x_2}$$

$$x \mapsto N_x = N^1_{x_1} \cup N^2_{x_2},$$

will be called as Smarandache similar relative to the subbigroup $H = H_1 \cup H_2 \subset G = G_1 \cup G_2$ in the sense that there is an invertible $n \times n$ bimatrix $S = S^1 \cup S^2$ with entries in K such that $N_x = S M_x S^{-1}$ i.e. $N^1_{x_1} \cup N^2_{x_2} = S^1 M^1_{x_1} (S^1)^{-1} \cup S^2 M^2_{x_2} (S^2)^{-1}$ for all $x = x_1 \cup x_2 \subset G = G_1 \cup G_2$. It is pertinent to mention that when a different H' is taken $H \neq H'$ i.e. $H^1 \cup H^2 \neq (H')^1 \cup (H')^2$ then we may have a different $m \times m$ bimatrix. Thus using a single S-neutrosophic bisemigroup we have very many such bimappings depending on each $H \subset G$. On the other hand one can begin with a bimapping $x \mapsto M_x$ from H into invertible $n \times n$ matrices with entries in K i.e. $x_1 \mapsto M^1_{x_1} \cup x_2 \mapsto M^2_{x_2}$ from $H = H_1 \cup H_2$ into invertible $n \times n$ matrices. Thus now one can reformulate the condition for two Smarandache birepresentations to be biisomorphic.

If one has two birepresentation of a fixed subbigroup $H = H_1 \cup H_2$, H a subbigroup of the S-neutrosophic bisemigroup G on two bivector spaces V and $W (V = V^1 \cup V^2$ and $W = W^1 \cup W^2)$ with the same scalar field K then these two Smarandache birepresentations are Smarandache biisomorphic if and only if the associated bimappings from $H = H_1 \cup H_2$ to invertible bimatrices as above, for any choice of basis on $V = V^1 \cup V^2$ and $W = W^1 \cup W^2$ are bisimilar with the bisimilarity bimatrix S having entries in K.

Now we proceed on to give a brief description of Smarandache biirreducible birepresentation, Smarandache biirreducible birepresentation and Smarandache bistable representation and so on. Now we proceed on to define Smarandache bireducibility of finite S-neutrosophic bisemigroups.

Let G be a finite neutrosophic S-bisemigroup when we say G is a S-finite bisemigroup or finite S-bisemigroup we only mean all proper subset in G which are subbigroups in $G = G_1 \cup G_2$ are of finite order V_H be a bivector space over a field K and ρH a birepresentation of H on V_H.

Suppose that there is a bivector space W_H of V_H such that $(\rho H)_x W_H \subseteq W_H$ here $W_H = W^1_{H_1} \cup W^2_{H_2}$ where $H = H_1 \cup H_2$ and $V_H = V^1_{H_1} \cup V^2_{H_2}$, $H = H_1 \cup H_2$, $\rho H = \rho^1 H_1 \cup \rho^2 H_2$ and $x = x_1 \cup x_2 \in H$ i.e. $x_1 \in H_1$ and $x_2 \in H_2$.

This is equivalent to saying that

$$(\rho H)_x \ (W_H) = W_H$$

i.e.

$$\left[\left(\rho^1 H_1\right)_{x_1} \cup \left(\rho^2 H_2\right)_{x_2}\right] \ [W^1_{H_1} \cup W^2_{H_2}] = W^1_{H_1} \cup W^2_{H_2},$$

for all $x = x_1 \cup x_2 \in H_1 \cup H_2$ as $(\rho H)_{x^{-1}} = [(\rho H)_x]^{-1}$,
i.e.

$$\left(\rho^1 H_1 \cup \rho^2 H_2\right)_{(x_1 \cup x_2)^{-1}} = \left[\left(\rho^1 H_1 \cup \rho^2 H_2\right)_{(x_1 \cup x_2)}\right]^{-1},$$

$$\left(\rho^1 H_1\right)_{x_1^{-1}} \cup \left(\rho^2 H_2\right)_{x_2^{-1}} = \left[\left(\rho^1 H_1\right)_{x_1}\right]^{-1} \cup \left[\left(\rho^2 H_2\right)_{x_2}\right]^{-1}.$$

We say $W_H = W^1_{H_1} \cup W^2_{H_2}$ is Smarandache biinvariant or Smarandache bistable under the birepresentation $\rho H = \rho^1 H_1 \cup \rho^2 H_2$.

We say the bisubspace $Z_H = Z^1_{H_1} \cup Z^2_{H_2}$ of $V_H = V^1_{H_1} \cup V^2_{H_2}$ to be a Smarandache bicomplement of a subbispace

$$W_H = W^1_{H_1} \cup W^2_{H_2} \text{ if } W_H \cap Z_H = \{0\}$$

and

$$W_H + Z_H = V_H \text{ i.e. } \left(W^1_{H_1} \cap Z^1_{H_1}\right) \cup \left(W^2_{H_2} \cap Z^2_{H_2}\right) = \{0\} \cup \{0\}$$

and

$$\left(W^1_{H_1} + Z^1_{H_1}\right) \cup \left(W^2_{H_2} + Z^2_{H_2}\right) = V^1_{H_1} + V^2_{H_2},$$

here $W^i_{H_i} + Z^i_{H_i} (i = 1, 2)$ denotes the bispan of W_H and Z_H which is a subbispace of V_H consisting of bivectors of the form $w + z = (w_1 + z_1) \cup (w_2 + z_2)$ where $w \in W_H$ and $z \in Z_H$. These conditions are equivalent to saying that every bivector $\nu = \nu_1 \cup \nu_2 \in V^1_{H_1} \cup V^2_{H_2}$ can be written in an unique way as $w + z = (w_1 + z_1) \cup (w_2 + z_2)$, $w_i \in W^i_{H_i}$ and $z_i \in Z^i_{H_i} (i = 1, 2)$.

Complementary bispaces always exists because of basis for a bivector subspace of a bivector space can be enlarged to a basis of a whole bivector space. If $Z_H = Z^1_{H_1} \cup Z^2_{H_2}$ and $W_H = W^1_{H_1} \cup W^2_{H_2}$ are complementary subbispaces (bisubspaces) of a bivector space $V_H = V^1_{H_1} \cup V^2_{H_2}$ then we get a linear bimapping $P_H = P^1_{H_1} \cup P^2_{H_2}$ on $V_H = V^1_{H_1} \cup V^2_{H_2}$ on to $W_H = W^1_{H_1} \cup W^2_{H_2}$ along $Z_H = Z^1_{H_1} \cup Z^2_{H_2}$ and is defined by $P_H(w + z)w$ for all $w \in W_H$ and $z \in Z_H$. Thus $I_H - -P_H$ is the biprojection of V_H on to Z_H along W_H where I_H denotes the identity bitransformation on $V_H = V^1_{H_1} \cup V^2_{H_2}$.

Note. $(P_H)^2 = \left(P^1_{H_1} \cup P^2_{H_2}\right)^2 = \left(P^1_{H_1}\right)^2 \cup \left(P^2_{H_2}\right)^2 = P^1_{H_1} \cup P^2_{H_2}$, when P_H is a biprojection.

Conversely, if P_H is a linear bioperator on V_H such that $(P_H)^2 = P_H$ then P_H is the biprojection of V_H on to the bisubspace of V_H which is the biimage of $P_H = P^1_{H_1} \cup P^2_{H_2}$ along the subspace of V_H which is the bikernel of $\rho H = \rho^1 H_1 \cup \rho^2 H_2$.

It is important to mention here unlike usual complements using a finite bigroup we see when we used pseudo S-neutrosophic bisemigroups. The situation is very varied. For each proper subset H of $G(H_1 \cup H_2 \subset G_1 \cup G_2)$ where H is a subbigroup of G we get several important S-bicomplements and several S-biinvariant or S-bistable or S-birepresentative of $\rho H = \rho^1 H_1 \cup \rho^2 H_2$.

Now we proceed on to define Smarandache biirreducible birepresentation. Let G be a S-finite neutrosophic bisemigroup, $V_H = V^1_{H_1} \cup V^2_{H_2}$ be a bivector space over a field K, $\rho H = \rho^1 H_1 \cup \rho^2 H_2$ be a birepresentation of H on V_H and W_H is a subbispace of $V_H = V^1_{H_1} \cup V^2_{H_2}$ which is invariant under $\rho H = \rho^1 H_1 \cup \rho^2 H_2$. Here we make an assumption that the field K has characteristic 0 or K has positive characteristic and the number of elements in each $H = H^1 \cup H^2$ is not divisible by the characteristic K, $H_1 \cup H_2 \subset G_1 \cup G_2$ is a S-bisemigroup.

Let us show that there is a bisubspace $Z_H = Z^1_{H_1} \cup Z^2_{H_2}$ of $V^1_{H_1} \cup V^2_{H_2} = V_H$ such that Z_H is a bicomplement of $W_H = W^1_{H_1} \cup W^2_{H_2}$ and Z_H is also biinvariant under the birepresentation ρH *of* H *i.e.* $\rho^1 H_1 \cup \rho^2 H_2$ of $H_1 \cup H_2$ on $V_H = V^1_{H_1} \cup V^2_{H_2}$. To do this we start with any bicomplements $(Z_H)_o = \left(Z^1_{H_1}\right)_o \cup \left(Z^2_{H_2}\right)_o$ of $W_H = W^1_{H_1} \cup W^2_{H_2}$ of $V_H = V^1_{H_1} \cup V^2_{H_2}$ and let $(P_H)_o = \left(P^1_{H_1} \cup P^2_{H_2}\right)_o : V_H = V^1_{H_1} \cup V^2_{H_2} \to V^1_{H_1} \cup V^2_{H_2}$ be the biprojection of $V_H = V^1_{H_1} \cup V^2_{H_2}$ on to $W^1_{H_1} \cup W^2_{H_2} = W_H$ along $(Z_H)_o$. Thus $(P_H)_o = \left(P^1_{H_1} \cup P^2_{H_2}\right)_o$ maps V to W and $(P_H)_o w = w$ for all $w \in W$.

Let $m = m_1 \cup m_2$ denotes the number of elements in $H = H_1 \cup H_2 \subset G_1 \cup G_2$ i.e. $|H_i| = m_i$ $(i = 1, 2)$. Define a linear bimapping

$$P_H : V_H \quad \to V_H$$

i.e.

$$P^1_{H_1} \cup P^2_{H_2} : V^1_{H_1} \cup V^2_{H_2} \to \quad V^1_{H_1} \cup V^2_{H_2}$$

by

$$\begin{aligned} P_H &= P^1_{H_1} \cup P^2_{H_2} \\ &= \frac{1}{m_1} \sum_{x_1 \in H_1} \left(\rho^1 H_1\right)_{x_1} \circ \left(P^1_{H_1}\right) \circ \left(\rho^1 H_1\right)^{-1}_{x_1} \cup \frac{1}{m_2} \sum_{x_2 \in H_2} \left(\rho^2 H_2\right)_{x_2} \circ \left(P^2_{H_2}\right) \circ \left(\rho^2 H_2\right)^{-1}_{x_2}, \end{aligned}$$

assumption on K implies that $\frac{1}{m_i}$ $(i = 1, 2)$ makes sense as an element of K i.e. as the multiplicative inverse of a sum of m 1's in K where 1 refers to the multiplicative identity element of K. This expression defines a linear bimapping on $V_H = V^1_{H_1} \cup V^2_{H_2}$ because $(\rho H)'_x\, s$ and $(P_H)_o$ are linear bimapping.

We actually have that $P_H = P^1_{H_1} \cup P^2_{H_2}$ bimaps V_H to W_H i.e. $V^1_{H_1} \cup V^2_{H_2}$ to $W^1_{H_1} \cup W^2_{H_2}$ and because the $(P_H)_o = \left(P^1_{H_1} \cup P^2_{H_2}\right)_o$ maps $V_H = V^1_{H_1} \cup V^2_{H_2}$ to $W_H = W^1_{H_1} \cup W^2_{H_2}$, and because the $(\rho_H)'_x\, s$ $\left(= \left(\rho^1_{H_1}\right)_{x_1} \cup \left(\rho^2_{H_2}\right)_{x_2}\right)$ maps $W_H = W^1_{H_1} \cup W^2_{H_2}$ to $W^1_{H_1} \cup W^2_{H_2}$. If $w \in W_H$ then

$$\begin{aligned} [(\rho H)_x]^{-1} w &= \left[\left(\rho^1 H_1\right)_{x_1} \cup \left(\rho^2 H_2\right)_{x_2}\right]^{-1} (w_1 \cup w_2) \\ &= \left(\rho^1 H_1\right)^{-1}_{x_1} (w_1) \cup \left(\rho^2 H_2\right)_{x_2} (w_2) \in W^1_{H_1} \cup W^2_{H_2}, \end{aligned}$$

for all $x = x_1 \cup x_2$ in $H = H_1 \cup H_2 \subset G = G_1 \cup G_2$ and then

$$\begin{aligned}(P_H)_o \left((\rho H)_x\right)^{-1} \omega &= (P_H)_o \left(\left(\rho^1 H_1\right)_{x_1}\right)^{-1} (w_1) \cup \left(P^2_{H_2}\right)_o \left(\left(\rho^2 H_2\right)_{x_2}\right)^{-1} (w_2) \\ &= \left(\left(\rho^1 H_1\right)_{x_1}\right)^{-1} (w_1) \ \cup \ \left(\left(\rho^2 H_2\right)_{x_2}\right)^{-1} (w_2).\end{aligned}$$

Thus we conclude that

$$(P_H)(w) = w \ \text{ i.e. } \ \left(P^1_{H_1}\right) \ (w_1) \ = \ w_1 \ ,$$

and

$$\left(P^2_{H_2}\right) \ (w_2) = w_2 \ \text{ i.e. } \ P_H = P^1_{H_1} \ \cup \ P^2_{H_2},$$

for all $w = (w_1 \cup w_2)$ in $W_H = W^1_{H_1} \ \cup \ W^2_{H_2}$ by the very definition of P_H.

The definition of P_H also implies that

$$(\rho H)_y \circ P_H \circ \left[(\rho H)_y\right]^{-1} = \mathrm{P}_H$$

i.e.

$$\left(\rho^1 H_1\right)_{y_1} \circ P^1_{H_1} \circ \left(\left(\rho^1 H_1\right)_{y_1}\right)^{-1} \cup \left(\rho^2 H_2\right)_{y_2} \circ P^1_{H_2} \circ \left(\left(\rho^2 H_2\right)_{y_2}\right)^{-1} = P^1_{H_1} \ \cup \ P^2_{H_2},$$

for all $y \in H = H_1 \cup H_2$.

The only case this does not occur is when $W_H = \{0\}$ i.e. $W^1_{H_1} \ \cup \ W^2_{H_2} = \{0\} \cup \{0\}$. Because $P_H(V_H) \subset W_H$ and $P_H(w) = w$ for all $w \in W_H = W^1_{H_1} \ \cup \ W^2_{H_2}$. $P_H = P^1_{H_1} \ \cup \ P^2_{H_2}$ is a biprojection of V_H onto W_H i.e. $P^i_{H_i}$ is a projection of $V^i_{H_i}$ onto $W^i_{H_i}$, $i = 1, 2$ along some bisubspace $Z_H = Z^1_{H_1} \ \cup \ Z^2_{H_2}$ of $V_H = V^1_{H_1} \ \cup \ V^2_{H_2}$. Specifically one should take $Z_H = Z^1_{H_1} \ \cup \ Z^2_{H_2}$ to be the bikernel of $P_H = P^1_{H_1} \ \cup \ P^2_{H_2}$. It is easy to see that $W_H \cap Z_H = \{0\}$ i.e. $W^1_{H_1} \cap Z^1_{H_1} = \{0\}$ and $W^2_{H_2} \cap Z^2_{H_2} = \{0\}$ since $P^i_{H_i}(w_i) = w_i$ for all $w_i \in W^i_{H_i}$, $i = 1, 2$.

On the other hand if $\nu = \nu_1 \cup \nu_2$ is any element of $V_H = V^1_{H_1} \ \cup \ V^2_{H_2}$ then we can write $\nu = \nu_1 \cup \nu_2$ as $P_H\ (\nu) = P^1_{H_1}(\nu_1) \ \cup \ P^2_{H_2}(\nu_2)$ so $P_H(\nu) + (V - -P_H(\nu))$.

Thus $\nu - -P_H(\nu)$ lies in Z_H, the bikernel of P_H. This shows that W_H and Z_H satisfies the essential bicomplement of W_H in V_H. The biinvariance of Z_H under the birepresentation ρH is evident.

Thus the Smarandache birepresentation ρH of H on V_H is biisomorphic to the direct sum of H on W_H and Z_H, that are the birestrictions of ρH to W_H and Z_H.

