Revisiting the Kronecker Array Transform

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***Abstract*—It is known that the calculation of a matrix-vector product can be accelerated if this matrix can be recast (or approximated) by the Kronecker product of two smaller matrices. In array signal processing the manifold matrix can be described as the Kronecker product of two other matrices if the sensor array displays a separable geometry. This forms the basis of the Kronecker Array Transform (KAT), which was previously introduced to speed up calculations of acoustic images with microphone arrays. If, however, the array has a quasi-separable geometry, e.g. an otherwise separable array with a missing sensor, then the KAT acceleration can no longer be applied. In this letter, we review the definition of the KAT and provide a much simpler derivation that relies on an explicit new relation developed between Kronecker and Khatri-Rao matrix products. Additionally, we extend the KAT to deal with quasi-separable arrays, alleviating the restriction on the need of perfectly separable arrays.**

***Index Terms*—Kronecker array transform, Khatri-Rao identity, fast acoustic imaging, microphone array**

I. Introduction

MICROPHONE arrays are commonly used either as superdirective microphones [1] or as acoustic cameras [2]. They differ in the fact that the first provides an estimate of the time signal arriving from a given direction while the second uses the estimate of the sound pressure levels at a number of directions to generate a noise map (presented as a color map).

The standard algorithm for array processing is the delay-and-sum beamformer (DAS) [1], [2]. Despite its simplicity, the angular resolution obtained with this method is rather low, which prompted several authors to propose improved estimation algorithms [3]–[5]. As a tradeoff, these algorithms have a higher computational complexity and, for a large number of microphones, the computational cost may become prohibitive [6].

The Kronecker Array Transform (KAT) was introduced in [6], [7] to accelerate the calculation of acoustic images, essentially, by reorganizing the matrix-vector multiplication structure for a special case of *separable* arrays. The original KAT [6] was designed for acoustic camera applications. This was latter extended to superdirective microphone algorithms in [8]. In this letter, we present two main contributions. First, we develop a general formulation for the KAT, which extends the derivation for superdirective microphones proposed in [8] to the original KAT for acoustic cameras. This results in a much more compact derivation than the original in [6], through the use of a new relation developed between the Kronecker [9] and the Khatri-Rao [10], [11] matrix products.

Furthermore, the KAT was originally only applicable to arrays with *separable* geometry [6]–[8]. The second main contribution we present in this letter is to relax this restriction, generalizing the KAT to deal with quasi-separable arrays, i.e., separable arrays in which some positions in the grid may be left empty. We conclude by analyzing the efficiency of the generalized KAT.

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II. Preliminaries

We consider a sensor array composed of M microphones at Cartesian coordinates $\mathbf{p}_0, \cdots, \mathbf{p}_{M-1} \in \mathbb{R}^3$ being irradiated by an arbitrary sound field which we wish to estimate. We model the sound field as the superposition of the wave fields generated by N acoustic point sources located at coordinates $\mathbf{q}_0, \cdots, \mathbf{q}_{N-1} \in \mathbb{R}^3$, where N is usually a large number in order to obtain an accurate model.

The time-domain samples of each microphone are segmented into frames of K samples, and each frame is converted to the frequency domain using the fast Fourier transform (FFT). In the presence of additive noise, the $M \times 1$ array output vector for a single frequency ω_k $(0 < k < K/2)$ on a single frame can be modeled, according to [12], as

$$\boldsymbol{x}(\omega_k) = \boldsymbol{V}(\omega_k)\,\boldsymbol{y}(\omega_k) + \boldsymbol{\eta}(\omega_k), \tag{1}$$

where $\boldsymbol{y}(\omega_k) = [f_0(\omega_k) \;\; f_1(\omega_k) \;\; \cdots \;\; f_{N-1}(\omega_k)]^T$ represents the source signals in the frequency domain, and $\boldsymbol{\eta}(\omega_k)$ represents additive noise in frequency-domain. The array manifold matrix $\boldsymbol{V}(\omega_k) = [\boldsymbol{v}(\mathbf{q}_0, \omega_k) \;\; \boldsymbol{v}(\mathbf{q}_1, \omega_k) \;\; \cdots \;\; \boldsymbol{v}(\mathbf{q}_{N-1}, \omega_k)]$, of size $M \times N$, describes the transfer function between each source n and each sensor m at frequency ω_k.

Assuming that the point sources are in the far field, we define the look direction for source n as $\mathbf{u}_n = -\mathbf{q}_n / \|\mathbf{q}_n\|$. The array manifold vector for source n, according to [6], is modelled as $\boldsymbol{v}(\mathbf{u}_n, \omega_k) = \left[e^{j(\omega_k/c)\mathbf{u}_n^T\mathbf{p}_0} \;\; \ldots \;\; e^{j(\omega_k/c)\mathbf{u}_n^T\mathbf{p}_{M-1}}\right]^T$, where c is the speed of sound.

