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Automorphisms of F_4 quadrangles

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Abstract. The goal of this article is to show that the automorphism group of a Moufang quadrangle of type F_4 is, up to field automorphisms, generated by the root groups.

Key words. exceptional Moufang quadrangles – root groups – quadratic forms

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1. Introduction

The irreducible spherical buildings of rank at least three have been classified by J. Tits in 1974 [3]. The irreducible spherical buildings of rank two – which are called generalized polygons – are too numerous to classify, but in the addenda of [3], the Moufang condition for spherical buildings was introduced, and it was observed that every thick irreducible spherical building of rank at least three as well as every irreducible residue of such a building satisfies the Moufang condition. In this sense, the Moufang polygons are the "building bricks" of any spherical building of rank at least three.

Recently, the classification of Moufang polygons has been completed by J. Tits and R. Weiss in [4]. As far as the quadrangles are concerned, it turns out that there are six different families of Moufang quadrangles. A uniform algebraic structure to parametrize all Moufang quadrangles has recently been obtained by the author; see [1].

An important problem in the study of Moufang polygons is to determine the structure of the automorphism group G modulo the subgroup $G^\dagger$ generated by all the root groups. In [4], this has been done for four of these families. The

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two classes which have been left open, are the cases of the so-called exceptional Moufang quadrangles. The quadrangles in the first of these classes arise from forms of simple algebraic groups of type E_6, E_7 and E_8; the quadrangles in the second class are related to buildings of type F_4; see [2].

The goal of this paper is to determine the quotient $G/G^\dagger$ for the latter case. More precisely, we will show that the automorphism group is, up to field automorphisms, generated by the root groups.

During the proof of this fact, we will determine the multipliers of the similitudes of a certain *non*-regular anisotropic quadratic form in characteristic 2, something which does not seem to appear often in the existing literature.

2. Preliminaries

We start by recalling the definition of a quadrangular system of type F_4 as given in the appendix of [1].

Definition 1. *Let K and L be two commutative fields with* $\mathrm{char}(K) = \mathrm{char}(L) = 2$*, such that K is a vector space over L and that L is a vector space over K. If t is an element of the field K, then we will denote the corresponding element of the vector space K by $[t]$; if s is an element of the field L, then we will denote the corresponding element of the vector space L by $[s]$. Let V be a vector space over K containing $[L]$ as a proper subspace, and let W be a vector space over L containing $[K]$ as a proper subspace. The scalars will always be written at the left.*

Suppose that q is an anisotropic quadratic form from V to K, with corresponding bilinear form f, and that $\hat{q}$ is an anisotropic quadratic form from W to L, with corresponding bilinear form $\hat{f}$, such that $[L] = \mathrm{Rad}(f)$ and $[K] = \mathrm{Rad}(\hat{f})$; in particular, $f \neq 0$ and $\hat{f} \neq 0$. Let $\epsilon := [1] \in [L] \subseteq V$ and $\delta := [1] \in [K] \subseteq W$. Finally, suppose that there is a map τ_V from $V \times W$ to V which is K-linear on V, and a map τ_W from $W \times V$ to W which is L-linear on W, both of which will be denoted by $\cdot$ or simply by juxtaposition, i.e. $\tau_V(v, w) = vw = v \cdot w$ and $\tau_W(w, v) = wv = w \cdot v$ for all $v \in V$ and all $w \in W$. Then $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a quadrangular system of type F_4 *if and only if the following axioms hold. For all $v \in V$, $w \in W$, $t \in K$ and $s \in L$,*

($\mathbf{F}_1$) $v[t] = tv$;
($\mathbf{F}_2$) $w[s] = sw$;
($\mathbf{F}_3$) $v \cdot sw = vw \cdot s\delta$;
($\mathbf{F}_4$) $w \cdot tv = wv \cdot t\epsilon$;
($\mathbf{F}_5$) $[t]v = [tq(v)]$;
($\mathbf{F}_6$) $[s]w = [s\hat{q}(w)]$;
($\mathbf{F}_7$) $vw \cdot w = v \cdot \hat{q}(w)\delta$;
($\mathbf{F}_8$) $wv \cdot v = w \cdot q(v)\epsilon$;

($\mathbf{F}_9$) $v \cdot wv = q(v)vw$;
($\mathbf{F}_{10}$) $w \cdot vw = \hat{q}(w)wv$;
($\mathbf{F}_{11}$) $v(w_1 + w_2) = vw_1 + vw_2 + [\hat{f}(w_1 v, w_2)]$;
($\mathbf{F}_{12}$) $w(v_1 + v_2) = wv_1 + wv_2 + [f(v_1 w, v_2)]$.

Remark 1. In writing down these axioms, we used the convention that the actions which are denoted by juxtaposition preceed those which are denoted by "·". Note, however, that there is no danger of confusion, since we have not defined a multiplication on V or on W. Hence we will often write vww instead of $vw \cdot w$, for example.

Once we have such a system $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$, the corresponding Moufang quadrangle $\mathcal{Q}(\Omega)$ is uniquely determined by its commutator relations as explained in [4, Chapters 7 and 8]: Let U_1 and U_3 be two groups parametrized by W via some isomorphisms x_1 and x_3, respectively, and let U_2 and U_4 be two groups parametrized by V via some isomorphisms x_2 and x_4, respectively. Then the commutator relations

$$\begin{aligned} [x_1(w_1), x_3(w_2)] &= x_2([\hat{f}(w_1, w_2)]) \;, \\ [x_2(v_1), x_4(v_2)] &= x_3([f(v_1, v_2)]) \;, \\ [x_1(w), x_4(v)] &= x_2(vw)x_3(wv) \;, \\ [U_i, U_{i+1}] &= 1 \quad \forall i \in \{1, 2, 3\} \;, \end{aligned} \tag{1}$$

for all $w, w_1, w_2 \in W$ and all $v, v_1, v_2 \in V$, define a unique Moufang quadrangle of type F_4.

Quadrangular systems of type F_4 have a very precise and well understood structure. In particular, we have the following theorem.

Theorem 1.