There can be smaller biinvariant bisubspaces within these biinvariant subbispaces so that one can repeat the process for each H, $H \subset G$. We say that the subbispaces

$$(W_H)_1, (W_H)_2, \cdots, (W_H)_t$$

of V_H, i.e.

$$\left(W^1_{H_1} \cup W^2_{H_2}\right)_1, \ \left(W^1_{H_1} \cup W^2_{H_2}\right)_2, \cdots, \ \left(W^1_{H_1} \cup W^2_{H_2}\right)_t,$$

of $V^1_{H_1} \ \cup \ V^2_{H_2}$ form an Smarandache biindependent system related to each subbigroup $H = H_1 \cup H_2 \subset G = G_1 \cup G_2$. If $(W_H)_j \neq (0)$ for each j and if $w_j \in (W_H)_j$, $1 \leq j \leq t$ and

$$\sum_{j=1}^{t} w_j = \sum_{j=1}^{t} w^1_j \cup \sum_{j=1}^{t} w^2_j = \ 0 \ \cup \ 0,$$

where $w_j = w_j^1 \cup w_j^2$, $w_j^1 \in W_{H_1}^1$ and $w_j^2 \in W_{H_2}^2$ imply $w_j^i = 0 (i = 1, 2;\ j = 1, 2, \cdots, t)$. If in addition it spans $(W_H)_1$, $(W_H)_2, ..., (W_H)_t = V_{H_1}^1 \cup V_{H_2}^2 = V_H$, then every bivector $\nu = \nu^1 \cup \nu^2$ on $V_{H_1}^1 \cup V_{H_2}^2$ can be written in a unique way as $\sum_{j=1}^{t} u_j$ with $u_j = u_j^1 \cup u_j^2 \in (W_{H_1}^1 \cup W_{H_2}^2)$ for each j.

Next we proceed on to give two applications to Smarandache Markov bichains and Smarandache Leontief economic bimodels.

Suppose a physical or a mathematical system is such that at any movement it can occupy one of a finite number of states when we view them as stochastic bioprocess or Markov bichains we make an assumption that the system moves with time from one state to another so that a schedule of observation times keep the states of the system at these times. But when we tackle real world problems say even for simplicity the emotions of a person it need not fall under the category of sad, cold, happy, angry, affectionate, disinterested, disgusting, many times the emotions of a person may be very unpredictable depending largely on the situation, and the mood of the person and its relation with another, so such study cannot fall under Markov chains, for at a time more than one emotion may be in a person and also such states cannot be included and given as next pair of observation, these changes and several feelings at least two at a time will largely affect the very transition bimatrix

$$P = P_1 \cup P_2 = \left[p_{ij}^1\right] \ \cup \ \left[p_{ij}^2\right],$$

with non negative entries for which each of the column sums are one and all of whose entries are positive. This has relevance as even the policy makers are humans and their view is ultimate and this rules the situation. Here it is still pertinent to note that all decisions are not always possible at times certain of the views may be indeterminate at that period of time and may change after a period of time but all our present theory have no place for the indeterminacy only the neutrosophy gives the place for the concept of indeterminacy, based on which we have built neutrosophic vector spaces, neutrosophic bivector spaces, then now the notion of Smarandache -neutrosophic bivector spaces and so on.

So to overcome the problem we have indecisive situations we give negative values and indeterminate situations we give negative values so that our transition neutrosophic bimatrices individual columns sums do not add to one and all entries may not be positive.

Thus we call the new transition neutrosophic bimatrix which is a square bimatrix which can have negative entries and I the indeterminate also falling in the set $[-1, 1] \cup \{I\}$ and whose column sums can also be less than 1 and I as the Smarandache neutrosophic transition bimatrix.

Further the Smarandache neutrosophic probability bivector will be a bicolumn vector which can take entries from $[-1, 1] \cup [-I, I]$ whose sum can lie in the biinterval $[-1, 1] \cup [-I, I]$. The Smarandache neutrosophic probability bivectors $x^{(n)}$ for $n = 0, 1, 2, \cdots$ are said to be the Smarandache state neutrosophic bivectors of a Smarandache neutrosophic Markov bioprocess. Clearly if P is a S-transition bimatrix of a Smarandache Markov bioprocess and $x^{(n)} = x_1^{(n_1)} \cup x_2^{(n_2)}$ is the Smarandache state neutrosophic bivectors at the n^{th} pair of observation then

$$x^{(n+1)} \neq px^{(n)}$$

$$\text{i.e. } x_1^{(n+1)} \cup x_2^{(n_2+1)} \neq p_1 x_1^{(n_1)} \cup p_2 x_2^{(n_2)}.$$

Further research in this direction is innovative and interesting.

Matrix theory has been very successful in describing the inter relation between prices outputs and demands in an economic model. Here we just discuss some simple bimodels based on the ideals of the Nobel laureate Massily Leontief. We have used not only bimodel structure based on bimatrices also we have used the factor indeterminacy. So our matrices would be only Neutrosophic bimatrices. Two types of models which we wish to discuss are the closed or input-output model and the open or production model each of which assumes some economics parameter which describe the inter relations between the industries in the economy under considerations. Using neutrosophic bimatrix theory we can combine and study the effect of price bivector. Before the basic equations of the input-output model are built we just recall the definition of fuzzy neutrosophic bimatrix. For we need this type of matrix in our bimodel.

Definition 3. Let $M_{nxm} = \{(a_{ij})/a_{ij} \in K(I)\}$, where $K(I)$, is a neutrosophic field. We call M_{nxm} to be the neutrosophic rectangular matrix.

Example 1. Let $Q(I) = \langle Q \cup I \rangle$ be the neutrosophic field.

$$M_{4\times 3} = \begin{pmatrix} 0 & 1 & I \\ -2 & 4I & 0 \\ 1 & -I & 2 \\ 3I & 1 & 0 \end{pmatrix},$$

is the neutrosophic matrix, with entries from rationals and the indeterminacy I.

We define product of two neutrosophic matrices and the product is defined as follows: let

$$A = \begin{pmatrix} -1 & 2 & -I \\ 3 & I & 0 \end{pmatrix}_{2\times 3} \quad \text{and} \quad B = \begin{pmatrix} I & 1 & 2 & 4 \\ 1 & I & 0 & 2 \\ 5 & -2 & 3I & -I \end{pmatrix}_{3\times 4}$$

$$AB = \begin{pmatrix} -6I+2 & -1+4I & -2-3I & I \\ -4I & 3+I & 6 & 12+2I \end{pmatrix}_{2\times 4}$$

(we use the fact $I^2 = I$).

Let $M_{n\times n} = \{(a_{ij}) | (a_{ij}) \in Q(I)\}$, $M_{n\times n}$ is a neutrosophic vector space over Q and a strong neutrosophic vector space over $Q(I)$.

Now we proceed onto define the notion of fuzzy integral neutrosophic matrices and operations on them, for more about these refer [43].

Definition 4. Let $N = [0,1] \cup I$, where I is the indeterminacy. The $m \times n$ matrices $M_{m\times n} = \{(a_{ij})/a_{ij} \in [0,1] \cup I\}$ is called the fuzzy integral neutrosophic matrices. Clearly the class of $m \times n$ matrices is contained in the class of fuzzy integral neutrosophic matrices.

Example 2. Let

$$A = \begin{pmatrix} I & 0.1 & 0 \\ 0.9 & 1 & I \end{pmatrix},$$

A is a 2×3 integral fuzzy neutrosophic matrix.

We define operation on these matrices. An integral fuzzy neutrosophic row vector is $1 \times n$ integral fuzzy neutrosophic matrix. Similarly an integral fuzzy neutrosophic column vector is a $m \times 1$ integral fuzzy neutrosophic matrix.

Example 3. $A = (0.1, 0.3, 1, 0, 0, 0.7, I, 0.002, 0.01, I, 0.12)$ is a integral row vector or a 1×11, integral fuzzy neutrosophic matrix.

Example 4. $B = (1, 0.2, 0.111, I, 0.32, 0.001, I, 0, 1)^T$ is an integral neutrosophic column vector or B is a 9×1 integral fuzzy neutrosophic matrix.

We would be using the concept of fuzzy neutrosophic column or row vector in our study.

Definition 5. Let $P = (p_{ij})$ be a $m \times n$ integral fuzzy neutrosophic matrix and $Q = (q_{ij})$ be a $n \times p$ integral fuzzy neutrosophic matrix. The composition map $P \bullet Q$ is defined by $R = (r_{ij})$ which is a $m \times p$ matrix where $r_{ij} = \max_k \min(p_{ik} q_{kj})$ with the assumption $\max(p_{ij}, I) = I$ and $\min(p_{ij}, I) = I$ where $p_{ij} \in [0, 1]$. $\min(0, I) = 0$ and $\max(1, I) = 1$.

Example 5. Let

$$P = \begin{bmatrix} 0.3 & I & 1 \\ 0 & 0.9 & 0.2 \\ 0.7 & 0 & 0.4 \end{bmatrix}, \text{Q} = (0.1,\ I,\ 0)^T$$

be two integral fuzzy neutrosophic matrices.

$$P \bullet Q = \begin{bmatrix} 0.3 & I & 1 \\ 0 & 0.9 & 0.2 \\ 0.7 & 0 & 0.4 \end{bmatrix} \bullet \begin{bmatrix} 0.1 \\ I \\ 0 \end{bmatrix} = (I,\ I,\ 0.1).$$

Example 6. Let

$$P = \begin{bmatrix} 0 & I \\ 0.3 & 1 \\ 0.8 & 0.4 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0.1 & 0.2 & 1 & 0 & I \\ 0 & 0.9 & 0.2 & 1 & 0 \end{bmatrix}.$$

One can define the max-min operation for any pair of integral fuzzy neutrosophic matrices with compatible operation.

Now we proceed onto define the notion of fuzzy neutrosophic matrices.

Let $N_s = [0, 1] \cup nI/n \in (0, 1]\}$, we call the set N_s to be the fuzzy neutrosophic set.

Definition 6. Let N_s be the fuzzy neutrosophic set. $M_{n \times m} = \{(a_{ij})/a_{ij} \in N_s\}$, we call the matrices with entries from N_s to be the fuzzy neutrosophic matrices.

Example 7. Let $N_s = [0, 1] \cup \{nI/n \in (0, 1]\}$ be the set

$$P = \begin{bmatrix} 0 & 0.2I & 0.31 & I \\ I & 0.01 & 0.7I & 0 \\ 0.31I & 0.53I & 1 & 0.1 \end{bmatrix}.$$

P is a 3×4 fuzzy neutrosophic matrix.

Example 8. Let $N_s = [0,1] \cup \{nI/n \in (0,1]\}$ be the fuzzy neutrosophic matrix. $A = [0, 0.12I, I, 1, 0.31]$ is the fuzzy neutrosophic row vector:

$$B = \begin{bmatrix} 0.5I \\ 0.11 \\ I \\ 0 \\ -1 \end{bmatrix},$$

is the fuzzy neutrosophic column vector.

Now we proceed on to define operations on these fuzzy neutrosophic matrices.

Let $M = (m_{ij})$ and $N = (n_{ij})$ be two $m \times n$ and $n \times p$ fuzzy neutrosophic matrices.

$$M \bullet N = R = (r_{ij})$$

where the entries in the fuzzy neutrosophic matrices are fuzzy indeterminates i.e. the indeterminates have degrees from 0 to 1 i.e. even if some factor is an indeterminate we try to give it a degree to which it is indeterminate for instance $0.9I$denotes the indeterminacy rate; it is high where as $0.01I$denotes the low indeterminacy rate. Thus neutrosophic matrices have only the notion of degrees of indeterminacy. Any other type of operations can be defined on the neutrosophic matrices and fuzzy neutrosophic matrices. The notion of these matrices have been used to define neutrosophic relational equations and fuzzy neutrosophic relational equations.

Here we give define the notion of neutrosophic bimatrix and illustrate them with examples. Also we define fuzzy neutrosophic matrices.

Definition 7. Let $A = A_1 \cup A_2$, where A_1 and A_2 are two distinct neutrosophic matrices with entries from a neutrosophic field. Then $A = A_1 \cup A_2$ is called the neutrosophic bimatrix.

It is important to note the following:

(1) If both A_1 and A_2 are neutrosophic matrices we call A a neutrosophic bimatrix.

(2) If only one of A_1 or A_2 is a neutrosophic matrix and other is not a neutrosophic matrix then we all $A = A_1 \cup A_2$ as the semi neutrosophic bimatrix. (It is clear all neutrosophic bimatrices are trivially semi neutrosophic bimatrices).

It both A_1 and A_2 are $m \times n$ neutrosophic matrices then we call $A = A_1 \cup A_2$ a $m \times n$ neutrosophic bimatrix or a rectangular neutrosophic bimatrix.

If $A = A_1 \cup A_2$ be such that A_1 and A_2 are both $n \times n$ neutrosophic matrices then we call $A = A_1 \cup A_2$ a square or a $n \times n$ neutrosophic bimatrix. If in the neutrosophic bimatrix $A = A_1 \cup A_2$ both A_1 and A_2 are square matrices but of different order say A_1 is a $n \times n$ matrix and A_2 a $s \times s$ matrix then we call $A = A_1 \cup A_2$ a mixed neutrosophic square bimatrix. (Similarly one can define mixed square semi neutrosophic bimatrix).

Likewise in $A = A_1 \cup A_2$, if both A_1 and A_2 are rectangular matrices say A_1 is a $m \times n$ matrix and A_2 is a $p \times q$ matrix then we call $A = A_1 \cup A_2$ a mixed neutrosophic rectangular bimatrix. (If $A = A_1 \cup A_2$ is a semi neutrosophic bimatrix then we call A the mixed rectangular semi neutrosophic bimatrix).

Just for the sake of clarity we give some illustration.

Notation. We denote a neutrosophic bimatrix by $A_N = A_1 \cup A_2$.

Example 9. Let

$$A_N = \begin{bmatrix} 0 & I & 0 \\ 1 & 2 & -1 \\ 3 & 2 & I \end{bmatrix} \cup \begin{bmatrix} 2 & I & 1 \\ I & 0 & I \\ 1 & 1 & 2 \end{bmatrix},$$

A_N is the 3×3 square neutrosophic bimatrix.

At times one may be interested to study the problem at hand (i.e. the present situation) and a situation at the r^{th} time period the predicted model.

All notion and concept at all times is not determinable. For at time a situation may exist for a industry that it cannot say the monetary value of the output of the i^{th} industry needed to satisfy the outside demand at one time, this notion may become an indeterminate (For instance with the advent of globalization the electronic goods manufacturing industries are facing a problem for in the Indian serenio when an exported goods is sold at a cheaper rate than manufactured Indian goods will not be sold for every one will prefer only an exported good, so in situation like this the industry faces only a indeterminacy for it cannot fully say anything about the movements of the manufactured goods in turn this will affect the σ_{ij}. σ_{ij} may also tend to become an indeterminate. So to study such situation simultaneously the neutrosophic bimatrix would be ideal we may have the newly redefined production vector which we choose to call as Smarandache neutrosophic production bivector which will have its values taken from +ve value or −ve value or an indeterminacy.

So Smarandache neutrosophic Leontief open model is got by permitting.

$$x \geq 0, d \geq 0, c \geq 0$$

$$x \leq 0, d \leq 0, c \leq 0$$

and x can be I, d can take any value and c can be a neutrosophic bimatrix. We can say $(1-c)^{-1} \geq 0$ productive $(1-c)^{-1} < 0$ non productive or not up to satisfaction and $(1-c^{-1}) = nI$, I the indeterminacy i.e. the productivity cannot be determined i.e. one cannot say productive or non productive but cannot be determined. $c = c_1 \cup c_2$ is the consumption neutrosophic bimatrix.

c_1 at time of study and c_2 after a stipulated time period. x, d, c can be greater than or equal to zero less than zero or can be an indeterminate.

$$x = \begin{bmatrix} x_1^1 \\ \vdots \\ x_k^1 \end{bmatrix} \cup \begin{bmatrix} x_1^2 \\ \vdots \\ x_k^2 \end{bmatrix},$$

production neutrosophic bivector at the times t_1 and t_2 the demand neutrosophic bivector $d = d^1 \cup d^2$

$$d = \begin{bmatrix} d_1^1 \\ \vdots \\ d_k^1 \end{bmatrix} \cup \begin{bmatrix} d_1^2 \\ \vdots \\ d_k^2 \end{bmatrix},$$

at time t_1 and t_2 respectively. Consumption neutrosophic bimatrix $c = c_1 \cup c_2$

$$c_1 = \begin{bmatrix} \sigma_{11}^1 & \cdots & \sigma_{1k}^1 \\ \sigma_{21}^1 & \cdots & \sigma_{2k}^1 \\ \vdots & & \\ \sigma_{k1}^1 & \cdots & \sigma_{kk}^1 \end{bmatrix}, \, c_2 = \begin{bmatrix} \sigma_{11}^2 & \cdots & \sigma_{1k}^2 \\ \sigma_{21}^2 & \cdots & \sigma_{2k}^2 \\ \vdots & & \\ \sigma_{k1}^2 & \cdots & \sigma_{kk}^2 \end{bmatrix}$$

at times t_1 and t_2 respectively.

$$\begin{aligned} & \sigma_{i1}\, x_1 \; + \sigma_{12}\, x_2 \; + ... + \sigma_{ik}\, x_k \\ = \; & \left(\sigma_{i1}^1\, x_1^1 \; + \; \sigma_{i2}^1\, x_2^1 + ... + \; \sigma_{ik}^1\, x_k^1\right) \cup \left(\sigma_{i1}^2 \; x_1^2 \; + \sigma_{i2}^2\, x_2^2 \; + ... + \sigma_{ik}^2 \; x_k^2\right) \end{aligned}$$

is the value of the output of the i^{th} industry needed by all k industries at the time periods t_1 and t_2 to produce a total output specified by the production neutrosophic bivector $x = x^1 \cup x^2$. Consumption neutrosophic bimatrix c is such that; production if $(1-c)^{-1}$ exists and $(1-c)^{-1} \geq 0$, i.e. $c = c_1 \cup c_2$ and $(1-c_1)^{-1} \cup (1-c_2)^{-1}$ exists and each of $(1-c_1)^{-1}$ and $(1-c_2)^{-1}$ is greater than or equal to zero. A consumption neutrosophic bimatrix c is productive if and only if there is some production bivector $x \geq 0$ such that

$$x > cx \text{ i.e. } x^1 \cup x^2 > c^1 x^1 \cup c^2 x^2.$$

A consumption bimatrix c is productive if each of its birow sum is less than one. A consumption bimatrix c is productive if each of its bicolumn sums is less the one. Non productive if bivector $x < 0$ such that $x < cx$.

