An algebraic structure for Moufang quadrangles

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Abstract

Very recently, the classification of Moufang polygons has been completed by Tits and Weiss. Moufang n-gons exist for $n \in \{3, 4, 6, 8\}$ only. For $n \in \{3, 6, 8\}$, the proof is nicely divided into two parts: first, it is shown that a Moufang n-gon can be parametrized by a certain interesting algebraic structure, and secondly, these algebraic structures are classified. The classification of Moufang quadrangles (n=4) is not organized in this way due to the absence of a suitable algebraic structure. The goal of this article is to present such a uniform algebraic structure for Moufang quadrangles, and to classify these structures without referring back to the original Moufang quadrangles from which they arise, thereby also providing a new proof for the classification of Moufang quadrangles, which does consist of the division into these two parts. We hope that these algebraic structures will prove to be interesting in their own right.

Received by the editor September 19, 2002.

2000 *Mathematics Subject Classification.* 51E12, 16W10, 20E42.

Key words and phrases. Moufang quadrangles, quadrangular systems.

The author is a Research Assistant of the Fund for Scientific Research - Flanders (Belgium) (F.W.O.-Vlaanderen).

CHAPTER 1

Introduction

The irreducible spherical buildings of rank at least three have been classified by J. Tits in 1974 [**15**]. The irreducible spherical buildings of rank two – which are called generalized polygons – are too numerous to classify, but in the addenda of [**15**], the Moufang condition for spherical buildings was introduced, and it was observed that every thick irreducible spherical building of rank at least three as well as every irreducible residue of such a building satisfies the Moufang condition. In this sense, the Moufang polygons are the "building bricks" of any spherical building of rank at least three.

Very recently, the classification of Moufang polygons has been completed by J. Tits and R. Weiss in [**20**]. It was first shown by J. Tits (see [**17**] and [**18**]) that Moufang n-gons exist for $n \in \{3, 4, 6, 8\}$ only; see also [**21**]. For $n \in \{3, 6, 8\}$, the proof is divided into two parts, namely (A) it is shown that a Moufang n-gon can be parametrized by a certain algebraic structure, and (B) these algebraic structures are classified.

More precisely, it was already shown in 1933 (but in a slightly different form; see [**2**] or [**5**]) by R. Moufang (see [**11**]) that all Moufang triangles can be described by an alternative division ring, a notion which had been introduced by M. Zorn (see [**22**]). These alternative division rings were classified by R. Bruck and E. Kleinfeld in 1951; see [**3**]. The Moufang hexagons are described by unital quadratic Jordan division algebras of degree three, also known as anisotropic cubic norm structures (see [**16**]). These structures have been classified in its full generality in 1986 by H. Petersson and M. Racine (see [**13**] and [**14**]), whose proof is built on earlier work by A. Albert [**1**], F.D. Jacobson and N. Jacobson [**6**], N. Jacobson [**7**], [**8**] and K. McCrimmon [**9**], [**10**]. The Moufang octagons, finally, can be described by a so-called octagonal system, as was shown by J. Tits in 1983 (see [**19**]); since these systems have a very simple description, there is no need for part (B) in this case.

The classification of Moufang quadrangles (n=4) in [**20**] is not organized in this way due to the absence of a suitable algebraic structure. Instead, there are six different parameter systems, and even then, the division of the proof into parts (A) and (B) is missing in the two cases which lead to the exceptional quadrangles. Surprisingly, one of these classes, namely the exceptional quadrangles of type F_4, had only recently been discovered by R. Weiss during the classification process; see also [**12**].

The goal of this article is to present a uniform algebraic structure for Moufang quadrangles. These "quadrangular systems" reveal some of the structure of Moufang quadrangles which is hard to see without them. For example, we have successfully used them to answer a basic question about the automorphism group of the Moufang quadrangles of type F_4 left open in (37.38) of [**20**]; see [**4**]. Moreover, it is possible to classify these structures without referring back to the original

Moufang quadrangles from which they arise, thereby providing a new proof for the classification of Moufang quadrangles, which does consist of the division into parts (A) and (B).

The Moufang hexagons all arise from forms of algebraic groups of type G_2, ${}^3D_4$, E_6 or E_8, or they are of mixed type associated with groups of type G_2. The Moufang quadrangles arise either from certain classical groups or from forms of algebraic groups of type E_6, E_7 or E_8 or are of mixed type associated with groups of type B_2 or F_4. The quadrangular systems parametrize the Moufang quadrangles in the same way that the Jordan algebras mentioned above parametrize the Moufang hexagons, and it is our hope that the quadrangular systems will turn out to be equally interesting objects of study.

We start by giving the (ad hoc) definition of the quadrangular systems. Considering the background of the Moufang quadrangles, it should not be too surprising that we need a large number of axioms to describe these systems. In the next chapter, we give some elementary properties of these systems. In chapter 4, we explain how to construct a Moufang quadrangle starting from an arbitrary quadrangular system. In chapter 5, we show that every Moufang quadrangle arises in this way. After a couple of remarks in chapter 6, we present a list of 6 examples of quadrangular systems, which corresponds to the 6 different classes of Moufang quadrangles as described in [**20**]. Finally, chapter 8 is devoted to the classification of the quadrangular systems. We conclude with an appendix in which we restate the axiom system for abelian quadrangular systems and for some specific subclasses of those.

Acknowledgment

I am very grateful to Richard Weiss, for providing me a copy of his book "Moufang Polygons" [**20**] prior to publication, and for the many illuminating discussions we have had on this topic.

CHAPTER 2

Definition

Throughout this article, we will use the following notation. If S is a group, then we define $S^* := S \setminus \{\text{neutral element}\}$. If S is a set which contains an element called "0", then we define $S^* := S \setminus \{0\}$. It will always be clear from the context which definition we mean.

Consider an abelian group $(V, +)$ and a (possibly non-abelian) group $(W, \boxplus)$. The inverse of an element $w \in W$ will be denoted by $\boxminus w$, and by $w_1 \boxminus w_2$, we mean $w_1 \boxplus (\boxminus w_2)$. Suppose that there is a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W, both of which will be denoted by $\cdot$ or simply by juxtaposition, i.e. $\tau_V(v, w) = vw = v \cdot w$ and $\tau_W(w, v) = wv = w \cdot v$ for all $v \in V$ and all $w \in W$. Consider a map F from $V \times V$ to W and a map H from $W \times W$ to V, both of which are "bi-additive" in the sense that

$$\begin{aligned} F(v_1 + v_2, v) &= F(v_1, v) \boxplus F(v_2, v)\,; \\ F(v, v_1 + v_2) &= F(v, v_1) \boxplus F(v, v_2)\,; \\ H(w_1 \boxplus w_2, w) &= H(w_1, w) + H(w_2, w)\,; \\ H(w, w_1 \boxplus w_2) &= H(w, w_1) + H(w, w_2)\,; \end{aligned}$$

for all $v, v_1, v_2 \in V$ and all $w, w_1, w_2 \in W$. Suppose furthermore that there exists a fixed element $\epsilon \in V^*$ and a fixed element $\delta \in W^*$, and suppose that, for each $v \in V^*$, there exists an element $v^{-1} \in V^*$, and for each $w \in W^*$, there exists an element $\kappa(w) \in W^*$, such that, for all $w, w_1, w_2 \in W$ and all $v, v_1, v_2 \in V$, the following axioms are satisfied. We define

$$\begin{aligned} \overline{v} &:= \epsilon F(\epsilon, v) - v \\ \mathrm{Rad}(F) &:= \{v \in V \mid F(v, V) = 0\} \\ \mathrm{Rad}(H) &:= \{w \in W \mid H(w, W) = 0\} \\ \mathrm{Im}(F) &:= F(V, V) \\ \mathrm{Im}(H) &:= H(W, W) \end{aligned}$$

($\mathbf{Q}_1$) $w\epsilon = w$.
($\mathbf{Q}_2$) $v\delta = v$.
($\mathbf{Q}_3$) $(w_1 \boxplus w_2)v = w_1 v \boxplus w_2 v$.
($\mathbf{Q}_4$) $(v_1 + v_2)w = v_1 w + v_2 w$.
($\mathbf{Q}_5$) $w(-\epsilon) \cdot v = w(-v)$.
($\mathbf{Q}_6$) $v \cdot w(-\epsilon) = vw$.
($\mathbf{Q}_7$) $\mathrm{Im}(F) \subseteq \mathrm{Rad}(H)$.
($\mathbf{Q}_8$) $[w_1, w_2 v]_{\boxplus} = F(H(w_2, w_1), v)$.
($\mathbf{Q}_9$) $\delta \in \mathrm{Rad}(H)$.
($\mathbf{Q}_{10}$) If $\mathrm{Rad}(F) \neq 0$, then $\epsilon \in \mathrm{Rad}(F)$.

($\mathbf{Q}_{11}$) $w(v_1 + v_2) = wv_1 \boxplus wv_2 \boxplus F(v_2 w, v_1)$.
($\mathbf{Q}_{12}$) $v(w_1 \boxplus w_2) = vw_1 + vw_2 + H(w_2, w_1 v)$.
($\mathbf{Q}_{13}$) $(v^{-1})^{-1} = v$ (if $v \neq 0$).
($\mathbf{Q}_{14}$) $\kappa(\boxminus\kappa(\boxminus w)) = w(-\epsilon)$ (if $w \neq 0$).
($\mathbf{Q}_{15}$) $wv \cdot v^{-1} = w$ (if $v \neq 0$).
($\mathbf{Q}_{16}$) $v^{-1} \cdot wv = -\overline{v(\boxminus w)}$ (if $v \neq 0$).
($\mathbf{Q}_{17}$) $F(v_1^{-1}, \overline{v_2})v_1 = F(v_1, v_2)$ (if $v_1 \neq 0$).
($\mathbf{Q}_{18}$) $v\kappa(w) \cdot (\boxminus w) = -v$ (if $w \neq 0$).
($\mathbf{Q}_{19}$) $w \cdot v\kappa(w) = \kappa(w)v$ (if $w \neq 0$).
($\mathbf{Q}_{20}$) $H(\kappa(w_1), w_2)w_1 = H(w_1, w_2)$ (if $w_1 \neq 0$).

Then we call the system $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ a *quadrangular system.* Note that we omit the maps F and H in our notation, as well as the maps $v \mapsto v^{-1}$ and $w \mapsto \kappa(w)$. The reason is that they are uniquely determined by V, W, τ_V, τ_W, ϵ and δ; see Theorem 3.7.

REMARK 2.1. We will sometimes think about the maps τ_V from $V \times W$ to V and τ_W from $W \times V$ to W as "actions", since it will turn out that, for every $w \in W^*$, the map from V to itself which maps v to vw for every $v \in V$ is an automorphism of V; similarly, for every $v \in V^*$, the map from W to itself which maps w to wv for every $w \in W$ is an automorphism of W; see Theorem 3.6. Note, however, that these maps are no group actions in the proper sense of the word, since $v(w_1 \boxplus w_2) \neq vw_1 \cdot w_2$ and $w(v_1 + v_2) \neq wv_1 \cdot v_2$ in general.

REMARK 2.2. In writing down these axioms, we used the convention that the maps which are denoted by juxtaposition preceed those which are denoted by "·". Note, however, that there is no danger of confusion, since we have not defined a multiplication on V or on W. Hence we will often write wvv^{-1} instead of $wv \cdot v^{-1}$, for example.

We will show in Theorem 3.8 below that the following two identities are satisfied for every quadrangular system, for all $v_1, v_2 \in V$ and all $w_1, w_2 \in W$.

($\mathbf{Q}_{21}$) $F(v_1, v_2) = F(v_2, v_1)$.
($\mathbf{Q}_{22}$) $H(w_1, w_2) = -\overline{H(w_2, w_1)}$.

REMARK 2.3. These two identities show that, in some sense, F is a symmetric form and H is a skew-hermitian form. Note, however, that V and W are *not* vector spaces in general.

Moreover, we will show in Theorem 6.1 that the following four identities are satisfied for every quadrangular system, for all $v, c \in V$ and all $w, z \in W$. We first introduce the notion of a *reflection*, which is a direct generalization of the classical notion of a reflection in a quadratic space:

$$\pi_v(c) := c - vF(v^{-1}, \overline{c}) \qquad \text{(if } v \neq 0\text{)}$$
$$\Pi_w(z) := z \boxplus w(-H(\kappa(w), z)) \qquad \text{(if } w \neq 0\text{)} .$$

Then

($\mathbf{Q}_{23}$) $v \cdot \Pi_w(z) = -\overline{\overline{\overline{\overline{v}(\boxminus w)}z}\kappa(w)}$ (if $w \neq 0$).
($\mathbf{Q}_{24}$) $w \cdot \overline{\pi_v(\epsilon)^{-1}} \cdot \overline{\pi_v(c)} = wvcv^{-1}$ (if $v \neq 0$).
($\mathbf{Q}_{25}$) $\pi_v(\overline{c \cdot \delta v})w = \pi_v(\overline{c \cdot wv})$ (if $v \neq 0$).
($\mathbf{Q}_{26}$) $\Pi_{\boxminus z}(w \cdot \epsilon z)v = \Pi_{\boxminus z}(w \cdot vz)$ (if $w \neq 0$).

Let $\Omega := (V, W, \tau_V, \tau_W, \epsilon, \delta)$ and $\Omega' := (V', W', \tau_{V'}, \tau_{W'}, \epsilon', \delta')$ be two quadrangular systems. We say that (ϕ, ψ) is a *morphism* from Ω to Ω' if and only if ϕ is a morphism from V to V' and ψ is a morphism from W to W' such that $\phi(\epsilon) = \epsilon'$, $\psi(\delta) = \delta'$, $\phi(vw) = \phi(v)\psi(w)$ and $\psi(wv) = \psi(w)\phi(v)$, for all $v \in V$ and all $w \in W$.

A morphism (ϕ, ψ) is called an *monomorphism* (respectively *epimorphism*, *isomorphism*) if and only if both ϕ and ψ are monomorphisms (respectively epimorphisms, isomorphisms). We call Ω and Ω' *isomorphic* if and only if there exists an isomorphism (ϕ, ψ) from Ω to Ω'.

CHAPTER 3

Some Identities

We will now prove some identities which we will use in the construction of the Moufang quadrangles in chapter 4, and which will also be used in the classification of quadrangular systems.

Definition 3.1. For each $w \in W^*$, we define $\lambda(w) := \boxminus\kappa(\boxminus w)$. Using this definition, $(\mathbf{Q}_{14})$ can be rephrased as $\kappa(\lambda(w)) = w(-\epsilon)$.

Lemma 3.2. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $w \in W^*$ and all $v \in V$, we have that*

(i) $vw\kappa(\boxminus w) = -v$;
(ii) $\kappa(w)(v(\boxminus w)) = w(-v)$.

Proof. If we plug in $\lambda(w)$ for w in $(\mathbf{Q}_{18})$, then it follows from $(\mathbf{Q}_{14})$ that $v(w(-\epsilon))(\boxminus\lambda(w)) = -v$, and by $(\mathbf{Q}_6)$ and the definition of λ, this is equivalent to $vw\kappa(\boxminus w) = -v$, which proves (i).

If we plug in $\lambda(w)$ for w in $(\mathbf{Q}_{19})$, then we get, again by $(\mathbf{Q}_{14})$, that $\lambda(w)(v \cdot w(-\epsilon)) = w(-\epsilon)v$. By $(\mathbf{Q}_6)$, $(\mathbf{Q}_5)$ and the definition of λ, this is equivalent to $\boxminus\kappa(\boxminus w)(vw) = w(-v)$. Replacing w by $\boxminus w$ now yields (ii). $\square$

Lemma 3.3. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then the following holds, for all $w \in W$ and all $v \in V$:*

(i) $wv = 0 \iff w = 0$ *or* $v = 0$;
(ii) $vw = 0 \iff v = 0$ *or* $w = 0$.

Proof. We will only prove statement (i); because of Lemma 3.2(i), the proof of (ii) is completely similar. By choosing $v_1 = v_2 = 0$ in $(\mathbf{Q}_{11})$, we get $w0 = w0 \boxplus w0$, from which it follows that $w0 = 0$. Similarly, it follows from $(\mathbf{Q}_3)$ that $0v = 0$.

On the other hand, suppose that $wv = 0$. If $v \neq 0$, then it follows from $(\mathbf{Q}_{15})$ that $w = wvv^{-1} = 0v^{-1} = 0$. $\square$

Lemma 3.4. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $w \in W$ and all $v \in V$, we have :*

(i) $(\boxminus w)v = \boxminus(wv)$;
(ii) $(-v)w = -(vw)$.

It follows that the notations $\boxminus wv$ and $-vw$ are unambiguous.

Proof. By putting $w_1 = w$ and $w_2 = \boxminus w$ in $(\mathbf{Q}_3)$, we get $0v = wv \boxplus (\boxminus w)v$, from which it follows that $(\boxminus w)v = \boxminus(wv)$. Similarly, (ii) follows from $(\mathbf{Q}_4)$. $\square$

Lemma 3.5. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then the following holds, for all $w \in W$ and all $v \in V$:*

(i) $w_1v = w_2v \iff w_1 = w_2$ *or* $v = 0$;

(ii) $v_1 w = v_2 w \iff v_1 = v_2$ *or* $w = 0$.

Proof. By ($\mathbf{Q}_3$) and Lemma 3.4(i), we have $(w_1 \boxminus w_2)v = w_1 v \boxminus w_2 v$, and so (i) is an immediate consequence of Lemma 3.3(i). Similarly, (ii) follows from Lemma 3.3(ii). □

Theorem 3.6. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then*

(i) *for every $w \in W^*$, the map from V to itself which maps v to vw for every $v \in V$ is an automorphism of V ;*

(ii) *for every $v \in V^*$, the map from W to itself which maps w to wv for every $w \in W$ is an automorphism of W .*

Proof. We will only show (i), the proof of (ii) being completely similar. So let $w \in W^*$ be arbitrary, and let α be the map from V to itself which maps v to vw for every $v \in V$. By Lemma 3.3(ii), we have that $\alpha(0) = 0$, and it follows from ($\mathbf{Q}_4$) that $\alpha(v_1 + v_2) = \alpha(v_1) + \alpha(v_2)$ for all $v_1, v_2 \in V$, so α is a group morphism. Since $w \neq 0$, it follows from Lemma 3.5(ii) that α is injective. Finally, it follows from ($\mathbf{Q}_{18}$) that $\alpha\big(-v\kappa(\boxminus w)\big) = v$ for all $v \in V$, hence α is surjective as well, and we are done. □

Theorem 3.7. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then the maps F and H and the maps $v \mapsto v^{-1}$ and $w \mapsto \kappa(w)$ are uniquely determined.*

Proof. By ($\mathbf{Q}_{11}$), $F(v_2, v_1) = \boxminus\delta v_2 \boxminus \delta v_1 \boxplus \delta(v_1 + v_2)$, so F is uniquely determined. Note that this implies that the map $v \mapsto \overline{v}$ is uniquely determined as well. By ($\mathbf{Q}_{12}$), $H(w_2, w_1) = -\epsilon w_2 - \epsilon w_1 + \epsilon(w_1 + w_2)$, so H is uniquely determined. Suppose that v^* were another "inverse" of v. Then it would follow from ($\mathbf{Q}_{16}$) that $v^*(wv) = v^{-1}(wv)$, but then Lemma 3.5 would imply that $v^* = v^{-1}$ after all. Similarly, it follows from Lemma 3.2(ii) that the map κ is uniquely determined. □

Theorem 3.8. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then the identities ($\mathbf{Q}_{21}$) and ($\mathbf{Q}_{22}$) are satisfied for all $v_1, v_2 \in V$ and all $w_1, w_2 \in W$.*

Proof. We will first show that ($\mathbf{Q}_{21}$) follows from ($\mathbf{Q}_8$), ($\mathbf{Q}_9$) and ($\mathbf{Q}_{11}$). Since V is abelian, $\delta(v_1 + v_2) = \delta(v_2 + v_1)$, and hence, by ($\mathbf{Q}_{11}$), we have that

$$\delta v_1 \boxplus \delta v_2 \boxplus F(v_2, v_1) = \delta v_2 \boxplus \delta v_1 \boxplus F(v_1, v_2) \ ,$$

for all $v_1, v_2 \in V$. In order to show ($\mathbf{Q}_{21}$), it thus suffices to show that δv_1 and δv_2 commute for all $v_1, v_2 \in V$. By ($\mathbf{Q}_8$) and ($\mathbf{Q}_9$),

$$[\delta v_1, \delta v_2]_{\boxplus} = F(H(\delta, \delta v_1), v_2) = 0$$

for all $v_1, v_2 \in V$, and hence ($\mathbf{Q}_{21}$) holds.

Similarly, we will show that ($\mathbf{Q}_{22}$) follows from ($\mathbf{Q}_{12}$), ($\mathbf{Q}_{15}$) and ($\mathbf{Q}_{16}$). By substituting $\boxminus w$ for w and ϵ for v in ($\mathbf{Q}_{16}$), we have that $\overline{\epsilon w} = -\epsilon^{-1}(\boxminus w)$ for all $w \in W$. Moreover, by ($\mathbf{Q}_{15}$), $w\epsilon^{-1} = w$ for all $w \in W$. By ($\mathbf{Q}_{12}$) and the fact that H is additive in both variables, we thus have that

$$\begin{aligned}
\overline{H(w_2, w_1)} &= \overline{\epsilon(w_1 \boxplus w_2)} - \overline{\epsilon w_1} - \overline{\epsilon w_2} \\
&= -\epsilon^{-1}(\boxminus w_2 \boxminus w_1) + \epsilon^{-1}(\boxminus w_1) + \epsilon^{-1}(\boxminus w_2) \\
&= -H(\boxminus w_1, \boxminus w_2 \epsilon^{-1}) \\
&= -H(w_1, w_2)
\end{aligned}$$

for all $w_1, w_2 \in W$, hence ($\mathbf{Q}_{22}$) holds. □

Lemma 3.9. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then the map $v \mapsto \overline{v}$ is additive. In particular, we have that $\overline{-v} = -\overline{v}$ for all $v \in V$. Moreover, for all $c \in V^*$, the map π_c is additive. In particular, we have that $\pi_c(-v) = -\pi_c(v)$ for all $v \in V$.*

Proof. It follows from ($\mathbf{Q}_7$) that $H(F(\epsilon, v_2), F(\epsilon, v_1)) = 0$, for all $v_1, v_2 \in V$. Hence

$$\begin{aligned} \epsilon F(\epsilon, v_1 + v_2) &= \epsilon(F(\epsilon, v_1) \boxplus F(\epsilon, v_2)) \\ &= \epsilon F(\epsilon, v_1) + \epsilon F(\epsilon, v_2) \ , \end{aligned}$$

by ($\mathbf{Q}_{12}$). Since $\overline{v} = \epsilon F(\epsilon, v) - v$, it follows from this that the map $v \mapsto \overline{v}$ is additive. Similarly, it follows from ($\mathbf{Q}_7$) that $H(F(c^{-1}, \overline{v_2}), F(c^{-1}, \overline{v_1})c) = 0$, for all $c \in V^*$ and all $v_1, v_2 \in V$. Since the map $v \mapsto \overline{v}$ is additive, it now follows, again by ($\mathbf{Q}_{12}$), that

$$\begin{aligned} \pi_c(v_1 + v_2) &= (v_1 + v_2) - cF(c^{-1}, \overline{v_1 + v_2}) \\ &= v_1 + v_2 - c(F(c^{-1}, \overline{v_1}) \boxplus F(c^{-1}, \overline{v_2})) \\ &= v_1 - cF(c^{-1}, \overline{v_1}) + v_2 - cF(c^{-1}, \overline{v_2}) \\ &= \pi_c(v_1) + \pi_c(v_2) \ , \end{aligned}$$

which is what we wanted to show. □

Lemma 3.10. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $w \in W^*$ and all $v \in V^*$, we have*

(i) $(-v)^{-1} = -(v^{-1})$;
(ii) $\kappa(\boxminus w) = \boxminus \lambda(w)$.