(i) *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system of type F_4. Let $d \in V \setminus [L]$ and $\xi \in W \setminus [K]$ be arbitrary. Then there exists an element $e \in V$ such that $f(d, e) = 1$ and $f(d, e\xi) = 0$. Let $A := \langle \xi, \xi e d^{-1}, \xi d^{-1}, \xi e \rangle$ and let $B := \langle d, e, d\xi, e\xi \rangle$. Then $\dim_K B = \dim_L A = 4$, and V and W have a decomposition $V = B \oplus [L]$ and $W = A \oplus [K]$, respectively.*

(ii) *L is isomorphic to a subfield of K; hence we will assume that L* is *a subfield of K. Let $\alpha := f(d\xi, e\xi)$ and let $\beta := q(d)^{-1}$. Then $\alpha \in L \setminus K^2$ and $\beta \in K \setminus L$.*

(iii) *Let E be the splitting field of the polynomial*

$$P_E(x) := q(d)x^2 + x + q(e)$$

over K. Then E/K is a separable quadratic extension; denote its norm by N. Let $\omega \in E$ be one of the roots of P_E. Then $D := E^2 L$ is the splitting field of the polynomial

$$P_D(x) := q(d)^2 x^2 + x + q(e)^2$$

over L, and D/L is also a separable quadratic extension. Its norm is the restriction of N to D, so we will also denote it by N. The non-trivial elements of the Galois groups $\mathrm{Gal}(E/K)$ *and* $\mathrm{Gal}(D/L)$ *will be denoted by $x \mapsto \bar{x}$. Then*

$$q(t_1 d + t_2 e + t_3 d\xi + t_4 e\xi + [s]) \\ = \beta^{-1}\big(N(t_1 + t_2\omega) + \alpha N(t_3 + t_4\omega)\big) + s$$

for all $t_1, t_2, t_3, t_4 \in K$ and all $s \in L$, and

$$\hat{q}(s_1\xi + s_2\xi e d^{-1} + s_3\xi d^{-1} + s_4\beta^2\xi e + [t]) \\ = \alpha\big(N(s_1 + s_2\omega^2) + \beta^2 N(s_3 + s_4\omega^2)\big) + t^2$$

for all $s_1, s_2, s_3, s_4 \in L$ and all $t \in K$.

Proof. See [1, Section 8.5]. □

Remark 2. With the convention of Theorem 1.(ii) that L is a subfield of K, we have that $s[t] = [st]$ and $t[s] = [t^2 s]$ for all $s \in L$ and all $t \in K$. Note that this implies that $v \cdot s\delta = sv$ and $w \cdot t\epsilon = t^2 w$ for all $v \in V$, $s \in L$, $w \in W$ and $t \in K$.

The following properties will be used very frequently, without being mentioned explicitly.

Lemma 1. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system of type F_4. For all $u, v \in V$ and all $w, z \in W$, we have that*

(i) $f(v, vw) = 0$*;*
(ii) $\hat{f}(w, wv) = 0$*;*
(iii) $f(u, vw) = f(uw, v)$*;*
(iv) $\hat{f}(w, zv) = \hat{f}(wv, z)$*;*
(v) $q(vw) = \hat{q}(w)q(v)$*;*
(vi) $\hat{q}(wv) = q(v)^2\hat{q}(w)$*.*

Proof. The first four properties follow immediately from ($\mathbf{F}_{11}$) and ($\mathbf{F}_{12}$) and the fact that $\mathrm{char}(K) = \mathrm{char}(L) = 2$. For the last two identities, see [1, Lemma 8.94]. □

Every element $v \in V^*$ and $w \in W^*$ has an *inverse*, denoted by v^{-1} and w^{-1}, respectively, and defined by $v^{-1} = q(v)^{-1}v$ and $w^{-1} = \hat{q}(w)^{-1}w$. It follows from ($\mathbf{F}_3$) and ($\mathbf{F}_7$) that

$$\begin{aligned} vww^{-1} &= vw \cdot \hat{q}(w)^{-1}w = vw \cdot w \cdot \hat{q}(w)^{-1}\delta \\ &= v \cdot \hat{q}(w)\delta \cdot \hat{q}(w)^{-1}\delta = v \cdot \hat{q}(w)^{-1}\hat{q}(w)\delta = v\ , \end{aligned}$$

and similarly that $wvv^{-1} = w$, for all $v \in V^*$ and all $w \in W^*$.

Remark 3. In [1], the inverses are defined in a different way (namely in terms of certain properties), but it is shown in [1, (8.86)] that our definitions coincide.

We recall the definition of a reflection corresponding to a quadratic form. For all $v \in V$, $c \in V^*$, $w \in W$ and $z \in W^*$, we define

$$\pi_c(v) := v + f(v,c)q(c)^{-1}c \,;$$
$$\hat{\pi}_z(w) := w + \hat{f}(w,z)\hat{q}(z)^{-1}z \,.$$

The reflections in a quadrangular system of type F_4 have the following properties.

Lemma 2. *For all $v \in V$, $c \in V^*$, $w \in W$ and $z \in W^*$, we have that*

(i) $w\pi_c(v) = wc^{-1}vc$*;*
(ii) $v\hat{\pi}_z(w) = vz^{-1}wz$*;*
(iii) $\pi_c(vw) = q(c)^{-1}\pi_c(v) \cdot wc$*;*
(iv) $\hat{\pi}_z(wv) = \hat{q}(z)^{-1}\hat{\pi}_z(w) \cdot vz$*.*

Proof. This follows from the identities ($\mathbf{Q}_{23}$) – ($\mathbf{Q}_{26}$) in [1, Chapter 2]. □

3. Main Theorem

Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be an arbitrary quadrangular system of type F_4, and let $\Gamma := \mathcal{Q}(\Omega)$ be the corresponding Moufang quadrangle. Let $\mathrm{Aut}(\Gamma)$ be the full automorphism group (that is, the correlation group) of Γ, and let G be its subgroup of type-preserving automorphisms. Then G is of index at most 2 in $\mathrm{Aut}(\Gamma)$. Let $G^\dagger$ be the subgroup of G generated by all the root groups of Γ.