There exist several techniques (e.g. [3], [4], [7], [13]) for estimating $\hat{\boldsymbol{y}}(\omega_k)$ (for superdirective microphone applications) or the average value of $|\hat{y}_i(\omega_k)|^2$ (for acoustic camera applications, where $\hat{y}_i(\omega)$ is the i-th entry of $\hat{\boldsymbol{y}}(\omega_k)$, $i = 0 \ldots N-1$) from the array output vector $\boldsymbol{x}(\omega_k)$, all of them requiring the calculation of certain matrix-vector products involving $\boldsymbol{V}(\omega_k)$.

Reference [6] argues that, especially for iterative algorithms and for large arrays, these matrix-vector products are a calculation bottleneck. To speed up these calculations, [6] proposes, based on the discussion in [14], to decompose the manifold matrix $\boldsymbol{V}(\omega_k)$ in the Kronecker product of two smaller matrices, which can be used to reorganize the computations and significantly accelerate the above listed operations.

III. Kronecker Array Transform

The KAT was introduced in [6] as a method to speed up calculation of a class of acoustic imaging algorithms (as discussed in Sec. IV). However, the reorganization of Kronecker products can be used to speed up any problem modeled as a matrix-vector product, as long as the matrix can be described as the Kronecker product of two smaller matrices [14]. In [8], the KAT was extended to be used with a larger group of acoustic imaging and superdirective microphone algorithms, as we show next.

As its name suggests, the KAT is obtained by applying the above acceleration strategy to sensor arrays, more specifically, to sensor arrays of *separable* geometry, which, as shown below, guarantees that its separable manifold matrix $\boldsymbol{V}_{\mathrm{s}}$ can be recast as a Kronecker product $\boldsymbol{V}_{\mathrm{s}} = \boldsymbol{V}_x \otimes \boldsymbol{V}_y$.

Let us first specify the enumeration $\mathbf{p}_0, \cdots, \mathbf{p}_{M-1}$ of the sensing positions. Suppose the arrays is designed as an $M_x \times M_y$ rectangular grid ($M = M_x M_y$). We enumerate the sensing positions ordered from top to bottom and from left to right. Breaking $\mathbf{p}_m$ into its Cartesian components results in $\mathbf{p}_m = \begin{bmatrix} p_x(\lfloor m/M_y \rfloor) & p_y(\mathrm{mod}(m, M_y)) & 0 \end{bmatrix}^T$, where $p_x(m)$ and $p_y(m)$ are the x and y coordinates of the m-th microphone, $\lfloor m \rfloor$ represents the largest integer smaller than m, and $\mathrm{mod}(\cdot)$ represents the modulo operation. For simplicity, we assumed the array to be horizontally oriented. The same enumeration is used for the look directions $\mathbf{u}_0, \cdots, \mathbf{u}_{N-1}$ distributed in a $N_x \times N_y$ rectangular grid ($N = N_x N_y$) with coordinates $u_x(n)$ and $u_y(n)$ in the parametrized U-space [6].

The elements of a far-field manifold matrix associated to the above listed separable array and U-space parametrized rectangular scan grid are then given by

$$\begin{aligned} v_{\mathrm{s}}(m,n) = \; & e^{j\frac{\omega_k}{c} u_x(\lfloor n/N_y \rfloor) p_x(\lfloor m/M_y \rfloor)} \\ & \times e^{j\frac{\omega_k}{c} u_y(\mathrm{mod}(n,N_y)) p_y(\mathrm{mod}(m,M_y))}. \end{aligned} \tag{2}$$

We now define two new manifold matrices $\boldsymbol{V}_x$ and $\boldsymbol{V}_y$, whose elements are given below

$$v_x(m_x, n_x) = e^{j\omega_k u_x(n_x) p_x(m_x)/c}, \tag{3a}$$

$$v_y(m_y, n_y) = e^{j\omega_k u_y(n_y) p_y(m_y)/c}. \tag{3b}$$

The horizontal array manifold matrix $\boldsymbol{V}_x$ has size $M_x \times N_x$, and the vertical array manifold matrix $\boldsymbol{V}_y$ has size $M_y \times N_y$. By comparing the inner structure of (2) with the inner structure of (3), it can be verified that

$$\boldsymbol{V}_{\mathrm{s}} = \boldsymbol{V}_x \otimes \boldsymbol{V}_y. \tag{4}$$

Recapitulating, as long as we have a planar sensor array with separable geometry and we define a separable scan grid, we guarantee that the corresponding far-field manifold matrix can be decomposed as the Kronecker product of two more compact manifold matrices, which will allow a speed up in calculations, as discussed in [8]. For better comprehension, we repeat the derivation of the *Adjoint Fast Transform* below.