Proof. If we replace w by δ in ($\mathbf{Q}_{16}$), we have that $-(v^{-1})(\delta v) = \overline{v(\boxminus\delta)}$. If we replace w by $\delta(-\epsilon)$ in the same identity ($\mathbf{Q}_{16}$), then we get, by ($\mathbf{Q}_5$) and ($\mathbf{Q}_6$) that $v^{-1}(\delta(-v)) = -\overline{v(\boxminus\delta)}$. If we replace v by $-v$ in this identity, then we get, using the fact that $\overline{-v} = -\overline{v}$, that $(-v)^{-1}(\delta v) = \overline{v(\boxminus\delta)}$. It follows that $(-v)^{-1}(\delta v) = -(v^{-1})(\delta v)$. Since δv is non-zero, this implies, by Lemma 3.5(i), that $(-v)^{-1} = -(v^{-1})$, which proves (i). Identity (ii) follows immediately from the definition of λ. □

Lemma 3.11. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $w \in W$ and all $v \in V$, we have*

$$wv(-\epsilon) = w(-v) \ .$$

Proof. Note that this identity is trivial if $v = 0$, so assume $v \neq 0$. By ($\mathbf{Q}_{15}$) and Lemma 3.10(i), we have that $wvv^{-1} = w(-v)(-v^{-1})$. It follows, by ($\mathbf{Q}_5$), that $wv(-\epsilon)(-v^{-1}) = w(-v)(-v^{-1})$. By Lemma 3.5(i), this implies that $wv(-\epsilon) = w(-v)$, for all $w \in W$ and all $v \in V$. □

Lemma 3.12. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $v \in V$, we have*

$$\overline{\overline{v}} = v \ .$$

Proof. Assume $v \neq 0$. By replacing w by $\boxminus\delta$ in ($\mathbf{Q}_{16}$), we see that $\overline{v} = -v^{-1}(\boxminus\delta v)$. If, on the other hand, we replace v by v^{-1} and w by δv in this same identity ($\mathbf{Q}_{16}$), then we get $v(\delta v v^{-1}) = -\overline{v^{-1}(\boxminus\delta v)}$. Combining those two equalities gives us the required identity $\overline{\overline{v}} = v$, since $v(\delta v v^{-1})$ is equal to v because of $(\mathbf{Q}_{15})$ and $(\mathbf{Q}_2)$. □

LEMMA 3.13. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $w \in W$ and all $v \in V$, we have*

(i) $w(-v) = F(vw, v) \boxminus wv$ *;*
(ii) $v(\boxminus w) = H(w, wv) - vw$ *.*

PROOF. If we put $v_1 = -v$ and $v_2 = v$ in $(\mathbf{Q}_{11})$, then we get that $w0 = w(-v) \boxplus wv \boxplus F(vw, -v)$. Since F is additive in both variables, this is equivalent to $w(-v) = F(vw, v) \boxminus wv$, which proves (i). Similarly, (ii) follows from $(\mathbf{Q}_{12})$. □

LEMMA 3.14. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $v \in V^*$, we have*

(i) $\kappa(\delta v) = \delta(\overline{v})^{-1}$ *;*
(ii) $\overline{v^{-1}} = (\overline{v})^{-1}$ *.*

PROOF. If we substitute δv for w and $-v^{-1}$ for v in Lemma 3.2(ii), then we get that

$$\kappa(\delta v) \cdot (-v^{-1}(\boxminus \delta v)) = \delta v v^{-1} ,$$

and hence, by $(\mathbf{Q}_{16})$ and $(\mathbf{Q}_{15})$,

$$\kappa(\delta v) \cdot \overline{v} = \delta .$$

By $(\mathbf{Q}_{15})$, it thus follows that $\kappa(\delta v) = \delta(\overline{v})^{-1}$, which shows (i). Note that it follows from Lemma 3.13(ii) that $v(\boxminus \delta) = -v$ for all $v \in V$, since $\delta \in \mathrm{Rad}(H)$ by $(\mathbf{Q}_9)$. By Lemma 3.2(i) with $\boxminus \delta \overline{v}$ in place of w, $(\mathbf{Q}_{16})$ with $\overline{v}$ in place of v and $\boxminus \delta$ in place of w, Lemma 3.12, and (i) with $\overline{v}$ in place of v, we now have that

$$\begin{aligned}(\overline{v})^{-1} &= -(\overline{v})^{-1} \cdot (\boxminus \delta \overline{v}) \cdot \kappa(\delta \overline{v}) \\ &= v \cdot \delta v^{-1} = -\overline{v^{-1}(\boxminus \delta)} = \overline{v^{-1}} ,\end{aligned}$$

which shows (ii). □

LEMMA 3.15. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $v_1, v_2 \in V$ and all $w_1, w_2 \in W$, we have that*

(i) $F(v_1, v_2)(-\epsilon) = F(v_1, v_2)$ *;*
(ii) $H(\kappa(w_1(-\epsilon)), w_2) = -H(\kappa(w_1), w_2)$ *.*

PROOF. If we substitute $\overline{v_2}$ for v_2 in $(\mathbf{Q}_{17})$, then we get, using $(\mathbf{Q}_{21})$, that $F(v_2, v_1^{-1})v_1 = F(v_1, \overline{v_2})$. Replacing v_1 by $-v_1$ in this last identity and applying Lemma 3.10(i) yields $F(v_2, v_1^{-1})(-v_1) = F(v_1, \overline{v_2})$. Thus, by $(\mathbf{Q}_5)$, $F(v_2, v_1^{-1})(-\epsilon)v_1 = F(v_2, v_1^{-1})v_1$, and it follows from Lemma 3.5(i) that $F(v_2, v_1^{-1})(-\epsilon) = F(v_2, v_1^{-1})$. Replacing v_1 by v_1^{-1} and using $(\mathbf{Q}_{13})$ completes the proof of (i).

The proof of (ii) is similar. If we substitute $w_1(-\epsilon)$ for w_1 in $(\mathbf{Q}_{20})$, then we get that $H(\kappa(w_1(-\epsilon)), w_2) \cdot w_1(-\epsilon) = H(w_1(-\epsilon), w_2)$. On the other hand, since $\mathrm{Im}(F) \subseteq \mathrm{Rad}(H)$ by $(\mathbf{Q}_7)$, it follows from Lemma 3.13(i) that $H(w_1(-\epsilon), w_2) = H(\boxminus w_1, w_2) = -H(w_1, w_2)$. Hence

$$H(\kappa(w_1(-\epsilon)), w_2) \cdot w_1(-\epsilon) = -H(w_1, w_2) ,$$

and it follows from $(\mathbf{Q}_6)$ and $(\mathbf{Q}_{20})$ that

$$H(\kappa(w_1(-\epsilon)), w_2) \cdot w_1 = -H(\kappa(w_1), w_2) \cdot w_1 .$$

It now follows from Lemma 3.5(ii) that (ii) holds. □

LEMMA 3.16. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $w \in W$ and all $v \in V$, we have*

(i) $F(\overline{vw}, v^{-1}) \boxminus w(-v)v^{-1} = w$ *(if $v \neq 0$);*
(ii) $H(w, \kappa(w)(-v)) + v\kappa(w)w = v$ *(if $w \neq 0$).*

PROOF. Putting $v_1 = v$ and $v_2 = vw$ in ($\mathbf{Q}_{17}$), and using ($\mathbf{Q}_{21}$), yields $F(\overline{vw}, v^{-1})v = F(vw, v)$, from which it follows, by ($\mathbf{Q}_{15}$), that $F(\overline{vw}, v^{-1}) = F(vw, v)v^{-1}$. It follows from Lemma 3.13(i) and from ($\mathbf{Q}_3$) that

$$\begin{aligned} w(-v)v^{-1} &= (F(vw, v) \boxminus wv)v^{-1} \\ &= F(vw, v)v^{-1} \boxminus wvv^{-1} \\ &= F(\overline{vw}, v^{-1}) \boxminus w \ , \end{aligned}$$

from which (i) follows, since $\mathrm{Im}(F) \subseteq Z(W)$ by ($\mathbf{Q}_7$) and ($\mathbf{Q}_8$).

If we plug in $v\kappa(w)$ for v in Lemma 3.13(ii), we get

$$v\kappa(w)(\boxminus w) = H(w, w \cdot v\kappa(w)) - v\kappa(w)w \ ,$$

and applying ($\mathbf{Q}_{18}$) and ($\mathbf{Q}_{19}$) yields $-v = H(w, \kappa(w)v) - v\kappa(w)w$. Replacing v by $-v$ gives us the required identity (ii). □

THEOREM 3.17. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $w_1 \in W^*$, $w_2 \in W$, $v_1 \in V^*$ and $v_2 \in V$, we have*

(i) $F(2v_2 - \overline{v_1 F(v_2, v_1^{-1})}, v_1^{-1}) = 0$*;*
(ii) $H(\kappa(w_1) \boxplus \lambda(w_1), w_2) + H(\lambda(w_1), w_1(-H(\kappa(w_1), w_2))) = 0$*.*

PROOF. By ($\mathbf{Q}_5$), ($\mathbf{Q}_{15}$) and Lemma 3.15(i), we have

$$\begin{aligned} F(v_2, v_1^{-1})(-v_1)v_1^{-1} &= F(v_2, v_1^{-1})(-\epsilon)v_1 v_1^{-1} \\ &= F(v_2, v_1^{-1}) \ . \end{aligned}$$

If we put $v = v_1$ and $w = F(v_2, v_1^{-1})$ in Lemma 3.16(i), we get

$$\begin{aligned} F(\overline{v_1 F(v_2, v_1^{-1})}, v_1^{-1}) &= F(v_2, v_1^{-1}) \boxplus F(v_2, v_1^{-1})(-v_1)v_1^{-1} \\ &= F(v_2, v_1^{-1}) \boxplus F(v_2, v_1^{-1}) \\ &= F(2v_2, v_1^{-1}) \ , \end{aligned}$$

from which (i) follows.

To prove (ii), we first observe that, by ($\mathbf{Q}_7$), it follows from Lemma 3.13(i) that

$$\begin{aligned} H(w_1(-v), w_2) &= H(F(vw_1, v) \boxminus w_1 v, w_2) \\ &= -H(w_1 v, w_2) \ , \end{aligned}$$

for all $w_1, w_2 \in W$ and all $v \in V$. We also observe that $\kappa(\lambda(w))(-v) = w(-\epsilon)(-v) = wv$ because of ($\mathbf{Q}_{14}$) and ($\mathbf{Q}_5$), and that $v\kappa(\lambda(w)) = v(w(-\epsilon)) = vw$ because of ($\mathbf{Q}_{14}$) and ($\mathbf{Q}_6$), for all $w \in W^*$ and all $v \in V$.

If we substitute $\lambda(w_1)$ for w and $-H(\kappa(w_1), w_2)$ for v in Lemma 3.16(ii), then we get, using these remarks, that

$$\begin{aligned} H(\lambda(w_1), w_1(-H(\kappa(w_1), w_2))) &= -H(\kappa(w_1), w_2) + H(\kappa(w_1), w_2)w_1\lambda(w_1) \\ &= -H(\kappa(w_1), w_2) + H(w_1, w_2)\lambda(w_1) \\ &= -H(\kappa(w_1), w_2) - H(w_1(-\epsilon), w_2)\lambda(w_1) \\ &= -H(\kappa(w_1), w_2) - H(\kappa(\lambda(w_1)), w_2)\lambda(w_1) \\ &= -H(\kappa(w_1), w_2) - H(\lambda(w_1), w_2) \\ &= -H(\kappa(w_1) \boxplus \lambda(w_1), w_2) \ , \end{aligned}$$

where we have used identity $(\mathbf{Q}_{20})$ twice. This completes the proof of (ii). □

LEMMA 3.18. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then, for all $v \in V^*$, $c \in V$, $w \in W^*$ and $z \in W$, we have that*

(i) $\pi_v(c) = c - vF(v^{-1}, \overline{c}) = c - \overline{v^{-1}F(v, c)}$;
(ii) $\Pi_w(z) = z \boxplus w(-H(\kappa(w), z)) = z \boxplus \lambda(w)H(w, z)$.

PROOF. By $(\mathbf{Q}_{17})$, $(\mathbf{Q}_{15})$ and $(\mathbf{Q}_{16})$, we have that

$$\begin{aligned} vF(v^{-1}, \overline{c}) &= v \cdot F(v, c)v^{-1} \\ &= -\overline{v^{-1}(\boxminus F(v, c))} \ . \end{aligned}$$

Since $\operatorname{Im}(F) \subseteq \operatorname{Rad}(H)$ by $(\mathbf{Q}_7)$, it follows from Lemma 3.13(ii) that $v^{-1}(\boxminus F(v, c)) = -v^{-1}F(v, c)$, and hence

$$vF(v^{-1}, \overline{c}) = \overline{v^{-1}F(v, c)} \ ,$$

which shows (i).

By $(\mathbf{Q}_{20})$, Lemma 3.2(i) and $(\mathbf{Q}_{19})$, we have that

$$\begin{aligned} w(-H(\kappa(w), z)) &= w \cdot (H(w, z)\kappa(\boxminus w)) \\ &= \boxminus\kappa(\boxminus w)H(w, z) \\ &= \lambda(w)H(w, z) \ , \end{aligned}$$

which shows (ii). □

In the sequel, we will use both expressions as definitions of π_v and Π_w, without explicitly referring to this lemma.

LEMMA 3.19. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system, and let $w \in \operatorname{Rad}(H)$ and $v \in V$. Then $wv \in \operatorname{Rad}(H)$ as well.*

PROOF. By $(\mathbf{Q}_8)$, $[w, w_2]_\boxplus = 0$ for all $w_2 \in W$, hence $v(w \boxplus w_2) = v(w_2 \boxplus w)$. It follows from $(\mathbf{Q}_{12})$ that $H(w_2, wv) = H(w, w_2v) = 0$ for all $w_2 \in W$, since $w \in \operatorname{Rad}(H)$. By $(\mathbf{Q}_{22})$, this implies that $H(wv, w_2) = 0$ for all $w_2 \in W$, hence $wv \in \operatorname{Rad}(H)$. □

LEMMA 3.20. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then we have that $vw = -\overline{\overline{v}(\delta\overline{v^{-1}})(\boxminus wv)}$ for all $v \in V^*$ and all $w \in W$.*

PROOF. It follows from $(\mathbf{Q}_{16})$ that $\overline{vw} = -v^{-1}(\boxminus wv)$ for all $v \in V^*$ and all $w \in W$. In particular, we have that $\overline{v} = -v^{-1}(\boxminus \delta v)$ for all $v \in V^*$, and hence that $\overline{v}\kappa(\delta v) = v^{-1}$ by Lemma 3.2(i). If we substitute this expression for v^{-1} in the first identity, then we get that $\overline{vw} = -\overline{v}\kappa(\delta v)(\boxminus wv)$ for all $v \in V^*$ and all $w \in W$. The result follows, since $\kappa(\delta v) = \delta\overline{v^{-1}}$ for all $v \in V^*$ by Lemma 3.14(i) and Lemma 3.14(ii). □

LEMMA 3.21. *Let* $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ *be a quadrangular system. Then* $\pi_v(vw) = v(\boxminus w)$ *for all* $v \in V^*$ *and all* $w \in W$.

PROOF. Let $v \in V^*$ and $w \in W$ be arbitrary. It follows from $(\mathbf{Q}_5)$ and $(\mathbf{Q}_{15})$ that $w(-v)v^{-1} = w(-\epsilon)$. It thus follows from Lemma 3.16(i) that $F(\overline{vw}, v^{-1}) = w \boxplus w(-\epsilon)$. Hence, by $(\mathbf{Q}_{12})$, $(\mathbf{Q}_5)$, $(\mathbf{Q}_6)$ and Lemma 3.13(ii),

$$\begin{aligned} vF(\overline{vw}, v^{-1}) &= v(w \boxplus w(-\epsilon)) \\ &= vw + v \cdot w(-\epsilon) + H(w(-\epsilon), wv) \\ &= vw + vw + H(w(-\epsilon), w(-\epsilon)(-v)) \\ &= vw + vw - v(\boxminus w(-\epsilon)) - v(w(-\epsilon)) \\ &= vw + vw - v(\boxminus w) - vw \\ &= vw - v(\boxminus w) \ , \end{aligned}$$

and hence

$$\pi_v(vw) = vw - vF(\overline{vw}, v^{-1}) = v(\boxminus w) \ ,$$

which is what we had to show. □

The following two lemmas generalize some properties of reflections in an ordinary quadratic space.

LEMMA 3.22. *Let* $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ *be a quadrangular system. Then*

(i) $F(v, \pi_v(c)) = F(v, -c)$ *for all* $v \in V^*$ *and all* $c \in V$ *;*
(ii) $H(\lambda(w), \Pi_w(z)) = -H(\kappa(w), z)$ *for all* $w \in W^*$ *and all* $z \in W$ *.*

PROOF. By Lemma 3.18(i) and Theorem 3.17(i) with v^{-1} in place of v_1 and c in place of v_2, we have that

$$\begin{aligned} F(v, \pi_v(c)) &= F(v, c - \overline{v^{-1}F(v, c)}) \\ &= F(2c - \overline{v^{-1}F(c, v)}, v) \boxminus F(c, v) \\ &= F(v, -c) \ , \end{aligned}$$

which shows (i). By Theorem 3.17(ii) with $w_1 = w$ and $w_2 = z$,

$$\begin{aligned} H(\lambda(w), \Pi_w(z)) &= H(\lambda(w), z \boxplus w(-H(\kappa(w), z))) \\ &= H(\kappa(w) \boxplus \lambda(w), z) + H(\lambda(w), w(-H(\kappa(w), z))) - H(\kappa(w), z) \\ &= -H(\kappa(w), z) \ , \end{aligned}$$

which shows (ii). □

LEMMA 3.23. *Let* $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ *be a quadrangular system. Then*

(i) $\pi_v(\pi_v(c)) = c$ *for all* $v \in V^*$ *and all* $c \in V$ *;*
(ii) $\Pi_{\boxminus w}(\Pi_w(z)) = z$ *for all* $w \in W^*$ *and all* $z \in W$ *.*

PROOF. By Lemma 3.22(i),

$$\begin{aligned}\pi_v(\pi_v(c)) &= \pi_v(c) - \overline{v^{-1}F(v, \pi_v(c))} \\ &= -\pi_v(-c) - \overline{v^{-1}F(v, -c)} \\ &= c + \overline{v^{-1}F(v, -c)} - \overline{v^{-1}F(v, -c)} \\ &= c \ ,\end{aligned}$$

which shows (i). By Lemma 3.10(ii) and Lemma 3.22(ii),

$$\begin{aligned}\Pi_{\boxminus w}(\Pi_w(z)) &= \Pi_w(z) \boxplus (\boxminus w)(-H(\kappa(\boxminus w), \Pi_w(z))) \\ &= \Pi_w(z) \boxminus wH(\lambda(w), \Pi_w(z)) \\ &= \Pi_w(z) \boxminus w(-H(\kappa(w), z) \\ &= z \ ,\end{aligned}$$

which shows (ii). ∎

CHAPTER 4

From Quadrangular Systems To Moufang Quadrangles

We will now describe how we can construct a Moufang quadrangle from a quadrangular system. We will use the method described in *Tits and Weiss* [**20**]. Therefore, we will describe 4 groups U_1, U_2, U_3 and U_4, and we will implicitly define the group $U_+ := \langle U_1, U_2, U_3, U_4 \rangle$ by giving the commutator relations between any two of those groups. It is possible to construct a graph $\mathcal{Q}$ out of the data $(U_+, U_1, U_2, U_3, U_4)$.

To prove that this construction will actually result in a Moufang quadrangle, it suffices to check that certain conditions $\mathcal{A}_k$, $\mathcal{B}_k$ and $\mathcal{C}_k$ are satisfied (see [**20**] for details), and that we can construct groups U_0 and U_5, such that, for every $a_0 \in U_0^*$, there exists some element $\mu(a_0) \in U_4^* a_0 U_4^*$, and for every $a_5 \in U_5^*$, there exists some element $\mu(a_5) \in U_1^* a_5 U_1^*$, for which certain conditions have to be satisfied (see again [**20**] for details).

Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Let U_1 and U_3 be two groups isomorphic to W, and let U_2 and U_4 be two groups isomorphic to V. Denote the corresponding isomorphisms by

$$\begin{aligned} x_1 &: W \to U_1 : w \mapsto x_1(w) \ ; \\ x_2 &: V \to U_2 : v \mapsto x_2(v) \ ; \\ x_3 &: W \to U_3 : w \mapsto x_3(w) \ ; \\ x_4 &: V \to U_4 : v \mapsto x_4(v) \ ; \end{aligned}$$

we say that U_1 and U_3 are *parametrized* by W and that U_2 and U_4 are *parametrized* by V. For all $1 \leq i < j \leq 4$, we will denote the group $\langle U_i, \ldots, U_j \rangle$ by $U_{[i,j]}$. Now, we implicitly define the group $U_+ = U_{[1,4]}$ by the following commutator relations:

$$\begin{aligned} [x_1(w_1), x_3(w_2)^{-1}] &= x_2(H(w_1, w_2)) \ , \\ [x_2(v_1), x_4(v_2)^{-1}] &= x_3(F(v_1, v_2)) \ , \\ [x_1(w), x_4(v)^{-1}] &= x_2(vw) x_3(wv) \ , \\ [U_i, U_{i+1}] &= 1 \quad \forall i \in \{1, 2, 3\} \ , \end{aligned}$$

for all $w, w_1, w_2 \in W$ and all $v, v_1, v_2 \in V$. We will denote the corresponding graph by $\mathcal{Q}(V, W, \tau_V, \tau_W, \epsilon, \delta)$. If we define

$$\begin{aligned} \xi_{13}(x_1(w_1), x_3(w_2)^{-1}) &= x_2(H(w_1, w_2)) \ , \\ \xi_{24}(x_2(v_1), x_4(v_2)^{-1}) &= x_3(F(v_1, v_2)) \ , \\ \xi_{14}(x_1(w), x_4(v)^{-1}) &= x_2(vw) x_3(wv) \ , \end{aligned}$$

then we can rephrase the conditions $\mathcal{A}_k$, $\mathcal{B}_k$ and $\mathcal{C}_k$ as follows.

For all $(i, j) \in \{(1, 3), (2, 4), (1, 4)\}$, the following conditions should hold, for all $a_i, b_i \in U_i$, for all $a_j, b_j \in U_j$, and for all $c \in U_{[i+1,j-1]}$.

$\mathcal{A}_{ij}$. $\xi_{ij}(a_ib_i, a_j^{-1}) = \xi_{ij}(a_i, a_j^{-1})^{b_i}\xi_{ij}(b_i, a_j^{-1})$.

$\mathcal{B}_{ij}$. $\xi_{ij}(a_i, (a_jb_j)^{-1}) = \xi_{ij}(a_i, a_j^{-1})\xi_{ij}(a_i, b_j^{-1})^{a_j^{-1}}$.

$\mathcal{C}_{ij}$. $c^{\xi_{ij}(a_i,a_j^{-1})} = c^{a_i^{-1}a_ja_ia_j^{-1}}$.

Theorem 4.1. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then the corresponding graph $\mathcal{Q}(V, W, \tau_V, \tau_W, \epsilon, \delta)$ satisfies all of the conditions $\mathcal{A}_{ij}$, $\mathcal{B}_{ij}$ and $\mathcal{C}_{ij}$.*

Proof. By plugging in the formulas for the functions ξ_{ij}, we get the following explicit conditions, which must hold for all $v, v', v_1, v_2 \in V$ and all $w, w', w_1, w_2 \in W$.

$\mathcal{A}_{13}$. $x_2(H(w_1 \boxplus w_2, w')) = x_2(H(w_1, w'))^{x_1(w_2)}x_2(H(w_2, w'))$.

$\mathcal{A}_{24}$. $x_3(F(v_1 + v_2, v')) = x_3(F(v_1, v'))^{x_2(v_2)}x_3(F(v_2, v'))$.

$\mathcal{A}_{14}$. $x_2(v(w_1 \boxplus w_2))x_3((w_1 \boxplus w_2)v) = (x_2(vw_1)x_3(w_1v))^{x_1(w_2)}\cdot(x_2(vw_2)x_3(w_2v))$.

$\mathcal{B}_{13}$. $x_2(H(w', w_1 \boxplus w_2)) = x_2(H(w', w_1))x_2(H(w', w_2))^{x_3(\boxminus w_2)}$.

$\mathcal{B}_{24}$. $x_3(F(v', v_1 + v_2)) = x_3(F(v', v_1))x_3(F(v', v_2))^{x_4(-v_2)}$.

$\mathcal{B}_{14}$. $x_2((v_1+v_2)w)x_3(w(v_1+v_2)) = (x_2(v_1w)x_3(wv_1))\cdot(x_2(v_2w)x_3(wv_2))^{x_4(-v_1)}$.

$\mathcal{C}_{13}$. $x_2(v)^{x_2(H(w,w'))} = x_2(v)^{x_1(\boxminus w)x_3(w')x_1(w)x_3(\boxminus w')}$.