Let K and L be as in Theorem 1. In particular, L is a subfield of K. Let $\mathrm{Aut}(K, L)$ denote the group of field automorphisms of K which map L to itself. Note that it follows from the fact that $\mathrm{char}(K) = \mathrm{char}(L) = 2$ that $\mathrm{Aut}(K, L) \cong \mathrm{Aut}(L, K^2)$.

We can now state the Main Theorem of this paper.

Main Theorem. *$G/G^\dagger$ is isomorphic to a subgroup of* $\mathrm{Aut}(K, L)$.

Remark 4. $[\mathrm{Aut}(\Gamma) : G]$ is 1 or 2, and both cases can actually occur. By [4, (35.12)], Γ has a correlation if and only if the quadratic space (K, V, q) is similar to $(L, W, \hat{q})$ (that is, if $\hat{q}$ is isomorphic to λq for some $\lambda \in K^*$). In [4, (14.25) and (14.26)], examples are given of quadratic spaces of type F_4 which do have this property and others which do not.

Now let H be the pointwise stabilizer of the (labeled) base apartment $\Sigma = \{0, \ldots, 7\}$ in the construction of Γ. (See again [4, Chapters 7 and 8] for details.) Moreover, let $H^\dagger := H \cap G^\dagger$. Then it follows from [4, (4.12)] that $G = HG^\dagger$, and hence

$$G/G^\dagger \cong H/H^\dagger \,. \tag{2}$$

4. Similarities

We will first translate the geometric problem of isomorphic Moufang quadrangles of type F_4 to the algebraic problem of similar quadrangular systems of type F_4.

Definition 2. *Let* $\Omega := (V, W, \tau_V, \tau_W, \epsilon, \delta)$ *and* $\Omega' := (V', W', \tau_{V'}, \tau_{W'}, \epsilon', \delta')$ *be two quadrangular systems of type* F_4*. We will say that* $(\varphi, \hat{\varphi})$ *is a* similarity *from* Ω *to* Ω' *if and only if* φ *and* $\hat{\varphi}$ *are group isomorphisms from* $(V, +)$ *to* $(V', +)$ *and from* $(W, +)$ *to* $(W', +)$*, respectively, for which there exist constants* $g \in K'^*$ *and* $\hat{g} \in L'^*$ *(called the* parameters *of the similarity) such that*

$$\varphi(vw) = g\varphi(v)\hat{\varphi}(w) \tag{3}$$

$$\hat{\varphi}(wv) = \hat{g}\hat{\varphi}(w)\varphi(v) \tag{4}$$

for all $v \in V$ *and all* $w \in W$*. A similarity from* Ω *to itself will be called a* self-similarity*. Moreover, if both* φ *and* $\hat{\varphi}$ *are vector space isomorphisms, then the self-similarity* $(\varphi, \hat{\varphi})$ *will be called* linear.

Remark 5. If $(\varphi_1, \hat{\varphi}_1)$ and $(\varphi_2, \hat{\varphi}_2)$ are two self-similarities of Ω with parameters $(g_1, \hat{g}_1)$ and $(g_2, \hat{g}_2)$, respectively, then their product $(\varphi_1\varphi_2, \hat{\varphi}_1\hat{\varphi}_2)$ is again a self-similarity. If both $(\varphi_1, \hat{\varphi}_1)$ and $(\varphi_2, \hat{\varphi}_2)$ are linear, then their product is also linear, and has parameters $(g_1g_2, \hat{g}_1\hat{g}_2)$.

Theorem 2. *Let* Ω *and* Ω' *be two quadrangular systems of type* F_4*. Let* $\mathcal{Q}(\Omega)$ *and* $\mathcal{Q}(\Omega')$ *be the corresponding Moufang quadrangles with labeled base apartments* $\Sigma = \{0, \ldots, 7\}$ *and* $\Sigma' = \{0', \ldots, 7'\}$*, respectively. Let* $H_{\Sigma,\Sigma'}$ *denote the set of isomorphisms from* $\mathcal{Q}(\Omega)$ *to* $\mathcal{Q}(\Omega')$ *mapping* i *to* i' *for all* $i \in \Sigma$*, and let* $X_{\Omega,\Omega'}$ *denote the set of similarities from* Ω *to* Ω'*. Then there is a natural one-to-one correspondence between* $H_{\Sigma,\Sigma'}$ *and* $X_{\Omega,\Omega'}$ *which is a group isomorphism if* $\Omega = \Omega'$ *and* $\Sigma = \Sigma'$*. In particular,* $\mathcal{Q}(\Omega)$ *and* $\mathcal{Q}(\Omega')$ *are isomorphic (in the type-preserving sense) if and only if* Ω *and* Ω' *are similar.*

Proof. For every object $\mathfrak{o}$ which we have defined for Ω, we will denote the corresponding object in Ω' by $\mathfrak{o}'$. For example, we will use the notations U_i', $\hat{f}'$, K', and so on.

Let $U_+ := \langle U_1, \ldots, U_4 \rangle$ and $U_+' := \langle U_1', \ldots, U_4' \rangle$, and let Y denote the set of isomorphisms from U_+ to U_+' mapping U_i to U_i' for all $i \in \{1, \ldots, 4\}$. By [4, (7.7)], there is a natural one-to-one correspondence between Y and $H_{\Sigma,\Sigma'}$, which is a group isomorphism if $\Omega = \Omega'$ and $\Sigma = \Sigma'$.