Now quasi productive if one of $x^1 \geq 0$ and $x^1 > c^1 x^1$ or $x^2 \geq 0$ and $x^1 > c^1 x^1$.

Now production is indeterminate if x is indeterminate x and cx are indeterminates or x is indeterminate and $c\,x$ is determinate. Production is quasi indeterminate if at t_1 or t_2, $x^i \geq 0$ and $x^i > c^i x^i$ are indeterminates quasi non productive and indeterminate if one of $x^i < 0$, $c^i x^i < 0$ and one of x^i and $I^i x^i$ are indeterminate. Quasi production if one of $c^i x^i > 0$ and $x^i > 0$ and $x^i < 0$ and $I^i x^i < 0$. Thus 6 possibilities can occur at anytime of study say at times t_1 and t_2 for it is but very natural as in any industrial problem the occurrences of any factor like demand or production is very much dependent on the people and the government policy and other external factors.

References

[1] Herstein, I.N., Abstract Algebra, John Wiley, 1990.

[2] Jacobson, N., Lectures in Abstract Algebra, D Van Nostrand Co, Princeton, 1953.

[3] Vasantha Kandasamy, W.B., Bialgebraic structures and Smarandache bialgebraic structures, American Research Press, Rehoboth, 2003.

[4] Vasantha Kandasamy, W.B., Linear Algebra and Smarandache Linear Algebra, Bookman Publishing, 2003.

[5] Vasantha Kandasamy, W.B., F. Smarandache and Ilanthenral, K., Introduction to Linear Bialgebra, Hexis, Arizona, 2005.

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The translational hull of superabundant semigroups with semilattice of idempotents[1]

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Abstract Superabundant semigroups with semilattice of idempotents are generalizations of inverse semigroups. This paper proves that the translational hull of such kind of semigroups is also of the same type.

Keywords Translational hull, superabundant semigroups, superabundant semigroup with semilattice of idempotents

§1. Introduction

A mapping λ which maps a semigroup S into itself is called a left translation of S if $\lambda(ab) = (\lambda a)b$, for all $a, b \in S$; a mapping ρ which maps S into itself is called a right translation of S if $(ab)\rho = a(b\rho)$, for all $a, b \in S$. A left translation λ and a right translation ρ of the semigroup S are linked if $a(\lambda b) = (a\rho)b$, for all $a, b \in S$. In this case, we call the pair of translations (λ, ρ) a bitranslation of S. The set $\Lambda(S)$ of all left translations and the set $P(S)$ of all right translations of the semigroup S form the semigroups under the composition of mappings. By the translational hull of a semigroup S, we mean a subsemigroup $\Omega(S)$ consisting of all bitranslations (λ, ρ) of $\Lambda(S) \times P(S)$. The concept of translational hull of a semigroup was first introduced by M.Petrich in 1970. Later on, J.E.Ault [1] studied the translational hull of an inverse semigroup in 1973. Recently, Guo, Shum and Ren, Shum have studied the translational hull of type-A semigroups [9] and strongly right or left adequate semigroups [6], respectively.

On a semigroup S the relation $\mathcal{L}^*$ is defined by $(a, b) \in \mathcal{L}^*$ if and only if the elements a, b of S are $\mathcal{L}$-related in some oversemigroup of S. The relation $\mathcal{R}^*$ is defined dually. The intersection of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{H}^*$. A semigroup S is called abundant if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class of S contains an idempotent [1]. A semigroup in which each $\mathcal{H}^*$-class contains an idempotent is called superabundant. It is easy to see that a superabundant semigroup is a natural generalization of a completely regular semigroup and a superabundant semigroup with semilattice of idempotents is the analogue of an inverse semigroup. In this

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paper we will consider the translational hull of superabundant semigroups with semilattice of idempotents and show that the translational hull of this kind of semigroups is of the same type.

§2. The translational hull

Lemma 2.1. [1] Let S be a semigroup and $a, b \in S$. Then the following statements hold:

(i) $(a, b) \in \mathcal{L}^*$, if and only if $ax = ay \Leftrightarrow bx = by$, for any $x, y \in S^1$;

(ii) for any idempotent e, $(e, a) \in \mathcal{L}^*$, if and only if $ae = a$ and $ax = ay \Rightarrow ex = ey$, for any $x, y \in S^1$.

The dual of Lemma 2.1 also hold.

Suppose that S is a superabundant semigroup with semilattice of idempotents. It is easy to verify that every $\mathcal{H}^*$-class of S contains a unique idempotent, denoted the idempotent in H_a^* containing a of S by a^*.

Lemma 2.2. Let S be a superabundant semigroup with semilattice of idempotents. Then every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class of S has a unique idempotent.

Proof. Suppose that $a\mathcal{L}^*b$ for any $a, b \in S$. Clearly, $a^*\mathcal{L}^*a\mathcal{L}^*b\mathcal{L}^*b^*$. Since a^* and b^* are both idempotents, we immediately have $b^* = b^*a^*$ and $a^* = a^*b^*$ and hence $b^* = b^*a^* = a^*b^* = a^*$ by hypothesis. This shows that every $\mathcal{L}^*$-class of S contains a unique idempotent. Similarly, every $\mathcal{R}^*$-class of S has a unique idempotent.

Lemma 2.3. Let S be a superabundant semigroup with semilattice of idempotents. Then for any $a, b \in S$, the following statements hold:

(i) $a^*a = a = aa^*$;

(ii) $(a^*b^*)^* = a^*b^*$;

(iii) $(ab)^* = a^*b^*$.

Proof. (i) and (ii)are straightforward from hypothesis.

(iii) Suppose that $a, b \in S$. Clearly, $b\mathcal{R}^*b^*$. Since $\mathcal{R}^*$ is a left congruence on S, we have that $ab\mathcal{R}^*ab^*$. By Lemma 2.2, since every $\mathcal{R}^*$-class contains a unique idempotent, we obtain that $(ab)^* = (ab^*)^*$. Similarly, since $a\mathcal{L}^*a^*$ and $\mathcal{L}^*$ is a right congruence, we also have $(ab^*)^* = (a^*b^*)^* = a^*b^*$. Thus $(ab)^* = a^*b^*$.

Lemma 2.4. Let S be a superabundant semigroup with semilattice of idempotents. Then the following statements hold:

(i) if λ_1 and λ_2 are left translations of S, then $\lambda_1 = \lambda_2$ if and only if $\lambda_1 e = \lambda_2 e$ for any $e \in E$;

(ii) if ρ_1 and ρ_2 are right translations of S, then $\rho_1 = \rho_2$ if and only if $e\rho_1 = e\rho_2$ for any $e \in E$.

Proof. We only need to show that (i) since the proof of (ii) is similar. The necessity part of (i) is clear. We now prove the sufficiency of (i). For any a in S, it is obvious that $a^*a = a$. We have that $\lambda_1 a = \lambda_1(a^*a) = (\lambda_1 a^*)a = (\lambda_2 a^*)a = \lambda_2(a^*a) = \lambda_2 a$ and hence $\lambda_1 = \lambda_2$.

Recall that a semigroup S is said to be an idempotent balanced semigroup [6] if for any $a \in S$, there exist idempotents e and f such that $a = ea = af$. It is clear from Lemma 2.3 that a superabundant semigroup S is an idempotent balanced semigroup.

Lemma 2.5. [6] Let S be an idempotent balanced semigroup. If $(\lambda_i, \rho_i) \in \Omega(S)(i = 1, 2)$, then the following statements are equivalent:

(i) $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$;

(ii) $\rho_1 = \rho_2$;

(iii) $\lambda_1 = \lambda_2$.

We always suppose below that S is a superabundant semigroup with semilattice of idempotents and suppose $(\lambda, \rho) \in \Omega(S)$. For any $a \in S$, define λ^*, ρ^* which mapping S into itself by the following rule that

$$\lambda^* a = (\lambda a^*)^* a, \quad a\rho^* = a(\lambda a^*)^*. \tag{1}$$

Lemma 2.6. Let S be a given semigroup. Then for any e in E, we have

(i) $e\rho^* = \lambda^* e$ and $\lambda^* e \in E$;

(ii) $\lambda^* e = (\lambda e)^*, e\rho^* = (e\rho)^*$.

Proof. (i) Suppose that e is an idempotent of S. Then $e\rho^* = e(\lambda e)^* = (\lambda e)^* e = \lambda^* e$, and

$$(\lambda^* e)^2 = (\lambda e)^* e \cdot (\lambda e)^* e = (\lambda e)^* (\lambda e)^* ee = (\lambda e)^* e = \lambda^* e.$$

Thus, $\lambda^* e$ is idempotent.

(ii) Since $\mathcal{L}^*$ is a left congruence on S and $(\lambda e)^* \mathcal{L}^* \lambda e$, we have $(\lambda e)^* e \mathcal{L}^* \lambda e$, that is, $\lambda^* e \mathcal{L}^* \lambda e$. On the other hand, since $\lambda^* e$ is idempotent and every $\mathcal{L}^*$-class has a unique idempotent, we get $\lambda^* e = (\lambda e)^*$. Similarly, we can also prove $e\rho^* = (e\rho)^*$.

Lemma 2.7. For $(\lambda, \rho) \in \Omega(S)$, (λ^*, ρ^*) defined above is an element of $E_{\Omega(S)}$.

Proof. (1) We first show that λ^* is a left translation of S. Suppose that $a, b \in S$. Then, by Lemma 2.3, we have

$$\begin{aligned}
\lambda^*(ab) &= (\lambda(ab)^*)^* ab = (\lambda a^* b^*)^* ab \\
&= (\lambda a^*(a^* b^*))^* ab = [(\lambda a^*)(ab)^*]^* ab \\
&= (\lambda a^*)^* (ab)^* ab = (\lambda a^*)^* ab \\
&= (\lambda^* a) b.
\end{aligned}$$

(2) To see that ρ^* is a right translation of S, we let $a, b \in S$. It is follows from Lemma 2.3 that

$$\begin{aligned}
(ab)\rho^* &= ab(\lambda(ab)^*)^* = ab(\lambda a^* b^*)^* \\
&= ab(\lambda(a^* b^*) b^*)^* = ab[\lambda(ab)^* b^*]^* \\
&= ab[(\lambda b^*)(ab)^*]^* = ab(\lambda b^*)^* (ab)^* \\
&= (ab)(ab)^* (\lambda b^*)^* = ab(\lambda b^*)^* \\
&= a(b\rho^*).
\end{aligned}$$

This completed the proof.

(3) Next, we want to prove that λ^* and ρ^* are also linked. For any $a, b \in S$, by Lemma 2.6 (i) and (2) above, we have

$$\begin{aligned} a(\lambda^* b) &= a(\lambda^* b^*)b = a(b^*\rho^*)b \\ &= aa^*(b^*\rho^*)b = a(a^*b^*)\rho^* b \\ &= a(b^*a^*)\rho^* b = ab^*(a^*\rho^*)b \\ &= a(a^*\rho^*)b^*b = (a\rho^*)b. \end{aligned}$$

Combining with these observations above, we have shown that $(\lambda^*, \rho^*) \in \Omega(S)$.

(4) For any $e \in E(S)$, $e(\rho^*)^2 = e(e\rho^*)\rho^* = (e\rho^*)(e\rho^*) = e\rho^*$, by Lemma 2.5, $(\lambda^*, \rho^*) = (\lambda^*, \rho^*)^2$. So, $(\lambda^*, \rho^*) \in E_{\Omega(S)}$.

Lemma 2.8. $(\lambda^*, \rho^*)\mathcal{H}^*(\lambda, \rho)$.

Proof. (1) In order to prove that $(\lambda^*, \rho^*)\mathcal{L}^*(\lambda, \rho)$, we first show that $(\lambda, \rho)(\lambda^*, \rho^*) = (\lambda, \rho)$, that is $(\lambda\lambda^*, \rho\rho^*) = (\lambda, \rho)$. By Lemma 2.5, we only need to show that $\lambda\lambda^* = \lambda$. Taking $e \in E$, we have

$$\begin{aligned} \lambda\lambda^* e &= \lambda(\lambda^* e) = \lambda(e\rho^*) \\ &= \lambda[e(e\rho^*)] = (\lambda e)(e\rho^*) \\ &= (\lambda e)(\lambda^* e) = (\lambda e)(\lambda e)^* = \lambda e. \end{aligned}$$

This shows by Lemma 2.4 that $\lambda = \lambda\lambda^*$ and hence, by Lemma 2.5, $(\lambda, \rho)(\lambda^*, \rho^*) = (\lambda, \rho)$.

Now we let $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$ and $(\lambda, \rho)(\lambda_1, \rho_1) = (\lambda, \rho)(\lambda_2, \rho_2)$. Clearly, $\rho\rho_1 = \rho\rho_2$. Then $(\lambda e)^*\rho\rho_1 = (\lambda e)^*\rho\rho_2$ for any e in E. Since $(\lambda e)^*\rho\mathcal{L}^*[(\lambda e)^*\rho]^*$, we have $[(\lambda e)^*\rho]^*\rho_1 = [(\lambda e)^*\rho]^*\rho_2$ by Lemma 2.1. Notice that $[(\lambda e)^*\rho]e = (\lambda e)^*(\lambda e) = \lambda e$, we have that $e[(\lambda e)^*\rho]^* = [(\lambda e)^*\rho]^* e = (\lambda e)^*$. Thus, we can deduce that $(\lambda e)^*\rho_1 = (\lambda e)^*\rho_2$. By Lemma 2.6, we immediately have

$$e\rho^*\rho_1 = (e\rho^*)\rho_1 = (\lambda^* e)\rho_1 = (\lambda^* e)\rho_2 = e\rho^*\rho_2.$$

This shows that $\rho^*\rho_1 = \rho^*\rho_2$ from Lemma 2.4. Again, using Lemma 2.5, we have that $(\lambda^*, \rho^*)(\lambda_1, \rho_1) = (\lambda^*, \rho^*)(\lambda_2, \rho_2)$. This implies that $(\lambda, \rho)\mathcal{L}^*(\lambda^*, \rho^*)$ from Lemma 2.1.

(2) To prove that $(\lambda^*, \rho^*)\mathcal{R}^*(\lambda, \rho)$, we first prove that $(\lambda^*, \rho^*)(\lambda, \rho) = (\lambda, \rho)$. By Lemma 2.5, we only need to show that $e\rho^*\rho = e\rho$ for any $e \in E$. It follows from Lemma 2.6 that

$$\begin{aligned} e\rho^*\rho &= \lambda^* e\rho = (\lambda^* e)(e\rho) \\ &= (e\rho^*)e\rho = (e\rho)^* e\rho = e\rho. \end{aligned}$$

This implies that $\rho = \rho^*\rho$. Again by Lemma 2.5, we have $(\lambda^*, \rho^*)(\lambda, \rho) = (\lambda, \rho)$.

Next, for any $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$, we suppose that $(\lambda_1, \rho_1)(\lambda, \rho) = (\lambda_2, \rho_2)(\lambda, \rho)$. Clearly, $\lambda_1\lambda = \lambda_2\lambda$ and hence $\lambda_1\lambda e = \lambda_2\lambda e$, for any e in E. Since $\lambda e\mathcal{R}^*(\lambda e)^*$ and $[\lambda_1(\lambda e)^*](\lambda e) = [\lambda_2(\lambda e)^*](\lambda e)$, by the dual of Lemma 2.1, we have that $[\lambda_1(\lambda e)^*](\lambda e)^* = [\lambda_2(\lambda e)^*](\lambda e)^*$. Thus, $\lambda_1(\lambda e)^* = \lambda_2(\lambda e)^*$ and

$$\lambda_1\lambda^* e = \lambda_1(\lambda^* e) = \lambda_1(\lambda e)^* = \lambda_2(\lambda e)^* = \lambda_2\lambda^* e.$$

Consequently, $\lambda_1\lambda^* = \lambda_2\lambda^*$. Again using Lemma 2.5, we immediately have that $(\lambda_1, \rho_1)(\lambda^*, \rho^*) = (\lambda_2, \rho_2)(\lambda^*, \rho^*)$. Hence $(\lambda, \rho)\mathcal{R}^*(\lambda^*, \rho^*)$ from the dual of Lemma 2.1.

Thus, we have completed the proof of $(\lambda^*, \rho^*)\mathcal{H}^*(\lambda, \rho)$.

Lemma 2.9. Let

$$\Psi_{\Omega(S)} = \{(\lambda, \rho) \in \Omega(S) | (\exists(\hat{\lambda}, \hat{\rho}) \in \Omega(S))(\forall a \in S)\lambda a = (\hat{\lambda}a^*)^* a, a\rho = a(\hat{\lambda}a^*)^*\}.$$

Then $\Psi_{\Omega(S)}$ is the set of all idempotents of $\Omega(S)$, that is, $\Psi_{\Omega(S)} = E_{\Omega(S)}$.