$\mathcal{C}_{24}$. $x_3(w)^{x_3(F(v,v'))} = x_3(w)^{x_2(-v)x_4(v')x_2(v)x_4(-v')}$.

$\mathcal{C}_{14,2}$. $x_2(v')^{x_2(vw)x_3(wv)} = x_2(v')^{x_1(\boxminus w)x_4(v)x_1(w)x_4(-v)}$.

$\mathcal{C}_{14,3}$. $x_3(w')^{x_2(vw)x_3(wv)} = x_3(w')^{x_1(\boxminus w)x_4(v)x_1(w)x_4(-v)}$.

Note that $[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = 1$; some of the conditions can be simplified by this observation.

Condition $(\mathcal{A}_{13})$ is equivalent to

$$x_2(H(w_1 \boxplus w_2, w')) = x_2(H(w_1, w'))x_2(H(w_2, w')) ,$$

which is, in turn, equivalent to the fact that H is additive in the first variable. Completely similarly, $(\mathcal{A}_{24})$, $(\mathcal{B}_{13})$ and $(\mathcal{B}_{24})$ also follow from the fact that F and H are additive in both variables.

By $(\mathbf{Q}_{12})$, the left hand side of $(\mathcal{A}_{14})$ can be rewritten as

$$x_2(vw_1 + vw_2 + H(w_2, w_1v))x_3((w_1 \boxplus w_2)v) .$$

Using the fact that $b^a = [a, b^{-1}]b$, we can rewrite the right hand side as

$$x_2(vw_1)[x_1(w_2), x_3(w_1v)^{-1}]x_3(w_1v)x_2(vw_2)x_3(w_2v)$$

which is also equal to

$$x_2(vw_1)x_2(H(w_2, w_1v))x_3(w_1v)x_2(vw_2)x_3(w_2v) .$$

Since $[U_2, U_2] = [U_2, U_3] = 1$, we can rewrite this once more as

$$x_2(vw_1 + vw_2 + H(w_2, w_1v))x_3(w_1v + w_2v) .$$

It now follows from $(\mathbf{Q}_3)$ that $(\mathcal{A}_{14})$ holds.

Similarly, $(\mathcal{B}_{14})$ follows from $(\mathbf{Q}_{11})$ and $(\mathbf{Q}_4)$; we additionally need the fact that $\mathrm{Im}(F) \leq Z(W)$, which follows from $(\mathbf{Q}_7)$ and $(\mathbf{Q}_8)$.

Since $[U_2, U_1] = [U_2, U_2] = [U_2, U_3] = 1$, $(\mathcal{C}_{13})$ becomes trivial. Because $[U_3, U_2] = [U_3, U_4] = 1$, we have that $(\mathcal{C}_{24})$ is equivalent to the condition $[w, F(v, v')]_{\boxplus} = 1$. Since $\mathrm{Im}(F) \leq Z(W)$, this is always satisfied.

To prove $(\mathcal{C}_{14,2})$, we need to show that

$$x_2(v') = x_2(v')^{x_4(v)x_1(w)x_4(-v)} ,$$

which is the same as

$$(x_2(v')^{x_4(v)})^{x_1(w)} = x_2(v')^{x_4(v)} .$$

Since $x_2(v')^{x_4(v)} = x_2(v')[x_2(v'), x_4(-v)^{-1}] = x_2(v')x_3(F(v', -v))$, we have that

$$\begin{aligned}(x_2(v')^{x_4(v)})^{x_1(w)} &= (x_2(v')x_3(F(v', -v)))^{x_1(w)} \\ &= x_2(v')[x_1(w), x_3(F(v', -v))^{-1}]x_3(F(v', -v)) \\ &= x_2(v')x_2(H(w, F(v', -v)))x_3(F(v', -v)) \\ &= x_2(v')x_3(F(v', -v)) \\ &= x_2(v')^{x_4(v)}\end{aligned}$$

since $\mathrm{Im}(F) \leq \mathrm{Rad}(H)$ by $(\mathbf{Q}_7)$. Thus $(\mathcal{C}_{14,2})$ holds.

The left hand side of $(\mathcal{C}_{14,3})$ is equal to

$$x_3(w')[x_3(w'), x_3(wv)] ,$$

which is, by $(\mathbf{Q}_8)$, also equal to

$$x_3(w' \boxplus F(H(w, w'), v)) .$$

The right hand side is equal to

$$\begin{aligned}&x_3(w')^{x_1(\boxminus w)x_4(v)x_1(w)x_4(-v)} \\ &= (x_2(-H(w, w'))x_3(w'))^{x_4(v)x_1(w)x_4(-v)} \\ &= (x_2(-H(w, w'))x_3(F(H(w, w'), v))x_3(w'))^{x_1(w)x_4(-v)} \\ &= (x_2(-H(w, w'))x_3(w' \boxplus F(H(w, w'), v)))^{x_1(w)x_4(-v)} \\ &= (x_2(-H(w, w'))x_2(H(w, w' \boxplus F(H(w, w'), v)))x_3(w' \boxplus F(H(w, w'), v)))^{x_4(-v)} \\ &= x_3(w' \boxplus F(H(w, w'), v))^{x_4(-v)} \\ &= x_3(w' \boxplus F(H(w, w'), v)) ,\end{aligned}$$

thus $(\mathcal{C}_{14,3})$ holds. This concludes the proof of this theorem. □

Let U_0 be a group parametrized by V (via a map x_0), and let U_5 be a group parametrized by W (via a map x_5). We define an action of U_0 on $U_{[1,3]}$ by the following commutator relations.

$$\begin{aligned}[U_0, U_1] &= 1 \\ [x_0(v_1), x_2(v_2)^{-1}] &= x_1(F(v_1, \overline{v_2})) \\ [x_0(v), x_3(w)^{-1}] &= x_1(wv)x_2(-\overline{v(\boxminus w)})\end{aligned}$$

for all $w \in W$ and all $v, v_1, v_2 \in V$. For each $x_4(v) \in U_4^*$, we define an element $\mu(x_4(v)) \in U_0^* x_4(v) U_0^*$ as

$$\mu(x_4(v)) = x_0(v^{-1})x_4(v)x_0(v^{-1}) .$$

We define an action of U_5 on $U_{[2,4]}$ by the following commutator relations.

$$\begin{aligned}[x_2(v), x_5(w)^{-1}] &= x_3(w(-v))x_4(-v(\boxminus w))\\ [x_3(w_1), x_5(w_2)^{-1}] &= x_4(H(w_2, w_1))\\ [U_4, U_5] &= 1\end{aligned}$$

for all $w, w_1, w_2 \in W$ and all $v \in V$. For each $x_1(w) \in U_1^*$, we define an element $\mu(x_1(w)) \in U_5^* x_1(w) U_5^*$ as

$$\mu(x_1(w)) = x_5(\kappa(w))x_1(w)x_5(\lambda(w)) \ .$$

Note that, by Lemma 3.10, $\mu(x_4(v)^{-1}) = \mu(x_4(v))^{-1}$, and $\mu(x_1(w)^{-1}) = \mu(x_1(w))^{-1}$.

It follows Theorem 4.1 that the graph $\mathcal{Q}(V, W, \tau_V, \tau_W, \epsilon, \delta)$ satisfies the conditions which are denoted by $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$ in [**20**]. In order to obtain a Moufang quadrangle, this graph additionally has to satisfy the conditions $(\mathcal{M}_3)$ and $(\mathcal{M}_4)$. In Theorem 4.2, we will show that $(\mathcal{M}_3)$ holds; the validity of $(\mathcal{M}_4)$ will be shown in Theorem 4.3.

THEOREM 4.2. *Let $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then the corresponding graph $\mathcal{Q}(V, W, \tau_V, \tau_W, \epsilon, \delta)$, together with the group U_0 and the map μ, satisfies the following conditions, for all $v \in V$.*

(i) $U_0^{\mu(x_4(\epsilon))} = U_4$ *, considered as subgroups of* $\mathrm{Aut}(U_{[1,3]})$ *;*

(ii) $U_1^{\mu(x_4(v))} = U_3$ *. More precisely, we have that $x_1(w)^{\mu(x_4(v))} = x_3(w(-v))$ for all $w \in W$ and all $v \in V^*$;*

(iii) $U_2^{\mu(x_4(v))} = U_2$ *. More precisely, we have that $x_2(v')^{\mu(x_4(v))} = x_2(\pi_v(v'))$ for all $v' \in V$ and all $v \in V^*$;*

(iv) $U_3^{\mu(x_4(v))} = U_1$ *. More precisely, we have that $x_3(w)^{\mu(x_4(v))} = x_1(wv^{-1})$ for all $w \in W$ and all $v \in V^*$;*

(v) $U_4^{\mu(x_4(\epsilon))} = U_0$ *, considered as subgroups of* $\mathrm{Aut}(U_{[1,3]})$ *.*

PROOF. For all $w \in W$ and all $v \in V^*$, we have

$$\begin{aligned}x_1(w)^{\mu(x_4(v))} &= x_1(w)^{x_0(v^{-1})x_4(v)x_0(v^{-1})}\\ &= x_1(w)^{x_4(v)x_0(v^{-1})}\\ &= (x_1(w)x_2(-vw)x_3(w(-v)))^{x_0(v^{-1})}\\ &= x_1(w)x_1(F(-\overline{vw}, v^{-1}))x_2(-vw)x_1(w(-v)v^{-1})\\ &\qquad\qquad \cdot x_2(-\overline{v^{-1}(\boxminus w(-v))})x_3(w(-v))\\ &= x_3(w(-v)) \ ,\end{aligned}$$

where we have used Lemma 3.16, $(\mathbf{Q}_{16})$ and Lemma 3.12 for the last equality. By substituting wv^{-1} for w and $-v$ for v, we also get

$$x_1(wv^{-1})^{\mu(x_4(-v))} = x_3(wv^{-1}v) \ ,$$

and since $\mu(x_4(-v)) = \mu(x_4(v))^{-1}$ and by $(\mathbf{Q}_{15})$, it follows that

$$x_3(w)^{\mu(x_4(v))} = x_1(wv^{-1}) \ .$$

So we have proved that $U_1^{\mu(x_4(v))} \subseteq U_3$ and $U_3^{\mu(x_4(v))} \subseteq U_1$. If we replace v by $-v$ in those two relations, and conjugate by $\mu(x_4(v))$, it also follows that $U_1 \subseteq U_3^{\mu(x_4(v))}$ and $U_3 \subseteq U_1^{\mu(x_4(v))}$. So (ii) and (iv) are proved.

We will now prove (iii). For all $v \in V^*$ and all $v' \in V$, we have

$$\begin{aligned} x_2(v')^{\mu(x_4(v))} &= x_2(v')^{x_0(v^{-1})x_4(v)x_0(v^{-1})} \\ &= (x_1(F(v^{-1},\overline{v'}))x_2(v'))^{x_4(v)x_0(v^{-1})} \\ &= (x_1(F(v^{-1},\overline{v'}))x_2(-vF(v^{-1},\overline{v'}))x_3(F(v^{-1},\overline{v'})(-v)) \\ &\qquad \cdot x_2(v')x_3(\boxminus F(v',v)))^{x_0(v^{-1})} \\ &= (x_1(F(v^{-1},\overline{v'}))x_2(v'-vF(v^{-1},\overline{v'})))^{x_0(v^{-1})} , \end{aligned}$$

where we have used ($\mathbf{Q}_{17}$) for the last equality. It follows that

$$\begin{aligned} x_2(v')^{\mu(x_4(v))} &= x_1(F(v^{-1},\overline{v'}))x_1(F(v^{-1},\overline{v'-vF(v^{-1},\overline{v'})}))x_2(v'-vF(v^{-1},\overline{v'})) \\ &= x_2(v'-vF(v^{-1},\overline{v'})) \\ &= x_2(\pi_v(v')) , \end{aligned}$$

where we have used Lemma 3.9 and Theorem 3.17(i). Hence $U_2^{\mu(x_4(v))} \subseteq U_2$, and again by replacing v by $-v$ and conjugating by $\mu(x_4(v))$, we get that $U_2 \subseteq U_2^{\mu(x_4(v))}$ as well, from which (iii) follows.

To prove (v), we will check that the action of $\mu(x_4(-\epsilon))x_4(v)\mu(x_4(\epsilon))$ on $U_{[1,3]}$ is the same as the action of $x_0(v)$ on $U_{[1,3]}$, for all $v \in V$. Note that we will use the fact that $w\epsilon^{-1} = w$, which follows by choosing $v = \epsilon$ in ($\mathbf{Q}_{15}$), and the fact that $F(\epsilon^{-1},\overline{v}) = F(\epsilon, v)$, which holds by substituting ϵ for v_1 in ($\mathbf{Q}_{17}$).

Using the definition of the map $v \mapsto \overline{v}$, we see that

$$\begin{aligned} x_2(v)^{\mu(x_4(\epsilon))} &= x_2(v-\epsilon F(\epsilon^{-1},\overline{v})) \\ &= x_2(v-\epsilon F(\epsilon,v)) \\ &= x_2(-\overline{v}) , \end{aligned}$$

for all $v \in V$. Since $-\overline{(-\overline{v})} = v$, replacing v by $-\overline{v}$ and conjugating by $\mu(x_4(-\epsilon))$ yields

$$x_2(v)^{\mu(x_4(-\epsilon))} = x_2(-\overline{v})$$

for all $v \in V$, as well. For the action on U_1, we have

$$\begin{aligned} x_1(w)^{\mu(x_4(-\epsilon))x_4(v)\mu(x_4(\epsilon))} &= x_3(w)^{x_4(v)\mu(x_4(\epsilon))} \\ &= x_3(w)^{\mu(x_4(\epsilon))} \\ &= x_1(w) \\ &= x_1(w)^{x_0(v)} ; \end{aligned}$$

for the action on U_2, we have

$$\begin{aligned} x_2(v')^{\mu(x_4(-\epsilon))x_4(v)\mu(x_4(\epsilon))} &= x_2(-\overline{v'})^{x_4(v)\mu(x_4(\epsilon))} \\ &= (x_2(-\overline{v'})x_3(F(\overline{v'},v)))^{\mu(x_4(\epsilon))} \\ &= x_2(v')x_1(F(v,\overline{v'})) \\ &= x_2(v')^{x_0(v)} . \end{aligned}$$

To check the action on U_3, we need ($\mathbf{Q}_6$), ($\mathbf{Q}_5$), ($\mathbf{Q}_{22}$), Lemma 3.9 and Lemma 3.13(ii) :

$$\begin{aligned} x_3(w)^{\mu(x_4(-\epsilon))x_4(v)\mu(x_4(\epsilon))} &= x_1(w(-\epsilon))^{x_4(v)\mu(x_4(\epsilon))} \\ &= (x_1(w(-\epsilon))x_2(-vw)x_3(wv))^{\mu(x_4(\epsilon))} \\ &= x_3(w)x_2(\overline{vw})x_1(wv) \\ &= x_1(wv)x_2(\overline{vw} + H(wv,w))x_3(w) \\ &= x_1(wv)x_2(\overline{vw - H(w,wv)})x_3(w) \\ &= x_1(wv)x_2(-\overline{v(\boxminus w)})x_3(w) \\ &= x_3(w)^{x_0(v)} \ . \end{aligned}$$

Thus (v) is proved.

To prove (i), we will check that the action of $\mu(x_4(\epsilon))x_4(v)\mu(x_4(-\epsilon))$ on $U_{[1,3]}$ is the same as the action of $x_0(v)$ on $U_{[1,3]}$. We can take a shortcut by observing that

$$\mu(x_4(\epsilon))x_4(v)\mu(x_4(-\epsilon)) = \mu(x_4(\epsilon))^2\mu(x_4(-\epsilon))x_4(v)\mu(x_4(\epsilon))\mu(x_4(-\epsilon))^2 \ .$$

We just have to do a short calculation to see that

$$\begin{aligned} x_1(w)^{\mu(x_4(\epsilon))^2} &= x_1(w)^{\mu(x_4(-\epsilon))^2} = x_1(w(-\epsilon)) \ ; \\ x_2(v)^{\mu(x_4(\epsilon))^2} &= x_2(v)^{\mu(x_4(-\epsilon))^2} = x_2(v) \ ; \\ x_3(w)^{\mu(x_4(\epsilon))^2} &= x_3(w)^{\mu(x_4(-\epsilon))^2} = x_3(w(-\epsilon)) \ . \end{aligned}$$

For the action on U_1, we have

$$\begin{aligned} x_1(w)^{\mu(x_4(\epsilon))x_4(v)\mu(x_4(-\epsilon))} &= x_1(w(-\epsilon))^{\mu(x_4(-\epsilon))x_4(v)\mu(x_4(\epsilon))\mu(x_4(-\epsilon))^2} \\ &= x_1(w(-\epsilon))^{\mu(x_4(-\epsilon))^2} \\ &= x_1(w) \\ &= x_1(w)^{x_0(v)} \ ; \end{aligned}$$

for the action on U_2, we have, by Lemma 3.15, that

$$\begin{aligned} x_2(v')^{\mu(x_4(\epsilon))x_4(v)\mu(x_4(-\epsilon))} &= x_2(v')^{\mu(x_4(-\epsilon))x_4(v)\mu(x_4(\epsilon))\mu(x_4(-\epsilon))^2} \\ &= (x_2(v')x_1(F(v,\overline{v'})))^{\mu(x_4(-\epsilon))^2} \\ &= x_2(v')x_1(F(v,\overline{v'})(-\epsilon)) \\ &= x_2(v')x_1(F(v,\overline{v'})) \\ &= x_2(v')^{x_0(v)} \ ; \end{aligned}$$

Finally, for the action on U_3, we have, using Lemma 3.11, that

$$\begin{aligned} x_3(w)^{\mu(x_4(\epsilon))x_4(v)\mu(x_4(-\epsilon))} &= x_3(w(-\epsilon))^{\mu(x_4(-\epsilon))x_4(v)\mu(x_4(\epsilon))\mu(x_4(-\epsilon))^2} \\ &= (x_1(w(-v))x_2(-\overline{v(\boxminus w)})x_3(w(-\epsilon)))^{\mu(x_4(-\epsilon))^2} \\ &= x_1(wv)x_2(-\overline{v(\boxminus w)})x_3(w) \\ &= x_3(w)^{x_0(v)} \ . \end{aligned}$$

So we have proved (i), and this completes the proof of this theorem. □

THEOREM 4.3. *Let* $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ *be a quadrangular system. Then the corresponding graph* $\mathcal{Q}(V, W, \tau_V, \tau_W, \epsilon, \delta)$*, together with the group* U_5 *and the map* μ*, satisfies the following conditions, for all* $w \in W$*.*

(i) $U_5^{\mu(x_1(\delta))} = U_1$ *, considered as subgroups of* $\mathrm{Aut}(U_{[2,4]})$ *;*

(ii) $U_4^{\mu(x_1(w))} = U_2$ *. More precisely, we have that* $x_4(v)^{\mu(x_1(w))} = x_2(vw)$ *for all* $v \in V$ *and all* $w \in W^*$ *;*

(iii) $U_3^{\mu(x_1(w))} = U_3$ *. More precisely, we have that* $x_3(w')^{\mu(x_1(w))} = x_3(\Pi_w(w'))$ *for all* $w' \in W$ *and all* $w \in W^*$ *;*

(iv) $U_2^{\mu(x_1(w))} = U_4$ *. More precisely, we have that* $x_2(v)^{\mu(x_1(w))} = x_4(-v\kappa(w))$ *for all* $v \in V$ *and all* $w \in W^*$ *;*

(v) $U_1^{\mu(x_1(\delta))} = U_5$ *, considered as subgroups of* $\mathrm{Aut}(U_{[2,4]})$ *.*

PROOF. The proof of this theorem is very similar to the previous one, so we will skip most of the calculations.

For all $w \in W^*$ and all $v \in V$, we have

$$\begin{aligned} x_2(v)^{\mu(x_1(w))} &= x_2(v)^{x_5(\kappa(w))x_1(w)x_5(\lambda(w))} \\ &= x_4(-v\kappa(w)) \ , \end{aligned}$$

where we have to use $(\mathbf{Q}_{19})$ and Lemma 3.16(ii). By substituting $-w$ for w and vw for v, we also get

$$x_2(vw)^{\mu(x_1(\boxminus w))} = x_4(-vw\kappa(\boxminus w)) \ ,$$

and since $\mu(x_1(\boxminus w)) = \mu(x_1(w))^{-1}$ and by Lemma 3.2(i), it follows that

$$x_4(v)^{\mu(x_1(w))} = x_2(vw) \ .$$

So we have proved that $U_4^{\mu(x_1(w))} \subseteq U_2$ and $U_2^{\mu(x_1(w))} \subseteq U_4$. If we replace w by $\boxminus w$ in those two relations, and conjugate by $\mu(x_1(w))$, it also follows that $U_4 \subseteq U_2^{\mu(x_1(w))}$ and $U_2 \subseteq U_4^{\mu(x_1(w))}$. So (ii) and (iv) are proved.

We will now prove (iii). For all $w \in W^*$ and all $w' \in W$, we have

$$\begin{aligned} x_3(w')^{\mu(x_1(w))} &= x_3(w')^{x_5(\kappa(w))x_1(w)x_5(\lambda(w))} \\ &= x_3(w' \boxplus w(-H(\kappa(w), w'))) \\ &= x_3(\Pi_w(w')) \ , \end{aligned}$$

where we have to use $(\mathbf{Q}_{20})$ and Theorem 3.17(ii). Hence $U_3^{\mu(x_1(w))} \subseteq U_3$, and again by replacing w by $-w$ and conjugating by $\mu(x_1(w))$, we get that $U_3 \subseteq U_3^{\mu(x_1(w))}$ as well, from which (iii) follows.

To prove (v), we will check that the action of $\mu(x_1(\boxminus\delta))x_1(w)\mu(x_1(\delta))$ on $U_{[2,4]}$ is the same as the action of $x_5(w)$ on $U_{[2,4]}$, for all $w \in W$. First of all, observe that it follows from Lemma 3.13(ii) and from the fact that $\delta \in \mathrm{Rad}(H)$ (by $(\mathbf{Q}_7)$) that $v(\boxminus\delta) = -v$. If we put $w = \delta$ in $(\mathbf{Q}_{18})$, it thus follows that $v\kappa(\delta) = v$; if we put $w = \boxminus\delta$ in this same identity $(\mathbf{Q}_{18})$, it follows that $v\kappa(\boxminus\delta) = -v$. Furthermore, if we put $w_1 = \delta$ in $(\mathbf{Q}_{20})$, it follows from $(\mathbf{Q}_7)$ that $H(\kappa(\delta), w) = 0$, for all $w \in W$.

Using these facts, we can prove that

$$\begin{aligned} x_4(v)^{\mu(x_1(\boxminus\delta))x_1(w)\mu(x_1(\delta))} &= x_4(v)^{x_5(w)} \ ; \\ x_3(w')^{\mu(x_1(\boxminus\delta))x_1(w)\mu(x_1(\delta))} &= x_3(w')^{x_5(w)} \ ; \\ x_2(v)^{\mu(x_1(\boxminus\delta))x_1(w)\mu(x_1(\delta))} &= x_2(v)^{x_5(w)} \ , \end{aligned}$$

where we have to use ($\mathbf{Q}_{21}$) and Lemma 3.13(i) as well. Thus (v) is proved.

To prove (i), we have to check that the action of $\mu(x_1(\delta))x_5(w)\mu(x_1(\boxminus\delta))$ on $U_{[2,4]}$ is the same as the action of $x_5(w(-\epsilon))$ on $U_{[2,4]}$. Again, we can take a shortcut by observing that

$$\mu(x_1(\delta))x_1(w)\mu(x_1(\boxminus\delta)) = \mu(x_1(\delta))^2\mu(x_1(\boxminus\delta))x_1(w)\mu(x_1(\delta))\mu(x_1(\boxminus\delta))^2 .$$

First, we observe that

$$\begin{aligned} x_2(v)^{\mu(x_1(\delta))^2} &= x_2(v)^{\mu(x_1(\boxminus\delta))^2} = x_2(-v) ; \\ x_3(w)^{\mu(x_1(\delta))^2} &= x_3(w)^{\mu(x_1(\boxminus\delta))^2} = x_3(w) ; \\ x_4(v)^{\mu(x_1(\delta))^2} &= x_4(v)^{\mu(x_1(\boxminus\delta))^2} = x_4(-v) . \end{aligned}$$

It now follows from a short calculation that

$$\begin{aligned} x_4(v)^{\mu(x_1(\delta))x_5(w)\mu(x_1(\boxminus\delta))} &= x_4(v)^{x_5(w(-\epsilon))} ; \\ x_3(w')^{\mu(x_1(\delta))x_5(w)\mu(x_1(\boxminus\delta))} &= x_3(w')^{x_5(w(-\epsilon))} ; \\ x_2(v)^{\mu(x_1(\delta))x_5(w)\mu(x_1(\boxminus\delta))} &= x_2(v)^{x_5(w(-\epsilon))} . \end{aligned}$$

So we have proved (i), and this completes the proof of this theorem. □

This completes the proof of the fact that the graph $\mathcal{Q}(V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a Moufang quadrangle.