A collection of group isomorphisms $\phi_i : U_i \to U_i'$ with $i \in \{1, \ldots, 4\}$ induces an element of Y (which we will then denote by $y(\phi_1, \ldots, \phi_4)$) if and only if it preserves the commutator relations. By (1), this amounts to the conditions

$$\phi_2([\hat{f}(w_1, w_2)]) = [\hat{f}'(\phi_1(w_1), \phi_3(w_2))] \;, \tag{5}$$
$$\phi_3([f(v_1, v_2)]) = [f'(\phi_2(v_1), \phi_4(v_2))] \;, \tag{6}$$
$$\phi_2(vw) = \phi_4(v)\phi_1(w) \;, \tag{7}$$
$$\phi_3(wv) = \phi_1(w)\phi_4(v) \;, \tag{8}$$

for all $v, v_1, v_2 \in V$ and all $w, w_1, w_2 \in W$. We will first show that (5) and (6) follow from (7) and (8). So assume that (7) and (8) hold, for all $v \in V$ and all $w \in W$. Then, by ($\mathbf{F}_{11}$),

$$\begin{aligned} \phi_2([\hat{f}(w_1, w_2)]) &= \phi_2\big(\epsilon(w_1 + w_2) + \epsilon w_1 + \epsilon w_2\big) \\ &= \phi_4(\epsilon)\phi_1(w_1 + w_2) + \phi_4(\epsilon)\phi_1(w_1) + \phi_4(\epsilon)\phi_1(w_2) \\ &= [\hat{f}'(\phi_1(w_1), \phi_1(w_2)\phi_4(\epsilon))] \\ &= [\hat{f}'(\phi_1(w_1), \phi_3(w_2))] \end{aligned}$$

for all $w_1, w_2 \in W$, which shows (5). The proof of (6) is similar.

Now, assume first that $(\varphi, \hat{\varphi}) \in X_{\Omega,\Omega'}$ is a similarity from Ω to Ω', with parameters $g \in K^*$ and $\hat{g} \in L^*$. Let $\phi_1(w) := \hat{\varphi}(w)$, $\phi_2(v) := g^{-1}\varphi(v)$, $\phi_3(w) := \hat{g}^{-1}\hat{\varphi}(w)$ and $\phi_4(v) := \varphi(v)$, for all $v \in V$ and all $w \in W$. Then it follows immediately from (3) and (4) that (7) and (8) hold; hence $(\phi_1, \dots, \phi_4)$ induce an element

$$y(\varphi, \hat{\varphi}) := y(\hat{\varphi}, g^{-1}\varphi, \hat{g}^{-1}\hat{\varphi}, \varphi) \in Y \;. \tag{9}$$

We have thus defined an injective map y from $X_{\Omega,\Omega'}$ to Y. Observe that it follows from (9) that y is a group homomorphism if $\Omega = \Omega'$.

It remains to show that the map y is onto. So let $z \in Y$ be arbitrary, and let $\phi_1, \dots, \phi_4$ denote the restriction of z to the groups $U_1, \dots, U_4$, respectively. Then (5) – (8) hold. It follows from (5) that $\phi_1(\mathrm{Rad}(\hat{f})) = \mathrm{Rad}(\hat{f}') = [K']$; hence $\phi_1(\delta) = [g]$ for some $g \in K'^*$. Similarly, it follows from (6) that $\phi_4(\epsilon) = [\hat{g}]$ for some $\hat{g} \in L'^*$. If we substitute δ for w in (7), then we get that $\phi_2(v) = g\phi_4(v)$ for all $v \in V$, and hence $\phi_4(vw) = g^{-1}\phi_4(v)\phi_1(w)$. Similarly, we have that $\phi_1(wv) = \hat{g}^{-1}\phi_1(w)\phi_4(v)$. Hence (ϕ_4, ϕ_1) is a similarity from Ω to Ω' with parameters $(g^{-1}, \hat{g}^{-1})$, and $z = y(\phi_4, \phi_1)$.

We conclude that there is a natural one-to-one correspondence between Y and $X_{\Omega,\Omega'}$, which is a group isomorphism if $\Omega = \Omega'$ and $\Sigma = \Sigma'$.

Finally, it follows from [4, (4.12)] that there exists a type-preserving isomorphism from $\mathcal{Q}(\Omega)$ to $\mathcal{Q}(\Omega')$ if and only if there exists an isomorphism in $H_{\Sigma,\Sigma'}$ from $\mathcal{Q}(\Omega)$ to $\mathcal{Q}(\Omega')$. The last statement now follows from the fact that $H_{\Sigma,\Sigma'} \neq \emptyset$ if and only if $X_{\Omega,\Omega'} \neq \emptyset$. □

As before, let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system of type F_4, and let H be the pointwise stabilizer of the labeled base apartment $\Sigma = \{0, \ldots, 7\}$ of $\mathcal{Q}(\Omega)$. Let X denote the group of self-similarities of Ω, and let X_ℓ denote its subgroup of linear self-similarities.

Lemma 3. (i) *H acts faithfully on $U_1 \times U_4$.*
(ii) *$H \cong X$.*

Proof. (i) See [4, (33.5)].
(ii) This follows from Theorem 2 with $\Omega = \Omega'$ and $\Sigma = \Sigma'$. □

5. Multipliers of similitudes of *q*

From now on, we will always assume that $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a quadrangular system of type F_4. The first step towards the examination of the automorphisms is the determination of the multipliers of the similitudes of q.

Definition 3. *Let (V_0, q_0) be any regular quadratic space over an arbitrary field K_0. An invertible linear map $T : V_0 \to V_0$ is called a* similitude *with* multiplier *$\lambda \in K_0^*$ if and only if $q_0(Tv) = \lambda q_0(v)$ for all $v \in V_0$. Not every $\lambda \in K_0^*$ can occur as multiplier of some similitude. The set of possible multipliers of similitudes of (V_0, q_0) is denoted by $G(V_0, q_0)$ or by $G(q_0)$ for short. Note that the similitudes with multiplier 1 are exactly the* isometries *of (V_0, q_0).*

The following lemma is crucial. We use a technique similar to the one in the proof of [4, (14.17)].