A. Adjoint Fast Transform

The simplest algorithm for superdirectional microphones is the DAS beamformer [1] that estimates the signals at each look-direction from $\hat{\boldsymbol{y}} = \boldsymbol{V}^H \boldsymbol{x}/M$. Given that the KAT conditions are met, we can substitute $\boldsymbol{V}$ by $\boldsymbol{V}_{\mathrm{s}}$ and apply (4) to the adjoint matrix-vector product, resulting in (apart from a constant M)

$$\hat{\boldsymbol{y}} \propto \boldsymbol{V}_{\mathrm{s}}^H \boldsymbol{x} = (\boldsymbol{V}_x \otimes \boldsymbol{V}_y)^H \boldsymbol{x}. \tag{5}$$

Using the well-known Kronecker product identity [9]

$$\mathrm{vec}(\boldsymbol{B}\boldsymbol{Q}\boldsymbol{A}^T) = (\boldsymbol{A} \otimes \boldsymbol{B})\,\mathrm{vec}\,(\boldsymbol{Q})\,, \tag{6}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{Q}$ are given matrices and $\mathrm{vec}(\cdot)$ denotes vectorization by stacking the columns of a matrix, we rewrite (5) as

$$\widehat{\boldsymbol{Y}} = \boldsymbol{V}_y^H \boldsymbol{X} \boldsymbol{V}_x^*, \tag{7}$$

where $\hat{\boldsymbol{y}} = \mathrm{vec}(\widehat{\boldsymbol{Y}})$ and $\boldsymbol{x} = \mathrm{vec}(\boldsymbol{X})$. The source signal matrix $\widehat{\boldsymbol{Y}} \in \mathbb{C}^{N_y \times N_x}$ contains all values of $\hat{\boldsymbol{y}}$ arranged in the same geometrical disposition as the scan grid, with the columns of the matrix representing the vertical y-axis and the rows of the matrix representing the horizontal x-axis. The same is valid for the sensor signal matrix $\boldsymbol{X} \in \mathbb{C}^{M_y \times M_x}$, which contains the values of $\boldsymbol{x}$ arranged in the same geometrical disposition as the sensors in the array.

We now discuss why the form in (7) is said to be a fast transform of the adjoint matrix-vector product (5). We can readily verify that calculation of the adjoint product $\boldsymbol{V}_{\mathrm{s}}^H \boldsymbol{x}$ requires $M_x M_y N_x N_y$ complex multiply-and-accumulate (MAC) operations. On the other hand, using (7) the required number of operations reduces to $M_y N_x N_y + M_x M_y N_x$ complex MACs when $\boldsymbol{X}\boldsymbol{V}_x^*$ is computed first, or to $M_x N_x N_y + M_x M_y N_y$ complex MACs when $\boldsymbol{V}_y^H \boldsymbol{X}$ is computed first.

If we assume that $N_x = N_y = \sqrt{N}$ and $M_x = M_y = \sqrt{M}$, and additionally, that the number of microphones contained in the array is substantially smaller than the number of scan points, i.e. $M \ll N$, than a rough estimate of the acceleration provided by the KAT lies in the order of $\sqrt{M}$, which is in agreement with the acceleration estimated in [14]. Further acceleration might be achieved using the NFFT and NNFFT algorithms together with the KAT, as discussed in [2], [8]. However, for the sake of brevity, we will refrain from this discussion here.

IV. KAT for CSM-based methods

Acoustic imaging algorithms generaly extract information not from the raw data $\boldsymbol{x}$, but from the array's narrow-band cross spectral matrix (CSM), defined as $\boldsymbol{R}_x = E\{\boldsymbol{x}\boldsymbol{x}^H\}$. The KAT, as presented in [6], was applicable only to this class of algorithms. We now present a new derivation for the KAT with CSM-based methods (CSM-KAT), relying upon the more general definition of the KAT presented in the previous section. We achieve a more compact and less cumbersome derivation then the derivation presented in [6] through the use of a, to the best of the authors knowledge, new relation developed between the Kronecker and the Khatri-Rao matrix products, presented in Appendix A.