CHAPTER 5

From Moufang Quadrangles To Quadrangular Systems

In this chapter, we will prove that *every* Moufang quadrangle can be obtained from the construction described in the previous chapters. We will make intensive use of Chapter 21 "Quadrangles" in [**20**]. Since we are dealing with the same objects as in [**20**], it should not be very surprising that we need these same properties. However, after recalling these facts, our approach will very quickly diverge from the one given in [**20**].

Let Γ be an arbitrary Moufang quadrangle, and consider a fixed apartment $\Sigma = (0, 1, \dots, 7)$, where the vertices are labeled modulo 8. We will denote its root groups $U(i, i+1, \dots, i+4)$ by U_i, for all $i \in \mathbb{Z}$. The following properties of the root groups are fundamental.

Theorem 5.1. (i) *$[U_i, U_j] \le U_{[i+1,j-1]}$, if $i < j < i + 4$.*

(ii) *For each i, the product map from $U_i \times U_{i+1} \times U_{i+2} \times U_{i+3}$ to $U_{[i,i+3]}$ is bijective.*

Proof. See, for example, [**20**, (5.5) and (5.6)]. □

Thanks to this theorem, we can use the following notation. Let $a_i \in U_i$ and $a_j \in U_j$, with $j \in \{i + 2, i + 3\}$. For each k such that $i < k < j$, we set

$$[a_i, a_j]_k = a_k \ ,$$

where a_k is the unique element of U_k appearing in the factorization of $[a_i, a_j] \in U_{[i+1,j-1]}$.

Let $V_i := [U_{i-1}, U_{i+1}] \le U_i$ and $Y_i := C_{U_i}(U_{i-2}) \le U_i$ for each i. It can be shown (see [**20**, (21.20.i)]) that $Y_i = C_{U_i}(U_{i+2})$ as well.

The following theorem defines the functions κ, λ and μ.

Theorem 5.2. *For each i, there exist unique functions $\kappa_i, \lambda_i : U_i^* \to U_{i+4}^*$, such that $(i-1)^{a_i\lambda_i(a_i)} = i + 1$ and $(i+1)^{\kappa_i(a_i)a_i} = i - 1$, for all $a_i \in U_i^*$. The product $\mu_i(a_i) := \kappa_i(a_i)a_i\lambda_i(a_i)$ fixes i and $i + 4$ and reflects Σ, and $U_j^{\mu_i(a_i)} = U_{2i+4-j}$ for each $a_i \in U_i^*$ and each j.*

Proof. See [**20**, (6.1)]. □

Since we will apply these functions only when it is clear in which U_i^* the argument lies, we will write κ, λ and μ in place of κ_i, λ_i and μ_i. Note that it follows from the last statement of this theorem, that U_i and U_j are conjugate (and hence isomorphic) whenever i and j have the same parity.

Lemma 5.3. *For all $a_i \in U_i^*$, we have :*

(i) *$\mu(a_i^{-1}) = \mu(a_i)^{-1}$;*

(ii) $\lambda(a_i^{-1}) = \kappa(a_i)^{-1}$;
(iii) $\mu(a_i^g) = \mu(a_i)^g$ *for every element* $g \in \mathrm{Aut}(\Gamma)$ *mapping* Σ *to itself.*

Proof. See [**20**, (6.2)]. □

The following "Shift Lemma" is essential.

Theorem 5.4. *Suppose, for some* i*, that* $[a_i, a_{i+3}^{-1}] = a_{i+1}a_{i+2}$*, with* $a_k \in U_k$ *for each* k*, and with* a_i *and* a_{i+3} *non-trivial. Then we have:*

(i) $a_i = a_{i+2}^{\mu(a_{i+3})}$ *and* $a_{i+1} = a_{i+3}^{\mu(a_i)}$;
(ii) $[\kappa(a_{i+3}), a_{i+2}^{-1}] = a_i a_{i+1}$;
(iii) $[a_{i+1}, \lambda(a_i)^{-1}] = a_{i+2}a_{i+3}$.

Proof. See [**20**, (21.19)]. □

The following theorem already puts strong restrictions on the root groups.

Theorem 5.5. *By relabeling the vertices of* Σ *by the transformation* $i \mapsto 5 - i$ *if necessary, we can assume the following :*

(i) $Y_i \neq 1$, $[U_i, U_i] \leq V_i \leq Y_i \leq Z(U_i)$, *for all odd* i ;
(ii) U_i *is abelian, for all even* i .

Proof. See [**20**, (21.28)]. □

From now on, we will assume that we have chosen the labeling of our apartment Σ in such a way that the statements of Theorem 5.5 hold.
We will also use the following results from [**20**].

Theorem 5.6. (see [**20**, (21.29)])
If $a_1 \in Y_1^*$*, then* $\kappa(a_1)$ *and* $\lambda(a_1)$ *both lie in* Y_5^*.

Theorem 5.7. (see [**20**, (21.33)])
Let $h = \mu(a_1)^2$*, for some* $a_1 \in Y_1^*$*. Then :*

(i) $a_3^h = a_3$*, for all* $a_3 \in U_3$;
(ii) $a_4^h = a_4^{-1}$*, for all* $a_4 \in U_4$.

Theorem 5.8. (see [**20**, (21.34)])
$\kappa(a_4) = \lambda(a_4)$*, for all* $a_4 \in U_4^*$.

Theorem 5.9. (see [**20**, (21.36)])
Let $a_1 \in U_1^*$*,* $a_2 \in U_2$*,* $a_3 \in U_3$ *and* $a_4 \in U_4^*$*. Then :*

(i) $a_2^{\mu(a_4)}a_2^{-1} = [[\lambda(a_4), a_2^{-1}], a_4]_2$;
(ii) $[[\lambda(a_4), a_2^{-1}], a_4]_3 = [a_2, a_4]^{-1}$;
(iii) $[a_1, [a_3, \kappa(a_1)]^{-1}]_2 = [a_1, a_3^{-1}]^{-1}$.

Theorem 5.10. (see [**20**, (21.37)])
$[\mu(a_4)^2, Y_1U_2Y_3U_4] = 1$*, for all* $a_4 \in U_4^*$.

Proof. For all the proofs of these theorems, see [**20**], except for Theorem 5.9(iii), for which the proof is completely similar to the proof of Theorem 5.9(ii). □

We can now start to build up our quadrangular systems. We start the construction by choosing an arbitrary parametrization of the group U_1 by some group $(W, \boxplus) \cong U_1$, and an arbitrary parametrization of the group U_4 by some group $(V, +) \cong U_4$. We will denote the isomorphisms from W to U_1 and from V to U_4 by

x_1 and x_4, respectively. Choose some fixed elements $e_1 = x_1(\delta) \in Y_1^*$ (note that Y_1^* is non-empty because of Theorem 5.5(i)) and $e_4 = x_4(\epsilon) \in U_4^*$, where we choose e_4 in Y_4^* if $Y_4 \neq 1$. Since U_3 is isomorphic to U_1, we can also have it parametrized by the same group $(W, \boxplus)$ by some isomorphism x_3, which we define by setting

$$x_3(w) := [x_1(w), e_4^{-1}]_3 \ ,$$

for all $w \in W$. Similarly, we let U_2 be parametrized by $(V, +)$, via the isomorphism x_2 defined by

$$x_2(v) := [e_1, x_4(v)^{-1}]_2 \ ,$$

for all $v \in V$. To parametrize U_0 and U_5, we choose the following isomorphisms x_0 and x_5 from V to U_0 and from W to U_5, respectively :

$$x_0(v) := x_4(v)^{\mu(e_4)} \ ,$$

$$x_5(w) := x_1(w)^{\mu(e_1)} \ ,$$

for all $w \in W$ and all $v \in V$. We will now define a map F from $V \times V$ to W and a map H from $W \times W$ on V, by setting

$$[x_1(w_1), x_3(w_2)^{-1}] = x_2(H(w_1, w_2)) \ ,$$

$$[x_2(v_1), x_4(v_2)^{-1}] = x_3(F(v_1, v_2)) \ ,$$

for all $w_1, w_2 \in W$ and all $v_1, v_2 \in V$. Furthermore, we define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W, both of which will be denoted by $\cdot$ or by juxtaposition, by setting

$$[x_1(w), x_4(v)^{-1}]_2 = x_2(\tau_V(v, w)) = x_2(vw) \ ,$$

$$[x_1(w), x_4(v)^{-1}]_3 = x_3(\tau_W(w, v)) = x_3(wv) \ ,$$

for all $w \in W$ and all $v \in V$. Finally, for each $w \in W^*$, we define two elements $\kappa(w), \lambda(w) \in W^*$ by setting

$$\kappa(x_1(w)) = x_5(\kappa(w)) \ ,$$

$$\lambda(x_1(w)) = x_5(\lambda(w)) \ ,$$

and for each $v \in V^*$, we define an element $v^{-1} \in V^*$, by setting

$$\kappa(x_4(v)) = x_0(v^{-1}) \ .$$

Note that, by Theorem 5.8, $\lambda(x_4(v)) = x_0(v^{-1})$ as well.

If we can now prove that these data satisfy all of the axioms $(\mathbf{Q}_1)$ – $(\mathbf{Q}_{20})$, then we have proved that every Moufang quadrangle can actually be obtained from the construction in the previous chapters, since we have started from an arbitrary Moufang quadrangle. At the same time, however, we will show that the identities $(\mathbf{Q}_{21})$ – $(\mathbf{Q}_{26})$ hold; see Theorem 6.1.

Remark 5.11. It is interesting to observe that the choice of δ and ϵ is arbitrary (up to some restrictions about the radical). This gives us some freedom for the choice of the base points for the parametrizing structure of an arbitrary Moufang quadrangle. See also Remark 6.4.

By Theorem 5.5(ii), the group U_4 is abelian. Since U_4 is parametrized by $(V, +)$, we have that V is abelian as well.

By the definition of the isomorphism x_3 and the definition of the map from $V \times W$ to V, we have, for all $w \in W$, that $x_3(w) = [x_1(w), e_4^{-1}]_3 = [x_1(w), x_4(\epsilon)^{-1}]_3 =$

$x_3(w\epsilon)$, from which it follows that $w = w\epsilon$, which proves ($\mathbf{Q}_1$). Similarly, we can prove that ($\mathbf{Q}_2$) holds.

We now take a look at the subgroups V_3 and Y_3 of U_3. By definition, we have $V_3 = [U_2, U_4] = [x_2(V), x_4(V)^{-1}] = x_3(F(V,V)) = x_3(\mathrm{Im}(F))$. The elements of U_3 which commute with every element of U_1, are exactly those elements $x_3(w) \in U_3$ such that $[x_1(w'), x_3(w)] = 1$, for all $w' \in W$, this is, such that $x_2(H(w',w)) = 1$ or equivalently $H(w',w) = 0$, for all $w' \in W$. This means that $Y_3 = C_{U_3}(U_1) = x_3(\mathrm{Rad}(H))$. It now follows from Theorem 5.5(i) that $[W,W] \leq \mathrm{Im}(F) \leq \mathrm{Rad}(H) \leq Z(W)$. In particular, we have proved ($\mathbf{Q}_7$). We have also proved that $[\mathrm{Im}(F), W] = 1$.

Completely similarly as in the previous paragraph, it follows from the definitions that $Y_1 = C_{U_1}(U_3) = x_1(\mathrm{Rad}(H))$ and that $Y_4 = C_{U_4}(U_2) = x_4(\mathrm{Rad}(F))$. It thus follows from $e_1 = x_1(\delta) \in Y_1^*$ that $\delta \in \mathrm{Rad}(H)^*$, and it follows from the fact that $e_4 = x_4(\epsilon)$ was chosen to lie in Y_4^* if $Y_4 \neq 1$ that $\epsilon \in \mathrm{Rad}(F)^*$ if $\mathrm{Rad}(F) \neq 0$. Hence we have shown ($\mathbf{Q}_9$) and ($\mathbf{Q}_{10}$).

Using the identity $[ab, c^{-1}] = [a, c^{-1}]^b [b, c^{-1}]$ and the fact that U_1 and U_2 commute (because of Theorem 5.1(i)), we can deduce that

$$\begin{aligned}
x_2(H(w_1 \boxplus w_2, w')) &= [x_1(w_1 \boxplus w_2), x_3(w')^{-1}] \\
&= [x_1(w_1)x_1(w_2), x_3(w')^{-1}] \\
&= [x_1(w_1), x_3(w')^{-1}]^{x_1(w_2)} [x_1(w_2), x_3(w')^{-1}] \\
&= x_2(H(w_1, w'))^{x_1(w_2)} x_2(H(w_2, w')) \\
&= x_2(H(w_1, w')) x_2(H(w_2, w')) \\
&= x_2(H(w_1, w') + H(w_2, w')) \ ,
\end{aligned}$$

for all $w_1, w_2, w' \in W$, so H is additive in the first variable. Similarly, it follows from the identity $[a, (bc)^{-1}] = [a, b^{-1}][a, c^{-1}]^{b^{-1}}$ that H is additive in the second variable. In the same way, we can deduce from those two identities that F is additive in both variables. Since we will use this fact very often from now on, we will not mention it explicitly anymore.

Using the same identity $[ab, c^{-1}] = [a, c^{-1}]^b [b, c^{-1}]$ and the fact that $[U_2, U_2] = 1$ (since V is abelian) and $[U_2, U_3] = 1$ (by Theorem 5.1(i)), we deduce that

$$\begin{aligned}
x_2(v(w_1 \boxplus w_2)) x_3((w_1 \boxplus w_2)v) &= [x_1(w_1 \boxplus w_2), x_4(v)^{-1}] \\
&= [x_1(w_1)x_1(w_2), x_4(v)^{-1}] \\
&= [x_1(w_1), x_4(v)^{-1}]^{x_1(w_2)} [x_1(w_2), x_4(v)^{-1}] \\
&= (x_2(vw_1)x_3(w_1 v))^{x_1(w_2)} x_2(vw_2) x_3(w_2 v) \\
&= x_2(vw_1) x_2(H(w_2, w_1 v)) x_3(w_1 v) x_2(vw_2) x_3(w_2 v) \\
&= x_2(vw_1 + vw_2 + H(w_2, w_1 v)) x_3(w_1 v \boxplus w_2 v) \ ,
\end{aligned}$$

for all $w_1, w_2 \in W$ and all $v_1, v_2 \in V$. By Theorem 5.1(ii), this implies that

$$\begin{aligned}
&x_2(v(w_1 \boxplus w_2)) = x_2(vw_1 + vw_2 + H(w_2, w_1 v)) \qquad \text{and} \\
&x_3((w_1 \boxplus w_2)v) = x_3(w_1 v \boxplus w_2 v) \ ,
\end{aligned}$$

for all $w_1, w_2 \in W$ and all $v_1, v_2 \in V$, from which it follows that ($\mathbf{Q}_{12}$) and ($\mathbf{Q}_3$) hold.

Similarly, it follows from the identity $[a,(bc)^{-1}] = [a,b^{-1}][a,c^{-1}]^{b^{-1}}$, the fact that $[\mathrm{Im}(F), W] = 1$ and the fact that $[U_2, U_3] = 1$ (because of Theorem 5.1(i)), that $(\mathbf{Q}_{11})$ and $(\mathbf{Q}_4)$ hold.

Now, we will *define* a map $v \mapsto \overline{v}$ from V to V, by setting

$$x_2(v)^{\mu(e_4)} = x_2(-\overline{v}) \ ,$$

for all $v \in V$; we will prove later on (see page 31) that $\overline{v} = \epsilon F(\epsilon, v) - v$. Note that, by Theorem 5.10, we have $x_2(v)^{\mu(e_4)^2} = x_2(v)$, and hence $-\overline{(-\overline{v})} = v$, for all $v \in V$. If we invert the identity $x_2(v)^{\mu(e_4)} = x_2(-\overline{v})$, then we get $x_2(-v)^{\mu(e_4)} = x_2(\overline{v})$; it follows that $\overline{-v} = -\overline{v}$, for all $v \in V$. Combining these two relations, we also get $\overline{\overline{v}} = v$, for all $v \in V$.

Theorem 5.12. *For all $w \in W$ and all $v \in V$, we have:*

(i) $x_0(v)^{\mu(e_4)} = x_4(v)$;
(ii) $x_1(w)^{\mu(e_4)} = x_3(w(-\epsilon))$;
(iii) $x_2(v)^{\mu(e_4)} = x_2(-\overline{v})$;
(iv) $x_3(w)^{\mu(e_4)} = x_1(w)$;
(v) $x_4(v)^{\mu(e_4)} = x_0(v)$;
(vi) $x_1(w)^{\mu(e_1)} = x_5(w)$;
(vii) $x_2(v)^{\mu(e_1)} = x_4(-v)$;
(viii) $x_3(w)^{\mu(e_1)} = x_3(w)$;
(ix) $x_4(v)^{\mu(e_1)} = x_2(v)$;
(x) $x_5(w)^{\mu(e_1)} = x_1(w(-\epsilon))$.

Proof. First of all, (iii), (v) and (vi) hold by definition. By Theorem 5.10, $x_4(v)^{\mu(e_4)^2} = x_4(v)$. So if we conjugate (v) by $\mu(e_4)$, we get (i). If we apply Theorem 5.4(i) on the identity

$$[x_1(w), x_4(v)^{-1}] = x_2(vw)x_3(wv) \ ,$$

we get that

$$x_1(w) = x_3(wv)^{\mu(x_4(v))} \quad \text{and}$$
$$x_2(vw) = x_4(v)^{\mu(x_1(w))} \ ,$$

for all $w \in W^*$ and all $v \in V^*$. If we choose $v = \epsilon$ in the first equality, we get, by $(\mathbf{Q}_1)$, that $x_1(w) = x_3(w)^{\mu(e_4)}$, which proves (iv). If we choose $v = -\epsilon$ in this same equality, we get

$$\begin{aligned} x_1(w) &= x_3(w(-\epsilon))^{\mu(e_4^{-1})} \\ &= x_3(w(-\epsilon))^{\mu(e_4)^{-1}} \ , \end{aligned}$$

by Lemma 5.3(i); conjugating by $\mu(e_4)$ yields (ii).

If we choose $w = \delta$ in the second equality, then it follows from $(\mathbf{Q}_2)$ that $x_2(v) = x_4(v)^{\mu(e_1)}$, which proves (ix). By Theorem 5.7(ii), we have that $x_4(v)^{\mu(e_1)^2} = x_4(-v)$. So if we conjugate (ix) by $\mu(e_1)$, we get (vii).

By Theorem 5.6, we know that $\mu(e_1) \in Y_5Y_1Y_5$. Since $Y_1 = C_{U_1}(U_3)$ and $Y_5 = C_{U_5}(U_3)$, it follows that $[\mu(e_1), U_3] = 1$, which implies (viii).

If we conjugate the identity

$$[x_1(w), x_4(v)^{-1}] = x_2(vw)x_3(wv)$$

by $\mu(e_1)^2$, we get, using (vi), (vii), (viii) and (ix), that

$$[x_5(w)^{\mu(e_1)}, x_4(-v)^{-1}] = x_2(-vw)x_3(wv) \ ,$$

for all $w \in W$ and all $v \in V$. If we choose $v = -\epsilon$, then this yields

$$[x_5(w)^{\mu(e_1)}, e_4^{-1}] = x_2(\epsilon w)x_3(w(-\epsilon)) \ .$$

It now follows from Theorem 5.4(i) and (iv) that

$$x_5(w)^{\mu(e_1)} = x_3(w(-\epsilon))^{\mu(e_4)} = x_1(w(-\epsilon)) \ ,$$

for all $w \in W$, which proves (x). □

So far, we have proved ($\mathbf{Q}_1$), ($\mathbf{Q}_2$), ($\mathbf{Q}_3$), ($\mathbf{Q}_4$), ($\mathbf{Q}_7$), ($\mathbf{Q}_9$), ($\mathbf{Q}_{10}$), ($\mathbf{Q}_{11}$) and ($\mathbf{Q}_{12}$). We now continue to prove the other axioms.

If we conjugate the identity

$$[x_1(w_1), x_3(w_2)^{-1}] = x_2(H(w_1, w_2))$$

by $\mu(e_4)$, we get, by Theorem 5.12, that

$$[x_3(w_1(-\epsilon)), x_1(w_2)^{-1}] = x_2(-\overline{H(w_1, w_2)}) \ ,$$

for all $w_1, w_2 \in W$. Using the fact that $[b, a] = [a, b]^{-1}$, it follows that

$$[x_1(\boxminus w_2), x_3(\boxminus w_1(-\epsilon))^{-1}] = x_2(\overline{H(w_1, w_2)}) \ ,$$

hence

$$x_2(H(\boxminus w_2, \boxminus w_1(-\epsilon))) = x_2(\overline{H(w_1, w_2)}) \ ,$$

for all $w_1, w_2 \in W$. Using the fact that H is additive, it follows from this last equality that $H(w_2, w_1(-\epsilon)) = \overline{H(w_1, w_2)}$, for all $w_1, w_2 \in W$. Note that it follows from ($\mathbf{Q}_{12}$) that $H(w_2, w_1(-\epsilon)) = -H(w_2, w_1)$, for all $w_1, w_2 \in W$, so we have that $-H(w_2, w_1) = \overline{H(w_1, w_2)}$, which proves ($\mathbf{Q}_{22}$).

Completely similarly, we can conjugate the identity

$$[x_2(v_1), x_4(v_2)^{-1}] = x_3(F(v_1, v_2))$$

by $\mu(e_1)$, and, again by Theorem 5.12, we find after a short calculation that $F(v_1, v_2) = F(v_2, v_1)$, for all $v_1, v_2 \in V$, which proves ($\mathbf{Q}_{21}$).

If we conjugate the identity

$$[x_1(w), x_4(-v)^{-1}] = x_2(-vw)x_3(w(-v))$$

by $\mu(e_1)^2$, then we get, by Theorem 5.12, that

$$[x_1(w(-\epsilon)), x_4(v)^{-1}] = x_2(vw)x_3(w(-v)) \ ,$$

for all $w \in W$ and all $v \in V$. But on the other hand, we have that

$$[x_1(w(-\epsilon)), x_4(v)^{-1}] = x_2(v(w(-\epsilon)))x_3(w(-\epsilon)v) \ ,$$

for all $w \in W$ and all $v \in V$. By Theorem 5.1(ii), this implies that $vw = v(w(-\epsilon))$ and $w(-v) = w(-\epsilon)v$, for all $w \in W$ and all $v \in V$. Thus we have proved ($\mathbf{Q}_6$) and ($\mathbf{Q}_5$).

If we conjugate the same identity

$$[x_1(w), x_4(v)^{-1}] = x_2(vw)x_3(wv)$$

by $\mu(e_4)^2$, we get, again by Theorem 5.12, that

$$[x_1(w(-\epsilon)), x_4(v)^{-1}] = x_2(vw)x_3(wv(-\epsilon)) \ ,$$

from which it follows immediately (by Theorem 5.1(ii)) that $w(-\epsilon)v = wv(-\epsilon)$, for all $w \in W$ and all $v \in V$. This means that $w(-v) = wv(-\epsilon)$ as well.