Lemma 4. *Let $a, d \in V \setminus [L]$ be arbitrary elements such that $f(a, d) = 0$. Let $\gamma := q(a)/q(d)$. If $\gamma \in L$, then $\gamma \in K^2 \cdot \hat{q}(W)$.*

Proof. Let $\xi \in W \setminus [K]$ be arbitrary, and let $e \in V$ be as in Theorem 1. By the same theorem, there exist elements $t_1, t_2, t_3, t_4 \in K$ and $s \in L$ such that $a = t_1 d + t_2 e + t_3 d\xi + t_4 e\xi + [s]$. Since $f(d, d) = f(d, d\xi) = f(d, e\xi) = 0$ and $f(d, e) = 1$, it follows from $f(a, d) = 0$ that $t_2 = 0$. Hence

$$
\begin{aligned}
\gamma &= q(a)/q(d) \\
&= \beta q(t_1 d + t_3 d\xi + t_4 e\xi + [s]) \\
&= \big(t_1^2 + \alpha N(t_3 + t_4\omega)\big) + \beta s\ .
\end{aligned}
$$

Since $\gamma \in L$, it follows that $N(t_3 + t_4\omega) \in L(\beta)$. By [4, (14.9)], we know that $L(\beta) \cap N(E) = K^2 \cdot N(D(\beta))$. In particular,

$$N(t_3 + t_4\omega) = \lambda^2 N(x + \beta y) = \lambda^2\big(N(x) + \beta^2 N(y) + \beta(x\bar{y} + \bar{x}y)\big)$$

for some $\lambda \in K$ and some $x, y \in D$. Hence

$$\gamma = t_1^2 + \alpha\lambda^2\big(N(x) + \beta^2 N(y)\big) + \beta\big(s + \lambda^2(x\bar{y} + \bar{x}y)\big) .$$

Since $\gamma \in L$, $t_1^2 + \alpha\lambda^2\big(N(x) + \beta^2 N(y)\big) \in L$ and $\big(s + \lambda^2(x\bar{y} + \bar{x}y)\big) \in L$, but $\beta \notin L$, it follows that $\big(s + \lambda^2(x\bar{y} + \bar{x}y)\big) = 0$, hence

$$\gamma = t_1^2 + \alpha\lambda^2\big(N(x) + \beta^2 N(y)\big) .$$

If $\lambda = 0$, this implies that $\gamma = t_1^2 = t_1^2\hat{q}(\delta) \in K^2 \cdot \hat{q}(W)$; if $\lambda \neq 0$, it follows that

$$\gamma = \lambda^2 \cdot \Big(\alpha\big(N(x) + \beta^2 N(y)\big) + (\lambda^{-1}t_1)^2\Big) \in K^2 \cdot \hat{q}(W) .$$

□

Lemma 5. *Let $a, d \in V$ be arbitrary elements such that $f(a, d) \neq 0$. Then there exists an element $w \in W$ such that $f(aw, d) = 0$.*

Proof. As in the previous lemma, let $\xi \in W \setminus [K]$ be arbitrary, and let $e \in V$ be as in Theorem 1; then there exist elements $t_1, t_2, t_3, t_4 \in K$ and $s \in L$ such that $a = t_1 d + t_2 e + t_3 d\xi + t_4 e\xi + [s]$. Since $f(a, d) = t_2$, it follows that $t_2 \neq 0$. Let $w := \xi + [\alpha t_4 t_2^{-1}] \in W$. Then, by ($\mathbf{F}_{11}$), $dw = d\xi + \alpha t_4 t_2^{-1} d$, since $[\alpha t_4 t_2^{-1}] \in \mathrm{Rad}(\hat{f})$. Hence

$$\begin{aligned} f(aw, d) &= f(dw, a) = f(d\xi, a) + \alpha t_4 t_2^{-1} f(d, a) \\ &= t_4 f(d\xi, e\xi) + \alpha t_4 t_2^{-1} t_2 = t_4\alpha + \alpha t_4 = 0 , \end{aligned}$$

which is what we had to show. □

Theorem 3. $G(q) = K^2 \cdot \hat{q}(W) \cdot \hat{q}(W) \setminus \{0\}$.

Proof. Let T be an arbitrary similitude of q, with multiplier $\lambda \in K^*$. Then $q(Tv) = \lambda q(v)$ for all $v \in V$. In particular, if $v \in \mathrm{Rad}(f)$, then we have $Tv \in \mathrm{Rad}(f)$ as well; hence $T\epsilon \in \mathrm{Rad}(f) = [L]$. Since $q(\epsilon) = 1$, it follows that $\lambda = q(T\epsilon) \in q([L]) = L$.

Again, let $d \in V \setminus [L]$ and $\xi \in W \setminus [K]$ be arbitrary. If $f(d, Td) = 0$, then it follows from Lemma 4 that $\lambda = q(Td)/q(d) \in K^2 \cdot \hat{q}(W)$, and we are done.

So we can assume that $f(d, Td) \neq 0$. Then it follows from Lemma 5 that there exists an element $w \in W$ such that $f(dw, Td) = 0$. Now consider the maps $T_1 : V \to V : v \mapsto vw$ and $T_2 := T \circ T_1^{-1}$. Then T_1 is a similitude with multiplier $\hat{q}(w)$, since $q(vw) = \hat{q}(w)q(v)$ for all $v \in V$. It follows that T_2 is a similitude as well, with multiplier $\lambda_2 := \lambda\hat{q}(w)^{-1}$. Then $T = T_2 \circ T_1$, and we have that $f(dw, T_2(dw)) = f(dw, Td) = 0$. By the previous paragraph with T_2 in place of T and dw in place of d, we get that $\lambda_2 \in K^2 \cdot \hat{q}(W)$, and it finally follows that $\lambda = \lambda_2\hat{q}(w) \in K^2 \cdot \hat{q}(W) \cdot \hat{q}(W)$.

On the other hand, for every $t \in K^*$ and every $w_1, w_2 \in W^*$, we can find a similitude with multiplier $t^2 \hat{q}(w_1)\hat{q}(w_2)$, namely

$$T_{t,w_1,w_2} : V \to V : v \mapsto tv \cdot w_1 \cdot w_2 \ ,$$

and we are done. □

6. Action on the root groups

We continue to assume that Ω is a quadrangular system of type F_4. Remember that H is the pointwise stabilizer of the labeled base apartment $\Sigma = \{0, \dots, 7\}$ of $\mathcal{Q}(\Omega)$, and that $H^\dagger = H \cap G^\dagger$.