Ignoring the presence of noise and defining $\boldsymbol{R}_y$ to be the CSM of $\boldsymbol{y}$, we verify from (1) that

$$\boldsymbol{R}_x = \boldsymbol{V}\boldsymbol{R}_y\boldsymbol{V}^H. \tag{8}$$

Usually, CSM-based methods assume that all sources present in the sound field are mutually uncorrelated [3], [13]. This assumption results in $\boldsymbol{R}_y$ being diagonal, and greatly simplifies the calculations. We take advantage of the fact that $\boldsymbol{R}_y$ is assumed to be diagonal in the new derivation of the CSM-KAT by using the identity [11]

$$\operatorname{vec}\left(\boldsymbol{B}\boldsymbol{P}\boldsymbol{A}^T\right) = (\boldsymbol{A} \odot \boldsymbol{B}) \operatorname{vecd}(\boldsymbol{P}), \tag{9}$$

where $\boldsymbol{P}$ is a given diagonal matrix, $\operatorname{vecd}(\cdot)$ is the vector formed from the diagonal elements of a square matrix, and $\boldsymbol{A} \odot \boldsymbol{B}$ is the columnwise Khatri-Rao product [15], defined as

$$\boldsymbol{A} \odot \boldsymbol{B} = \left[\ \boldsymbol{a}_1 \otimes \boldsymbol{b}_1 \ \vdots\ \boldsymbol{a}_2 \otimes \boldsymbol{b}_2 \ \vdots\ \cdots \ \vdots\ \boldsymbol{a}_q \otimes \boldsymbol{b}_q \ \right], \tag{10}$$

where $\boldsymbol{a}_q$ and $\boldsymbol{b}_q$, represent the q-th column of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively.

Assuming that the KAT conditions are met, we substitute $\boldsymbol{V}$ by $\boldsymbol{V}_{\mathrm{s}}$ and apply equality (4) into (8). Furthermore, we assume that $\boldsymbol{R}_y$ is a diagonal matrix and apply identity (9) to the previous result, which leads to

$$\operatorname{vec}(\boldsymbol{R}_x) = \left[(\boldsymbol{V}_x \otimes \boldsymbol{V}_y)^* \odot (\boldsymbol{V}_x \otimes \boldsymbol{V}_y)\right] \operatorname{vecd}(\boldsymbol{R}_y). \tag{11}$$

We use the Kronecker Khatri-Rao identity (28) and the definitions $\widetilde{\boldsymbol{V}}_x \equiv (\boldsymbol{V}_x^* \odot \boldsymbol{V}_x)$ and $\widetilde{\boldsymbol{V}}_y \equiv \left(\boldsymbol{V}_y^* \odot \boldsymbol{V}_y\right)$ to rewrite (11) as

$$\operatorname{vec}(\boldsymbol{R}_x) = \boldsymbol{\Xi}\,[(\widetilde{\boldsymbol{V}}_x \otimes \widetilde{\boldsymbol{V}}_y) \operatorname{vecd}(\boldsymbol{R}_y)] \triangleq \boldsymbol{\Xi} \operatorname{vec}(\boldsymbol{Z}), \tag{12}$$

where $\boldsymbol{Z}$ is such that $\operatorname{vec}(\boldsymbol{Z})$ is the term between brackets, and $\boldsymbol{\Xi}$ is a permutation matrix.

We now define $\operatorname{vec}\{\boldsymbol{\mathcal{Y}}\} = \operatorname{vecd}(\boldsymbol{R}_y)$, where $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{N_y \times N_x}$ is the acoustic image sensed by a separable array. To conclude, we apply identity (6) to matrix $\boldsymbol{Z}$, defined in (12), resulting in

$$\boldsymbol{Z} = \widetilde{\boldsymbol{V}}_y \boldsymbol{\mathcal{Y}} \widetilde{\boldsymbol{V}}_x^T, \qquad \operatorname{vec}(\boldsymbol{R}_x) = \boldsymbol{\Xi} \operatorname{vec}(\boldsymbol{Z}), \tag{13}$$

which is exactly the *CSM fast direct transform* presented in [6].

We can further verify that the CSM adjoint transform

$$\operatorname{vecd}(\widehat{\boldsymbol{R}}_y) = [\boldsymbol{\Xi}(\widetilde{\boldsymbol{V}}_x \otimes \widetilde{\boldsymbol{V}}_y)]^H \operatorname{vec}(\boldsymbol{R}_x), \tag{14}$$

can be recast as the *CSM fast adjoint transform* [6], allowing us to efficiently obtain the estimated acoustic image $\widehat{\boldsymbol{\mathcal{Y}}}$ from

$$\operatorname{vec}(\boldsymbol{Z}) = \boldsymbol{\Xi}^T \operatorname{vec}(\boldsymbol{R}_x), \qquad \widehat{\boldsymbol{\mathcal{Y}}} = \widetilde{\boldsymbol{V}}_y^H \boldsymbol{Z} \widetilde{\boldsymbol{V}}_x^*. \tag{15}$$