We will now prove ($\mathbf{Q}_8$). We will make use of the identity $[a,b]=a^{-1}a^b$ and of the identity $[abc,d]=[a,d]^{bc}[b,d]^c[c,d]$.

$$\begin{aligned} x_3(F(H(w_2,w_1),v)) &= [x_2(H(w_2,w_1)),x_4(v)^{-1}] \\ &= [[x_1(w_2),x_3(w_1)^{-1}],x_4(v)^{-1}] \\ &= [x_1(w_2),x_3(w_1)^{-1}]^{-1}[x_1(w_2),x_3(w_1)^{-1}]^{x_4(v)^{-1}} \\ &= [x_1(w_2),x_3(w_1)^{-1}]^{-1}[x_1(w_2)x_2(vw_2)x_3(w_2v),x_3(w_1)^{-1}] \ . \end{aligned}$$

If $a_1\in U_1$, $a_2\in U_2$ and $a_3,b_3\in U_3$, then $[a_2,b_3]\in[U_2,U_3]=1$ and $[a_1,b_3]\in[U_1,U_3]\le U_2$ (by Theorem 5.1(i)), and since $[U_2,U_2U_3]=1$, we have that $[a_1,b_3]^{a_2a_3}=[a_1,b_3]$. Therefore $[a_1a_2a_3,b_3]=[a_1,b_3][a_3b_3]$. Hence

$$\begin{aligned} x_3(F(H(w_2,w_1),v)) &= [x_1(w_2),x_3(w_1)^{-1}]^{-1}[x_1(w_2),x_3(w_1)^{-1}][x_3(w_2v),x_3(w_1)^{-1}] \\ &= [x_3(w_2v),x_3(w_1)^{-1}] \\ &= x_3(\boxminus w_2v\boxplus w_1\boxplus w_2v\boxminus w_1) \ , \end{aligned}$$

and since $\mathrm{Im}(F)\le Z(W)$, we have that

$$\begin{aligned} x_3(F(H(w_2,w_1),v)) &= x_3(\boxminus w_1\boxminus w_2v\boxplus w_1\boxplus w_2v) \\ &= x_3([w_1,w_2v]_\boxplus) \end{aligned}$$

as well, for all $w_1,w_2\in W$ and all $v\in V$, which proves ($\mathbf{Q}_8$).

We will now apply the Shift Lemma 5.4(ii) on the identity

$$[x_1(w),x_4(v)^{-1}]=x_2(vw)x_3(wv) \ .$$

This gives us the identity

$$[\kappa(x_4(v)),x_3(wv)^{-1}]=x_1(w)x_2(vw) \ .$$

Note that, by definition, we have $\kappa(x_4(v))=x_0(v^{-1})$. If we conjugate this identity by $\mu(e_4)$, we thus get, by Theorem 5.12, that

$$[x_4(v^{-1}),x_1(wv)^{-1}]=x_3(w(-\epsilon))x_2(-\overline{vw}) \ .$$

Inverting this identity and replacing w by $\boxminus w$ yields

$$[x_1(wv),x_4(-(v^{-1}))^{-1}]=x_2(\overline{v(\boxminus w)})x_3(w(-\epsilon)) \ ,$$

for all $w\in W$ and all $v\in V^*$. But on the other hand, we have

$$[x_1(wv),x_4(-(v^{-1}))^{-1}]=x_2(-(v^{-1})(wv))x_3(wv(-(v^{-1}))) \ ,$$

for all $w\in W$ and all $v\in V^*$. By Theorem 5.1(ii), this implies that

$$\begin{aligned} \overline{v(\boxminus w)} &= -(v^{-1})(wv) \ , \\ w(-\epsilon) &= wv(-(v^{-1})) \ , \end{aligned}$$

for all $w\in W$ and all $v\in V^*$. If we apply the identity $w(-v)=wv(-\epsilon)$ on the second equality, we can conclude that this is equivalent to

$$\begin{aligned} v^{-1}(wv) &= -\overline{v(\boxminus w)} \ , \\ wvv^{-1} &= w \ , \end{aligned}$$

for all $w\in W$ and all $v\in V^*$. So we have proved ($\mathbf{Q}_{16}$) and ($\mathbf{Q}_{15}$).

If we replace v by v^{-1} and w by wv in ($\mathbf{Q}_{16}$), then we get

$$(v^{-1})^{-1}(wvv^{-1})=-\overline{v^{-1}(\boxminus wv)} \ ,$$

for all $w \in W$ and all $v \in V^*$. Using ($\mathbf{Q}_{15}$) and ($\mathbf{Q}_{16}$) once again, and using the fact that $-\overline{(-\overline{v})} = v$ for all $v \in V$, we get

$$(v^{-1})^{-1}w = vw \ ,$$

for all $w \in W$ and all $v \in V^*$. If we choose $w = \delta$, it follows that $(v^{-1})^{-1} = v$, for all $v \in V^*$, which proves ($\mathbf{Q}_{13}$).

If we take $a_i = x_4(v)$ in Lemma 5.3(ii), then we get that $\lambda(x_4(-v)) = \kappa(x_4(v))^{-1}$, for all $v \in V^*$. By the definition of v^{-1}, this is equivalent to $x_0((-v)^{-1}) = x_0(-(v^{-1}))$, from which it follows that $(-v)^{-1} = -(v^{-1})$, for all $v \in V^*$.

Similarly, if we choose $a_i = x_1(w)$ in Lemma 5.3(ii), then we get that $\lambda(x_1(\boxminus w)) = \kappa(x_1(w))^{-1}$, for all $w \in W^*$. By the definition of κ and λ, this is equivalent to $x_5(\lambda(\boxminus w)) = x_5(\boxminus\kappa(w))$, from which it follows that $\lambda(\boxminus w) = \boxminus\kappa(w)$, for all $w \in W^*$.

If we apply the Shift Lemma 5.4(iii) on the identity

$$[x_1(w), x_4(v)^{-1}] = x_2(vw)x_3(wv) \ ,$$

then we get that

$$[x_2(vw), \lambda(x_1(w))^{-1}] = x_3(wv)x_4(v) \ ,$$

for all $w \in W^*$ and all $v \in V$. By definition, we have $\lambda(x_1(w)) = x_5(\lambda(w))$. If we conjugate this identity by $\mu(e_1)^{-1}$, we thus get, by Theorem 5.12, that

$$[x_4(vw), x_1(\lambda(w))^{-1}] = x_3(wv)x_2(-v) \ ,$$

for all $w \in W^*$ and all $v \in V$. We can rewrite this identity as

$$[x_1(\boxminus\lambda(w)), x_4(-vw)^{-1}] = x_2(v)x_3(\boxminus wv) \ ,$$

for all $w \in W^*$ and all $v \in V$. On the other hand, we also have that

$$[x_1(\boxminus\lambda(w)), x_4(-vw)^{-1}] = x_2(-vw(\boxminus\lambda(w)))x_3(\boxminus\lambda(w)(-vw)) \ ,$$

for all $w \in W^*$ and all $v \in V$. It follows from Theorem 5.1(ii) that

$$\begin{aligned} v &= -vw(\boxminus\lambda(w)) \ , \\ wv &= \lambda(w)(-vw) \ , \end{aligned}$$

for all $w \in W^*$ and all $v \in V$. If we replace w by $\boxminus\lambda(w)$ and v by vw in the second equality, then we get

$$\boxminus\lambda(w)(vw) = \lambda(\boxminus\lambda(w))(-vw(\boxminus\lambda(w))) \ ,$$

for all $w \in W^*$ and all $v \in V$. If we use these same equalities once again, then we can simplify this to

$$\boxminus w(-v) = \lambda(\boxminus\lambda(w))v \ ,$$

for all $w \in W^*$ and all $v \in V$. If we choose $v = \epsilon$, then we get $\lambda(\boxminus\lambda(w)) = \boxminus w(-\epsilon)$, for all $w \in W^*$. Since $\lambda(\boxminus w) = \boxminus\kappa(w)$, for all $w \in W^*$, this is the same as $\kappa(\lambda(w)) = w(-\epsilon)$, for all $w \in W^*$, so we have proved ($\mathbf{Q}_{14}$). If we replace w by $\boxminus w$, then we get $\lambda(\kappa(w)) = w(-\epsilon)$ as well. Now, we substitute $\kappa(w)$ for w in the equations

$$\begin{aligned} v &= -vw(\boxminus\lambda(w)) \ , \\ wv &= \lambda(w)(-vw) \ ; \end{aligned}$$

this gives us, using the fact that $\lambda(\kappa(w)) = w(-\epsilon)$, that

$$\begin{aligned} v &= -v\kappa(w)(\boxminus w(-\epsilon)) \ , \\ \kappa(w)v &= w(-\epsilon)(-v\kappa(w)) \ , \end{aligned}$$

for all $w \in W^*$ and all $v \in V$. It suffices to use $(\mathbf{Q}_6)$ and $(\mathbf{Q}_5)$ to see that those two equations are equivalent to $(\mathbf{Q}_{18})$ and $(\mathbf{Q}_{19})$, respectively.

If we put $a_2 = x_2(v_2)$ and $a_4 = x_4(v_1)$ in Theorem 5.9(ii), then we get

$$[[\lambda(x_4(v_1)), x_2(v_2)^{-1}], x_4(v_1)]_3 = [x_2(v_2), x_4(v_1)]^{-1} \ ,$$

for all $v_1 \in V^*$ and all $v_2 \in V$. First of all, we have that

$$\begin{aligned} [\lambda(x_4(v_1)), x_2(v_2)^{-1}] &= [x_0(v_1^{-1}), x_2(v_2)^{-1}] \\ &= [x_4(v_1^{-1}), x_2(-\overline{v_2})^{-1}]^{\mu(e_4)} \\ &= ([x_2(\overline{v_2}), x_4(-v_1^{-1})^{-1}]^{-1})^{\mu(e_4)} \\ &= (x_3(F(\overline{v_2}, -v_1^{-1}))^{-1})^{\mu(e_4)} \\ &= x_3(F(\overline{v_2}, v_1^{-1}))^{\mu(e_4)} \\ &= x_1(F(\overline{v_2}, v_1^{-1})) \ , \end{aligned}$$

for all $v_1 \in V^*$ and all $v_2 \in V$. So it follows from this identity that

$$[x_1(F(\overline{v_2}, v_1^{-1})), x_4(-v_1)^{-1}]_3 = [x_2(v_2), x_4(-v_1)^{-1}]^{-1} \ ,$$

for all $v_1 \in V^*$ and all $v_2 \in V$, from which it follows that

$$F(\overline{v_2}, v_1^{-1})(-v_1) = F(v_2, v_1) \ ,$$

for all $v_1 \in V^*$ and all $v_2 \in V$. If we now replace v_1 by $-v_1$, then we get, using the fact that $(-v_1)^{-1} = -(v_1^{-1})$ and $(\mathbf{Q}_{21})$, that $(\mathbf{Q}_{17})$ holds.

If we choose $v_1 = \epsilon$ in $(\mathbf{Q}_{17})$, then we get that $F(\epsilon^{-1}, \overline{v}) = F(\epsilon, v)$, for all $v \in V$. If we put $a_2 = x_2(v)$ and $a_4 = e_4$ in Theorem 5.9(i), then we get

$$x_2(v)^{\mu(e_4)} x_2(v)^{-1} = [[\lambda(e_4), x_2(v)^{-1}], e_4]_2 \ ,$$

for all $v \in V$. We have that

$$\begin{aligned} [\lambda(e_4), x_2(v)^{-1}] &= x_1(F(\overline{v}, \epsilon^{-1})) \\ &= x_1(F(\epsilon, v)) \ , \end{aligned}$$

for all $v \in V$. Thus we have

$$\begin{aligned} x_2(v)^{\mu(e_4)} x_2(v)^{-1} &= [[\lambda(e_4), x_2(v)^{-1}], e_4]_2 \\ &= [x_1(F(\epsilon, v)), x_4(-\epsilon)^{-1}]_2 \\ &= x_2(-\epsilon F(\epsilon, v)) \ , \end{aligned}$$

for all $v \in V$. Since $x_2(v)^{\mu(e_4)} x_2(v)^{-1} = x_2(-\overline{v} - v)$, we conclude that

$$\overline{v} = \epsilon F(\epsilon, v) - v \ ,$$

for all $v \in V$; see page 27.

If we put $a_1 = x_1(w_1)$ and $a_3 = x_3(w_2)$ in Theorem 5.9(iii), then we get

$$[x_1(w_1), [x_3(w_2), \kappa(x_1(w_1))]^{-1}]_2 = [x_1(w_1), x_3(w_2)^{-1}]^{-1} \ ,$$

for all $w_1 \in W$ and all $w_2 \in W^*$. First of all, we have that

$$\begin{aligned}[x_3(w_2), \kappa(x_1(w_1))] &= [x_3(w_2), x_5(\kappa(w_1))] \\ &= [x_3(w_2), x_1(\kappa(w_1))]^{\mu(e_1)} \\ &= ([x_1(\kappa(w_1)), x_3(\boxminus w_2)^{-1}]^{-1})^{\mu(e_1)} \\ &= (x_2(-H(\kappa(w_1), w_2))^{-1})^{\mu(e_1)} \\ &= x_2(H(\kappa(w_1), w_2))^{\mu(e_1)} \\ &= x_4(-H(\kappa(w_1), w_2)) \ ,\end{aligned}$$

for all $w_1 \in W$ and all $w_2 \in W^*$. So it follows from this identity that

$$[x_1(w_1), x_4(-H(\kappa(w_1), w_2))^{-1}]_2 = [x_1(w_1), x_3(w_2)^{-1}]^{-1} \ ,$$

for all $w_1 \in W$ and all $w_2 \in W^*$, from which it follows that

$$-H(\kappa(w_1), w_2)w_1 = -H(w_1, w_2) \ ,$$

for all $w_1 \in W$ and all $w_2 \in W^*$. So we have proved ($\mathbf{Q}_{20}$).

Since we have shown all of the identities ($\mathbf{Q}_1$) – ($\mathbf{Q}_{20}$), we can conclude that every Moufang quadrangle can be obtained from a quadrangular system.

In particular, we are now allowed to use the results of chapter 4 as well. We thus continue to show that the identities ($\mathbf{Q}_{23}$) – ($\mathbf{Q}_{26}$) hold.

In order to show ($\mathbf{Q}_{23}$), we will calculate the expression

$$x_2(v)^{[\mu(x_1(\delta))\mu(x_1(z))]^{\mu(x_3(w))\mu(x_3(\delta))}}$$

with $v \in V$ and $w, z \in W^*$ in two different ways. We have shown in Theorem 4.3(iii) that

$$x_3(z)^{\mu(x_1(w))} = x_3(\Pi_w(z))$$

for all $w, z \in W^*$. If we let $\mu(e_4)$ act on both sides of this equality, then it follows by Lemma 5.3(iii) and Theorem 5.12 that

$$x_1(z)^{\mu(x_3(w(-\epsilon)))} = x_1(\Pi_w(z))$$

and it thus follows by substituting $w(-\epsilon)$ for w and by Lemma 5.3(iii) that

$$\mu(x_1(z))^{\mu(x_3(w))} = \mu(x_1(\Pi_{w(-\epsilon)}(z)))$$

for all $w, z \in W^*$. By Lemma 3.15(ii), $\Pi_{w(-\epsilon)}(z) = \Pi_w(z)$ for all $w, z \in W^*$. Since $\delta \in \mathrm{Rad}(H)$ by ($\mathbf{Q}_9$), it now follows that

$$\mu(x_1(z))^{\mu(x_3(w))\mu(x_3(\delta))} = \mu(x_1(\Pi_w(z)))$$

and hence, since $\Pi_w(\delta) = \delta$,

$$[\mu(x_1(\delta))\mu(x_1(z))]^{\mu(x_3(w))\mu(x_3(\delta))} = \mu(x_1(\delta))\mu(x_1(\Pi_w(z)))$$

for all $w, z \in W^*$. Note that $v\kappa(\delta) = v$ for all $v \in V$. Since we have shown in Theorem 4.3 that

$$\begin{aligned}x_2(v)^{\mu(x_1(w))} &= x_4(-v\kappa(w)) \quad \text{and} \\ x_4(v)^{\mu(x_1(w))} &= x_2(vw)\end{aligned}$$

for all $v \in V$ and all $w \in W^*$, it thus follows, by Lemma 5.3(i), that

$$\begin{aligned} x_2(v)^{[\mu(x_1(\delta))\mu(x_1(z))]^{\mu(x_3(w))\mu(x_3(\delta))}} &= x_2(v)^{\mu(x_1(\delta))\mu(x_1(\Pi_w(z)))} \\ &= x_4(-v)^{\mu(x_1(\Pi_w(z)))} \\ &= x_2(-v \cdot \Pi_w(z)) \end{aligned}$$

for all $w, z \in W^*$.

On the other hand, if we let $\mu(e_4)$ act on both sides of the identity $x_4(v)^{\mu(x_1(w))} = x_2(vw)$, then we can deduce that

$$\begin{aligned} x_0(v)^{\mu(x_3(w))} &= x_2(-\overline{vw}) \quad \text{and} \\ x_2(v)^{\mu(x_3(w))} &= x_0(\overline{v}\kappa(w)) \end{aligned}$$

for all $v \in V$ and all $w \in W^*$. Hence, by Lemma 5.3(i),

$$\begin{aligned} x_2(v)^{[\mu(x_1(\delta))\mu(x_1(z))]^{\mu(x_3(w))\mu(x_3(\delta))}} &= x_2(v)^{\mu(x_3(\boxminus\delta))\mu(x_3(\boxminus w))\mu(x_1(\delta))\mu(x_1(z))\mu(x_3(w))\mu(x_3(\delta))} \\ &= x_2\big(\overline{\overline{v}(\boxminus w)}\big)^{\mu(x_1(\delta))\mu(x_1(z))\mu(x_3(w))\mu(x_3(\delta))} \\ &= x_2\big(-\overline{\overline{v}(\boxminus w)}z\big)^{\mu(x_3(w))\mu(x_3(\delta))} \\ &= x_2\big(\overline{\overline{\overline{\overline{v}(\boxminus w)}}z\kappa(w)}\big) \end{aligned}$$

for all $w, z \in W^*$. Hence we have shown that $(\mathbf{Q}_{23})$ holds.

The proof of $(\mathbf{Q}_{24})$ follows in a completely similar way by calculating the expression

$$x_3(w)^{[\mu(x_4(\epsilon))\mu(x_4(c))]^{\mu(x_2(v))\mu(x_2(\epsilon))}}$$

with $w \in W$ and $v, c \in V^*$ in two different ways.

We will now show $(\mathbf{Q}_{25})$. Let $c \in V$, $v \in V^*$ and $w \in W^*$ be arbitrary. This time, we will calculate the expression

$$x_2(c)^{[\mu(x_1(\delta))\mu(x_1(w))]^{\mu(x_4(-v))}}$$

in two different ways. By Theorem 4.2(ii),

$$x_1(w)^{\mu(x_4(v))} = x_3(w(-v))$$

for all $v \in V^*$ and all $w \in W$, and hence, by Lemma 5.3(iii),

$$[\mu(x_1(\delta))\mu(x_1(w))]^{\mu(x_4(-v))} = \mu(x_3(\delta v))\mu(x_3(wv)) \ .$$

It follows that

$$\begin{aligned} x_2(c)^{[\mu(x_1(\delta))\mu(x_1(w))]^{\mu(x_4(-v))}} &= x_2(c)^{\mu(x_3(\delta v))\mu(x_3(wv))} \\ &= x_2(\overline{c} \cdot \kappa(\delta v))^{\mu(x_3(wv))} \\ &= x_2(-\overline{\overline{c} \cdot \kappa(\delta v) \cdot wv}) \ . \end{aligned}$$

On the other hand, we have shown in Theorem 4.2(iii) that

$$x_2(u)^{\mu(x_4(v))} = x_2(\pi_v(u))$$

for all $u \in V$ and all $v \in V^*$, and hence, by Lemma 5.3(i),

$$\begin{aligned} x_2(c)^{[\mu(x_1(\delta))\mu(x_1(w))]^{\mu(x_4(-v))}} &= x_2(c)^{\mu(x_4(v))\mu(x_1(\delta))\mu(x_1(w))\mu(x_4(-v))} \\ &= x_2(\pi_v(c))^{\mu(x_1(\delta))\mu(x_1(w))\mu(x_4(-v))} \\ &= x_2(-\pi_v(c)w)^{\mu(x_4(-v))} \\ &= x_2(-\pi_{-v}(\pi_v(c)w)) \ . \end{aligned}$$

Since $\pi_{-v}(u) = \pi_v(u)$ for all $u \in V$ and all $v \in V^*$, it follows by comparing these two expressions that

$$\overline{\overline{c} \cdot \kappa(\delta v) \cdot wv} = \pi_v(\pi_v(c)w) \ .$$

If we substitute $\overline{c \cdot \delta v}$ for c in the last identity and apply π_v on both sides, then we get, by Lemma 3.23(i), that

$$\pi_v(\overline{c \cdot \delta v \cdot \kappa(\delta v) \cdot wv}) = \pi_v(\overline{c \cdot \delta v})w \ .$$

Since $\delta v \in \mathrm{Rad}(H)$ by $(\mathbf{Q}_9)$ and Lemma 3.19, it follows by Lemma 3.13(ii) and Lemma 3.2(i) that

$$c \cdot \delta v \cdot \kappa(\delta v) = -c \cdot (\boxminus \delta v) \cdot \kappa(\delta v) = c \ ,$$

which completes the proof of $(\mathbf{Q}_{25})$.

The proof of $(\mathbf{Q}_{26})$ follows in a completely similar way by calculating the expression

$$x_3(w)^{[\mu(x_4(\epsilon))\mu(x_4(-v))]^{\mu(x_1(z))}}$$

with $w \in W$, $z \in W^*$ and $v \in V^*$ in two different ways.

This concludes the proof of all of the identities $(\mathbf{Q}_1)$ – $(\mathbf{Q}_{26})$.

CHAPTER 6

Some Remarks

We start by pointing out that we have really shown that the identities $(\mathbf{Q}_{23})$ – $(\mathbf{Q}_{26})$ follow from the axioms $(\mathbf{Q}_1)$ – $(\mathbf{Q}_{20})$.

THEOREM 6.1. *Let* $\Omega := (V, W, \tau_V, \tau_W, \epsilon, \delta)$ *be a quadrangular system. Then the identities* $(\mathbf{Q}_{23})$ – $(\mathbf{Q}_{26})$ *hold, for all* $v, c \in V$ *and all* $w, z \in W$.

PROOF. Let $\Omega := (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. Then it follows from chapter 4 that we can construct a Moufang quadrangle Γ starting from Ω. In chapter 5, it is shown that every Moufang quadrangle can be constructed from a quadrangular system for which additionally the identities $(\mathbf{Q}_{23})$ – $(\mathbf{Q}_{26})$ hold. In particular, Γ can be constructed from a quadrangular system, which can be chosen to coincide with the quadrangular system Ω that we started with, since the choice of the parametrization of the groups U_1 and U_4 and of the elements ϵ and δ was arbitrary. (Note that the parametrization of the groups U_2 and U_3 and the definition of the maps F and H then automatically coincide by construction.) This shows that Ω is a quadrangular system for which additionally the identities $(\mathbf{Q}_{23})$ – $(\mathbf{Q}_{26})$ hold. Since Ω was arbitrary, these identities hold for every quadrangular system. □

REMARK 6.2. One might wonder why we pay so much attention to these last four identities $(\mathbf{Q}_{23})$, $(\mathbf{Q}_{24})$, $(\mathbf{Q}_{25})$ and $(\mathbf{Q}_{26})$. The reason is that these identities turn out to be essential for the classification of the quadrangular systems, but still, we are not aware of a direct proof for the fact that they follow from the other axioms.

REMARK 6.3. Although every quadrangular system gives rise to a Moufang quadrangle and every Moufang quadrangle can be constructed from a quadrangular system, it is *not* true that there is a bijection between the set of classes of isomorphic quadrangular systems and the set of classes of isomorphic Moufang quadrangles. In particular, two non-isomorphic quadrangular systems can give rise to isomorphic Moufang quadrangles. However, two isomorphic quadrangular systems will always give rise to isomorphic Moufang quadrangles.

REMARK 6.4. We could as well have defined a quadrangular system without axiom $(\mathbf{Q}_{10})$. The reason that we added this axiom has to do with the classification of the so-called *wide* quadrangular systems which are the extension of a quadrangular system of *quadratic form type*. Without axiom $(\mathbf{Q}_{10})$, one would have to define a *translate* of a quadrangular system of type F_4 in order to describe all possible quadrangular systems (up to isomorphism), which is not needed now because of this extra axiom. (See section 8.5 for more details.)

On the other hand, if there are no quadrangles of type F_4 involved in a certain application, then it can often be more convenient to drop this axiom $(\mathbf{Q}_{10})$, since it gives more freedom in the choice of the base point $\epsilon \in V^*$. See also Remark 5.11 and Remark A.1.

CHAPTER 7

Examples

We will now present a list of six examples of quadrangular systems. These examples correspond to the six different classes of Moufang quadrangles in [**20**]. The goal of the next chapter is to prove that, up to isomorphism, this list is complete.

In each case, we will describe a *parametrization* for the groups V and W, that is, we will describe V and W as groups which are isomorphic to certain other groups $\tilde{V}$ and $\tilde{W}$, respectively; we will denote the isomorphisms from $\tilde{V}$ to V and from $\tilde{W}$ to W by square brackets: $a \in \tilde{V} \mapsto [a] \in V$ and $b \in \tilde{W} \mapsto [b] \in W$.