Lemma 6. *$H^\dagger = X_1 X_4$, where $X_i := \langle \mu(U_i^*)\mu(U_i^*) \rangle$ for $i \in \{1, 4\}$.*

Proof. See [4, (33.9)]. The explicit definition of the fundamental maps μ can be found, for example, in [4, (6.1)]. □

Remembering that H acts faithfully on $U_1 \times U_4$, we can now proceed by explicitly calculating the action of $H^\dagger$ on $U_1 \times U_4$. Very similarly as in [4, (33.11)], we get, using [1, Theorems 4.2 and 4.3] that the action of $\mu_1^{(z)} := \mu(x_1(\delta))\mu(x_1(z))$ on $U_1 \times U_4$ corresponds to the map

$$\hat{\theta}_z : (w, v) \mapsto (\hat{q}(z)\hat{\pi}_z(w), vz^{-1})$$

from $W \times V$ to itself, and the action of $\mu_4^{(c)} := \mu(x_4(\epsilon))\mu(x_4(c))$ on $U_1 \times U_4$ corresponds to the map

$$\theta_c : (w, v) \mapsto (wc^{-1}, q(c)\pi_c(v))$$

from $W \times V$ to itself. By Lemma 6, we also know that

$$H^\dagger = \left\langle \mu_1^{(z)}, \mu_4^{(c)} \mid z \in W, c \in V \right\rangle \ .$$

The maps $\hat{\theta}_z$ and θ_c can also be interpreted as self-similarities of Ω. Using the definition of the isomorphism y in (9), we thus get that

$$H^\dagger \cong X^\dagger := \left\langle \hat{\theta}_z, \theta_c \mid z \in W^*, c \in V^* \right\rangle \ .$$

Note that it now follows from Lemma 3.(ii) that

$$H/H^\dagger \cong X/X^\dagger \ . \tag{10}$$

We will now define some useful elements in $X^\dagger$.

Definition 4. *For all $z \in W^*$ and all $c \in V^*$, we define the self-similarities*

$$\hat{\chi}_z := \theta_{[\hat{q}(z)]}\hat{\theta}_z \qquad \textit{and} \qquad \chi_c := \hat{\theta}_{[q(c)]}\theta_c \ .$$

It follows from the fact that $\pi_{[s]} = 1$ and $\hat{\pi}_{[t]} = 1$ for all $s \in L^*$ and all $t \in K^*$ that

$$\hat{\chi}_z(v, w) = (vz, \hat{\pi}_z(w)) \ ,$$
$$\chi_c(v, w) = (\pi_c(v), wc) \ ,$$

for all $c \in V^*, z \in W^*$ and all $v \in V, w \in W$.

Lemma 7. *For all $z \in W^*$, $\hat{\chi}_z$ is a self-similarity of Ω with parameters $(1, \hat{q}(z)^{-1})$; for all $c \in V^*$, χ_c is a self-similarity of Ω with parameters $(q(c)^{-1}, 1)$.*

Proof. We will only show the second statement, the first one being completely similar. So let $c \in V^*$ be arbitrary, and let $\varphi : V \to V : v \mapsto \pi_c(v)$ and $\hat{\varphi} : W \to W : w \mapsto wc$. Then, by Lemma 2.(i and iii),

$$\varphi(vw) = \pi_c(vw) = q(c)^{-1}\pi_c(v) \cdot wc = q(c)^{-1}\varphi(v)\hat{\varphi}(w) \ ,$$
$$\hat{\varphi}(wv) = wvc = (wc)c^{-1}vc = wc \cdot \pi_c(v) = \hat{\varphi}(w)\varphi(v) \ ,$$

for all $v \in V$ and all $w \in W$, and we are done. □

6.1. Field automorphisms

Recall that X_ℓ is the subgroup of X consisting of the linear self-similarities, and that $\mathrm{Aut}(K, L)$ is the group of field automorphisms of K which map L to itself. We will first show that X/X_ℓ is isomorphic to a subgroup of $\mathrm{Aut}(K, L)$.

So let us consider an arbitrary element $\eta := (\varphi, \hat{\varphi}) \in X$, with parameters $(g, \hat{g})$. Recall that

$$\varphi(vw) = g\varphi(v)\hat{\varphi}(w) \qquad (11)$$
$$\hat{\varphi}(wv) = \hat{g}\hat{\varphi}(w)\varphi(v) \qquad (12)$$

for all $v \in V$ and all $w \in W$. If we set $w = \delta$ in (11), then we get that $\varphi(v) = g\varphi(v)\hat{\varphi}(\delta)$, and hence

$$\varphi(v)\hat{\varphi}(\delta) = g^{-1}\varphi(v) = \varphi(v) \cdot [g^{-1}] \ .$$

Since $[g^{-1}] \in \mathrm{Rad}(\hat{f})$, it follows from ($\mathbf{F}_{11}$) that $\varphi(v) \cdot (\hat{\varphi}(\delta) + [g^{-1}]) = 0$, and hence $\hat{\varphi}(\delta) = [g^{-1}]$.

If we apply $\hat{\varphi}$ on ($\mathbf{F}_{12}$), then it follows from (12) that

$$\hat{\varphi}([f(v_1 w, v_2)]) = \hat{g}[f(\varphi(v_1)\hat{\varphi}(w), \varphi(v_2))] \in [K]$$

for all $v_1, v_2 \in V$ and all $w \in W$. Since $\mathrm{Im}(f) = K$, it now follows that $\hat{\varphi}([K]) \subseteq [K]$. We can thus define a map $\rho = \rho_\eta : K \to K$ such that

$$\hat{\varphi}([t]) = [g^{-1}\rho(t)] \tag{13}$$

for all $t \in K$. In particular, $\rho(0) = 0$ and $\rho(1) = 1$.

If we now set $w = [t]$ in (11) for some $t \in K$, then we get that

$$\varphi(tv) = g\varphi(v)\hat{\varphi}([t]) = g\varphi(v)[g^{-1}\rho(t)] = \rho(t)\varphi(v) \tag{14}$$

for all $v \in V$. In particular, it follows from (14) that ρ is multiplicative; hence ρ is a field automorphism of K. Similarly, there is a field automorphism $\hat{\rho} = \hat{\rho}_\eta$ of L such that

$$\hat{\varphi}(sw) = \hat{\rho}(s)\hat{\varphi}(w) \tag{15}$$

for all $s \in L$ and all $w \in W$.