By combining (12) and (14) and using the Kronecker "mixed product rule" [16]—note that $\boldsymbol{\Xi}^T\boldsymbol{\Xi}$ equals the identity matrix as the permutation matrix is an orthogonal matrix—we obtain the direct-adjoint transform in the form

$$\begin{aligned}\operatorname{vecd}(\widehat{\boldsymbol{R}}_y) &= (\widetilde{\boldsymbol{V}}_x \otimes \widetilde{\boldsymbol{V}}_y)^H (\widetilde{\boldsymbol{V}}_x \otimes \widetilde{\boldsymbol{V}}_y) \operatorname{vecd}(\boldsymbol{R}_y) \\ &= [(\widetilde{\boldsymbol{V}}_x^H \widetilde{\boldsymbol{V}}_x) \otimes (\widetilde{\boldsymbol{V}}_y^H \widetilde{\boldsymbol{V}}_y)] \operatorname{vecd}(\boldsymbol{R}_y),\end{aligned} \tag{16}$$

which can be recast in the *CSM fast direct-adjoint transform*

$$\widehat{\boldsymbol{\mathcal{Y}}} = \widetilde{\boldsymbol{V}}_y^H \widetilde{\boldsymbol{V}}_y \boldsymbol{\mathcal{Y}} \widetilde{\boldsymbol{V}}_x^T \widetilde{\boldsymbol{V}}_x^*. \tag{17}$$

As discussed in [6], implementing the direct-adjoint transform as $\widehat{\boldsymbol{\mathcal{Y}}} = (\widetilde{\boldsymbol{V}}_y^H \widetilde{\boldsymbol{V}}_y)\boldsymbol{\mathcal{Y}}(\widetilde{\boldsymbol{V}}_x^T \widetilde{\boldsymbol{V}}_x^*)$ can be much faster than using a composition of the direct and adjoint CSM-KAT as for large problems one can precompute $\widetilde{\boldsymbol{V}}_y^H \widetilde{\boldsymbol{V}}_y$ and $\widetilde{\boldsymbol{V}}_x^T \widetilde{\boldsymbol{V}}_x^*$, which are real-valued matrices. In fact, letting $\boldsymbol{v}_{y,i}$ denote the i-th column of $\boldsymbol{V}_y$, the (i,j)-th element of $\widetilde{\boldsymbol{V}}_y$ is

$$\begin{aligned}\left[\widetilde{\boldsymbol{V}}_y^H \widetilde{\boldsymbol{V}}_y\right]_{i,j} &= \left(\boldsymbol{v}_{y,i}^* \otimes \boldsymbol{v}_{y,i}\right)^H \left(\boldsymbol{v}_{y,j}^* \otimes \boldsymbol{v}_{y,j}\right) \\ &= \left(\boldsymbol{v}_{y,i}^H \boldsymbol{v}_{y,j}\right)^* \otimes \left(\boldsymbol{v}_{y,i}^H \boldsymbol{v}_{y,j}\right) \\ &= \left|\boldsymbol{v}_{y,i}^H \boldsymbol{v}_{y,j}\right|^2.\end{aligned} \tag{18}$$

Note that in the last equality, we used the fact that $\boldsymbol{v}_{y,i}^H \boldsymbol{v}_{y,j}$ is a scalar, so the Kronecker product reduces to a regular product.

V. Generalized KAT for quasi-separable arrays

The KAT was developed under the assumption that the microphone array possesses separable geometry, as discussed in section III. However, an array with separable geometry may not be available, e.g., because some elements of a separable array were damaged and, thus, need to be discarded, resulting in an array with quasi-separable geometry.

To apply the KAT in such cases, we define a "virtual" output vector $\boldsymbol{x}_{\mathrm{s}} = \boldsymbol{V}_{\mathrm{s}}\boldsymbol{y}$ generated from a "virtual" separable array with the least number of sensors M that contain all M' elements of the non-separable array of interest (with output vector $\boldsymbol{x}$), such that

$$\boldsymbol{x} = \boldsymbol{\Gamma}\boldsymbol{x}_{\mathrm{s}} = \boldsymbol{\Gamma}\boldsymbol{V}_{\mathrm{s}}\boldsymbol{y} = \boldsymbol{\Gamma}\left(\boldsymbol{V}_x \otimes \boldsymbol{V}_y\right)\boldsymbol{y}. \tag{19}$$

The matrix $\boldsymbol{\Gamma} \in \mathbb{R}^{M' \times M}$ is a *selection* matrix built by setting $[\boldsymbol{\Gamma}]_{m,n} = 1$ when the m^{th} element of $\boldsymbol{x}$ is equivalent to the n^{th} element of $\boldsymbol{x}_{\mathrm{s}}$ and $[\boldsymbol{\Gamma}]_{m,n} = 0$ otherwise.