7.1. Quadrangular Systems of Quadratic Form Type

Consider a non-trivial anisotropic quadratic space (K, V_0, q), that is, a commutative field K (of arbitrary characteristic), a non-trivial vector space V_0 over K, and a quadratic form q from V_0 to K such that $q(v) = 0$ if and only if $v = 0$. Choose an arbitrary element $\epsilon \in V_0$, and replace q by $q(\epsilon)^{-1}q$. Then $q(\epsilon) = 1$. Denote the corresponding bilinear form of q by f, i.e. $f(u,v) := q(u+v) - q(u) - q(v)$. We define a map $v \mapsto \overline{v}$ by setting $\overline{v} := f(\epsilon, v)\epsilon - v$ for all $v \in V$. (This map is the negative of the reflection about the base point ϵ.) Let V be parametrized by $(V_0, +)$, and let W be parametrized by the additive group of K. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$\tau_V([v],[t]) := [v][t] := [tv] \ ,$$
$$\tau_W([t],[v]) := [t][v] := [tq(v)] \ ,$$

for all $v \in V_0$ and all $t \in K$. Then $(V, W, \tau_V, \tau_W, [\epsilon], [1])$ is a quadrangular system. One can check that

$$F([u],[v]) = [f(u,v)] \ ,$$
$$H([s],[t]) = [0] \ ,$$

for all $u, v \in V_0$ and all $s, t \in K$, and that

$$[v]^{-1} = [q(v)^{-1}\overline{v}] \ ,$$
$$\kappa([t]) = [t^{-1}] \ ,$$

for all $v \in V_0^*$ and all $t \in K^*$. Note that $\overline{[v]} = \epsilon F(\epsilon, [v]) - [v] = [\epsilon][f(\epsilon, v)] - [v] = [\overline{v}]$ for all $v \in V$.

These are the quadrangular systems *of quadratic form type.* They will be denoted by $\Omega_Q(K, V_0, q)$.

7.2. Quadrangular Systems of Involutory Type

Following [**20**], we define an *involutory set* as a triple (K, K_0, σ), where K is a field or a skew-field, σ is an involution of K, and K_0 is an additive subgroup of K

containing 1 such that

$$K_\sigma \subseteq K_0 \subseteq \mathrm{Fix}_K(\sigma) \quad \text{and}$$
$$a^\sigma K_0 a \subseteq K_0 \quad \text{for all } a \in K \ ,$$

where $K_\sigma := \{a + a^\sigma \mid a \in K\}$. Note that if $\mathrm{char}(K) \neq 2$, then $K_\sigma = \mathrm{Fix}_K(\sigma)$, and hence $K_0 = K_\sigma$ as well, so the second condition is superfluous in this case.

Let V be parametrized by the additive group of K, and let W be parametrized by K_0. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$\tau_V([a],[t]) := [a][t] := [ta] \ ,$$
$$\tau_W([t],[a]) := [t][a] := [a^\sigma ta] \ ,$$

for all $a \in K$ and all $t \in K_0$. Then $(V, W, \tau_V, \tau_W, [1], [1])$ is a quadrangular system. One can check that

$$F([a],[b]) = [a^\sigma b + b^\sigma a] \ ,$$
$$H([s],[t]) = [0] \ ,$$

for all $a, b \in K$ and all $s, t \in K_0$, and that

$$[a]^{-1} = [a^{-1}] \ ,$$
$$\kappa([t]) = [t^{-1}] \ ,$$

for all $a \in K^*$ and all $t \in K_0^*$. Note that $\overline{[a]} = \epsilon F(\epsilon, [a]) - [a] = [1][a + a^\sigma] - [a] = [a^\sigma]$ for all $a \in K$.

These are the quadrangular systems *of involutory type*. They will be denoted by $\Omega_I(K, K_0, \sigma)$.

7.3. Quadrangular Systems of Indifferent Type

Again following [**20**], we define an *indifferent set* as a triple (K, K_0, L_0), where K is a commutative field of characteristic 2, and K_0 and L_0 are additive subgroups of K both containing 1, such that

$$K_0^2 L_0 \subseteq L_0 \ ,$$
$$L_0 K_0 \subseteq K_0 \ ,$$
$$K_0 \text{ generates } K \text{ as a ring.}$$

Let V be parametrized by L_0, and let W be parametrized by K_0. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$\tau_V([a],[t]) := [a][t] := [t^2 a] \ ,$$
$$\tau_W([t],[a]) := [t][a] := [ta] \ ,$$

for all $a \in L_0$ and all $t \in K_0$. Then $(V, W, \tau_V, \tau_W, [1], [1])$ is a quadrangular system. One can check that

$$F([a],[b]) = [0] \ ,$$
$$H([s],[t]) = [0] \ ,$$

for all $a, b \in L_0$ and all $s, t \in K_0$, and that

$$[a]^{-1} = [a^{-1}] \ ,$$
$$\kappa([t]) = [t^{-1}] \ ,$$

for all $a \in K^*$ and all $t \in K_0^*$. Note that $\overline{[a]} = [a]$ for all $a \in K$.

These are the quadrangular systems *of indifferent type*. They will be denoted by $\Omega_D(K, K_0, L_0)$.

7.4. Quadrangular Systems of Pseudo-quadratic Form Type

Let K be an arbitrary field or skew-field, let σ be an involution of K (which may be trivial), and let V_0 be a right vector space over K. A map h from $V_0 \times V_0$ to K is called a *sesquilinear form* (with respect to σ) if and only if h is additive in both variables, and $h(at, bs) = t^\sigma h(a, b)s$, for all $a, b \in V_0$ and all $t, s \in K$. A form $h : V_0 \times V_0 \to K$ is called *hermitian*, respectively *skew-hermitian*, (with respect to σ) if and only if h is sesquilinear with respect to σ and $h(a, b)^\sigma = h(b, a)$, respectively $h(a, b)^\sigma = -h(b, a)$, for all $a, b \in V_0$.

Let (K, K_0, σ) be an involutory set, let V_0 be a right vector space over K and let p be a map from V_0 to K. Then p is an *anisotropic pseudo-quadratic* form on V (with respect to K_0 and σ) if there is a form h on V_0 which is skew-hermitian with respect to σ such that

$$p(a + b) \equiv p(a) + p(b) + h(a, b) \pmod{K_0} \ ,$$
$$p(at) \equiv t^\sigma p(a)t \pmod{K_0} \ ,$$
$$p(a) \in K_0 \iff a = 0 \ ,$$

for all $a, b \in V_0$ and all $t \in K$.

As in [**20**], we define an *anisotropic pseudo-quadratic space* as a quintuple (K, K_0, σ, V_0, p) such that (K, K_0, σ) is an involutory set, V_0 is a right vector space over K and p is an anisotropic pseudo-quadratic form on V_0 with respect to K_0 and σ.

Let (K, K_0, σ, V_0, p) be an arbitrary anisotropic pseudo-quadratic space with corresponding skew-hermitian form h. We define a group $(T, \boxplus)$ as

$$T := \{(a, t) \in V_0 \times K \mid p(a) - t \in K_0\} \ ,$$

where the group action is given by

$$(a, t) \boxplus (b, s) := (a + b, t + s + h(b, a)) \ ,$$

for all $(a, t), (b, s) \in T$. One can check that T is indeed a group with neutral element $(0, 0)$, and with the inverse given by $\boxminus(a, t) = (-a, -t + h(a, a))$, for all $(a, t) \in T$.

Let V be parametrized by the additive group of K, and let W be parametrized by T. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$\tau_V([v], [a, t]) := [v][a, t] := [tv] \ ,$$
$$\tau_W([a, t], [v]) := [a, t][v] := [av, v^\sigma tv] \ ,$$

for all $v \in K$ and all $(a, t) \in T$. Then $(V, W, \tau_V, \tau_W, [1], [0, 1])$ is a quadrangular system. One can check that

$$F([u], [v]) = [0, u^\sigma v + v^\sigma u] \ ,$$
$$H([a, t], [b, s]) = [h(a, b)] \ ,$$

for all $u, v \in K$ and all $(a, t), (b, s) \in T$, and that

$$[v]^{-1} = [v^{-1}] \ ,$$
$$\kappa([a, t]) = [at^{-\sigma}, t^{-\sigma}] \ ,$$

for all $v \in K^*$ and all $(a, t) \in T^*$. Note that

$$\overline{[v]} = [1]F([1], [v]) - [v] = [1][0, v + v^\sigma] - [v] = [v^\sigma]$$

for all $v \in K$.

These are the quadrangular systems *of pseudo-quadratic form type.* They will be denoted by $\Omega_P(K, K_0, \sigma, V_0, p)$.

7.5. Quadrangular Systems of Type E_6, E_7 and E_8

We first recall the notion of a *norm splitting*, which has been introduced in [**20**, (12.9)]. Let K be an arbitrary commutative field, let V_0 be a $2d$-dimensional vector space over K, and let q be a regular quadratic form from V_0 to K. First of all, observe that, if E/K is a separable quadratic extension with norm N, then N is a 2-dimensional anisotropic regular quadratic form from E (as a vector space over K) to K. We say that $(E, \{v_1, \dots, v_d\})$ is a norm splitting of q if and only if V_0 is a vector space over E, where the scalar multiplication from $E \times V_0$ to V_0 is an extension of the scalar multiplication from $K \times V_0$ to V_0, such that $\{v_1, \dots, v_d\}$ is a basis of V_0 over E for which

$$q(t_1 v_1 + \cdots + t_d v_d) = s_1 N(t_1) + \cdots + s_d N(t_d) \ ,$$

for all $t_1, \dots, t_d \in E$. Observe that $s_i = q(v_i)$, for all $i \in \{1, \dots, d\}$. The constants $s_1, s_2, \dots, s_d$ are called the *constants of the norm splitting.* Note that these constants are all non-zero, since q is regular. It follows that a regular quadratic form which has a norm splitting is anisotropic.

Remark 7.1. This definition is equivalent to the assumption that q has an orthogonal decomposition $q \simeq q_1 \perp q_2 \perp \cdots \perp q_d$, where each q_i is a 2-dimensional regular quadratic form with the same non-trivial discriminant.

Now let (K, V_0, q) be an arbitrary anisotropic quadratic space with corresponding bilinear map f. An automorphism T of V_0 is called a *norm splitting map* of q if and only if there exist constants $\alpha, \beta \in K$ with $\alpha = 0$ if $\mathrm{char}(K) \neq 2$ and $\alpha \neq 0$ if $\mathrm{char}(K) = 2$, and with $\beta \neq 0$ in all characteristics, such that

$$q(T(v)) = \beta q(v) \ ,$$
$$f(v, T(v)) = \alpha q(v) \ ,$$
$$T(T(v)) + \alpha T(v) + \beta v = 0 \ ,$$

for all $v \in V_0$. For each norm splitting map T, we can define a corresponding norm splitting map $\overline{T}$, defined by the relation $\overline{T}(v) := \alpha v - T(v)$ for all $v \in V_0$. It is straightforward to check that $\overline{T}$ is a norm splitting map with the same parameters α and β as the original norm splitting map T.

DEFINITION 7.2. Let K be an arbitrary commutative field, let V_0 be a vector space over K, and let q be an anisotropic quadratic form from V_0 to K. Then

- q is a quadratic form *of type* E_6 if and only if $dim_K V_0 = 6$ and q has a norm splitting $q \simeq s_1 N \perp s_2 N \perp s_3 N$.
- q is a quadratic form *of type* E_7 if and only if $dim_K V_0 = 8$ and q has a norm splitting $q \simeq s_1 N \perp s_2 N \perp s_3 N \perp s_4 N$ such that $s_1 s_2 s_3 s_4 \notin N(E)$.
- q is a quadratic form *of type* E_8 if and only if $dim_K V_0 = 12$ and q has a norm splitting $q \simeq s_1 N \perp \cdots \perp s_6 N$ such that $-s_1 s_2 s_3 s_4 s_5 s_6 \in N(E)$.

An anisotropic quadratic space (K, V_0, q) is called of type E_6, E_7 or E_8 if and only if q is a quadratic form of type E_6, E_7 or E_8, respectively.

THEOREM 7.3. *Let (K, V_0, q) be a quadratic space of type E_k with $k \in \{6, 7, 8\}$, and suppose that ϵ is an element of V_0 such that $q(\epsilon) = 1$. Let T be a norm splitting map of q, and let X_0 be a vector space over K of dimension 2^{k-3}. Then there exists a unique map $(a, v) \mapsto av$ from $X_0 \times V_0$ to X_0 and an element $\xi \in X_0^*$ such that*

$$at = a(t\epsilon) \ ,$$
$$(av)\overline{v} = aq(v) \ ,$$
$$\xi T(v) = (\xi T(\epsilon))v \ ,$$

for all $a \in X_0$, $t \in K$ and $v \in V_0$.

PROOF. This follows from [**20**, (12.56) and (13.11)]. □

From now on, we let T be a fixed arbitrary norm splitting map of q, and we let X_0 be a fixed vector space over K of dimension 2^{k-3}. We apply Theorem 7.3 with these choices of T and X_0. Note that ξ is *not* uniquely determined; see [**20**, (13.12)].

REMARK 7.4. The first two conditions of Theorem 7.3 say that X_0 is a $C(q, \epsilon)$-module, where $C(q, \epsilon)$ is the Clifford algebra of q with base point ϵ.

THEOREM 7.5. *We can choose the norm splitting $(E, \{v_1, \dots, v_d\})$ in such a way that $v_1 = \epsilon$ (and hence $s_1 = 1$). Furthermore, if $k = 8$, then we can choose it in such a way that $\xi v_2 v_3 v_4 v_5 v_6 = \xi$ as well.*

PROOF. This follows from [**20**, (27.20) and (27.13)]. □

So assume that the norm splitting satisfies the conditions of this Theorem. Then we can now define a subspace M_0 of X_0 as follows.
If $k = 6$, then we set

$$M_0 := \{\xi t v_2 v_3 \ \mid \ t \in E\} \ ;$$

If $k = 7$, then we set

$$M_0 := \{\xi t_1 v_2 v_3 + \xi t_2 v_1 v_3 + \xi t_3 v_1 v_2 + \xi t v_1 v_2 v_3 \ \mid \ t_1, t_2, t_3, t \in E\} \ ;$$

If $k = 8$, then we set

$$M_0 := \left\{ \sum_{\substack{i,j \in \{2,\dots,6\} \\ i<j}} \xi t_{ij} v_i v_j \ \mid \ t_{ij} \in E \right\} .$$

THEOREM 7.6. *$X_0 = \xi V_0 \oplus M_0$.*

PROOF. See [**20**, (13.14)]. □

THEOREM 7.7. *There is a unique map h from $X_0 \times X_0$ to V_0 which is bilinear over K, such that*

(i) $h(\xi, \xi v) = T(v) - \overline{T}(v)$, *for all* $v \in V_0$ *;*
(ii) $h(\xi, a) = 0$, *for all* $a \in M_0$ *;*
(iii) $h(a, b) = -\overline{h(b, a)}$, *for all* $a, b \in X_0$ *;*
(iv) $h(a, bv) = h(b, av) + f(h(a, b), \epsilon)v$, *for all* $a, b \in X_0$ *and all* $v \in V_0$ *.*

PROOF. See [**20**, (13.15)]. □

We now define an element $\zeta \in V_0$ as follows. Note that, if $\mathrm{char}(K) = 2$, then $f(\epsilon, T(\epsilon)) = \alpha \neq 0$ by the definition of T.

$$\zeta := \begin{cases} \epsilon/2 & \text{if } \mathrm{char}(K) \neq 2 \\ T(\epsilon)/f(\epsilon, T(\epsilon)) & \text{if } \mathrm{char}(K) = 2 \end{cases} .$$

Next, let g be the bilinear form from $X_0 \times X_0$ to K given by

$$g(a, b) := f(h(b, a), \zeta)$$

for all $a, b \in X_0$. Set

$$v^* := \begin{cases} 0 & \text{if } \mathrm{char}(K) \neq 2 \\ f(v, \zeta)\epsilon + f(v, \epsilon)\zeta + v & \text{if } \mathrm{char}(K) = 2 \end{cases} ,$$

for all $v \in V_0$.

THEOREM 7.8. *There is a unique map θ from $X_0 \times V_0$ to V_0 satisfying the following conditions, for all $a, b \in X_0$ and all $u, v \in V_0$:*

(i) $\theta(\xi, v) = T(v)$ *;*
(ii) $\theta(a + b, v) = \theta(a, v) + \theta(b, v) + h(b, av) - g(a, b)v$ *;*
(iii) $\theta(av, w) = \overline{\theta(a, \overline{w})}q(v) - \overline{\theta(a, v)}f(w, \overline{v}) + f(\theta(a, v), \overline{w})\overline{v} + f(\theta(a, v^*), v)w$ *.*

PROOF. See [**20**, (13.30), (13.31) and (13.37)]. □

Let φ be the map from $X_0 \times V_0$ to K defined as

$$\varphi(a, v) := f(\theta(a, v^*), v) ,$$

for all $a \in X_0$ and all $v \in V_0$.

Finally, we define a group $(S, \boxplus)$ as $S := X_0 \times K$ where the group action is given by

$$(a, t) \boxplus (b, s) := (a + b, t + s + g(a, b)) ,$$

for all $(a, t), (b, s) \in S$. One can check that S is indeed a group with neutral element $(0, 0)$, and with the inverse given by $\boxminus(a, t) = (-a, -t + g(a, a))$, for all $(a, t) \in S$.

Let V be parametrized by $(V_0, +)$, and let W be parametrized by S. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$\tau_V([v], [a, t]) := [v][a, t] := [\theta(a, v) + tv] ,$$
$$\tau_W([a, t], [v]) := [a, t][v] := [av, tq(v) + \varphi(a, v)] ,$$

for all $v \in V$ and all $(a, t) \in S$. Then $(V, W, \tau_V, \tau_W, [\epsilon], [0, 1])$ is a quadrangular system. One can check that

$$F([u], [v]) = [0, f(u, v)] ,$$
$$H([a, t], [b, s]) = [h(a, b)] ,$$

for all $u, v \in V$ and all $(a,t),(b,s) \in S$, and that

$$[v]^{-1} = [q(v)^{-1}\overline{v}] \ ,$$

$$\kappa([a,t]) = \left[\frac{a\theta(a,\epsilon)+ta}{q(\theta(a,\epsilon)+t\epsilon)}, \frac{t}{q(\theta(a,\epsilon)+t\epsilon)}\right] \ ,$$

for all $v \in K^*$ and all $(a,t) \in S^*$.

REMARK 7.9. It is not obvious at all to verify that this is a quadrangular system. Quite a lot of identities involving these functions h, g, θ and φ are needed. We will omit these calculations, since our main interest here is to give a *classification*, and not to prove *existence*. However, see [**20**, Chapter 13 and (32.2)] for more details about these identities.

These are the quadrangular systems *of type* E_6*,* E_7 *and* E_8. They will be denoted by $\Omega_E(K, V_0, q)$.

7.6. Quadrangular Systems of Type F_4

Consider an anisotropic space (K, V_0, q) with $\mathrm{char}(K) = 2$ and with non-trivial radical $R := \mathrm{Rad}(f) = \mathrm{Def}(q) = \{v \in V_0 \mid f(v, V_0) = 0\} \neq 0$. Suppose that there is an element $\epsilon \in R$ such that $q(\epsilon) = 1$. Then this quadratic space is said to be *of type* F_4 if and only if $L := q(R)$ is a subfield of K, and there is a complement S of R in V_0 such that the restriction of q to the subspace S has a norm splitting $(E, \{v_1, v_2\})$ with constants $s_1, s_2 \in K^*$ such that $s_1 s_2 \in L^*$.

From now on, we will assume that (K, V_0, q) is of type F_4. Since $t^2 = q(t\epsilon) \in q(R) = L$ for all $t \in K$, we have that $K^2 \subseteq L \subseteq K$. Denote the restriction of q to S by q_1. Denote the norm of the extension E/K by N, and denote the non-trivial element of $\mathrm{Gal}(E/K)$ by $u \mapsto \overline{u}$ (not to be confused with the map $v \mapsto \overline{v}$ in the definition of a quadrangular system). Set $B_0 := E \oplus E$. Then B_0 is a 4-dimensional vector space over K which can be identified with S by the relation

$$(u, v) \in B_0 \ \longleftrightarrow \ uv_1 + vv_2 \in S \ .$$

In particular, we will write $q_1(u, v) = s_1 N(u) + s_2 N(v)$ for all $(u, v) \in B_0$.

Next, we define a commutative field $D := E^2 L = \{u^2 s \mid u \in E, s \in L\}$. Then $E^2 \subseteq D \subseteq E$, D/L is a separable quadratic extension, and $D \cap K = L$. The non-trivial element of $\mathrm{Gal}(D/L)$ is precisely the restriction of the map $u \mapsto \overline{u}$ to D; hence we will also denote it by $x \mapsto \overline{x}$. Also, the norm of D is precisely the restriction of N to D, and so we will denote it by N as well. Now set $A_0 := D \oplus D$; then A_0 is a 4-dimensional vector space over L. Observe that both $s_1^{-1}s_2$ and $s_1^{-3}s_2$ are elements of L. We now define a quadratic form q_2 on A_0 given by

$$q_2(x, y) := s_1^{-1}s_2 N(x) + s_1^{-3}s_2 N(y)$$

for all $(x, y) \in A_0$. If we set $\alpha := s_1^{-1}s_2 \in L$ and $\beta := s_1^{-1} \in K$, then we have

$$q_1(u, v) = \beta^{-1} \cdot (N(u) + \alpha N(v)) \quad \text{for all } (u, v) \in B_0 \ .$$

$$q_2(x, y) = \alpha \cdot (N(x) + \beta^2 N(y)) \qquad \text{for all } (x, y) \in A_0 \ .$$

We will denote the bilinear forms corresponding to q_1 and q_2 by f_1 and f_2, respectively.

THEOREM 7.10. *For all* $(u, v) \in B_0$ *and all* $(x, y) \in A_0$ *we have:*

(i) $q_1(u, v) \in L \iff (u, v) = (0, 0)$*;*

(ii) $q_2(x,y) \in K^2 \iff (x,y) = (0,0)$;
(iii) $\alpha \in L \setminus K^2$;
(iv) $\beta \in K \setminus L$.

Proof. See [**20**, (14.8)]. □

Note that it follows from (iii) and (iv) of this Theorem that $K^2 \subset L \subset K$. In particular, K is *not* perfect.

Since $L \subseteq K$, we can consider K as a (left) vector space over L by the trivial scalar multiplication $s \cdot t := st$ for all $s \in L$ and all $t \in K$. Since $K^2 \subseteq L$ and $\mathrm{char}(K) = 2$, we can also consider L as a (left) vector space over K by the scalar multiplication $t * s := t^2 s$ for all $t \in K$ and all $s \in L$. One can check that in this sense, q is a vector space isomorphism from R to $L = q(R)$.

From now on, we will identify R with L via q, and we still identify S with $B_0 = E \oplus E$. Combining those two identifications, we have actually identified V_0 with $B_0 \oplus L$. Then $\epsilon = (0,1)$, and we have $q(b,s) = q_1(b) + s$, for all $(b,s) \in V_0$.

Now set $W_0 := A_0 \oplus K$. Then W_0 is a vector space over L, and we can define a quadratic form $\hat{q}$ from W_0 to f given by $\hat{q}(a,t) = q_2(a) + t^2$ for all $(a,t) \in W_0$. It follows from Theorem 7.10(ii) that $\hat{q}$ is anisotropic as well. One can actually check that $(L, W_0, \hat{q})$ is again a quadratic form of type F_4.

Finally, we define a map Θ from $A_0 \oplus B_0$ to B_0, a map Υ from $A_0 \oplus B_0$ to A_0, a map ν from $A_0 \oplus B_0$ to K, and a map ψ from $A_0 \oplus B_0$ to L as follows.

$$\begin{aligned}
\Theta((x,y),(u,v)) &:= (\alpha \cdot (\bar{x}v + \beta y\bar{v}), xu + \beta y\bar{u}) \ , \\
\Upsilon((x,y),(u,v)) &:= (y\bar{u}^2 + \alpha\bar{y}v^2, \beta^{-2} \cdot (xu^2 + \alpha\bar{x}v^2)) \ , \\
\nu((x,y),(u,v)) &:= \alpha \cdot (\beta^{-1} \cdot (xu\bar{v} + \bar{x}\bar{u}v) + y\bar{u}\bar{v} + \bar{y}uv) \ , \\
\psi((x,y),(u,v)) &:= \alpha \cdot (x\bar{y}u^2 + \bar{x}y\bar{u}^2 + \alpha \cdot (xy\bar{v}^2 + \bar{x}\bar{y}v^2)) \ ,
\end{aligned}$$

for all $(x,y) \in A_0 = D \oplus D$ and all $(u,v) \in B_0 = E \oplus E$.