Now let $s \in L$ and $v \in V$ be arbitrary. By (14), we have that $\varphi(sv) = \rho(s)\varphi(v)$. On the other hand, it follows from Remark 2 that $sv = v \cdot s\delta$, and hence, by (11), (15) and Remark 2,

$$\begin{aligned}\varphi(sv) &= \varphi(v \cdot s\delta) = g\varphi(v)\hat{\varphi}(s\delta) = g\varphi(v) \cdot \hat{\rho}(s)\hat{\varphi}(\delta) \\ &= g\varphi(v) \cdot \hat{\rho}(s)[g^{-1}] = g\varphi(v) \cdot [\hat{\rho}(s)g^{-1}] = \hat{\rho}(s)\varphi(v)\ ,\end{aligned}$$

and it follows that $\rho(s) = \hat{\rho}(s)$, for all $s \in L$. Hence $\hat{\rho}$ is the restriction of ρ to L; in particular, $\rho(L) = L$, hence $\rho \in \mathrm{Aut}(K, L)$.

Now let Φ be the map from X to $\mathrm{Aut}(K, L)$ which maps every self-similarity $\eta = (\varphi, \hat{\varphi}) \in X$ to the corresponding $\rho_\eta \in \mathrm{Aut}(K, L)$. Then it follows from the fact that $\varphi(tv) = \rho_\eta(t)\varphi(v)$ for all $t \in K$ and all $v \in V$ that Φ is a group homomorphism. The kernel of Φ consists of the self-similarities η for which the corresponding field automorphisms ρ_η and hence also $\hat{\rho}_\eta$ are trivial – those are precisely the linear self-similarities. Hence $\mathrm{Ker}(\Phi) = X_\ell$, and it follows that

$$X/X_\ell \cong \mathrm{Im}(\Phi) \leq \mathrm{Aut}(K, L)\ , \tag{16}$$

which is what we wanted to show.

From now on, let us assume that $\eta = (\varphi, \hat{\varphi}) \in X_\ell$. Our goal is to show that $\eta \in X^\dagger$, and we will do this in several steps. In each case, we will multiply this given element by other elements in $X^\dagger$ to reduce to the next case.

6.2. Reduction to the case $\hat{g} = 1$

Since $\eta \in X_\ell$, the map ρ is now the identity map, so it follows from (13) that

$$\hat{\varphi}([t]) = [g^{-1}t] \tag{17}$$

for all $t \in K$. Similarly, we have that $\varphi([s]) = [\hat{g}^{-1}s]$ for all $s \in L$; in particular, $\varphi(\epsilon) = [\hat{g}^{-1}]$.

If we set $w = \delta$ in (12), we get by ($\mathbf{F}_5$) that $\hat{\varphi}([q(v)]) = \hat{g}[g^{-1}]\varphi(v)$, and hence, by (17) and ($\mathbf{F}_5$), $[g^{-1}q(v)] = \hat{g}[g^{-1}q(\varphi(v))] = [\hat{g}g^{-1}q(\varphi(v))]$ (see Remark 2). We conclude that

$$q(\varphi(v)) = \hat{g}^{-1}q(v) \tag{18}$$

for all $v \in V$, and hence φ is a similitude of q with multiplier $\hat{g}^{-1}$. It now follows from Theorem 3 that $\hat{g}^{-1} = t_0^2\hat{q}(w_1)\hat{q}(w_2)$ for some $t_0 \in K^*$ and some $w_1, w_2 \in W^*$.

By Lemma 7 and Remark 5, $\zeta := \hat{\chi}_{[t_0]}^{-1}\hat{\chi}_{w_1}^{-1}\hat{\chi}_{w_2}^{-1}$ is a self-similarity of Ω with parameters $(1, t_0^2\hat{q}(w_1)\hat{q}(w_2)) = (1, \hat{g}^{-1})$. Moreover, $\zeta \in X^\dagger$. Again by Remark 5, $\eta\zeta$ is a linear self-similarity with parameters $(g, 1)$, and $\eta \in X^\dagger$ if and only if $\eta\zeta \in X^\dagger$.

We have thus reduced the problem to the case where $\hat{g} = 1$, and we will assume this from now on.

6.3. Reduction to the case $\varphi = 1$

By (18), we now have that $q(\varphi(v)) = q(v)$ for all $v \in V$, that is, φ is an isometry of q. Moreover, $\varphi([s]) = [s]$ for all $s \in L$, that is, φ acts trivially on $[L]$. This allows us to prove a Dieudonné-Cartan-type theorem for φ, even though q is *not* regular and not necessarily finite dimensional. Note, however, that q *is* anisotropic.

Theorem 4. *φ is the product of at most 4 reflections in V.*

Proof. The proof is completely similar to the proof in the regular anisotropic case.

Let J denote the fixed point set of φ. Note that $[L] \subseteq J$, so by Theorem 1.(i), J has codimension at most 4 in V. If $\varphi = 1$, then we are done, so we can assume that $J \neq V$; choose an arbitrary element $d \in V \setminus J$. Let $b := \varphi(d) + d \in V$; then $b \neq 0$. Then $q(d) = q(\varphi(d)) + q(b) + f(b, \varphi(d))$, which implies that $q(b) = f(b, \varphi(d))$. Hence

$$\pi_b\varphi(d) = \varphi(d) + f(\varphi(d), b)q(b)^{-1}b = \varphi(d) + b = d\,,$$

so $\pi_b\varphi$ fixes the element d. On the other hand, if c is an arbitrary element of J, then

$$\pi_b\varphi(c) = \pi_b(c) = c + f(c, b)q(b)^{-1}b = c\,,$$

since

$$f(c, b) = f(c, \varphi(d) + d) = f(\varphi(c), \varphi(d)) + f(c, d) = 0\,.$$

Hence $\pi_b\varphi$ acts trivially on $\langle J, d\rangle$, since φ is K-linear. Since $d \notin J$, we have that $\operatorname{codim}_K\langle J, d\rangle = \operatorname{codim}_K J - 1$. By induction, it now follows that φ is the product of at most 4 reflections. □