From (19) we can easily verify that the *generalized direct fast transform* is now given by

$$\boldsymbol{X}_{\mathrm{s}} = \boldsymbol{V}_y \boldsymbol{Y} \boldsymbol{V}_x^T, \qquad \boldsymbol{x} = \boldsymbol{\Gamma} \operatorname{vec}\{\boldsymbol{X}_{\mathrm{s}}\}. \tag{20}$$

Note that $\boldsymbol{X}_{\mathrm{s}}$ contains $M - M'$ dummy entries, that are computed by the KAT, but are discarded when computing $\boldsymbol{x}$. As we show below, this procedure is efficient when M' is large enough.

The adjoint transform $\boldsymbol{y} = [\boldsymbol{\Gamma}\left(\boldsymbol{V}_x \otimes \boldsymbol{V}_y\right)]^H \boldsymbol{x}$ results in the *generalized adjoint fast transform*

$$\operatorname{vec}\{\boldsymbol{X}_{\mathrm{s}}\} = \boldsymbol{\Gamma}^T \boldsymbol{x}, \qquad \widehat{\boldsymbol{Y}} = \boldsymbol{V}_y^H \boldsymbol{X}_{\mathrm{s}} \boldsymbol{V}_x^*. \tag{21}$$

Finally, combining (20) and (21) results in the *generalized direct-adjoint fast transform*

$$\begin{gathered}\boldsymbol{X}_{\mathrm{s}} = \boldsymbol{V}_y \boldsymbol{Y} \boldsymbol{V}_x^T, \qquad \widehat{\boldsymbol{X}}_{\mathrm{s}} = \boldsymbol{G} \circ \boldsymbol{X}_{\mathrm{s}}, \\ \widehat{\boldsymbol{Y}} = \boldsymbol{V}_y^H \widehat{\boldsymbol{X}}_{\mathrm{s}} \boldsymbol{V}_x^*,\end{gathered} \tag{22}$$

where $\circ$ is the Hadamard-Schur matrix product, and $\operatorname{vec}\{\boldsymbol{G}\} = \operatorname{vecd}\{\boldsymbol{\Gamma}^T\boldsymbol{\Gamma}\}$. Please note it is not possible to inlay the influence of $\boldsymbol{\Gamma}$ into $\boldsymbol{V}_x$ or $\boldsymbol{V}_y$ to arrive in a formulation similar to (17).

A. Generalized CSM-KAT

In the same manner described in the previous section, we can also generalize the application of the CSM-KAT for quasi-separable arrays. To do so, we observe that

$$\begin{aligned}\operatorname{vec}(\boldsymbol{R}_x) &= (\boldsymbol{V}^* \otimes \boldsymbol{V}) \operatorname{vec}(\boldsymbol{R}_y) \\ &= \left[(\boldsymbol{\Gamma}\boldsymbol{V}_{\mathrm{s}})^* \otimes (\boldsymbol{\Gamma}\boldsymbol{V}_{\mathrm{s}})\right] \operatorname{vec}(\boldsymbol{R}_y) \\ &= (\boldsymbol{\Gamma}^* \otimes \boldsymbol{\Gamma})(\boldsymbol{V}_{\mathrm{s}}^* \otimes \boldsymbol{V}_{\mathrm{s}}) \operatorname{vec}(\boldsymbol{R}_y).\end{aligned} \tag{23}$$

Equation (23) shows us that, for CSM based methods, the calculation with a quasi-separable array can be directly related to the calculations with a separable array by the selection matrix $\boldsymbol{W} \equiv \boldsymbol{\Gamma}^* \otimes \boldsymbol{\Gamma} = \boldsymbol{\Gamma} \otimes \boldsymbol{\Gamma}$. This result allows the use of the CSM-KAT with quasi-separable arrays, resulting in the *generalized CSM fast direct transform*

$$\boldsymbol{Z} = \widetilde{\boldsymbol{V}}_y \boldsymbol{\mathcal{Y}} \widetilde{\boldsymbol{V}}_x^T, \qquad \mathrm{vec}\left(\boldsymbol{R}_x\right) = \boldsymbol{W}\boldsymbol{\Xi}\,\mathrm{vec}\left(\boldsymbol{Z}\right), \tag{24}$$

the *generalized CSM fast adjoint transform*

$$\mathrm{vec}(\boldsymbol{Z}) = \boldsymbol{\Xi}^T \boldsymbol{W}^T\,\mathrm{vec}\left(\boldsymbol{R}_x\right), \qquad \widehat{\boldsymbol{\mathcal{Y}}} = \widetilde{\boldsymbol{V}}_y^H \boldsymbol{Z} \widetilde{\boldsymbol{V}}_x^*, \tag{25}$$

and the *generalized CSM fast direct-adjoint transform*

$$\begin{aligned} \boldsymbol{Z} = \widetilde{\boldsymbol{V}}_y \boldsymbol{\mathcal{Y}} \widetilde{\boldsymbol{V}}_x^T, \quad \mathrm{vec}(\widehat{\boldsymbol{Z}}) = \boldsymbol{\Xi}^T \boldsymbol{W}^T \boldsymbol{W} \boldsymbol{\Xi}\,\mathrm{vec}\left(\boldsymbol{Z}\right), \\ \widehat{\boldsymbol{\mathcal{Y}}} = \widetilde{\boldsymbol{V}}_y^H \widehat{\boldsymbol{Z}} \widetilde{\boldsymbol{V}}_x^*. \end{aligned} \tag{26}$$