Let V be parametrized by $(V_0, +)$, and let W be parametrized by $(W_0, +)$. We define a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W as follows:

$$\begin{aligned}
\tau_V([b,s],[a,t]) &:= [b,s][a,t] := [\Theta(a,b) + tb, \hat{q}(a,t)s + \psi(a,b)] \ , \\
\tau_W([a,t],[b,s]) &:= [a,t][b,s] := [\Upsilon(a,b) + sa, q(b,s)t + \nu(a,b)] \ ,
\end{aligned}$$

for all $(b,s) \in V_0$ and all $(a,t) \in W_0$. Then $(V, W, \tau_V, \tau_W, [0,1], [0,1])$ is a quadrangular system. One can check that

$$\begin{aligned}
F([b,s],[b',s']) &= [0, f_1(b,b')] \ , \\
H([a,t],[a',t']) &= [0, f_2(a,a')] \ ,
\end{aligned}$$

for all $(b,s),(b',s') \in V_0$ and all $(a,t),(a',t') \in W_0$, and that

$$\begin{aligned}
[b,s]^{-1} &= [q(b,s)^{-1}b, q(b,s)^{-2}s] \ , \\
\kappa([a,t]) &= [\hat{q}(a,t)^{-1}a, \hat{q}(a,t)^{-1}t] \ ,
\end{aligned}$$

for all $(b,s) \in V_0^*$ and all $(a,t) \in W_0^*$.

Remark 7.11. It would be a very tedious job to check that this is indeed a quadrangular system by only using the definitions of the different functions involved. However, it is not very hard to prove the following list of twelve identities, after which the verification of the axioms for the quadrangular systems is straightforward.

THEOREM 7.12. *For all $a, a' \in A_0$ and all $b, b' \in B_0$, we have that*

(i) $\nu(a, b + b') = \nu(a, b) + \nu(a, b') + f_1(\Theta(a, b), b')$;
(ii) $\psi(a + a', b) = \psi(a, b) + \psi(a', b) + f_2(\Upsilon(a, b), a')$;
(iii) $\Upsilon(\Upsilon(a, b), b) = q_1(b)^2 a$;
(iv) $\Theta(a, \Theta(a, b)) = q_2(a)b$;
(v) $\Theta(\Upsilon(a, b), b) + b\nu(a, b) = q_1(b)\Theta(a, b)$;
(vi) $\Upsilon(a, \Theta(a, b)) + a\psi(a, b) = q_2(a)\Upsilon(a, b)$;
(vii) $\nu(\Upsilon(a, b), b) = q_1(b)\nu(a, b)$;
(viii) $\psi(a, \Theta(a, b)) = q_2(a)\psi(a, b)$;
(ix) $\psi(\Upsilon(a, b), b) = q_1(b)^2\psi(a, b)$;
(x) $\nu(a, \Theta(a, b)) = q_2(a)\nu(a, b)$;
(xi) $q_1(\Theta(a, b)) = q_1(b)q_2(a) + \psi(a, b)$;
(xii) $q_2(\Upsilon(a, b)) = q_1(b)^2 q_2(a) + \nu(a, b)^2$.

These are the quadrangular systems *of type F_4*. They will be denoted by $\Omega_F(K, V_0, q)$.

REMARK 7.13. Although it can be very useful to have these explicit formulas to calculate with, this description is not very clarifying. The description in terms of the quadrangular systems ought to give more insight in the structure of these F_4-quadrangles. See section A.3.

CHAPTER 8

The Classification

We will now start the classification of the quadrangular systems. We start with some definitions.

Definition 8.1. A quadrangular system $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is called *indifferent* if $F \equiv 0$ and $H \equiv 0$, *reduced* if $F \not\equiv 0$ and $H \equiv 0$ and *wide* if $F \not\equiv 0$ and $H \not\equiv 0$.

Remark 8.2. We will prove that if $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a quadrangular system with $F \equiv 0$ and $H \not\equiv 0$, then $\Omega^* := (W, V, \tau_W, \tau_V, \delta, \epsilon)$ is a reduced quadrangular system; see Theorem 8.13.

Remark 8.3. Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. If $X \subseteq V$ and $Y \subseteq W$, then the restriction of τ_V to $X \times Y$ and the restriction of τ_W to $Y \times X$ will also be denoted by τ_V and τ_W, respectively.

Definition 8.4. Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a wide quadrangular system. Set $Y := \mathrm{Rad}(H)$. We will show below that $\Gamma := (V, Y, \tau_V, \tau_W, \epsilon, \delta)$ is a reduced quadrangular system; see Theorem 8.14. We then say that Ω is an *extension* of Γ.

Definition 8.5. Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a reduced quadrangular system. Then Ω is said to be *normal* if and only if for all $w_1, w_2, \ldots, w_i \in W$, there exists a $w \in W$ such that $\epsilon w_1 w_2 \ldots w_i = \epsilon w$.

Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be an arbitrary quadrangular system. The classification will be divided up into the following five theorems.

Theorem 8.6. *If Ω is reduced but not normal, then $\Omega \cong \Omega_I(K, K_0, \sigma)$ for some involutory set (K, K_0, σ) such that $\sigma \neq 1$ and K is generated by K_0 as a ring.*

Theorem 8.7. *If Ω is normal, then $\Omega \cong \Omega_Q(K, V_0, q)$ for some anisotropic quadratic space (K, V_0, q).*

Theorem 8.8. *If Ω is indifferent, then $\Omega \cong \Omega_D(K, K_0, L_0)$ for some indifferent set (K, K_0, L_0).*

Theorem 8.9. *If Ω is an extension of the reduced quadrangular system $\Gamma = \Omega_I(K, K_0, \sigma)$ for some involutory set (K, K_0, σ) such that $\sigma \neq 1$ and K is generated by K_0 as a ring, then $\Omega \cong \Omega_P(K, K_0, \sigma, V_0, p)$ for some anisotropic pseudo-quadratic space (K, K_0, σ, V_0, p).*

Theorem 8.10. *If Ω is an extension of the reduced quadrangular system $\Gamma = \Omega_Q(K, V_0, q)$ for some anisotropic quadratic space (K, V_0, q), then one of the following holds:*

- *There exists*

(a) *a multiplication on V_0 making the vector space V_0 into an algebra over K such that either V_0 is a field and V_0/K is a separable quadratic extension with norm q or V_0 is a quaternion division algebra over K with norm q ,*
(b) *an involution σ of V_0 (which is the unique non-trivial element of $\mathrm{Gal}(V_0/K)$ if $\dim_K V_0 = 2$ and which is the standard involution of V_0 if $\dim_K V_0 = 4$),*
(c) *a non-trivial right vector space X over V_0 ,*
(d) *a pseudo-quadratic form π on X ,*

such that (V_0, K, σ, X, π) is an anisotropic pseudo-quadratic space, $\Gamma \cong \Omega_I(V_0, K, \sigma)$ and $\Omega \cong \Omega_P(V_0, K, \sigma, X, \pi)$.

- *(K, V_0, q) is a quadratic space of type E_6, E_7 or E_8, and $\Omega \cong \Omega_E(K, V_0, q)$.*
- *(K, V_0, q) is a quadratic space of type F_4, and $\Omega \cong \Omega_F(K, V_0, q)$.*

We will now prove the two theorems which we mentioned in the above remarks. But first, we make an easy but useful observation.

LEMMA 8.11. *If $\mathrm{Rad}(F) \neq 0$, then W is abelian, and all elements of V and W have order 1 or 2. Furthermore, $\pi_r(v) = v$ for all $r \in \mathrm{Rad}(F)^*$ and all $v \in V$, and $\overline{v} = v$ for all $v \in V$.*

PROOF. Let r be an arbitrary non-zero element of $\mathrm{Rad}(F)$. If we substitute r for v in $(\mathbf{Q}_8)$, then we get that $[w_1, w_2 r]_{\boxplus} = 0$ for all $w_1, w_2 \in W$. Substituting $w_2 r^{-1}$ for w_2 shows, by $(\mathbf{Q}_{15})$, that W is abelian.

If we substitute r for v in 3.13(i), then we get that $w(-r) = F(rw, r) \boxminus wr = \boxminus wr$ for all $w \in W$. By $(\mathbf{Q}_5)$, this implies that $w(-\epsilon)r = \boxminus wr$, hence $w(-\epsilon) = \boxminus w$ for all $w \in W$. It thus follows from $(\mathbf{Q}_6)$ that $v(\boxminus w) = vw$ for all $v \in V$ and all $w \in W$. In particular, $v(\boxminus \delta) = v$. On the other hand, it follows from 3.13(ii) that $v(\boxminus \delta) = H(\delta, \delta v) - v\delta = -v$, for all $v \in V$. Hence $v = -v$ for all $v \in V$, so every element of V has order 1 or 2.

In particular, $\epsilon = -\epsilon$, hence $w = w(-\epsilon) = \boxminus w$ for all $w \in W$, that is, every element of W has order 1 or 2.

Finally, $\pi_r(v) = v + \overline{r^{-1}F(r, v)} = v$ for all $r \in \mathrm{Rad}(F)^*$ and all $v \in V$. Since it follows from $(\mathbf{Q}_{10})$ that $\epsilon \in \mathrm{Rad}(F)$, we have in particular that $\overline{v} = \pi_\epsilon(v) = v$ for all $v \in V$. □

REMARK 8.12. Apart from the last statement, we have avoided to use $(\mathbf{Q}_{10})$. We thereby want to stress the fact that this axiom is not essential, and is really only needed to simplify the list of the wide quadrangular systems which have $\mathrm{Rad}(F) \neq 0$. See Remark 6.4.

THEOREM 8.13. *Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system with $F \equiv 0$ and $H \not\equiv 0$. Then $\Omega^* := (W, V, \tau_W, \tau_V, \delta, \epsilon)$ is a reduced quadrangular system.*

PROOF. Since $F \equiv 0$, it follows from Lemma 8.11 that W is abelian and that all elements of V and W have order 1 or 2. In particular, we will write $+$ in place of $\boxplus$ and $\boxminus$. We define $\overline{w} := \delta H(\delta, w) + w$, for all $w \in W$. Then it follows from $(\mathbf{Q}_9)$ that $\overline{w} = w$, for all $w \in W$. We also set $w^{-1} := \kappa(w)$ for all $w \in W^*$ and $\kappa(v) := v^{-1}$ for all $v \in V^*$. Let $F^* \equiv H$ and $H^* \equiv F \equiv 0$.

We will denote the axioms that we have to prove for Ω^* by $(\mathbf{Q}_i)^*$.

Since all elements of V and W have order 1 or 2, the axioms $(\mathbf{Q}_5)^*$ and $(\mathbf{Q}_6)^*$ are trivial. Note that $\overline{v} = v$ for all $v \in V$ and $\overline{w} = w$ for all $w \in W$. We now prove the remaining axioms. We observe that $(\mathbf{Q}_1)^* \equiv (\mathbf{Q}_2)$, $(\mathbf{Q}_2)^* \equiv (\mathbf{Q}_1)$, $(\mathbf{Q}_3)^* \equiv (\mathbf{Q}_4)$, $(\mathbf{Q}_4)^* \equiv (\mathbf{Q}_3)$, $(\mathbf{Q}_{13})^* \equiv (\mathbf{Q}_{14})$, $(\mathbf{Q}_{14})^* \equiv (\mathbf{Q}_{13})$, $(\mathbf{Q}_{17})^* \equiv (\mathbf{Q}_{20})$ and $(\mathbf{Q}_{20})^* \equiv (\mathbf{Q}_{17})$. It follows from $F \equiv 0$ that $\mathrm{Rad}(F) = V$ and hence $\mathrm{Im}(H) \subseteq \mathrm{Rad}(F) \ni \epsilon$; this shows $(\mathbf{Q}_7)^*$ and $(\mathbf{Q}_9)^*$. Now $(\mathbf{Q}_8)^*$ follows from the fact that V is abelian and that $F \equiv 0$; $(\mathbf{Q}_{10})^*$ follows from $(\mathbf{Q}_9)$. By $(\mathbf{Q}_{22})$, $H(w_1, w_2) = H(w_2, w_1)$ for all $w_1, w_2 \in W$. Since W is abelian, it follows from $(\mathbf{Q}_{12})$ that $v(w_1 + w_2) = v(w_2 + w_1) = vw_2 + vw_1 + H(w_1, w_2v) = vw_1 + vw_2 + H(w_2v, w_1)$ for all $v \in V$ and all $w_1, w_2 \in W$. This proves $(\mathbf{Q}_{11})^*$. Vice versa, it follows from $(\mathbf{Q}_{11})$ that $w(v_1 + v_2) = wv_1 + wv_2$ for all $w \in W$ and all $v_1, v_2 \in V$, which proves $(\mathbf{Q}_{12})^*$. By $(\mathbf{Q}_{13})$, we have that $(\mathbf{Q}_{15})$ is equivalent to $wv^{-1}v = w$. It follows that $(\mathbf{Q}_{15})^* \equiv (\mathbf{Q}_{18})$ and $(\mathbf{Q}_{18})^* \equiv (\mathbf{Q}_{15})$. Again by $(\mathbf{Q}_{13})$, we have that $(\mathbf{Q}_{16})$ is equivalent to $v(wv^{-1}) = v^{-1}w$. Hence $(\mathbf{Q}_{16})^* \equiv (\mathbf{Q}_{19})$ and $(\mathbf{Q}_{19})^* \equiv (\mathbf{Q}_{16})$. It follows that Ω^* is a quadrangular system, which is reduced since $H^* \equiv 0$ and $F^* \not\equiv 0$. □

THEOREM 8.14. *Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a wide quadrangular system. Set $Y := \mathrm{Rad}(H)$. Then $\Gamma := (V, Y, \tau_V, \tau_W, \epsilon, \delta)$ is a reduced quadrangular system; see Remark 8.3.*

PROOF. First of all, we observe that Y is a subgroup of W: if $w_1, w_2 \in Y$, then $H(w_1 \boxplus w_2, w) = H(w_1, w) + H(w_2, w) = 0$ for all $w \in W$, so $w_1 \boxplus w_2 \in Y$ as well. It only remains to show that $\tau_W(Y \times V) \subseteq Y$, $F(V, V) \subseteq Y$ and $\kappa(Y^*) \subseteq Y$. So let w be an arbitrary element of Y, and let v be an arbitrary element of V. Then $[w, w_2]_\boxplus = 0$ for all $w_2 \in W$ by $(\mathbf{Q}_8)$, and therefore $v(w \boxplus w_2) = v(w_2 \boxplus w)$. It follows from $(\mathbf{Q}_{12})$ that $H(w_2, wv) = H(w, w_2v) = 0$ for all $w_2 \in W$. By $(\mathbf{Q}_{22})$, this implies that $H(wv, w_2) = 0$ for all $w_2 \in W$, hence $wv \in Y$. So we have proved that $\tau_W(Y \times V) = Y \cdot V \subseteq Y$.

It follows from $(\mathbf{Q}_7)$ that $F(V, V) \subseteq Y$. Now let w be an arbitrary element of Y^*. Substituting $-\epsilon$ for v in 3.2(ii) yields $\kappa(w)(-\epsilon(\boxminus w)) = w$, hence $\kappa(w) = w(-\epsilon(\boxminus w))^{-1} \in Y \cdot V \subseteq Y$. So Γ is a quadrangular system, which is reduced since H restricted to $Y \times Y$ is identically zero. □

8.1. Quadrangular Systems of Involutory Type

Our goal in this section is to classify the quadrangular systems which are reduced but not normal.

Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system. For the moment, we only assume that $H \equiv 0$, so Ω is reduced or indifferent.

Since $H \equiv 0$, it follows from $(\mathbf{Q}_8)$ that W is abelian. We will write $+$ and $-$ in place of $\boxplus$ and $\boxminus$, respectively. It follows from 3.13(ii) that $v(-w) = -vw$ for all $v \in V$ and all $w \in W$. If we replace w_2 by $-w_2$ in $(\mathbf{Q}_{12})$, we get $v(w_1 - w_2) = vw_1 - vw_2$, for all $v \in V$ and all $w_1, w_2 \in W$. In particular, it follows from $\epsilon w_1 = \epsilon w_2$ that $w_1 = w_2$, by 3.3(ii).

By $(\mathbf{Q}_{18})$, we have $\epsilon\kappa(w)w = \epsilon$ for all $w \in W$. If we replace w by $-w$, we get that $\epsilon(-\kappa(-w))w = \epsilon$ for all $w \in W$. Hence $\epsilon\kappa(w)w = \epsilon(-\kappa(-w))w$, so $\epsilon\kappa(w) = \epsilon(-\kappa(-w))$ by 3.5(ii) and hence $\kappa(w) = -\kappa(-w)$ for all $w \in W$ by the previous paragraph. Moreover, by $(\mathbf{Q}_6)$ and the result of the previous paragraph, we have that $w(-\epsilon) = w$ for all $w \in W$. It now follows from $(\mathbf{Q}_{14})$ that $\kappa(\kappa(w)) = w$, for all $w \in W$. Hence we will write w^{-1} in place of $\kappa(w)$, for all $w \in W$. Note that it follows from 3.2(i) that $vww^{-1} = v$, for all $v \in V$ and all $w \in W^*$.

For all $w_1, w_2, \dots, w_n \in W^*$, let $m = w_1 \bullet w_2 \bullet \cdots \bullet w_n$ be the automorphism of V which maps v to $vw_1w_2 \dots w_n$ for all $v \in V$; see Theorem 3.6(i). Let M be the set of all such automorphisms. Then $(M, \bullet)$ is a group with neutral element δ. We denote the action of an element $m \in M$ by right juxtaposition, i.e. $vm = vw_1w_2 \dots w_n$. Let K be the set of homomorphisms from V to itself (additively) generated by a finite number of elements of M. We write $k = m_1 + \cdots + m_\ell$, where $m_1, \dots, m_\ell \in M$. Again, we denote the action of an element $k \in K$ by right juxtaposition, so we have $vk = vm_1 + \cdots + vm_\ell$. Then K with this $+$ as addition and with $\bullet$ as multiplication is a ring with multiplicative identity δ. Note that, by $(\mathbf{Q}_{12})$, $v(w_1 + w_2) = vw_1 + vw_2$, hence the notation $w_1 + w_2$ is unambiguous. Let σ be the automorphism of M which maps $m = w_1 \bullet \cdots \bullet w_n$ to $m^\sigma := w_n \bullet \cdots \bullet w_1$. We extend σ to K by setting $k^\sigma := m_1^\sigma + \cdots + m_\ell^\sigma$ for all $k = m_1 + \cdots + m_\ell \in K$. Let $E := \epsilon K$.

Lemma 8.15. *For all $v \in V$, $w \in W$ and $m \in M$, we have that $w \cdot \epsilon m \cdot v = w \cdot vm$.*

Proof. Let $m = w_1 \bullet \cdots \bullet w_n$ be an arbitrary element of M, so $w_1, \dots, w_n$ are elements of W^*. We will show the lemma by induction on n.

Note that $\Pi_{\boxminus z} \equiv \mathrm{id}_W$ since $H \equiv 0$; hence by $(\mathbf{Q}_{26})$, the lemma holds for $n = 1$. Assume that $w \cdot \epsilon w_1 \dots w_{n-1} \cdot v = w \cdot vw_1 \dots w_{n-1}$ for all $w \in W$ and all $v \in V$. Then, by repeated use of $(\mathbf{Q}_{26})$,

$$\begin{aligned} w \cdot \epsilon w_1 \dots w_n \cdot v &= w \cdot (\epsilon w_1 \dots w_{n-1} \cdot w_n) \cdot v \\ &= w \cdot (\epsilon w_n) \cdot \epsilon w_1 \dots w_{n-1} \cdot v \\ &= w \cdot (\epsilon w_n) \cdot vw_1 \dots w_{n-1} \\ &= w \cdot (vw_1 \dots w_{n-1} \cdot w_n) \\ &= w \cdot vw_1 \dots w_n \end{aligned}$$

for all $w \in W$ and all $v \in V$, and we are done. □

Lemma 8.16. *For all $v \in V$ and all $w, z \in W$, we have that $\overline{\overline{v}w}z = \overline{\overline{v}z}w$.*

Proof. Let $v \in V$ and $w, z \in W$. We may assume that $w \neq 0$. Note that $\Pi_w(z) = z$ since $H \equiv 0$. Since $v(-w) = -vw$, it follows from $(\mathbf{Q}_{23})$ that $vz = \overline{\overline{\overline{\overline{v}w}}zw^{-1}}$. Since $(w^{-1})^{-1} = w$ and by 3.12, it follows from this that $\overline{\overline{v}z}w = \overline{\overline{v}w}z$. □

Lemma 8.17. *For all $v \in V$, all $w \in W$ and all $m \in M$, we have that $\overline{\overline{v}w}m = \overline{\overline{\overline{v}m}w}$ and $\overline{\overline{v}m}w = \overline{\overline{\overline{v}w}m}$.*

Proof. Let $m = w_1 \bullet \cdots \bullet w_n$ be an arbitrary element of M, so $w_1, \dots, w_n$ are elements of W^*. We will prove by induction on n that $\overline{\overline{v}w}w_1 \dots w_n = \overline{\overline{vw_1 \dots w_n}w}$. For $n = 1$, this was shown in Lemma 8.16. Assume that we have proved the current lemma for $n - 1$. Then

$$\begin{aligned} \overline{\overline{v}w}w_1 \dots w_n &= \overline{\overline{v}w}w_1 \dots w_{n-1} \cdot w_n \\ &= \overline{\overline{vw_1 \dots w_{n-1}}ww_n} \\ &= \overline{\overline{vw_1 \dots w_n}w}\ , \end{aligned}$$

again by Lemma 8.16 with $vw_1 \dots w_{n-1}$ in place of v. This proves the first identity; the second then follows from the first by substituting $\overline{v}$ for v. □

LEMMA 8.18. *For all $v \in V$ and all $k_1, k_2 \in m$, we have that $\overline{\overline{v}k_1}k_2 = \overline{\overline{vk_2}k_1}$.*

PROOF. Since both sides are additive in k_1 and k_2, it suffices to show this for $k_1 = m_1 \in M$ and $k_2 = m_2 \in M$. Let $m_2 = z_1 \bullet \cdots \bullet z_n$ for some $z_1, \dots, z_n \in W^*$. By repeated use of Lemma 8.17, we have that

$$\begin{aligned} \overline{\overline{v}m_1}m_2 &= \overline{\overline{v}m_1}z_1 \dots z_n \\ &= \overline{\overline{vz_1}m_1}z_2 \dots z_n \\ &= \dots \\ &= \overline{\overline{vz_1 \dots z_n}m_1} \\ &= \overline{\overline{vm_2}m_1} \ , \end{aligned}$$

which proves the lemma. □

THEOREM 8.19. *$\overline{\epsilon k} = \epsilon k^\sigma$, for all $k \in K$.*

PROOF. It is sufficient to prove this for $k \in M$. Let $k = w_1 \bullet \cdots \bullet w_n$ be an arbitrary element of M, so $w_1, \dots, w_n \in W^*$. We will prove by induction on n that $\overline{\epsilon w_1 \dots w_n} = \epsilon w_n \dots w_1$.

First assume that $n = 1$. Since $v(-w) = -vw$ for all $v \in V$ and all $w \in W$, it follows from 3.21 that $\overline{\epsilon w_1} = -\pi_\epsilon(\epsilon w_1) = -\epsilon(-w_1) = \epsilon w_1$.