Proof. "$\subseteq$". Let $(\lambda, \rho) \in \Psi_{\Omega(S)}$, then there exists $(\hat{\lambda}, \hat{\rho}) \in \Omega(S)$ such that $\lambda a = (\hat{\lambda}a^*)^* a$ and $a\rho = a(\hat{\lambda}a^*)^*$ for any $a \in S$. By equation 2.1, we know $\lambda = \hat{\lambda}^*$, $\rho = \hat{\rho}^*$, and then from Lemma 2.7 we easily get $\Psi_{\Omega(S)} \subseteq E_{\Omega(S)}$.

"$\supseteq$". Let $(\lambda, \rho) \in E_{\Omega(S)}$, and $(\hat{\lambda}, \hat{\rho}) \in \mathcal{H}^*{}_{(\lambda,\rho)}$. Then we can get $(\hat{\lambda}^*, \hat{\rho}^*)$ which satisfies $\hat{\lambda}^* a = (\hat{\lambda}a^*)^* a$ and $a\hat{\rho}^* = a(\hat{\lambda}a^*)^*$. By Lemma 2.7 and Lemma 2.8 we know $(\hat{\lambda}^*, \hat{\rho}^*) \in E_{\Omega(S)}$ and $(\hat{\lambda}^*, \hat{\rho}^*)\mathcal{H}^*(\hat{\lambda}, \hat{\rho})$. So$(\hat{\lambda}^*, \hat{\rho}^*)\mathcal{H}^*(\lambda, \rho)$. That is $(\hat{\lambda}^*, \hat{\rho}^*)\mathcal{L}^*(\lambda, \rho)$ and $(\hat{\lambda}^*, \hat{\rho}^*)\mathcal{R}^*(\lambda, \rho)$. From Lemma 2.1 and its dual we get $(\hat{\lambda}^*, \hat{\rho}^*)(\lambda, \rho) = (\hat{\lambda}^*, \hat{\rho}^*)$ and $(\hat{\lambda}^*, \hat{\rho}^*)(\lambda, \rho) = (\lambda, \rho)$, so $(\hat{\lambda}^*, \hat{\rho}^*) = (\lambda, \rho)$. This show that for $(\lambda, \rho) \in E_{\Omega(S)}$, there exists $(\hat{\lambda}, \hat{\rho})$ such that $\lambda a = \hat{\lambda}^* a = (\hat{\lambda}a^*)^* a$ and $a\rho = a\hat{\rho}^* = a(\hat{\lambda}a^*)^*$. Hence $(\lambda, \rho) \in \Psi_{\Omega(S)}$, and then $\Psi_{\Omega(S)} \supseteq E_{\Omega(S)}$.

Summarizing the above Lemma 2.7-2.9, we have proved that $\Omega(S)$ is a superabundant semigroup.

Now we can obtain the following main theorem of this paper as follows:

Theorem. If S is superabundant semigroup with semilattice of idempotents. Then $\Omega(S)$ is still of the same type.

Proof. By using Lemma 2.7, Lemma 2.8 and Lemma 2.9, we immediately know $\Omega(S)$ is a superabundant semigroup. It remains to show that idempotents of $\Omega(S)$ form a semilattice. Suppose that (λ^*, ρ^*) and (λ', ρ') in $E_{\Omega(S)}$ and $e \in E$. Since $\lambda^*\lambda' e = \lambda^*(\lambda' e)e = \lambda^* e(\lambda' e)$, by Lemma 2.6, we know that $\lambda^* e$ and $\lambda' e$ are idempotent. Thus,

$$\lambda^* e(\lambda' e) = (\lambda' e)(\lambda^* e) = \lambda' e(\lambda^* e) = \lambda'(\lambda^* e)e = \lambda'\lambda^* e,$$

This implies that $\lambda^*\lambda' = \lambda'\lambda^*$ by Lemma 2.4 and hence, by Lemma 2.5, $(\lambda^*, \rho^*)(\lambda', \rho') = (\lambda', \rho')(\lambda^*, \rho^*)$. This shows that all idempotents of $E_{\Omega(S)}$ commute and so the proof of theorem is completed.

References

[1] J.B. Fountain, Abundant semigroups, Proc. London Math. Soc., **44**(1982), No.3, 103-129.

[2] J.M. Howie, Fundamentals of semigroup theory, Clarendon Press, Oxford, 1995.

[3] J.B. Fountain and M.V. Lawson, The translational hull of an adequate semigroup, Semigroup Forum, **32**(1985), 79-86.

[4] J.E. Ault, Translational hull of an inverse semigroup, Semigroup Forum, **4**(1972), 105-168.

[5] X.M.Ren and K.P.Shum, The strcture of superabundant semigroups, Science in China (Series A), **47**(2004), No.5, 756-771.

[6] X.M.Ren and K.P.Shum, The translational hull of a strongly right or left adequate semigroup, Vietnam Journal of mathematics, **34**(2006), No.4, 1-7.

[7] X.J. Guo and Y.Q. Guo, The translational hull of strongly type-A semigroups(In Chinese), Science in China (Series A), **11**(1999), 1002-1008.

[8] M. Petrich, The translational hull in semigroups and rings, Semigroup Forum, **1**(1970), 283-360.

[9] X.J. Guo and K.P.Shum, On translational hull of type-A semigroups, J. of Algebra, **269**(2003), 240-249.

[10] J.E.Ault, The translational hull of an inverse semigroup, Glasgow Math. J., **14**(1973), 56-64.

[11] J.B. Fountain and M.V.Lawson, The translational hull of an adequate semigroup, Semigroup Forum, **32**(1985), 79-86.

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Neutrosophic applications in finance, economics and politics—a continuing bequest of knowledge by an exemplarily innovative mind

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It is not very common for a young PhD aspirant to select a topic for his dissertation that makes exploratory forays into a fledgling science — one that is still in the process of finding feet within the ramparts of academia. It would be considered a highly risky venture to say the least given that through his dissertation the PhD aspirant would need to not only convince his examiners on the merit of his own research on the topic but also present a strong case on behalf of the topic itself.

Sukanto Bhattacharya's doctoral thesis entitled "Utility, Rationality and Beyond — From Behavioral Finance to Informational Finance" not only succeeded in earning him a PhD degree but also went on to arguably become recognized as the first comprehensive published work of its kind on the application of neutrosophic logic in theoretical finance.

Bhattacharya postulated that when the long-term price of a market-traded derivative security (e.g. an exchange-traded option) is observed to deviate from the theoretical price; three possibilities should be considered:

(1) The theoretical pricing model is inadequate or inaccurate, which implies that the observed market price may very well be the true price of the derivative security, or

(2) A temporary upheaval has occurred in the market possibly triggered by psychological forces like mass cognitive dissonance that has pushed the market price "out of sync" with the theoretical price as the latter is based on the assumptions of rational economic behavior, or

(3) The nature of the deviation is indeterminate and could be either due to (1) or (2) or a mix of both (1) and (2) or is merely a random fluctuation with no apparent causal connection.

The systematic risk associated with transactions in financial markets is termed resolvable risk in Bhattacharya's dissertation. Since a financial market can only be as informationally efficient as the actual information it gets to process, if the information about the true price of the derivative security is misconstrued (perhaps due to an inadequate pricing model), the market cannot be expected to reconcile it just because it is operating at a certain level of informational efficiency.

Bhattacharya's statement that "financial markets can't be expected to pull jack-rabbits out of empty hats" beautifully encapsulates the neutrosophic element of market risk.

Bhattacharya coined the term irresolvable risk to represent the perceived risk resulting from the imprecision associated with decidedly irrational psycho-cognitive forces that subjectively interpret information and process the same in decision-making. He demonstrated that the neutrosophic probability of the true price of the derivative security being given by any theoretical pricing model is obtainable as $NP\ (H \bigcap M^C)$; where NP stands for neutrosophic probability, $H = \{p : p$ is the true price determined by the theoretical pricing model $\}$, $M = \{p : p$ is the true option price determined by the prevailing market price $\}$ and the C superscript is the complement operator.

Bhattacharya has since made significant contributions, either independently or collaboratively, to neutrosophic applications in various financial and economic problems ranging from financial fraud detection to portfolio choice and optimization.

However, arguably perhaps Bhattacharya's most significant contribution to the science of neutrosophy so far is the extension of the fuzzy game paradigm to a neutrosophic game paradigm and then successfully applying the same to model the vexing Israel-Palestine political problem, in collaboration with Florentin Smarandache—the father of neutrosophic logic.

Although he has written a few purely abstract pieces mainly on the forms of Smarandache geometries, Bhattacharya's major works are highly application-oriented and stand out in their brilliant innovation and real-world connection to business and the social sciences.

A bibliographical list of Bhattacharya's significant published works till date on neutrosophic applications in finance, economics and the social sciences is appended below:

[1] Bhattacharya S., Khoshnevisan M. and Smarandache F., Artificial intelligence and responsive optimization, Xiquan, Phoenix, U.S.A. (ISBN 1-931233-77-2), 2003. (Cited by International Statistical Institute in "Short Book Reviews", **23**(2003), No. 2, pp. 35.

[2] Bhattacharya S., Neutrosophic information fusion applied to the options market, Investment Management and Financial Innovations, **2**(2005), No. 1, pp. 139-145.

[3] Bhattacharya S., F. Smarandache and K. Kumar, Conditional probability of actually detecting a financial fraud-a neutrosophic extension to Benford's law, International Journal of Applied Mathematics, **17**(2005), No. 1, pp. 7-14.

[4] Bhattacharya S., Utility, Rationality and Beyond-From Behavioral Finance to Informational Finance, American Research Press, (ISBN 1-931233-85-3), 2005.

[5] Bhattacharya S., Smarandache F. and Khoshnevisan M., The Israel-Palestine question—A case for application of neutrosophic game theory", Octogon Mathematical Magazine, **14**(2006), No. 1, pp. 165-173.

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An equation involving the Smarandache function

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Abstract Let n be any positive integer, the famous Smarandache function $S(n)$ is defined by $S(n) = \min\{m : n|m!\}$. The main purpose of this paper is using the elementary method to prove that for some special positive integers k and m, the equation

$$mS(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinity positive integer solutions $(m_1, m_2, \cdots, m_k)$.

Keywords Vinogradov's three-primes theorem, equation, solutions

§1. Introduction

For any positive integer n, the famous Smarandache function $S(n)$ is defined as follows:

$$S(n) = \min\{m : n|m!\}.$$

For example, $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $\cdots$. About this function, many people had studied its properties, see [1-4]. Let $p(n)$ denotes the greatest prime divisor of n, it is clear that $S(n) \geq p(n)$.

In fact, $S(n) = p(n)$ for all most n, as noted by [5]. This means that the number of $n \leq x$ for which $S(n) \neq p(n)$, denoted by $N(x)$, is $o(x)$. Xu Zhefeng [7] studied the mean square value for $S(n) - p(n)$, and obtained an asymptotic formula as follows:

$$\sum_{n\leq x}(S(n) - p(n))^2 = \frac{2\zeta(\frac{3}{2})x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right).$$

It is easily to show that $S(p) = p$ and $S(n) < n$ except for the case $n = 4$, $n = p$. So there have a closely relationship between $S(n)$ and $\pi(x)$:

$$\pi(x) = -1 + \sum_{n=2}^{[x]}\left[\frac{S(n)}{n}\right],$$

where $\pi(x)$ denotes the number of primes up to x, and $[x]$ denotes the greatest integer less than or equal to x.

Recently, Lu Yaming [8] studied the solvability of the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k),$$

and proved that for each positive integer k, this equation has infinity positive integer solutions. J. Sandor [9] obtained some inequalities involving the Smarandache function. That is, he proved that for each $k \geq 2$, there are infinitely many positive integers m_1, m_2, $\cdots$, m_k and n_1, n_2, $\cdots$, n_k such that

$$S(m_1 + m_2 + \cdots + m_k) > S(m_1) + S(m_2) + \cdots + S(m_k)$$

and

$$S(n_1 + n_2 + \cdots + n_k) < S(n_1) + S(n_2) + \cdots + S(n_k).$$

This paper, as note of [8] and [9], we shall prove the following main conclusion:

Theorem. If positive integers k and m satisfy one of the following three conditions:

(i) $k > 2$ and $m \geq 1$ are both odd numbers;

(ii) $k \geq 5$ is an odd number and $m \geq 2$ is an even number;

(iii) any even number $k > 3$ and any integer $m \geq 1$, then the equation

$$mS(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinity positive integer solutions $(m_1, m_2, \cdots, m_k)$.

It is clear that our Theorem is a generalization of [8] and [9]. In fact, if we take $m = 1$, then from our Theorem we can deduce the conclusion of [8]. If we take $m > 1$, then we can get one of the two inequalities in [9].

§2. Proof of Theorem

To complete the proof of the theorem, we need the famous Vinogradov's three-primes theorem which was stated as follows:

Lemma 1. Let c be an odd integer large enough, then c can be expressed as a sum of three odd primes.

Proof. (See Theorem 20.2 and 20.3 of [10]).

Lemma 2. Let odd integer $k \geq 3$, then any sufficiently large odd integer n can be expressed as a sum of k odd primes. That is,

$$n = p_1 + p_2 + \cdots + p_k,$$

where p_1, p_2, $\cdots$, p_k are primes.

Proof. The proof of this Lemma follows from Lemma 1 and the mathematical induction.

Now we use these two Lemmas to prove our Theorem. Let $k \geq 3$ be an odd number, then from Lemma 2 we know that for any fixed odd number m and every sufficiently large prime p, there exist k primes p_1, p_2, $\cdots$, p_k such that

$$mp = p_1 + p_2 + \cdots + p_k.$$

Note that $S(p_i) = p_i$ and $S(mp) = p$ if $p > m$. So from the above formula we have

$$mS(mp) = mp = p_1 + p_2 + \cdots + p_k = S(p_1) + S(p_2) + \cdots + S(p_k).$$

This means that for any odd number $k > 2$ and any odd number m, the theorem is correct if we take $m_1 = p_1$, $m_2 = p_2$, $\cdots$, $m_k = p_k$. If $m \geq 2$ be an even number, then $mp - 2 - 3$ be an odd number. So for any odd number $k \geq 5$, from the above conclusion we have

$$mp - 2 - 3 = p_1 + p_2 + \cdots + p_{k-2}$$

or

$$mp = p_1 + p_2 + \cdots + p_{k-2} + 2 + 3.$$

Taking $m_1 = p_1$, $m_2 = p_2$, $\cdots$, $m_{k-2} = p_{k-2}$, $m_{k-1} = 2$, $m_k = 3$ in the theorem we may immediately get

$$mS(m_1 + m_2 + \cdots + m_k) = mS(mp) = mp = p_1 + p_2 + \cdots + p_k = S(m_1) + S(m_2) + \cdots + S(m_k).$$

If $k \geq 4$ and m are even numbers, then for every sufficiently large prime p, $mp - 3$ be an odd number, so from Lemma 2 we have

$$mp - 3 = p_1 + p_2 + \cdots + p_{k-1}$$

or

$$mp = p_1 + p_2 + \cdots + p_{k-1} + 3.$$

That is to say,

$$mS(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots S(m_k)$$

with $m_1 = p_1$, $m_2 = p_2$, $\cdots$, $m_{k-1} = p_{k-1}$ and $m_k = 3$.

If $k \geq 4$ be an even number and m be an odd number, then we can write

$$mp - 2 = p_1 + p_2 + \cdots + p_{k-1}$$

or

$$mp = p_1 + p_2 + \cdots + p_{k-1} + 2.$$

This means that the theorem is also true if we take $m_1 = p_1$, $m_2 = p_2$, $\cdots$, $m_{k-1} = p_{k-1}$ and $m_k = 2$.

This completes the proof of Theorem.

References

[1] C. Ashbacher, Some properties of the Smarandache-Kurepa and Smarandache-Wagstaff function, Mathematics and Informatics Quarterly, **7**(1997), pp. 114-116.

[2] A. Begay, Smarandache ceil functions, Bulletin of Pure and Applied Sciences, **16E**(1997), pp. 227-229.

[3] Mark Earris and Patrick Mitchell, Bounding the Smarandache function, Smarandache Notions Journal, **13**(2002), pp. 37-42.

[4] Kevin Ford, The normal behavior of the Smarandache function, Smarandache Notions Journal, **10**(1999), pp. 81-86.

[5] P.Erdös, Problem 6674, American Math. Monthly, **98**(1991), pp. 965.

[6] F. Smarandache, Only problems, not solutions. Xiquan Publishing House, Chicago, 1993.

[7] Xu Zhefeng, On the value distribution of the Smarandache function, Acta Mathematica Sinica (Chinese), 49(2006), pp. 1009-1012.

[8] Lu Yaming, On the solutions of an equation involving the Smarandache function, Scientia Magna, 2(2006), No.1, 76-79.

[9] J. Sándor, On certain inequalities involving the Smarandache function, Scientia Magna, 2(2006), No.3, 78-80.

[10] Pan Chengdong and Pan Chengbiao, Element of the analytic number theory, Science Press, Beijing, 1997.

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The relationship between $S_p(n)$ and $S_p(kn)$

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Abstract For any positive integer n, let $S_p(n)$ denotes the smallest positive integer such that $S_p(n)!$ is divisible by p^n, where p be a prime. The main purpose of this paper is using the elementary methods to study the relationship between $S_p(n)$ and $S_p(kn)$, and give an interesting identity.