Here, again, we cannot eliminate the influence of $\boldsymbol{\Xi}^T \boldsymbol{W}^T \boldsymbol{W} \boldsymbol{\Xi}$, nor inlay its influence in $\widetilde{\boldsymbol{V}}_x$ or $\widetilde{\boldsymbol{V}}_y$, being necessary to use a composition of the previously presented direct and adjoint transforms to calculate the direct-adjoint transform.

VI. Efficiency of the Generalized KAT

To evaluate the performance improvement obtained with the generalized KAT we simulate[1] a directional microphone using DAS beamformer and compare calculation time of $\boldsymbol{V}^H \boldsymbol{x}$ and (21). We define a separable scan grid with $N_x = N_y = \sqrt{N}$ directions and use an array with M sensors placed on a grid with separable geometry where L random sensors are defective, thus resulting in an array with $M' = M - L$ sensors placed in a quasi-separable geometry. Fig. 1 compares the average calculation time for an array with $M_x = M_y = 8$, making evident the advantage of the generalized KAT.

Analytically, we verify that the matrix-vector calculation requires $M'N$ complex MAC operations, while the generalized KAT requires $\sqrt{M'}N + M\sqrt{N}$ complex MAC operations. The acceleration obtained with the generalized KAT is estimated as

$$\frac{M'}{\sqrt{M'} + {}^{M}\!/\!\sqrt{N}} \leq \frac{M'}{2\max\left\{\sqrt{M'}, {}^{M}\!/\!\sqrt{N}\right\}}. \tag{27}$$

If we assume that $M \ll N$, we see that the obtained acceleration is roughly in the order of $M'/(2\sqrt{M'})$. This suggests that it is preferable to use the generalized adjoint fast transform as long as $L \lesssim M - 2\sqrt{M}$.

It is important to observe that the generalized KAT will likely not be effective if applied to a generic non-separable array. If we repeat the previous simulation with a random array containing M' sensors placed with no repetition in both x- and y-coordinates we will need a "virtual" separable array with $M = M' \times M'$ positions. Direct calculation still requires $M'N$ operations while the generalized KAT will now require $M'N + M'^2\sqrt{N}$ complex MAC operations. This shows that for general non-separable arrays the use of the generalized KAT would not be advisable, as no acceleration is achieved.

VII. Conclusion

The contributions presented in this letter are twofold: (i) a new (shorter) derivation of the CSM-KAT, linking it with the (more general) KAT through an explicit new relation between Kronecker and Khatri-Rao matrix products; and (ii) a generalization of the KAT to deal with quasi-separable arrays, which allows the use of the fast transforms when microphones in a separable array go defective.

[1]All MATLAB files necessary to recreate this simulation are available at http://ieeexplore.ieee.org, provided by the authors.

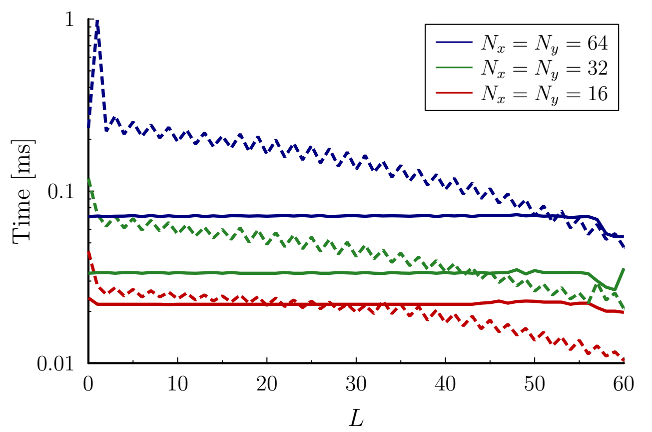

Fig. 1. Average calculation time for the DAS algorithm when applied to a separable array with $M_x = M_y = 8$ sensors positions and L (randomly chosen) missing sensors. The dashed line represents calculation with the adjoint matrix-vector product (5) while the straight line represents calculation with the generalized adjoint fast transform (21). Simulation was done in Matlab R2014a using a Intel Core2 Duo PC (3.16 GHz) with 1000 realizations per point.