By Theorem 4, there exist four elements $c_1, c_2, c_3, c_4 \in V$ such that $\varphi = \pi_{c_1}\pi_{c_2}\pi_{c_3}\pi_{c_4}$ (note that $\pi_\epsilon = 1$). On the other hand,

$$\zeta' := \chi_{c_4}\chi_{c_3}\chi_{c_2}\chi_{c_1} : (v, w) \mapsto (\pi_{c_4}\pi_{c_3}\pi_{c_2}\pi_{c_1}(v), wc_1c_2c_3c_4) \ .$$

Again, note that $\zeta' \in X^\dagger$, and hence $\eta \in X^\dagger$ if and only if $\zeta'\eta \in X^\dagger$. Since

$$\zeta'\eta : (v, w) \mapsto (v, \hat{\varphi}(w)c_1c_2c_3c_4) \ ,$$

we have reduced the problem to the case where $\varphi = 1$, and we will assume this from now on.

6.4. Determination of $\hat{\varphi}$

Our next goal is to show that $g \in L^*$. We start with a lemma.

Lemma 8. *Let $\xi \in W \setminus [K]$ and $z \in W$ be arbitrary elements such that $\hat{f}(\xi v, z) = 0$ for all $v \in V$. Then there exist an element $s \in L$ and an element $t \in K$ such that $z = s\xi + [t]$.*

Proof. Let $d \in V \setminus [L]$ be arbitrary, and let e be as in Theorem 1. Then $z = s_1\xi + s_2\xi ed^{-1} + s_3\xi d^{-1} + s_4\beta^2\xi e + [t]$ for some $s_1, s_2, s_3, s_4 \in L$ and some $t \in K$. It follows from $\hat{f}(\xi, z) = 0$, $\hat{f}(\xi d^{-1}, z) = 0$ and $\hat{f}(\xi e, z) = 0$ that $s_2 = 0$, $s_4 = 0$ and $s_3 = 0$, respectively. We conclude that $z = s_1\xi + [t]$, which is what we had to show. □

We now pick up our examination of $\eta = (1, \hat{\varphi}) \in X_\ell$. Let $w \in W \setminus [K]$ be arbitrary, and let $z := \hat{\varphi}(w)$. Then it follows from (11) that

$$vw = gvz \tag{19}$$

for all $v \in V$. If we apply q on both sides of (19), then we get that

$$\hat{q}(w) = g^2\hat{q}(z) \ . \tag{20}$$

Lemma 9. $\hat{f}(w, z) = 0$.

Proof. Let $v \in V \setminus [L]$ be arbitrary. By Lemma 2.(ii), (19), (20), ($\mathbf{F}_7$) and Remark 2,

$$\begin{aligned} v\hat{\pi}_z(w) &= vz^{-1}wz = \hat{q}(z)^{-1}vzwz = \hat{q}(z)^{-1}g^{-1}vwwz \\ &= \hat{q}(z)^{-1}g^{-1}\hat{q}(w)vz = \hat{q}(z)^{-1}g^{-2}\hat{q}(w)vw = vw \ , \end{aligned}$$

and hence, by ($\mathbf{F}_{11}$),

$$\begin{aligned}[\hat{f}(wv, \hat{\pi}_z(w))] &= v(w + \hat{\pi}_z(w)) + vw + v\hat{\pi}_z(w) \\ &= v \cdot \hat{f}(w, z)\hat{q}(z)^{-1}z \ ,\end{aligned}$$

which implies that $v \cdot \hat{f}(w, z)\hat{q}(z)^{-1}z \in [L]$. Since $v \notin [L]$, it follows that $\hat{f}(w, z) = 0$. □

Lemma 10. $\hat{f}(wv, z) = 0$ *for all* $v \in V$.

Proof. By Lemma 9, $\hat{\pi}_z(w) = w$. It thus follows from Lemma 2.(iv) that

$$wv + \hat{f}(wv, z)\hat{q}(z)^{-1}z = \hat{q}(z)^{-1}w \cdot vz \ .$$

By (19), Remark 2, (20) and ($\mathbf{F}_{10}$),

$$\hat{q}(z)^{-1}w \cdot vz = \hat{q}(z)^{-1}g^{-2}w \cdot vw = \hat{q}(w)^{-1}\hat{q}(w)wv = wv \ ,$$

and it follows that $\hat{f}(wv, z)\hat{q}(z)^{-1}z = 0$, which implies that $\hat{f}(wv, z) = 0$. □

By Lemma 10, we are now ready to invoke Lemma 8. It follows that $z = sw + [t]$ for some $s \in L$ and some $t \in K$. If we plug in this expression for z in (19), then we get that

$$vw = gv(sw + [t]) = (sg)vw + (tg)v \ .$$

But since we chose $w \notin [K]$, the elements v and vw are linearly independent (for if there were an $x \in K$ such that $vw = xv$, then $v(w + [x]) = 0$ by ($\mathbf{F}_{11}$) and hence $w = [x] \in [K]$). It thus follows that $tg = 0$ and $sg = 1$, hence s is invertible, $g = s^{-1} \in L^*$, and $z = g^{-1}w$.

Since $w \in W \setminus [K]$ was arbitrary, it follows that $\hat{\varphi}(w) = g^{-1}w$ for all $w \in W \setminus [K]$. Moreover, since $g^{-1} \in L$, it follows from (17) and Remark 2 that $\hat{\varphi}([t]) = g^{-1}[t]$ for all $w = [t] \in [K]$ as well. So $\hat{\varphi}$ is just scalar multiplication by the element $g^{-1} \in L^*$.

It now suffices to observe that $\pi_{[g^{-1}]} = 1$ to conclude that $\eta = \chi_{[g^{-1}]}$. Since $\chi_{[g^{-1}]} \in X^\dagger$, we have shown that $X_\ell = X^\dagger$.

It finally follows from (2), (10) and (16) that

$$G/G^\dagger \cong H/H^\dagger \cong X/X^\dagger = X/X_\ell \cong \mathrm{Im}(\Phi) \leq \mathrm{Aut}(K, L) \ ,$$

which proves the Main Theorem.

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