Keywords The primitive numbers of power p, properties, identity

§1. Introduction and Results

Let p be a prime and n be any positive integer. Then we define the primitive numbers of power p (p be a prime) $S_p(n)$ as the smallest positive integer m such that $m!$ is divided by p^n. For example, $S_3(1) = 3,\ S_3(2) = 6,\ S_3(3) = S_3(4) = 9,\ \cdots$. In problem 49 of book [1], Professor F.Smarandache asked us to study the properties of the sequence $\{S_p(n)\}$. About this problem, Zhang Wenpeng and Liu Duansen [3] had studied the asymptotic properties of $S_p(n)$, and obtained an interesting asymptotic formula for it. That is, for any fixed prime p and any positive integer n, they proved that

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p}\ln n\right).$$

Yi Yuan [4] had studied the asymptotic property of $S_p(n)$ in the form $\frac{1}{p}\sum_{n\leq x}|S_p(n+1) - S_p(n)|$, and obtained the following result: for any real number $x \geq 2$, let p be a prime and n be any positive integer,

$$\frac{1}{p}\sum_{n\leq x}|S_p(n+1) - S_p(n)| = x\left(1-\frac{1}{p}\right) + O\left(\frac{\ln x}{\ln p}\right).$$

Xu Zhefeng [5] had studied the relationship between the Riemann zeta-function and an infinite series involving $S_p(n)$, and obtained some interesting identities and asymptotic formulae for $S_p(n)$. That is, for any prime p and complex number s with $\mathrm{Re}s > 1$, we have the identity:

$$\sum_{n=1}^{\infty}\frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s - 1},$$

where $\zeta(s)$ is the Riemann zeta-function.

And, let p be a fixed prime, then for any real number $x \geq 1$ he got

$$\sum_{\substack{n=1\\ S_p(n)\leq x}}^{\infty} \frac{1}{S_p(n)} = \frac{1}{p-1}\left(\ln x + \gamma + \frac{p\ln p}{p-1}\right) + O(x^{-\frac{1}{2}+\varepsilon}),$$

where γ is the Euler constant, ε denotes any fixed positive number.

Chen Guohui [7] had studied the calculation problem of the special value of famous Smarandache function $S(n) = \min\{m :\ m \in N,\ n|m!\}$. That is, let p be a prime and k an integer with $1 \leq k < p$. Then for polynomial $f(x) = x^{n_k} + x^{n_{k-1}} + \cdots + x^{n_1}$ with $n_k > n_{k-1} > \cdots > n_1$, we have:

$$S(p^{f(p)}) = (p-1)f(p) + pf(1).$$

And, let p be a prime and k an integer with $1 \leq k < p$, for any positive integer n, we have:

$$S\left(p^{kp^n}\right) = k\left(\phi(p^n) + \frac{1}{k}\right)p,$$

where $\phi(n)$ is the Euler function. All these two conclusions above also hold for primitive function $S_p(n)$ of power p.

In this paper, we shall use the elementary methods to study the relationships between $S_p(n)$ and $S_p(kn)$, and get some interesting identities. That is, we shall prove the following:

Theorem. Let p be a prime. Then for any positive integers n and k with $1 \leq n \leq p$ and $1 < k < p$, we have the identities:

$S_p(kn) = kS_p(n)$, if $1 < kn < p$;

$S_p(kn) = kS_p(n) - p\left[\dfrac{kn}{p}\right]$, if $p < kn < p^2$, where $[x]$ denotes the integer part of x.

§2. Two simple Lemmas

To complete the proof of the theorem, we need two simple lemmas which stated as following:

Lemma 1. For any prime p and any positive integer $2 \leq l \leq p-1$, we have:

(1) $S_p(n) = np$, if $1 \leq n \leq p$;

(2) $S_p(n) = (n-l+1)p$, if $(l-1)p + l - 2 < n \leq lp + l - 1$.

Proof. First we prove the case (1) of Lemma 1. From the definition of $S_p(n) = \min\{m : p^n|m!\}$, we know that to prove the case (1) of Lemma 1, we only to prove that $p^n \| (np)!$. That is, $p^n|(np)!$ and $p^{n+1} \dagger (np)!$. According to Theorem 1.7.2 of [6] we only to prove that $\displaystyle\sum_{j=1}^{\infty}\left[\frac{np}{p^j}\right] = n$.

In fact, if $1 \leq n < p$, note that $\left[\dfrac{n}{p^i}\right] = 0,\ i = 1,\ 2, \cdots$, we have

$$\sum_{j=1}^{\infty}\left[\frac{np}{p^j}\right] = \sum_{j=1}^{\infty}\left[\frac{n}{p^{j-1}}\right] = n + \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots = n.$$

This means $S_p(n) = np$. If $n = p$, then $\displaystyle\sum_{j=1}^{\infty}\left[\frac{np}{p^j}\right] = n+1$, but $p^p \dagger (p^2-1)!$ and $p^p|p^2!$. This prove the case (1) of Lemma 1. Now we prove the case (2) of Lemma 1. Using the same method of proving the case (1) of Lemma 1 we can deduce that if $(l-1)p+l-2 < n \le lp+l-1$, then

$$\left[\frac{n-l+1}{p}\right] = l-1,\ \left[\frac{n-l+1}{p^i}\right] = 0,\ i = 2, 3, \cdots .$$

So we have

$$\begin{aligned}\sum_{j=1}^{\infty}\left[\frac{(n-l+1)p}{p^j}\right] &= \sum_{j=1}^{\infty}\left[\frac{n-l+1}{p^{j-1}}\right] \\ &= n-l+1+\left[\frac{n-l+1}{p}\right]+\left[\frac{n-l+1}{p^2}\right]+\cdots \\ &= n-l+1+l-1 = n.\end{aligned}$$

From Theorem 1.7.2 of reference [6] we know that if $(l-1)p+l-2 < n \le lp+l-1$, then $p^n \| ((n-l+1)p)!$. That is, $S_p(n) = (n-l+1)p$. This proves Lemma 1.

Lemma 2. For any prime p, we have the identity $S_p(n) = (n-p+1)p$, if $p^2-2 < n \le p^2$.

Proof. It is similar to Lemma 1, we only need to prove $p^n \| ((n-p+1)p)!$. Note that if $p^2-2 < n \le p^2$, then $\left[\frac{n-p+1}{p}\right] = p-1$, $\left[\frac{n-p+1}{p^i}\right] = 0$, $i = 2,\ 3, \cdots$. So we have

$$\begin{aligned}\sum_{j=1}^{\infty}\left[\frac{(n-p+1)p}{p^j}\right] &= \sum_{j=1}^{\infty}\left[\frac{n-p+1}{p^{j-1}}\right] \\ &= n-p+1+\left[\frac{n-p+1}{p}\right]+\left[\frac{n-p+1}{p^2}\right]+\cdots \\ &= n-p+1+p-1 = n.\end{aligned}$$

From Theorem 1.7.2 of [6] we know that if $p^2-2 < n \le p^2$, then $p^n \| ((n-p+1)p)!$. That is, $S_p(n) = (n-p+1)p$. This completes the proof of Lemma 2.

§3. Proof of Theorem

In this section, we shall use above Lemmas to complete the proof of our theorem.

Since $1 \le n \le p$ and $1 < k < p$, therefore we deduce $1 < kn < p^2$. We can divide $1 < kn < p^2$ into three interval $1 < kn < p$, $(m-1)p+m-2 < kn \le mp+m-1$ $(m = 2,\ 3, \cdots, p-1)$ and $p^2-2 < kn \le p^2$. Here, we discuss above three interval of kn respectively:

i) If $1 < kn < p$, from the case (1) of Lemma 1 we have

$$S_p(kn) = knp = kS_p(n).$$

ii) If $(m-1)p+m-2 < kn \le mp+m-1$ $(m = 2, 3, \cdots, p-1)$, then from the case (2) of Lemma 1 we have

$$S_p(kn) = (kn-m+1)p = knp-(m-1)p = kS_p(n)-(m-1)p.$$

In fact, note that if $(m-1)p+m-2 < kn < mp+m-1$ $(m = 2, 3, \cdots, p-1)$, then $m-1+\left[\frac{m-2}{p}\right] < \left[\frac{kn}{p}\right] < m+\left[\frac{m-1}{p}\right]$. Hence, $\left[\frac{kn}{p}\right] = m-1$. If $kn = mp+m-1$, then $\left[\dfrac{kn}{p}\right] = m$, but $p^{mp+m-1} \dagger ((mp+m-1)p-1)!$ and $p^{mp+m-1}|((mp+m-1)p)!$. So we immediately get

$$S_p(kn) = kS_p(n) - p\left[\frac{kn}{p}\right].$$

iii) If $p^2 - 2 < kn \leq p^2$, from Lemma 2 we have

$$S_p(kn) = (kn - p + 1)p = knp - (p-1)p.$$

Similarly, note that if $p^2 - 2 < kn \leq p^2$, then $p - \left[\dfrac{2}{p}\right] < \left[\dfrac{kn}{p}\right] \leq p$. That is, $\left[\dfrac{kn}{p}\right] = p - 1$. So we may immediately get

$$S_p(kn) = kS_p(n) - p\left[\frac{kn}{p}\right].$$

This completes the proof of our Theorem.

References

[1] F. Smarandache, Only problems, not solutions, Chicago, Xiquan Publishing House, 1993.

[2] Tom M. Apostol, Introduction to analytic number theory, New York, Springer-Verlag, 1976.

[3] Zhang Wenpeng and Liu Duansen, On the primitive numbers of power p and its asymptotic property, Smarandache Notions Journal, **13**(2002), pp. 173-175.

[4] Yi Yuan, On the primitive numbers of power p and its asymptotic property, Scientia Magna, **1**(2005), No. 1, pp. 175-177.

[5] Xu Zhefeng, Some arithmetical properties of primitive numbers of power p, Scientia Magna, **2**(2006), No. 1, pp. 9-12.

[6] Pan Chengdong and Pan Chengbiao, The elementary number theory, Beijing University Press, Beijing, 2003.

[7] Chen Guohui, Some exact calculating formulas for the Smarandache function, Scientia Magna, **2**(2006), No. 3, pp. 95-97.

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On the integer part of the M-th root and the largest M-th power not exceeding N

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Abstract The main purpose of this paper is using the elementary methods to study the properties of the integer part of the m-th root and the largest m-th power not exceeding n, and give some interesting identities involving these numbers.

Keywords The integer part of the m-th root, the largest m-th power not exceeding n, Dirichlet series, identities.

§1. Introduction and Results

Let m be a fixed positive integer. For any positive integer n, we define the arithmetical function $a_m(n)$ as the integer part of the m-th root of n. That is, $a_m(n) = [n^{\frac{1}{m}}]$, where $[x]$ denotes the greatest integer not exceeding to x. For example, $a_2(1) = 1$, $a_2(2) = 1$, $a_2(3) = 1$, $a_2(4) = 2$, $a_2(5) = 2$, $a_2(6) = 2$, $a_2(7) = 2$, $a_2(8) = 2$, $a_2(9) = 3$, $a_2(10) = 3$, $\cdots$. In [1], Professor F. Smarandache asked us to study the properties of the sequences $\{a_k(n)\}$. About this problem, Z. H. Li [2] studied its mean value properties, and given an interesting asymptotic formula:

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}_k}} a_m(n) = \frac{1}{\zeta(k)}\frac{m}{m+1}x^{\frac{m+1}{m}} + O(x),$$

where $\mathcal{A}_k$ denotes the set of all k-th power free numbers, $\zeta(k)$ is the Riemann zeta-function. X. L. He and J. B. Guo [3] also studied the mean value properties of $\sum\limits_{n\le x} a(n)$, and proved that

$$\sum_{n\le x} a(n) = \sum_{n\le x}[x^{\frac{1}{k}}] = \frac{k}{k+1}x^{\frac{k+1}{k}} + O(x).$$

Let n be a positive integer. It is clear that there exists one and only one integer k such that

$$k^m \le n < (k+1)^m.$$

Now we define $b_m(n) = k^m$. That is, $b_m(n)$ is the largest m-th power not exceeding n. If $m = 2$, then $b_2(1) = 1$, $b_2(2) = 1$, $b_2(3) = 1$, $b_2(4) = 4$, $b_2(5) = 4$, $b_2(6) = 4$, $b_2(7) = 4$, $b_2(8) = 4$, $b_2(9) = 9$, $b_2(10) = 9$, $\cdots$. In problem 40 and 41 of [1], Professor F. Smarandache asked us to study the properties of the sequences $\{b_2(n)\}$ and $\{b_3(n)\}$. For these problems, some people had studied them, and obtained many results. For example, W. P. Zhang [4] gave an useful asymptotic formula:

$$\sum_{n\le x} d(u(n)) = \frac{2}{9\pi^4} Ax\ln^3 x + Bx\ln^2 x + Cx\ln x + Dx + O\left(x^{\frac{5}{6}+\varepsilon}\right),$$

where $u(n)$ denotes the largest cube part not exceeding n, $A = \prod_p (1 - \frac{1}{(p+1)^2})$, B, C and D are constants, ε denotes any fixed positive number.

And in [5], J. F. Zheng made further studies for $\sum_{n\le x} d(b_m(n))$, and proved that

$$\sum_{n\le x} d(b_m(n)) = \frac{1}{kk!}\left(\frac{6}{k\pi^2}\right)^{k-1} A_0 x\ln^k x + A_1 x\ln^{k-1} x + \cdots + A_{k-1} x\ln x + A_k x + O\left(x^{1-\frac{1}{2k}+\varepsilon}\right),$$

where A_0, A_1,$\cdots$ A_k are constants, especially when k equals to 2, $A_0 = 1$.

In this paper, we using the elementary methods to study the convergent properties of two Dirichlet serieses involving $a_m(n)$ and $b_m(n)$, and give some interesting identities. That is, we shall prove the following conclusions:

Theorem 1. Let m be a fixed positive integer. Then for any real number $s > 1$, the Dirichlet series $f(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{a_m^s(n)}$ is convergent and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{a_m^s(n)} = \left(\frac{1}{2^{s-1}} - 1\right)\zeta(s),$$

where $\zeta(s)$ is the Riemann zeta-function.

Theorem 2. Let m be a fixed positive integer. Then for any real number $s > \frac{1}{m}$, the Dirichlet series $g_m(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{b_m^s(n)}$ is convergent and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{b_m^s(n)} = \left(\frac{1}{2^{ms-1}} - 1\right)\zeta(ms).$$

From our Theorems, we may immediately deduce the following:

Corollary 1. Taking $s = 2$ or $s = 3$ in Theorem 1, then we have the identities

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{a_m^2(n)} = -\frac{\pi^2}{12} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{a_m^3(n)} = -\frac{3}{4}\zeta(3).$$

Corollary 2. Taking $m = 2$ and $s = 2$ or $m = 2$ and $s = 3$ in Theorem 2, then we have the identities

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{b_2^2(n)} = -\frac{7}{720}\pi^4 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{b_2^3(n)} = -\frac{31}{30240}\pi^6.$$

Corollary 3. Taking $m = 3$ and $s = 2$ or $m = 3$ and $s = 3$ in Theorem 2, then we have the identities

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{b_3^2(n)} = -\frac{31}{30240}\pi^6 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{b_3^3(n)} = -\frac{255}{256}\zeta(9).$$

Corollary 4. For any positive integer s and $m \geq 2$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{b_m^s(n)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{b_s^m(n)}.$$

§2. Proof of the theorems

In this section, we shall complete the proof of our Theorems. For any positive integer n, let $a_m(n) = k$. It is clear that there are exactly $(k+1)^m - k^m$ integer n such that $a_m(n) = k$. So we may get

$$f(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{a_m^s(n)} = \sum_{k=1}^{\infty} \sum_{\substack{n=1 \\ a_m(n)=k}}^{\infty} \frac{(-1)^n}{k^s},$$

where if k be an odd number, then $\displaystyle\sum_{\substack{n=1 \\ a_m(n)=k}}^{\infty} \frac{(-1)^n}{k^s} = \frac{-1}{k^s}$. And if k be an even number, then $\displaystyle\sum_{\substack{n=1 \\ a_m(n)=k}}^{\infty} \frac{(-1)^n}{k^s} = \frac{1}{k^s}$. Combining the above two cases we have

$$\begin{aligned} f(s) &= \sum_{\substack{t=1 \\ k=2t}}^{\infty} \frac{1}{(2t)^s} + \sum_{\substack{t=1 \\ k=2t-1}}^{\infty} \frac{-1}{(2t-1)^s} \\ &= \sum_{t=1}^{\infty} \frac{1}{(2t)^s} - \left(\sum_{t=1}^{\infty} \frac{1}{t^s} - \sum_{t=1}^{\infty} \frac{1}{(2t)^s} \right) \\ &= \sum_{t=1}^{\infty} \frac{2}{2^s t^s} - \sum_{t=1}^{\infty} \frac{1}{t^s}. \end{aligned}$$

From the integral criterion, we know that $f(s)$ is convergent if $s > 1$. If $s > 1$, then $\zeta(s) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$, so we have

$$f(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{a_m^s(n)} = \left(\frac{1}{2^{s-1}} - 1 \right) \zeta(s).$$

This completes the proof of Theorem 1.