Appendix A
Proof of the Kronecker Khatri-Rao Identity

Theorem 1. *Let the matrices* $\boldsymbol{A} = \left[\, \boldsymbol{a}_1 \,\vdots\, \boldsymbol{a}_2 \,\vdots\, \cdots \,\vdots\, \boldsymbol{a}_p \,\right]$, $\boldsymbol{B} = \left[\, \boldsymbol{b}_1 \,\vdots\, \boldsymbol{b}_2 \,\vdots\, \cdots \,\vdots\, \boldsymbol{b}_q \,\right]$, $\boldsymbol{C} = \left[\, \boldsymbol{c}_1 \,\vdots\, \boldsymbol{c}_2 \,\vdots\, \cdots \,\vdots\, \boldsymbol{c}_p \,\right]$, *and* $\boldsymbol{D} = \left[\, \boldsymbol{d}_1 \,\vdots\, \boldsymbol{d}_2 \,\vdots\, \cdots \,\vdots\, \boldsymbol{d}_q \,\right]$ *be compatibly partitioned matrices, then*

$$(\boldsymbol{A} \otimes \boldsymbol{B}) \odot (\boldsymbol{C} \otimes \boldsymbol{D}) = \boldsymbol{\Xi}\left[(\boldsymbol{A} \odot \boldsymbol{C}) \otimes (\boldsymbol{B} \odot \boldsymbol{D})\right], \tag{28}$$

where $\boldsymbol{\Xi}$ *is a permutation matrix.*

Proof: Using the definitions of the Kronecker product and the Khatri-Rao product we verify that

$$\begin{aligned} (\boldsymbol{A} \otimes \boldsymbol{B}) \odot (\boldsymbol{C} \otimes \boldsymbol{D}) = \big[\, & (\boldsymbol{a}_1 \otimes \boldsymbol{b}_1) \otimes (\boldsymbol{c}_1 \otimes \boldsymbol{d}_1) \,\vdots\, \cdots \\ & \vdots\, (\boldsymbol{a}_1 \otimes \boldsymbol{b}_2) \otimes (\boldsymbol{c}_1 \otimes \boldsymbol{d}_2) \,\vdots\, \cdots \,\vdots\, (\boldsymbol{a}_1 \otimes \boldsymbol{b}_q) \otimes (\boldsymbol{c}_1 \otimes \boldsymbol{d}_q) \,\vdots\, \cdots \\ & \vdots\, (\boldsymbol{a}_2 \otimes \boldsymbol{b}_1) \otimes (\boldsymbol{c}_2 \otimes \boldsymbol{d}_1) \,\vdots\, \cdots \,\vdots\, (\boldsymbol{a}_2 \otimes \boldsymbol{b}_q) \otimes (\boldsymbol{c}_2 \otimes \boldsymbol{d}_q) \,\vdots\, \cdots \\ & \vdots\, (\boldsymbol{a}_p \otimes \boldsymbol{b}_1) \otimes (\boldsymbol{c}_p \otimes \boldsymbol{d}_1) \,\vdots\, \cdots \,\vdots\, (\boldsymbol{a}_p \otimes \boldsymbol{b}_q) \otimes (\boldsymbol{c}_p \otimes \boldsymbol{d}_q) \,\big]. \end{aligned} \tag{29}$$

The Kronecker product is associative but not commutative. However, according to [10],

$$(\boldsymbol{a} \otimes \boldsymbol{b}) = \boldsymbol{P}(\boldsymbol{b} \otimes \boldsymbol{a}), \tag{30}$$

where $\boldsymbol{P}$ is a permutation matrix. Therefore, we verify that

$$\begin{aligned} (\boldsymbol{a}_i \otimes \boldsymbol{b}_j) \otimes (\boldsymbol{c}_i \otimes \boldsymbol{d}_j) &= \boldsymbol{I}\boldsymbol{a}_i \otimes \boldsymbol{P}(\boldsymbol{c}_i \otimes \boldsymbol{b}_j) \otimes \boldsymbol{I}\boldsymbol{d}_j = \\ &= (\boldsymbol{I} \otimes \boldsymbol{P} \otimes \boldsymbol{I})[\boldsymbol{a}_i \otimes (\boldsymbol{c}_i \otimes \boldsymbol{b}_j) \otimes \boldsymbol{d}_j] \equiv \\ &\equiv \boldsymbol{\Xi}[(\boldsymbol{a}_i \otimes \boldsymbol{c}_i) \otimes (\boldsymbol{b}_j \otimes \boldsymbol{d}_j)]. \end{aligned} \tag{31}$$

Applying equality (31) to (29) results in (28).

Starting from $[(\boldsymbol{A} \odot \boldsymbol{C}) \otimes (\boldsymbol{B} \odot \boldsymbol{D})]$ and applying identity (31) will also result into (28), which concludes the proof. ■

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