Using the same method of proving Theorem 1 we have

$$\begin{aligned} g_m(s) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{b_m^s(n)} \\ &= \sum_{k=1}^{\infty} \sum_{\substack{n=1 \\ b_m(n)=k^m}}^{\infty} \frac{(-1)^n}{k^{ms}} \\ &= \sum_{\substack{t=1 \\ k=2t}}^{\infty} \frac{1}{(2t)^{ms}} + \sum_{\substack{t=1 \\ k=2t-1}}^{\infty} \frac{-1}{(2t-1)^{ms}} \\ &= \sum_{t=1}^{\infty} \frac{2}{2^{ms} t^{ms}} - \sum_{t=1}^{\infty} \frac{1}{t^{ms}}. \end{aligned}$$

From the integral criterion, we know that $g_m(s)$ is also convergent if $s > \frac{1}{m}$. If $s > 1$, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, so we may easily deduce

$$g_m(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{b_m^s(n)} = \left(\frac{1}{2^{ms-1}} - 1 \right) \zeta(ms).$$

This completes the proof of Theorem 2.

From our two Theorems, and note that $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{\pi^6}{945}$ (see [6]), we may immediately deduce Corollary 1, 2, and 3. Then, Corollary 4 can also be obtained from Theorem 2.

References

[1] F. Smarandache, Only problems, not solutions, Xiquan Publ. House, Chicago, 1993.

[2] Li Zhanhu, On the intrger part of the m-th root and the k-th power free number, Research on Smarandache problems in number theory (Vol.II), Hexis, 2005, 41-43.

[3] He Xiaolin and Guo Jinbao, On the 80th problem of the F.Smarandache (I), Smarandache Notions Journal, **14**(2004), 70-73.

[4] Zhang Wenpeng, On the cube part sequence of a positive integer, Journal of Xian Yang Teacher's College, **18**(2003), 5-7.

[5] Zheng Jianfeng, On the inferior and superior k-th power part of a positive integer and divisor function, Smarandache Notions Journal, **14**(2004), 88-91.

[6] T. M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.

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On the ten's complement factorial Smarandache function

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Sequence A110396 by Amarnath Murthy in the on-line encyclopedia of integer sequences [1] is defined as "the 10's complement factorial of n." Let $t(n)$ denote the difference between n and the next power of 10. This is the ten's complement of a number. E.g., $t(27) = 73$, because $100 - 27 = 73$. Hence the $10's$ complement factorial simply becomes

$$\begin{aligned} tcf(n) \quad = \quad & (10's \text{ complement } of\ n) * (10's \text{ complement } of\ n-1) \cdots \\ & (10's \text{ complement } of\ 2) * (10's \text{ complement } of\ 1). \end{aligned}$$

How would the Smarandache function behave if this variation of the factorial function were used in place of the standard factorial function? The Smarandache function $S(n)$ is defined as the smallest integer m such that n evenly divides m factorial. Let $TS(n)$ be the smallest integer m such that n divides the ($10's$ complement factorial of m.)

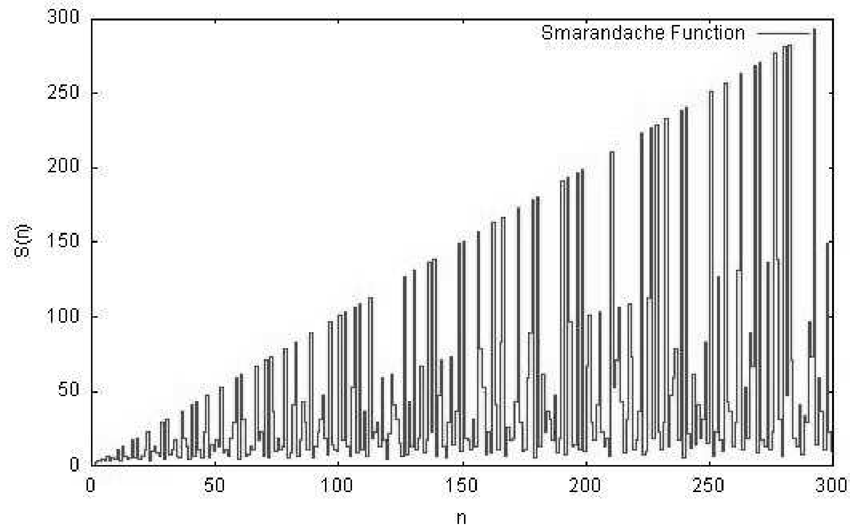
This new $TS(n)$ function produces the following sequence (which is A109631 in the OEIS [2]).

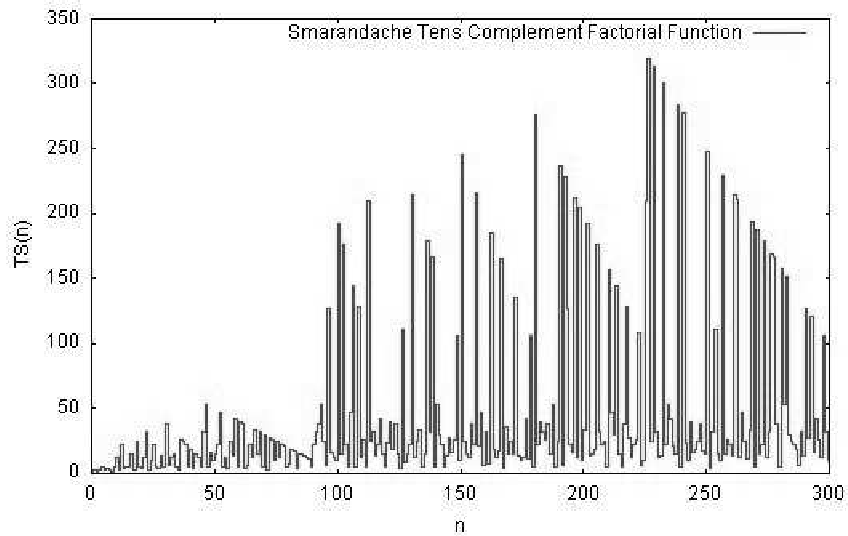
$n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 \cdots,$

$TS(n) = 1, 2, 1, 2, 5, 2, 3, 2, 1, 5, 12, 2, 22, 3, 5, 4, 15, 2, 24, 5 \cdots.$

For example, $TS(7) = 3$, because 7 divides $(10 - 3) * (10 - 2) * (10 - 1)$; and 7 does not divide (10's complement factorial of m) for $m < 3$.

Not surprisingly, the $TS(n)$ function differs significantly from the standard Smarandache function. Here are graphs displaying the behavior of each for the first 300 terms:





Four Problems Concerning the New $TS(n)$ Function

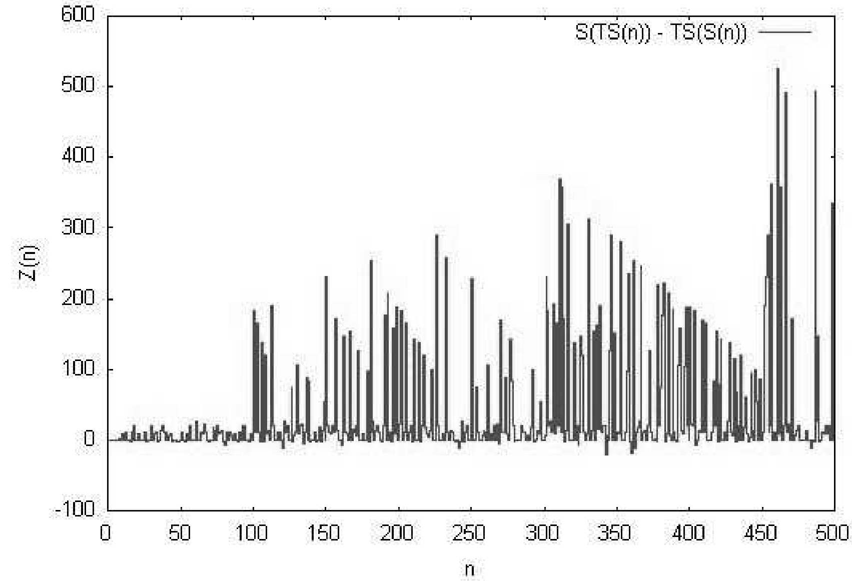
1. The Smarandache function and the ten's complement factorial Smarandache function have many values in common. Here are the initial solutions to $S(n) = TS(n)$:

1, 2, 5, 10, 15, 20, 25, 30, 40, 50, 60, 75, 100, 120, 125, 128, 150, 175, 200, 225, 250, 256, 300, 350, 375, 384, 400, 450, 500, 512, 525, 600, 625, 640, 675, 700, 750, 768, $\cdots$.

Why are most of the solutions multiples of 5 or 10? Are there infinitely many solutions?

2. After a computer search for all values of n from 1 to 1000, the only solution found for $TS(n) = TS(n + 1)$ is 374. We conjecture there is at least one more solution. But are there infinitely many?

3. Let $Z(n) = TS(S(n)) - S(TS(n))$. Is $Z(n)$ positive infinitely often? Negative infinitely often? The $Z(n)$ sequence seems highly chaotic with most of its values positive. Here is a graph of the first 500 terms:



4. The first four solutions to $TS(n)+TS(n+1) = TS(n+2)$ are 128, 186, 954, and 1462. Are there infinitely many solutions?

References

[1] Sequence A110396, The on-line encyclopedia of integer sequences, published electronically at http://www.research.att.com/ njas/sequences/ Editor: N. J. A. Sloane.

[2] Sequence A109631, The on-line encyclopedia of integer sequences, published electronically at http://www.research.att.com/ njas/sequences/ Editor: N. J. A. Sloane.

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The hybrid mean value of the Smarandache function and the Mangoldt function

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Abstract For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n|m!$. The main purpose of this paper is using the elementary methods to study the hybrid mean value of the Smarandache function $S(n)$ and the Mangoldt function $\Lambda(n)$, and prove an interesting hybrid mean value formula for $S(n)\Lambda(n)$.
Keywords F. Smarandache function, Mangoldt function, hybrid mean value, asymptotic formula

§1. Introduction

For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n|m!$. That is, $S(n) = \min\{m : \ n|m!, \ m \in N\}$. From the definition of $S(n)$ one can easily deduce that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the factorization of n into prime powers, then $S(n) = \max_{1 \leq i \leq k} \{S(p_i^{\alpha_i})\}$. From this formula we can easily get $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $S(11) = 11$, $S(12) = 4$, $S(13) = 13$, $S(14) = 7$, $S(15) = 5$, $S(16) = 6$, $\cdots$. About the elementary properties of $S(n)$, many people had studied it, and obtained some important results. For example, Wang Yongxing [2] studied the mean value properties of $S(n)$, and obtained that:

$$\sum_{n \leq x} S(n) = \frac{\pi^2}{12} \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Lu Yaming [3] studied the positive integer solutions of an equation involving the function $S(n)$, and proved that for any positive integer $k \geq 2$, the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinity positive integer solutions $(m_1, m_2, \cdots, m_k)$.

Jozsef Sandor [4] obtained some inequalities involving the F.Smarandache function. That is, he proved that for any positive integer $k \geq 2$, there exists infinite positive integer $(m_1, m_2, \cdots, m_k)$ such that the inequalities

$$S(m_1 + m_2 + \cdots + m_k) > S(m_1) + S(m_2) + \cdots + S(m_k).$$

$(m_1, m_2, \cdots, m_k)$ such that

$$S(m_1 + m_2 + \cdots + m_k) < S(m_1) + S(m_2) + \cdots + S(m_k).$$

On the other hand, Dr. Xu Zhefeng [5] proved: Let $P(n)$ denotes the largest prime divisor of n, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} (S(n) - P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right) x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ denotes the Riemann zeta-function.

The main purpose of this paper is using the elementary methods to study the hybrid mean value of the Smarandache function $S(n)$ and the Mangoldt function $\Lambda(n)$, which defined as follows:

$$\Lambda(n) = \begin{cases} \ln p, & \text{if } n = p^\alpha,\ p \text{ be a prime, } \alpha \text{ be any positive integer;} \\ 0, & \text{otherwise.} \end{cases}$$

and prove a sharper mean value formula for $\Lambda(n)S(n)$. That is, we shall prove the following conclusion:

Theorem. Let k be any fixed positive integer. Then for any real number $x > 1$, we have

$$\sum_{n\leq x} \Lambda(n)S(n) = x^2 \cdot \sum_{i=0}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 0,\ 1,\ 2,\ \cdots,\ k)$ are constants, and $c_0 = 1$.

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. In fact from the definition of $\Lambda(n)$ we have

$$\begin{aligned} \sum_{n\leq x} \Lambda(n)S(n) &= \sum_{\alpha\leq \frac{\ln x}{\ln 2}} \sum_{p\leq x^{\frac{1}{\alpha}}} \Lambda(p^\alpha)S(p^\alpha) = \sum_{\alpha\leq \frac{\ln x}{\ln 2}} \sum_{p\leq x^{\frac{1}{\alpha}}} S(p^\alpha)\ln p \\ &= \sum_{p\leq x} p\cdot \ln p + \sum_{2\leq\alpha\leq \frac{\ln x}{\ln 2}} \sum_{p\leq x^{\frac{1}{\alpha}}} S(p^\alpha)\ln p. \end{aligned} \quad (1)$$

For any positive integer k, from the prime theorem we know that

$$\pi(x) = \sum_{p\leq x} 1 = x \cdot \sum_{i=1}^{k} \frac{a_i}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right), \quad (2)$$

where a_i ($i = 1,\ 2, \cdots,\ k$) are constants, and $a_1 = 1$.

From the Abel's identity (see [6] Theorem 4.2) and (2) we have

$$\begin{aligned}
&\sum_{p\le x} p\cdot \ln p = \pi(x)\cdot x\cdot \ln x - \int_2^x \pi(y)(\ln y+1)dy\\
= \;& x\ln x\cdot x\cdot\left(\sum_{i=1}^{k}\frac{a_i}{\ln^i x}+O\left(\frac{x}{\ln^{k+1}x}\right)\right)-\int_2^x\left(\sum_{i=1}^{k}\frac{a_i}{\ln^i y}+O\left(\frac{y}{\ln^{k+1}y}\right)\right)(\ln y+1)dy\\
= \;& x^2\cdot\sum_{i=0}^{k}\frac{c_i}{\ln^i x}+O\left(\frac{x}{\ln^{k+1}x}\right), \qquad (3)
\end{aligned}$$

where c_i ($i=0,\ 1,\ 2,\cdots,\ k$) are constants, and $c_0=1$.

On the other hand, applying the estimate

$$S(p^{\alpha})\ll \alpha\cdot\ln p,$$

we have

$$\sum_{2\le\alpha\le\frac{\ln x}{\ln 2}}\sum_{p\le x^{\frac{1}{\alpha}}} S(p^{\alpha})\ln p \ll \sum_{2\le\alpha\le\frac{\ln x}{\ln 2}}\sum_{p\le x^{\frac{1}{2}}}\alpha\cdot p\cdot\ln p\ll x\cdot\ln^2 x. \qquad (4)$$

Combining (1)-(4) we have

$$\sum_{n\le x}\Lambda(n)S(n)=x^2\cdot\sum_{i=0}^{k}\frac{c_i\cdot x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right),$$

where c_i ($i=0,\ 1,\ 2,\ \cdots,\ k$) are constants, and $c_0=1$.

This completes the proof of the theorem.

References

[1] Smarandache F., Only problems, not solutions, Chicago: Xiquan Publ. House, 1993.

[2] Wang Yongxing, On the Smarandache function, Research on Smarandache Problem In Number Theory (Edited by Zhang Wenpeng, Li Junzhuang and Liu Duansen), Hexis, Vol.II(2005), 103-106.

[3] Lu Yaming, On the solutions of an equation involving the Smarandache function, Scientia Magna, **2**(2006), No. 1, 76-79.

[4] Jozsef Sandor, On certain inequalities involving the Smarandache function, Scientia Magna, **2**(2006), No. 3, 78-80.

[5] Xu Zhefeng, On the value distribution of the Smarandache function, Acta Mathematica Sinica (Chinese Series), **49**(2006), No.5, 1009-1012.

[6] Tom M. Apostol, Introduction to analytic number theory, New York: Springer-Verlag, 1976.

[7] Pan Chengdong and Pan Chengbiao, Elementary number theory, Beijing: Beijing University Press, 1992.

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Palindrome Studies
(Part I)
The Palindrome Concept and Its Applications to Prime Numbers

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Abstract This article originates from a proposal by M. L. Perez of American Research Press to carry out a study on Smarandache generalized palindromes [1]. The prime numbers were chosen as a first set of numbers to apply the development of ideas and computer programs on. The study begins by exploring regular prime number palindromes. To continue the study it proved useful to introduce a new concept, that of extended palindromes with the property that the union of regular palindromes and extended palindromes form the set of Smarandache generalized palindromes. An interesting observation is proved in the article, namely that the only regular prime number palindrome with an even number of digits is 11.

Keywords Equation, solutions, Mersenne prime, perfect number

§1. Regular Palindromes

Definition. A positive integer is a palindrome if it reads the same way forwards and backwards.

Using concatenation we can write the definition of a regular palindrome A in the form

$$A = x_1x_2x_3\ldots x_n\ldots x_3x_2x_1 \text{ or } x_1x_2x_3\ldots x_nx_n\ldots x_3x_2x_1$$

where $x_k \in \{0, 1, 2, \ldots 9\}$ for $k = 1, 2, 3, \ldots n$, except $x_1 \neq 0$

Examples and Identification. The digits 1, 2, ⋯, 9 are trivially palindromes. The only 2-digit palindromes are 11, 22, 33, ⋯ 99. Of course, palindromes are easy to identify by visual inspection. We see at once that 5493945 is a palindrome. In this study we will also refer to this type of palindromes as regular palindromes since we will later define another type of palindromes.

As we have seen, palindromes are easily identified by visual inspection, something we will have difficulties to do with, say prime numbers. Nevertheless, we need an algorithm to identify palindromes because we can not use our visual inspection method on integers that occur in computer analysis of various sets of numbers. The following routine, written in Ubasic, is built into various computer programs in this study:

```
10   'Palindrome identifier, Henry Ibstedt, 031021
20   input "N";N
30   s=n\10:r=res
40   while s>0
50   s=s\10:r=10*r+res
60   wend
70   print n,r
80   end
```

This technique of reversing a number is quite different from what will be needed later on in this study. Although very simple and useful it is worth thinking about other methods depending on the nature of the set of numbers to be examined. Let's look at prime number palindromes.

§2. Prime Number Palindromes

We can immediately list the prime number palindromes which are less than 100, they are: 2, 3, 5, 7 and 11. We realize that the last digit of any prime number except 2 must be 1, 3, 7 or 9. A three digit prime number palindrome must therefore be of the types: $1x1$,$3x3$, $7x7$ or $9x9$ where $x0, 1, \ldots, 9$. Here, numbers have been expressed in concatenated form. When there is no risk of misunderstanding we will simply write $2x2$, otherwise concatenation will be expressed 2_x_2 while multiplication will be made explicit by $2 \cdot x \cdot 2$.

In explicit form we write the above types of palindromes: $101 + 10x$, $303 + 10x$, $707 + 10x$ and $909 + 10x$ respectively.

A 5-digit palindrome axyxa can be expressed in the form:

$a\_000\_a + x \cdot 1010 + y \cdot 100$ where $a \in \{1, 3, 7, 9\}$, $x \in \{0, 1, \ldots, 9\}$ and $y \in \{0, 1, \ldots, 9\}$

This looks like complicating things, but not so. Implementing this in a Ubasic program will enable us to look for which palindromes are primes instead of looking for which primes are palindromes. Here is the corresponding computer code (C5):

```
10    'Classical 5-digit Prime Palindromes (C5)
20    'October 2003, Henry Ibstedt
30    dim V(4), U(4)
40    for I=1 to 4 : read V(I): next
50    data 1,3,7,9
60    T=10001
70    for I=1 to 4
80    U=0:'Counting prime palindromes
90    A=V(I)*T
100    for J=0 to 9
```

```
110    B = A + 1010 ⋆ J
120    for K=0 to 9
130    C = B + 100 ⋆ K
140    if nxtprm(C-1)=C then print C : inc U
150    next : next
160    U(I)=U
170    next
180    for I=1 to 4 : print U(I): next
190    end
```

Before implementing this code the following theorem will be useful.

Theorem. A palindrome with an even number of digits is divisible by 11.

Proof. We consider a palindrome with $2n$ digits which we denote $x_1, x_2, \ldots, x_n$. Using concatenation we write the palindrome

$$A = x_1 x_2 \ldots x_n x_n \ldots x_2 x_1.$$

We express A in terms of $x_1, x_2, \ldots, x_n$ in the following way:

$$A = x_1(10^{2n-1} + 1) + x_2(10^{2n-2} + 10) + x_3(10^{2n-3} + 10^2) + \ldots + x_n(10^{2n-n} + 10^{n-1})$$

or

$$A = \sum_{k=1}^{n} x_k(10^{2n-k} + 10^{k-1}) \qquad (1)$$

We will now use the following observation:

$10^q - 1 \equiv 0(\mathrm{mod} 11)$ for $q \equiv 0(\mathrm{mod} 2)$

and

$10^q + 1 \equiv 0(\mathrm{mod} 11)$ for $q \equiv 1(\mathrm{mod} 2)$

We re-write (1) in the form:

$A = \sum_{k=1}^{n} x_k(10^{2n-k} \pm 1 + 10^{k-1} \mp 1)$ where the upper sign applies if $k \equiv 1(\mathrm{mod} 2)$ and the lower sign if $k \equiv 0(\mathrm{mod} 2)$.

From this we see that $A \equiv 0(\mathrm{mod} 11)$ for $n \equiv 0(\mathrm{mod} 2)$.

Corollary. From this theorem we learn that the only prime number palindrome with an even number of digits is 11.

This means that we only need to examine palindromes with an odd number of digits for primality. Changing a few lines in the computer code C5 we obtain computer codes (C3, C7 and C9) which will allow us to identify all prime number palindromes less than 10^{10} in less than 5 minutes. The number of prime number palindromes in each interval was registered in a file. The result is displayed in Table 1.

Table 1. Number of prime number palindromes

		Number of			
Number		Palindromes			
of		of type			
digits	1...1	3...3	7...7	9...9	Total
3	5	4	4	2	15
5	26	24	24	19	93
7	190	172	155	151	668
9	1424	1280	1243	1225	5172

Table 2. Three-digit prime number palindromes

(Total 15)

interval		Prime	Number	Palindromes	
100-199	101	131	151	181	191
300-399	313	353	373	383	
700-799	727	757	787	797	
900-999	919	929			

Table 3. Five-digit prime number palindromes

(Total 93)

10301	10501	10601	11311	11411	12421	12721	12821	13331
13831	13931	14341	14741	1545	1555	16061	16361	16561
16661	17471	17971	18181	18481	19391	19891	19991	
30103	30203	30403	30703	30803	31013	31513	32323	32423
33533	34543	34843	35053	35153	35353	35753	36263	36563
37273	37573	38083	38183	38783	39293			
70207	70507	70607	71317	71917	72227	72727	73037	73237
73637	74047	74747	75557	76367	76667	77377	77477	77977
78487	78787	78887	79397	79697	79997			
90709	91019	93139	93239	93739	94049	94349	94649	94849
94949	95959	96269	96469	96769	97379	97579	97879	98389
98689								

Table 4. Seven-digit prime number palindromes

(Total 668)

1003001	1008001	1022201	1028201	1035301	1043401	1055501	1062601
1065601	1074701	1082801	1085801	1092901	1093901	1114111	1117111
1120211	1123211	1126211	1129211	1134311	1145411	1150511	1153511
1160611	1163611	1175711	1177711	1178711	1180811	1183811	1186811
1190911	1193911	1196911	1201021	1208021	1212121	1215121	1218121
1221221	1235321	1242421	1243421	1245421	1250521	1253521	1257521
1262621	1268621	1273721	1276721	1278721	1280821	1281821	1286821
1287821	1300031	1303031	1311131	1317131	1327231	1328231	1333331
1335331	1338331	1343431	1360631	1362631	1363631	1371731	1374731
1390931	1407041	1409041	1411141	1412141	1422241	1437341	1444441
1447441	1452541	1456541	1461641	1463641	1464641	1469641	1486841
1489841	1490941	1496941	1508051	1513151	1520251	1532351	1535351
1542451	1548451	1550551	1551551	1556551	1557551	1565651	1572751
1579751	1580851	1583851	1589851	1594951	1597951	1598951	1600061
1609061	1611161	1616161	1628261	1630361	1633361	1640461	1643461
1646461	1654561	1657561	1658561	1660661	1670761	1684861	1685861
1688861	1695961	1703071	1707071	1712171	1714171	1730371	1734371
1737371	1748471	1755571	1761671	1764671	1777771	1793971	1802081
1805081	1820281	1823281	1824281	1826281	1829281	1831381	1832381
1842481	1851581	1853581	1856581	1865681	1876781	1878781	1879781
1880881	1881881	1883881	1884881	1895981	1903091	1908091	1909091
1917191	1924291	1930391	1936391	1941491	1951591	1952591	1957591
1958591	1963691	1968691	1969691	1970791	1976791	1981891	1982891
1984891	1987891	1988891	1993991	1995991	1998991		

3001003	3002003	3007003	3016103	3026203	3064603	3065603	3072703
3073703	3075703	3083803	3089803	3091903	3095903	3103013	3106013
3127213	3135313	3140413	3155513	3158513	3160613	3166613	3181813
3187813	3193913	3196913	3198913	3211123	3212123	3218123	3222223
3223223	3228223	3233323	3236323	3241423	3245423	3252523	3256523
3258523	3260623	3267623	3272723	3283823	3285823	3286823	3288823
3291923	3293923	3304033	3305033	3307033	3310133	3315133	3319133
3321233	3329233	3331333	3337333	3343433	3353533	3362633	3364633
3365633	3368633	3380833	3391933	3392933	3400043	3411143	3417143
3424243	3425243	3427243	3439343	3441443	3443443	3444443	3447443
3449443	3452543	3460643	3466643	3470743	3479743	3485843	3487843
3503053	3515153	3517153	3528253	3541453	3553553	3558553	3563653
3569653	3586853	3589853	3590953	3591953	3594953	3601063	3607063
3618163	3621263	3627263	3635363	3643463	3646463	3670763	3673763
3680863	3689863	3698963	3708073	3709073	3716173	3717173	3721273

3722273	3728273	3732373	3743473	3746473	3762673	3763673	3765673
3768673	3769673	3773773	3774773	3781873	3784873	3792973	3793973
3799973	3804083	3806083	3812183	3814183	3826283	3829283	3836383
3842483	3853583	3858583	3863683	3864683	3867683	3869683	3871783
3878783	3893983	3899983	3913193	3916193	3918193	3924293	3927293
3931393	3938393	3942493	3946493	3948493	3964693	3970793	3983893
3991993	3994993	3997993	3998993				
7014107	7035307	7036307	7041407	7046407	7057507	7065607	7069607
7073707	7079707	7082807	7084807	7087807	7093907	7096907	7100017
7114117	7115117	7118117	7129217	7134317	7136317	7141417	7145417
7155517	7156517	7158517	7159517	7177717	7190917	7194917	7215127
7226227	7246427	7249427	7250527	7256527	7257527	7261627	7267627
7276727	7278727	7291927	7300037	7302037	7310137	7314137	7324237
7327237	7347437	7352537	7354537	7362637	7365637	7381837	7388837
7392937	7401047	7403047	7409047	7415147	7434347	7436347	7439347
7452547	7461647	7466647	7472747	7475747	7485847	7486847	7489847
7493947	7507057	7508057	7518157	7519157	7521257	7527257	7540457
7562657	7564657	7576757	7586857	7592957	7594957	7600067	7611167
7619167	7622267	7630367	7632367	7644467	7654567	7662667	7665667
7666667	7668667	7669667	7674767	7681867	7690967	7693967	7696967
7715177	7718177	7722277	7729277	7733377	7742477	7747477	7750577
7758577	7764677	7772777	7774777	7778777	7782877	7783877	7791977
7794977	7807087	7819187	7820287	7821287	7831387	7832387	7838387
7843487	7850587	7856587	7865687	7867687	7868687	7873787	7884887
7891987	7897987	7913197	7916197	7930397	7933397	7935397	7938397
7941497	7943497	7949497	7957597	7958597	7960697	7977797	7984897
7985897	7987897	7996997					
9002009	9015109	9024209	9037309	9042409	9043409	9045409	9046409
9049409	9067609	9073709	9076709	9078709	9091909	9095909	9103019
9109019	9110119	9127219	9128219	9136319	9149419	9169619	9173719
9174719	9179719	9185819	9196919	9199919	9200029	9209029	9212129
9217129	9222229	9223229	9230329	9231329	9255529	9269629	9271729
9277729	9280829	9286829	9289829	9318139	9320239	9324239	9329239
9332339	9338339	9351539	9357539	9375739	9384839	9397939	9400049
9414149	9419149	9433349	9439349	9440449	9446449	9451549	9470749
9477749	9492949	9493949	9495949	9504059	9514159	9526259	9529259
9547459	9556559	9558559	9561659	9577759	9583859	9585859	9586859
9601069	9602069	9604069	9610169	9620269	9624269	9626269	9632369

9634369	9645469	9650569	9657569	9670769	9686869	9700079	9709079
9711179	9714179	9724279	9727279	9732379	9733379	9743479	9749479
9752579	9754579	9758579	9762679	9770779	9776779	9779779	9781879
9782879	9787879	9788879	9795979	9801089	9807089	9809089	9817189
9818189	9820289	9822289	9836389	9837389	9845489	9852589	9871789
9888889	9889889	9896989	9902099	9907099	9908099	9916199	9918199
9919199	9921299	9923299	9926299	9927299	9931399	9932399	9935399
9938399	9957599	9965699	9978799	9980899	9981899	9989899	

Of the 5172 nine-digit prime number palindromes only a few in the beginning and at the end of each type are shown in table 5.

Table 5a. Nine-digit prime palindromes of type 1_1

(Total 1424)

100030001	100050001	100060001	100111001	100131001	100161001
100404001	100656001	100707001	100767001	100888001	100999001
101030101	101060101	101141101	101171101	101282101	101292101
101343101	101373101	101414101	101424101	101474101	101595101
101616101	101717101	101777101	101838101	101898101	101919101
101949101	101999101	102040201	102070201	102202201	102232201
102272201	102343201	102383201	102454201	102484201	102515201
102676201	102686201	102707201	102808201	102838201	103000301
103060301	103161301	103212301	103282301	103303301	103323301
103333301	103363301	103464301	103515301	103575301	103696301
195878591	195949591	195979591	196000691	196090691	196323691
196333691	196363691	196696691	196797691	196828691	196878691
197030791	197060791	197070791	197090791	197111791	197121791
197202791	197292791	197343791	197454791	197525791	197606791
197616791	197868791	197898791	197919791	198040891	198070891
198080891	198131891	198292891	198343891	198353891	198383891
198454891	198565891	198656891	198707891	198787891	198878891
198919891	199030991	199080991	199141991	199171991	199212991
199242991	199323991	199353991	199363991	199393991	199494991
199515991	199545991	199656991	199767991	199909991	199999991

Table 5b. Nine-digit prime palindromes of type 3_3

(Total 1280)

300020003	300080003	300101003	300151003	300181003	300262003
300313003	300565003	300656003	300808003	300818003	300848003
300868003	300929003	300959003	301050103	301111103	301282103
301434103	301494103	301555103	301626103	301686103	301818103
301969103	302030203	302070203	302202203	302303203	302313203
302333203	302343203	302444203	302454203	302525203	302535203
302555203	302646203	302676203	302858203	302898203	302909203
303050303	303121303	303161303	303272303	303292303	303373303
303565303	303616303	303646303	303757303	303878303	303929303
303979303	304050403	304090403	304131403	304171403	304191403
394191493	394212493	394333493	394494493	394636493	394696493
394767493	395202593	395303593	395363593	395565593	395616593
395717593	395727593	395868593	395898593	396070693	396191693
396202693	396343693	396454693	396505693	396757693	396808693
396919693	396929693	397141793	397242793	397333793	397555793
397666793	397909793	398040893	398111893	398151893	398232893
398252893	398363893	398414893	398474893	398616893	398666893
398676893	398757893	398838893	398898893	399070993	399191993
399262993	399323993	399464993	399484993	399575993	399595993
399616993	399686993	399707993	399737993	399767993	399878993

Table 5c. Nine-digit prime palindromes of type 7_7

(Total 1243)

700020007	700060007	700090007	700353007	700363007	700404007
700444007	700585007	700656007	700666007	700717007	700737007
700848007	700858007	700878007	700989007	701000107	701141107
701151107	701222107	701282107	701343107	701373107	701393107
701424107	701525107	701595107	701606107	701636107	701727107
701747107	701838107	701919107	701979107	701999107	702010207
702070207	702080207	702242207	702343207	702434207	702515207
702575207	702626207	702646207	702676207	702737207	702767207
702838207	702919207	702929207	702989207	703000307	703060307
703111307	703171307	703222307	703252307	703393307	703444307

795848597	795878597	796060697	796080697	796222697	796252697
796353697	796363697	796474697	796494697	796515697	796636697
796666697	796707697	796717697	796747697	796848697	796939697
797262797	797363797	797393797	797444797	797525797	797595797
797676797	797828797	797898797	797939797	797949797	798040897
798181897	798191897	798212897	798292897	798373897	798383897
798454897	798535897	798545897	798646897	798676897	798737897
798797897	798818897	798838897	798919897	798989897	799050997
799111997	799131997	799323997	799363997	799383997	799555997
799636997	799686997	799878997	799888997	799939997	799959997

Table 5d. Nine-digit prime palindromes of type 9_9
(Total 1225)

900010009	900050009	900383009	900434009	900484009	900505009
900515009	900565009	900757009	900808009	900838009	900878009
900919009	900929009	901060109	901131109	901242109	901252109
901272109	901353109	901494109	901585109	901606109	901626109
901656109	901686109	901696109	901797109	901929109	901969109
902151209	902181209	902232209	902444209	902525209	902585209
902757209	902828209	902888209	903020309	903131309	903181309
903292309	903373309	903383309	903424309	903565309	903616309
903646309	903727309	903767309	903787309	903797309	903878309
903979309	904080409	904090409	904101409	904393409	904414409
994969499	995070599	995090599	995111599	995181599	995303599
995343599	995414599	995555599	995696599	995757599	995777599
996020699	996101699	996121699	996181699	996242699	996464699
996494699	996565699	996626699	996656699	996686699	996808699
996818699	996878699	996929699	996949699	996989699	997030799
997111799	997393799	997464799	997474799	997555799	997737799
997818799	997909799	997969799	998111899	998121899	998171899
998202899	998282899	998333899	998565899	998666899	998757899
998898899	998939899	998979899	999070999	999212999	999272999
999434999	999454999	999565999	999676999	999686999	999727999

An idea about the strange distribution of prime number palindromes is given in diagram 1. In fact the prime number palindromes are spread even thinner than the diagram makes believe because the horizontal scale is in interval numbers not in decimal numbers, i.e. (100-200) is given the same length as $(1.1 \cdot 10^9 - 1.2 \cdot 10^9)$.

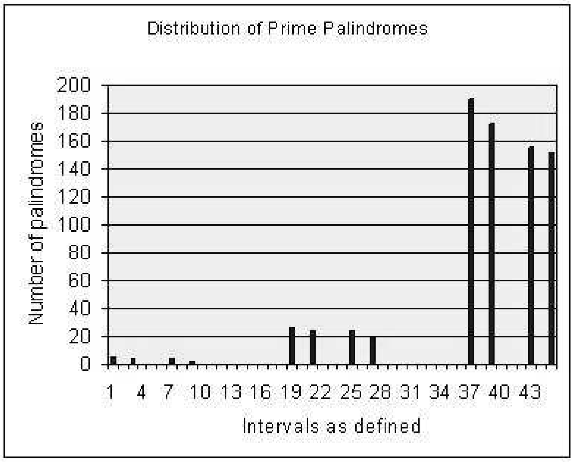


Diagram 1.

Intervals 1-9: 3-digit numbers divided into 9 equal intervals.
Intervals 11-18: 4-digit numbers divided into 9 equal intervals
Intervals 19-27: 5-digit numbers divided into 9 equal intervals
Intervals 28-36: 6-digit numbers divided into 9 equal intervals
Intervals 37-45: 7-digit numbers divided into 9 equal intervals

§3. Smarandache Generalized Palindromes

Definition. A Smarandache Generalized Palindrome **(SGP)** is any integer of the form

$$x_1x_2x_3 \ldots x_n \ldots x_3x_2x_1 \text{ or } x_1x_2x_3 \ldots x_nx_n \ldots x_3x_2x_1,$$

where $x_1, x_2, x_3, \ldots, x_n$ are natural numbers. In the first case we require $n > 1$ since otherwise every number would be a **SGP.**

Briefly speaking $x_k \in 0, 1, 2, \ldots, 9$ has been replaced by $x_k \in N$ (where N is the set of natural numbers).

Addition. To avoid that the same number is described as a **SGP** in more than one way this study will require the x_k to be maximum as a first priority and n to be maximum as a second priority (cf. examples below).

Interpretations and examples. Any regular palindrome **(RP)** is a Smarandache Generalized Palindrome **(SGP)**, i.e. **RP**$\subset$ **SGP**.

3 is a **RP** and also a **SGP**

123789 is neither **RP** nor **SGP**

123321 is **RP** as well as **SGP**

123231 is not a **RP** but it is a **SGP** 1_23_23_1 The **SGP** 334733 can be written in three ways: 3_3_47_3_3, 3_3473_3 and 33_47_33. Preference will be given to 33_47_33, (in compliance with the addition to the definition).
780978 is a **SGP** 78_09_78, i.e. we will permit natural numbers with leading zeros when they occur inside a **GSP**.

How do we identify a **GSP** generated by some sort of a computer application where we can not do it by visual inspection? We could design and implement an algorithm to identify **GSP**s directly. But it would of course be an advantage if methods applied in the early part of this study to identify the **RP**s could be applied first followed by a method to identify the **GSP**s which are not **RP**s. Even better we could set this up in such a way that we leave the **RP**s out completely. This leads to us to define in an operational way those **GSP**s which are not **RP**s, let us call them Extended Palindromes **(EP)**. The set of **EP**s must fill the condition

$$\{\mathbf{RP}\} \cup \{\mathbf{EP}\} = \{\mathbf{GSP}\}$$

§4. Extended Palindromes

Definition. An Extended Palindrome **(EP)** is any integer of the form

$$x_1x_2x_3\ldots x_n\ldots x_3x_2x_1 \text{ or } x_1x_2x_3\ldots x_nx_n\ldots x_3x_2x_1,$$

where $x_1, x_2, x_3, \ldots, x_n$ are natural numbers of which at least one is greater than or equal to 10 or has one or more leading zeros. x_1 is not allowed to not have leading zeros. Again x_k should be maximum as a first priority and n maximum as a second priority.

Computer Identification of EPs.

The number **A** to be examined is converted to a string **S** of length **L** (leading blanks are removed first). The symbols composing the string are compared by creating substrings from left L_1 and right R_1. If L_1 and R_1 are found so that $L_1 = R_1$ then **A** is confirmed to be an **EP**. However, the process must be continued to obtain a complete split of the string into substrings as illustrated in diagram 2.

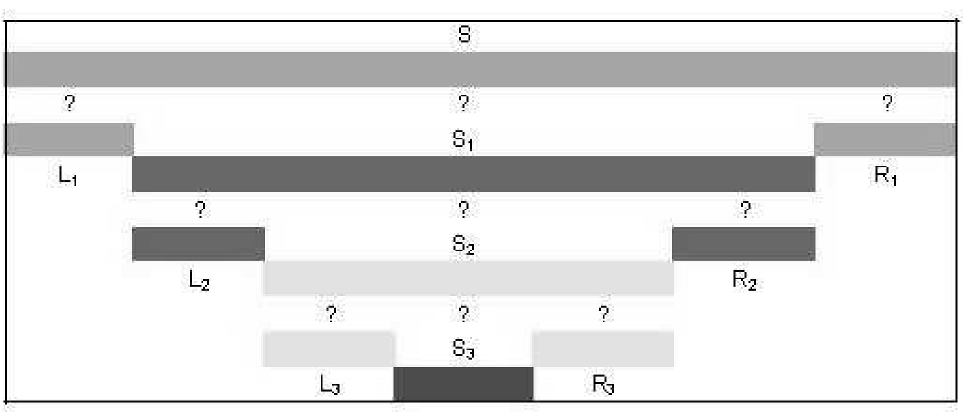


Diagram 2.

Diagram 2 illustrates the identification of extended palindromes up to a maximum of 4 elements. This is sufficient for our purposes since a 4 element extended palindrome must have a minimum of 8 digits. A program for identifying extended palindromes corresponding to diagram 2 is given below. Since we have $L_k = R_k$ we will use the notation Z_k for these in the program. The program will operate on strings and the deconcatenation into extended palindrome elements will be presented as strings, otherwise there would be no distinction between 690269 and 692269 which would both be presented as 69_2 (only distinct elements will be recorded) instead of 69_02 and 69_2 respectively.

Comments on the program

It is assumed that the programming in basic is well known. Therefore only the main structure and the flow of data will be commented on:

Lines 20 - 80: Feeding the set of numbers to be examined into the program. In the actual program this is a sequence of prime numbers in the interval $a_1 < a < a_2$.

Lines 90 - 270: On line 130 **A** is sent off to a subroutine which will exclude **A** if it happens to be a regular palindrome. The routine will search sub-strings from left and right. If no equal substrings are found it will return to the feeding loop otherwise it will print **A** and the first element Z_1 while the middle string S_1 will be sent of to the next routine (lines 280 - 400). The flow of data is controlled by the status of the variable u and the length of the middle string.

Lines 280 - 400: This is more or less a copy of the above routine. S_1 will be analyzed in the same way as **S** in the previous routine. If no equal substrings are found it will print S_1 otherwise it will print Z_2 and send S_2 to the next routine (lines 410 - 520).

Lines 410 - 520: This routine is similar to the previous one except that it is equipped to terminate the analysis. It is seen that routines can be added one after the other to handle extended palindromes with as many elements as we like. The output from this routine consists in writing the terminal elements, i.e. S_2 if **A** is a 3-element extended palindrome and Z_3 and S_3 if **A** is a 4-element extended palindrome.

Lines 530 - 560: Regular palindrome identifier described earlier.

```
10    'EPPRSTR, 031028
20    input "Search interval a1 to a2:"; A1, A2
30    A=A1
40    while A<A2
50    A=nxtprm(A)
60    gosub 90
70    wend
80    end
90    S=str(A)
100     M=len(S)
110     if M=2 then goto 270
120     S=right(S,M-1)
130     U=0:gosub 530
140     if U=1 then goto 270
150     I1=int((M-1)/2)
```

```
160 U=0
170 for I=1 to I1
180 if left(S,I)=right(S,I) then
190 :Z1=left(S,I)
200 : M1 = M - 1 - 2 * I : S1 = mid(S, I + 1, M1)
210 :U=1
220 endif
230 next
240 if U=0 then goto 270
250 print A;" ";Z1;
260 if M1 > 0 then gosub 280
270 return
280 I2=int(M1/2)
290 U=0
300 for J=1 to I2
310 if left(S1,J)=right(S1,J) then
320 :Z2=left(S1,J)
330 : M2 = M1 - 2 * J : S2 = mid(S1, J + 1, M2)
340 :U=1
350 endif
360 next
370 if U=0 then print " ";S1:goto 400
380 print " ";Z2;
390 if M2 > 0 then gosub 410 else print
400 return
410 I3=int(M2/2)
420 U=0
430 for K=1 to I3
440 if left(S2,K)=right(S2,K) then
450 :Z3=left(S2,K)
460 : M3 = M2 - 2 * K : S3 = mid(S2, K + 1, M3)
470 :U=1
480 endif
490 next
500 if U=0 then print " ";S2:goto 520
510 print " ";Z3;" ";S3
520 return
530 T=""
540 for I=M to 1 step -1:T=T+mid(S,I,1):next
550 if T=S then U=1:'print "a=";a;"is a RP"
560 return
```

§5.Extended Prime Number Palindromes

The computer program for identification of extended palindromes has been implemented to find extended prime number palindromes. The result is shown in tables 7 to 9 for prime numbers $< 10^7$. In these tables the first column identifies the interval in the following way: 1 - 2 in the column headed$x$ 10 means the interval $1 \cdot 10$ to $2 \cdot 10$. **EP** stands for the number of extended prime number palindromes, **RP** is the number regular prime number palindromes and **P** is the number of prime numbers. As we have already concluded the first extended prime palindromes occur for 4-digit numbers and we see that primes which begin and end with one of the digits 1, 3, 7 or 9 are favored. In table 8 the pattern of behavior becomes more explicit. Primes with an even number of digits are not regular palindromes while extended prime palindromes occur for even as well as odd digit primes. It is easy to estimate from the tables that about 25% of the primes of types $1 \ldots 1, 3 \ldots 3, 7 \ldots 7$ and $9 \ldots 9$ are extended prime palindromes. There are 5761451 primes less than 10^8, of these 698882 are extended palindromes and only 604 are regular palindromes.

Table 7. Extended and regular palindromes

Intervals 10-100, 100-1000 and 1000-10000

$\times 10$	EP	RP	P	$\times 10^2$	EP	RP	P	$\times 10^3$	EP	RP	P
1-2	0	1	4	1-2	0	5	21	1-2	33		135
2-3	0		2	2-3	0		16	2-3	0		127
3-4	0		2	3-4	0	4	16	3-4	28		120
4 - 5	0		3	4 - 5	0		17	4 - 5	0		119
5 - 6	0		2	5 - 6	0		14	5 - 6	0		114
6 - 7	0		2	6 - 7	0		16	6 - 7	0		117
7 - 8	0		3	7 - 8	0	4	14	7 - 8	30		107
8 - 9	0		2	8 - 9	0		15	8 - 9	0		110
9 - 10	0		1	9 - 10	0	2	14	9 - 10	27		112

Table 8. Extended and regular palindromes

Intervals $10^4 - 10^5$ and $10^5 - 10^6$

$\times 10^4$	EP	RP	P	$\times 10^5$	EP	RP	P
1 - 2	242	26	1033	1 - 2	2116		8392
2 - 3	12		983	2 - 3	64		8013
3 - 4	230	24	958	3 - 4	2007		7863
4 - 5	9		930	4 - 5	70		7678

5 - 6	10		924	5 - 6	70		7560
6 - 7	9		878	6 - 7	69		7445
7 - 8	216	24	902	7 - 8	1876		7408
8 - 9	10		876	8 - 9	63		7323
9 - 10	203	19	879	9 - 10	1828		7224

Table 9. Extended and regular palindromes

Intervals $10^5 - 10^6$ and $10^6 - 10^7$

$\times 10^6$	EP	RP	P	$\times 10^7$	EP	RP	P
1 - 2	17968	190	70435	1 - 2	156409		606028
2 - 3	739		67883	2 - 3	6416		587252
3 - 4	16943	172	66330	3 - 4	148660	575795	
4 - 5	687		65367	4 - 5	6253		567480
5 - 6	725		64336	5 - 6	6196		560981
6 - 7	688		63799	6 - 7	6099		555949
7 - 8	16133	155	63129	7 - 8	142521		551318
8 - 9	694		62712	8 - 9	6057		547572
9 - 10	15855	151	62090	9 - 10	140617		544501

We recall that the sets of regular palindromes and extended palindromes together form the set of Smarandache Generalized Palindromes. Diagram 3 illustrates this for 5-digit primes.

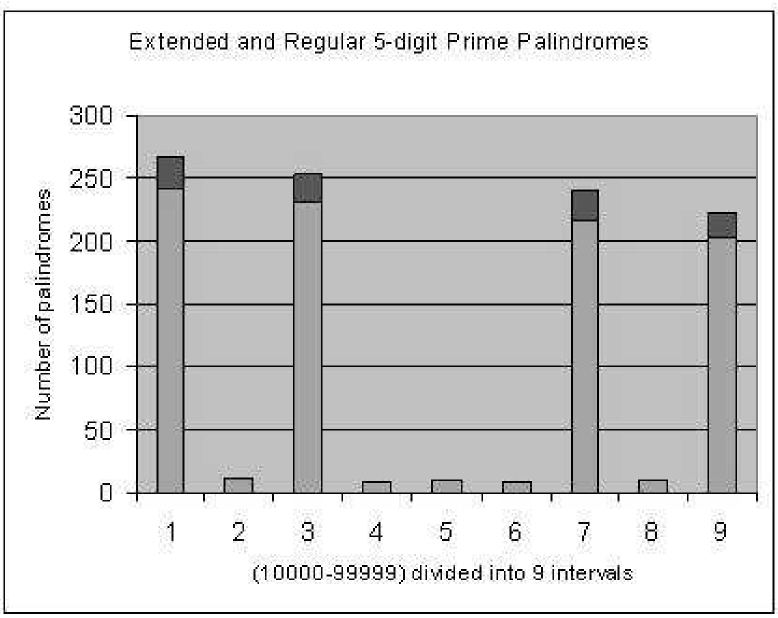
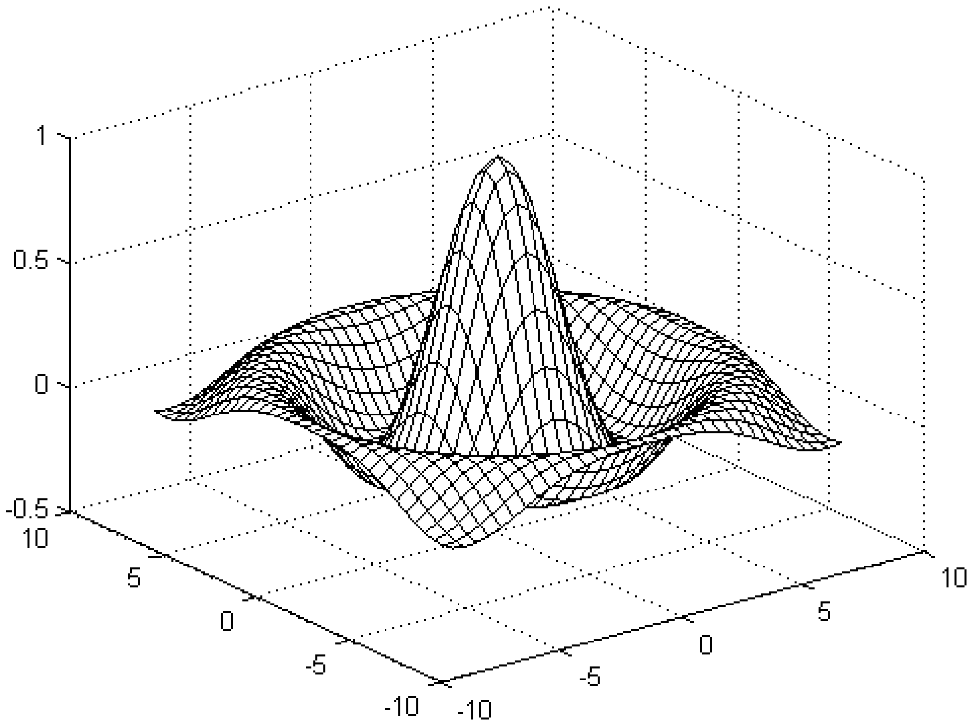


Diagram 3. Extended palindromes shown with blue color, regular with red.

Part II of this study is planned to deal with palindrome analysis of other number sequences.

References

[1] F. Smarandache, Generalized Palindromes, Arizona State University Special Collections, Tempe.



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