Now assume that we have proved that $\overline{\epsilon w_1 \dots w_{n-1}} = \epsilon w_{n-1} \dots w_1$, for all $w_1, \dots, w_{n-1} \in W^*$. Then it follows from 3.12 and Lemma 8.17 that

$$\begin{aligned} \overline{\epsilon w_1 \dots w_n} &= \overline{\overline{\overline{\epsilon w_1 \dots w_{n-1}}}w_n} \\ &= \overline{\overline{\epsilon w_{n-1} \dots w_1}w_n} \\ &= \overline{\overline{\epsilon}w_n}w_{n-1} \dots w_1 \\ &= \epsilon w_n w_{n-1} \dots w_1 \ , \end{aligned}$$

since $\overline{\overline{\epsilon}w_n} = \overline{\epsilon w_n} = \epsilon w_n$. □

THEOREM 8.20. *$\Delta := (E, W, \tau_V, \tau_W, \epsilon, \delta)$ is a quadrangular system; see Remark 8.3.*

PROOF. First of all, we observe that $E = \epsilon K$ is a subgroup of V, since $\epsilon k_1 + \epsilon k_2 = \epsilon(k_1 + k_2)$ for all $k_1, k_2 \in K$. It only remains to show that $\tau_V(E \times W) \subseteq E$, $H(W, W) \subseteq E$ and $(E^*)^{-1} \subseteq E$. Since $K \bullet W = K$, we have that $\tau_V(E \times W) = \epsilon K \cdot W = \epsilon(K \bullet W) = \epsilon K = E$. Since $H \equiv 0$, it is obvious that $H(W, W) \subseteq E$. Finally, if we substitute δ for w in ($\mathbf{Q}_{16}$) and apply the fact that $vww^{-1} = v$, we get that $v^{-1} = \overline{v}(\delta v)^{-1}$ for all $v \in V^*$. In particular, we get that $(\epsilon k)^{-1} = \overline{\epsilon k}(\delta \cdot \epsilon k)^{-1} = \epsilon(k^\sigma \bullet (\delta \cdot \epsilon k)^{-1}) \in \epsilon K$ for all $k \in K^*$, where we have used Theorem 8.19. Thus $\Delta := (E, W, \tau_V, \tau_W, \epsilon, \delta)$ is a quadrangular system. □

LEMMA 8.21. *If $vk = 0$ for some $v \in V$ and some $k \in K$, then $vk_2k = 0$ for all $k_2 \in K$.*

PROOF. We may assume that $v \neq 0$. We will first show the lemma for $k_2 = w \in W$. Since $v(-w) = -vw$, it follows from 3.20 that $vw = \overline{\overline{v}m}$ where $m = \delta\overline{v^{-1}} \bullet wv \in M$. By Lemma 8.18, it follows from $vk = 0$ that $vwk = \overline{\overline{v}m}k = \overline{\overline{vk}m} = 0$, which proves the lemma in this case.

Now let $k_2 = m = w_1 \bullet \cdots \bullet w_n$ be an arbitrary element of M, then it follows by induction on n that $vmk = 0$.

Finally, let $k_2 = m_1 + \cdots + m_\ell$ be an arbitrary element of K, then it follows from the previous paragraph that $vk_2k = vm_1k + \cdots + vm_\ell k = 0$, which completes the proof of this lemma. □

LEMMA 8.22. *For all $v_1, v_2 \in V$ and all $k \in K$, we have that $F(v_1k, v_2) = F(v_1, v_2k^\sigma)$.*

PROOF. Since both V and W are abelian, it follows from $(\mathbf{Q}_{21})$ and $(\mathbf{Q}_{11})$ that $F(v_1w, v_2) = F(v_1, v_2w)$ for all $v_1, v_2 \in V$ and all $w \in W$. For $m \in M$, say $m = w_1 \bullet \cdots \bullet w_n$, it follows by induction on n that $F(v_1m, v_2) = F(v_1, v_2m^\sigma)$. Since F is additive in both variables, it now follows that $F(v_1k, v_2) = F(v_1, v_2k^\sigma)$ for any $k = m_1 + \cdots + m_\ell \in K$. □

Our first goal is to prove that Δ is a quadrangular system of involutory type. Assume from now on that $V = E$, that is, that $\Omega = \Delta$.

LEMMA 8.23. (i) *Let $k \in K$. If $\epsilon k = 0$, then $k = 0$.*
(ii) *Let $k_1, k_2 \in K$. If $\epsilon k_1 = \epsilon k_2$, then $k_1 = k_2$.*

PROOF. Let $k \in K$ be such that $\epsilon k = 0$. By Lemma 8.21, it follows that $\epsilon k_2 k = 0$ for all $k_2 \in K$, and hence $Vk = Ek = \epsilon Kk = 0$, which implies that $k = 0$ (remember that the elements of K are endomorphisms of V). This proves (i); (ii) now follows from (i) by substituting $k_1 - k_2$ for k. □

DEFINITION 8.24. For each $k \in K^*$, we define k' as the (unique!) element in K such that $(\epsilon k)^{-1} = \epsilon k'$.

LEMMA 8.25. *For all $k \in K^*$ and all $w \in W$, we have that $k \bullet w = w(\epsilon k) \bullet (k')^\sigma$.*

PROOF. By $(\mathbf{Q}_{16})$, we have that

$$\begin{aligned}(\epsilon k)w &= \overline{(\epsilon k)^{-1} \cdot w(\epsilon k)} \\ &= \overline{\epsilon \cdot k' \cdot w(\epsilon k)} \\ &= \overline{\epsilon \cdot (k' \bullet w(\epsilon k))} \\ &= \epsilon \cdot (w(\epsilon k) \bullet (k')^\sigma) \ ,\end{aligned}$$

where we have used Theorem 8.19. It follows by Lemma 8.23(ii) that $k \bullet w = w(\epsilon k) \bullet (k')^\sigma$. □

LEMMA 8.26. *For all $k \in K^*$ and all $w \in W^*$, we have that $(k \bullet w)' = w^{-1} \bullet k'$.*

PROOF. By Lemma 8.25 with $k \bullet w$ in place of k and w^{-1} in place of w and by $(\mathbf{Q}_{19})$, we have that

$$\begin{aligned}k \bullet w &= ((k \bullet w) \bullet w^{-1}) \bullet w \\ &= w^{-1}(\epsilon(k \bullet w)) \bullet (k \bullet w)'^\sigma \bullet w \\ &= w(\epsilon k) \bullet (k \bullet w)'^\sigma \bullet w \ ,\end{aligned}$$

which, together with Lemma 8.25, implies that

$$w(\epsilon k) \bullet (k')^\sigma \bullet w^{-1} = w(\epsilon k) \bullet (k \bullet w)'^\sigma \ .$$

If we apply σ to both sides, we get that

$$w^{-1} \bullet k' \bullet w(\epsilon k) = (k \bullet w)' \bullet w(\epsilon k) \ ,$$

from which it follows that $w^{-1} \bullet k' = (k \bullet w)'$ since $w(\epsilon k)$ is invertible in K. □

LEMMA 8.27. *For all $w_1, \dots, w_n \in W^*$, we have that*

$$(w_1 \bullet \cdots \bullet w_n)' = w_n^{-1} \bullet \cdots \bullet w_1^{-1} \ .$$

PROOF. By Lemma 8.26 with $k = \delta$, we have that $w_1' = w_1^{-1}$. Again by Lemma 8.26, it now follows by induction on n that

$$\begin{aligned}(w_1 \bullet \cdots \bullet w_n)' &= w_n^{-1} \bullet (w_1 \bullet \cdots \bullet w_{n-1})' \\ &= w_n^{-1} \bullet w_{n-1}^{-1} \bullet \cdots \bullet w_1^{-1} \ ,\end{aligned}$$

which is what we wanted to show. □

Since it follows from this lemma that $m \bullet m' = m' \bullet m = \delta$, we will from now on write m^{-1} in place of m' for all $m \in M$.

LEMMA 8.28. *For all $m \in M$ and all $w \in W$, we have that $w(\epsilon m) = m \bullet w \bullet m^\sigma$.*

PROOF. First of all, observe that it follows from Lemma 8.27 that $(m^{-1})^\sigma = (m^\sigma)^{-1}$. By Lemma 8.25, we have that $m \bullet w = w(\epsilon m) \bullet (m^{-1})^\sigma$. It follows that $m \bullet w \bullet m^\sigma = w(\epsilon m)$. □

LEMMA 8.29. *For all $k_1, k_2 \in K$, we have that $F(\epsilon k_1, \epsilon k_2) = k_1 \bullet k_2^\sigma + k_2 \bullet k_1^\sigma$.*

PROOF. Since $v + \overline{v} = \epsilon F(\epsilon, v)$ for all $v \in V$, we have that $\epsilon F(\epsilon, \epsilon k) = \epsilon k + \overline{\epsilon k} = \epsilon k + \epsilon k^\sigma$ and hence $F(\epsilon, \epsilon k) = k + k^\sigma$, for all $k \in K$. It now follows from Lemma 8.22 that

$$\begin{aligned}F(\epsilon k_1, \epsilon k_2) &= F(\epsilon, \epsilon k_2 k_1^\sigma) \\ &= F(\epsilon, \epsilon(k_2 \bullet k_1^\sigma)) \\ &= k_2 \bullet k_1^\sigma + k_1 \bullet k_2^\sigma \ ,\end{aligned}$$

which proves the lemma. □

THEOREM 8.30. *For all $k \in K$ and all $w \in W$, we have that $w(\epsilon k) = k \bullet w \bullet k^\sigma$.*

PROOF. In Lemma 8.28, we have shown this theorem for all $k \in M$. Now suppose that the theorem holds for $k_1, k_2 \in K$. We will show that it then holds for $k_1 + k_2$ as well, which will prove the theorem for all $k \in K$.

It follows from $(\mathbf{Q}_{11})$ and Lemma 8.29 that

$$\begin{aligned}w(\epsilon(k_1 + k_2)) &= w(\epsilon k_1 + \epsilon k_2) \\ &= w(\epsilon k_1) + w(\epsilon k_2) + F(\epsilon k_2 w, \epsilon k_1) \\ &= k_1 \bullet w \bullet k_1^\sigma + k_2 \bullet w \bullet k_2^\sigma + (k_2 \bullet w) \bullet k_1^\sigma + k_1 \bullet (k_2 \bullet w)^\sigma \\ &= k_1 \bullet w \bullet k_1^\sigma + k_2 \bullet w \bullet k_2^\sigma + k_2 \bullet w \bullet k_1^\sigma + k_1 \bullet w \bullet k_2^\sigma \\ &= (k_1 + k_2) \bullet w \bullet (k_1 + k_2)^\sigma \ ,\end{aligned}$$

which is what we had to show. □

THEOREM 8.31. *$K_{+,\bullet}$ is a field or a skew-field.*

PROOF. We already know that $K_{+,\bullet}$ is a ring. Let k be an arbitrary element of K^*. We will show that $k' \bullet k = k \bullet k' = \delta$.

By ($\mathbf{Q}_{15}$), Theorem 8.30 and Lemma 8.25, we have that

$$\begin{aligned}\delta &= \delta(\epsilon k)(\epsilon k)^{-1} \\ &= \delta(\epsilon k)(\epsilon k') \\ &= k' \bullet \delta(\epsilon k) \bullet (k')^\sigma \\ &= k' \bullet (k \bullet \delta) \\ &= k' \bullet k \ ,\end{aligned}$$

and if we substitute k' for k, then we get that $\delta = k \bullet k'$ as well, since it follows from the definition of k' that $k'' = k$.

Hence every non-zero element $k \in K^*$ is invertible with inverse $k^{-1} = k'$. It follows that $K_{+,\bullet}$ is a field or a skew-field. □

For technical reasons which will be clear in a moment, we let $K_{+,\cdot} := K^{\text{op}}_{+,\bullet}$, that is, we set $k_1 k_2 := k_2 \bullet k_1$ for all $k_1, k_2 \in K$.

Theorem 8.32. *(K, W, σ) is an involutory set. Furthermore, K is generated by W as a ring.*

Proof. We have just shown that K is a field or a skew-field. It is obvious from the definition that $\sigma^2 = 1$ and that $(k_1 k_2)^\sigma = (k_2 \bullet k_1)^\sigma = k_1^\sigma \bullet k_2^\sigma = k_2^\sigma k_1^\sigma$ for all $k_1, k_2 \in K$, so σ is an involution of K. W is an additive subgroup of K containing δ. By Lemma 8.29, $k + k^\sigma = F(\epsilon, \epsilon k) \in \text{Im}(F) \subseteq W$ for all $k \in K$, hence $K_\sigma \subseteq W$, and by the definition of σ, all elements of W are fixed by σ. Finally, it follows from Theorem 8.30 that $k^\sigma W k = k \bullet W \bullet k^\sigma \subseteq W(\epsilon k) \subseteq W$ for all $k \in K$. Thus (K, W, σ) is an involutory set.

The fact that K is generated by W as a ring follows immediately from the definition of the ring K. □

Theorem 8.33. *$(E, W, \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_I(K, W, \sigma)$.*

Proof. Let ϕ be the isomorphism from $[K]$ to E which maps $[k]$ to ϵk for all $k \in K$, and let ψ be the isomorphism from $[W]$ to W which maps $[w]$ to w for all $w \in W$. Then $\phi([\delta]) = \epsilon\delta = \epsilon$ and $\psi([\delta]) = \delta$. Furthermore,

$$\phi([k][w]) = \phi([wk]) = \epsilon(wk) = \epsilon(k \bullet w) = \epsilon k \cdot w = \phi([k])\psi([w]) \ , \text{ and}$$
$$\psi([w][k]) = \psi(k^\sigma w k) = k^\sigma w k = k \bullet w \bullet k^\sigma = w(\epsilon k) = \psi([w])\phi([k]) \ ,$$

for all $w \in W$ and all $k \in K$. Hence (ϕ, ψ) is an isomorphism from $\Omega_I(K, W, \sigma)$ to $(E, W, \tau_V, \tau_W, \epsilon, \delta)$. □

The next lemma shows the "$\sigma \neq 1$" part of Theorem 8.6.

Lemma 8.34. *If (K, K_0, σ) is an involutory set with $\sigma = 1$, then $\Omega_I(K, K_0, \sigma)$ is normal or indifferent.*

Proof. Since σ is an involution, $ab = (ab)^\sigma = b^\sigma a^\sigma = ba$ for all $a, b \in K$, hence K is abelian. It follows that $F([a], [b]) = [2ab]$ for all $a, b \in K$. If $\text{char}(K) = 2$, then $F \equiv 0$, hence $\Omega_I(K, K_0, \sigma)$ is indifferent. So we can assume that $\text{char}(K) \neq 2$. But then $K_\sigma = \{2a \mid a \in K\} = K$, and hence $K_0 = K$. It follows that for all elements $t_1, t_2, \ldots, t_n \in K_0$, the product $t_1 t_2 \ldots t_n$ lies in K_0 as well, and hence $[1][t_1][t_2] \ldots [t_n] = [1][t_1 t_2 \ldots t_n]$, which implies that $\Omega_I(K, K_0, \sigma)$ is normal. □

From now on, we drop our assumption that $V = E$ (but we still assume that $H \equiv 0$). Our next goal is to show that if Ω is reduced but not normal, then $V = E$ after all.

We start with a generalization of Lemma 8.18:

LEMMA 8.35. *For all $v \in V^*$, all $c \in V$ and all $k_1, k_2 \in K$, we have that*

$$\pi_v(\pi_v(c)k_1)k_2 = \pi_v(\pi_v(ck_2)k_1) \ .$$

PROOF. If we substitute $\overline{c}(\delta v)^{-1}$ for c in $(\mathbf{Q}_{25})$, then we get that $\pi_v(c)w = \pi_v\big(\overline{\overline{c}(\delta v)^{-1}(wv)}\big)$, and hence $\pi_v(\pi_v(c)w) = \overline{\overline{c}(\delta v)^{-1}(wv)}$ for all $c, v \in V$ and all $w \in W$. It thus follows by 3.23(i) and Lemma 8.18 that

$$\begin{aligned}\pi_v(\pi_v(c)w_1)w_2 &= \overline{\overline{c}(\delta v)^{-1}(w_1 v)}w_2\\ &= \overline{\overline{cw_2}(\delta v)^{-1}(w_1 v)}\\ &= \pi_v(\pi_v(cw_2)w_1) \ ,\end{aligned}$$

which shows the lemma for all $k_1, k_2 \in W$. In the same way as in Lemma 8.17 and Lemma 8.18, we can use induction to deduce from this that the lemma holds for all $k_1, k_2 \in M$. Since π_v is additive, it then follows that the lemma holds for all $k_1, k_2 \in K$. □

LEMMA 8.36. *For all $v \in V^*$ and all $k \in K$, we have that $\pi_v(vk) = -vk^\sigma$.*

PROOF. Since π_v is additive, it suffices to show that $\pi_v(vm) = -vm^\sigma$ for all $m = w_1 \bullet \cdots \bullet w_n \in M$, which we will do by induction on n.

It already follows from 3.21 that $\pi_v(vw_1) = -vw_1$, which shows the statement for $n = 1$. Now assume that $\pi_v(vw_1 \dots w_{n-1}) = -vw_{n-1} \dots w_1$ for all $w_1, \dots, w_{n-1} \in W$. Then it follows by Lemma 8.35 that

$$\begin{aligned}\pi_v(vw_1 \dots w_n) &= -\pi_v(\pi_v(vw_{n-1} \dots w_1)w_n)\\ &= -\pi_v(\pi_v(v)w_n)w_{n-1} \dots w_1\\ &= -vw_n w_{n-1} \dots w_1 \ ,\end{aligned}$$

since $\pi_v(\pi_v(v)w_n) = -\pi_v(vw_n) = vw_n$. □

LEMMA 8.37. *For all $v \in V$ and all $w \in W$, we have that $vww = v \cdot \delta(\epsilon w)$.*

PROOF. We may assume that $v \neq 0$. Since $H \equiv 0$, it follows from $(\mathbf{Q}_{26})$ that $\delta(\epsilon w)v = \delta(vw)$, and hence $\delta(\epsilon w) = \delta(vw)v^{-1}$. By $(\mathbf{Q}_{16})$, it follows that

$$v \cdot \delta(\epsilon w) = v \cdot \big(\delta(vw)v^{-1}\big) = \overline{v^{-1} \cdot \delta(vw)} \ .$$

If we substitute vw for v, v^{-1} for c and w^{-1} for w in $(\mathbf{Q}_{25})$, then we get, by $(\mathbf{Q}_{19})$, $(\mathbf{Q}_{16})$ and 3.21, that

$$\begin{aligned}\pi_{vw}(\overline{v^{-1} \cdot \delta(vw)})w^{-1} &= \pi_{vw}(\overline{v^{-1} \cdot w^{-1}(vw)})\\ &= \pi_{vw}(\overline{v^{-1} \cdot wv})\\ &= \pi_{vw}(vw)\\ &= -vw \ ,\end{aligned}$$

and hence $\pi_{vw}(v \cdot \delta(\epsilon w)) = -vww$, from which it follows, by 3.23(i), that $v \cdot \delta(\epsilon w) = -\pi_{vw}(vww) = vww$ by Lemma 8.36, which is what we had to show. □

LEMMA 8.38. *For all $v \in V$ and all $w, z \in W$, we have that $vwzw = v \cdot z(\epsilon w)$.*

Proof. We may assume that $v \neq 0$. Since $H \equiv 0$, it follows from ($\mathbf{Q}_{26}$) that $z(\epsilon w)v = z(vw)$. By ($\mathbf{Q}_{16}$), it follows that

$$v \cdot z(\epsilon w) = v \cdot \big(z(vw)v^{-1}\big) = \overline{v^{-1} \cdot z(vw)} \ .$$

If we substitute vw for v, v^{-1} for c and z for w in ($\mathbf{Q}_{25}$), then we get that

$$\pi_{vw}(\overline{v^{-1} \cdot \delta(vw)})z = \pi_{vw}(\overline{v^{-1} \cdot z(vw)}) \ ,$$

and hence

$$\pi_{vw}(v \cdot \delta(\epsilon w))z = \pi_{vw}(v \cdot z(\epsilon w)) \ .$$

It now follows from Lemma 8.37 and Lemma 8.36 that

$$\begin{aligned} v \cdot z(\epsilon w) &= \pi_{vw}(\pi_{vw}(vww)z) \\ &= \pi_{vw}(-vwwz) \\ &= vwzw \ , \end{aligned}$$

which proves the lemma. □

Lemma 8.39. *For all $v \in V$, all $z \in W$ and all $m \in M$, we have that $vmzm^\sigma = v \cdot z(\epsilon m)$. In particular, $vmm^\sigma = v \cdot \delta(\epsilon m)$.*

Proof. Let $m = w_1 \bullet \cdots \bullet w_n$ with $w_1, \dots, w_n \in W^*$. We will prove the lemma by induction on n.

We have already shown in Lemma 8.38 that the current lemma holds for $n = 1$. Now assume that $vw_1 \dots w_{n-1} y w_{n-1} \dots w_1 = v \cdot y(\epsilon w_1 \dots w_{n-1})$ for all $y \in W$. Then by Lemma 8.38 and ($\mathbf{Q}_{26}$), we have that

$$\begin{aligned} vw_1 \dots w_n z w_n \dots w_1 &= \big((vw_1 \dots w_{n-1})w_n z w_n\big) w_{n-1} \dots w_1 \\ &= vw_1 \dots w_{n-1} \cdot z(\epsilon w_n) \cdot w_{n-1} \dots w_1 \\ &= v \cdot z(\epsilon w_n)(\epsilon w_1 \dots w_{n-1}) \\ &= v \cdot z(\epsilon w_1 \dots w_{n-1} w_n) \ , \end{aligned}$$

and we are done. □

Lemma 8.40. *For all $v \in V$ and all $w_1, w_2, w_3 \in W$, we have that*

(i) $F(vw_1, vw_2w_3) = F(\epsilon w_1, \epsilon w_2 w_3)v$ *;*
(ii) $vw_1w_2w_3 + vw_3w_2w_1 = vF(\epsilon w_1, \epsilon w_3 w_2)$ *.*

Proof. By ($\mathbf{Q}_{21}$), ($\mathbf{Q}_{11}$) and ($\mathbf{Q}_{26}$), we have that

$$\begin{aligned} F(vw_1, vw_2w_3) &= w_3(vw_1 + vw_2) - w_3(vw_1) - w_3(vw_2) \\ &= w_3 \cdot v(w_1 + w_2) - w_3 \cdot vw_1 - w_3 \cdot vw_2 \\ &= w_3 \cdot \epsilon(w_1 + w_2) \cdot v - w_3 \cdot \epsilon w_1 \cdot v - w_3 \cdot \epsilon w_2 \cdot v \\ &= (w_3 \cdot (\epsilon w_1 + \epsilon w_2) - w_3 \cdot \epsilon w_1 - w_3 \cdot \epsilon w_2) \cdot v \\ &= F(\epsilon w_1, \epsilon w_2 w_3)v \ , \end{aligned}$$

which proves (i). By Lemma 8.36, the definition of π_v, Lemma 8.22, (i) and ($\mathbf{Q}_{16}$), we have that

$$\begin{aligned} vw_1w_2w_3 + vw_3w_2w_1 &= vw_3w_2w_1 - \pi_v(vw_3w_2w_1) \\ &= \overline{v^{-1}F(v, vw_3w_2w_1)} \\ &= \overline{v^{-1}F(vw_1, vw_3w_2)} \\ &= \overline{v^{-1} \cdot F(\epsilon w_1, \epsilon w_3w_2)v} \\ &= vF(\epsilon w_1, \epsilon w_3w_2)\ , \end{aligned}$$

which proves (ii). □

Lemma 8.41. *Let $w_1, w_2, w_3 \in W$ be arbitrary. Let $k = w_1 + w_2 \bullet w_3 \in K$. Then*

(i) *$vkk^\sigma = v \cdot \delta(\epsilon k)$ for all $v \in V$;*
(ii) *If $\epsilon k = 0$, then $k = 0$.*

Proof. By Lemma 8.39, Lemma 8.40(ii) and ($\mathbf{Q}_{11}$), we have that

$$\begin{aligned} vkk^\sigma &= v(w_1 + w_2 \bullet w_3)(w_1 + w_3 \bullet w_2) \\ &= vw_1w_1 + vw_2w_3w_3w_2 + vw_1w_3w_2 + vw_2w_3w_1 \\ &= v \cdot \delta(\epsilon w_1) + v \cdot \delta(\epsilon w_2w_3) + vF(\epsilon w_1, \epsilon w_2w_3) \\ &= v \cdot (\delta(\epsilon w_1) + \delta(\epsilon w_2w_3) + F(\epsilon w_1, \epsilon w_2w_3)) \\ &= v \cdot \delta(\epsilon w_1 + \epsilon w_2w_3) \\ &= v \cdot \delta(\epsilon k) \end{aligned}$$

for all $v \in V$, which proves (i). Now suppose that $\epsilon k = \epsilon w_1 + \epsilon w_2w_3 = 0$. Then it follows, by ($\mathbf{Q}_{11}$), Lemma 8.15 and Lemma 8.40(i), that

$$\begin{aligned} 0 &= \delta \cdot (\epsilon w_1 + \epsilon w_2w_3) \cdot v \\ &= \delta \cdot \epsilon w_1 \cdot v + \delta \cdot \epsilon w_2w_3 \cdot v + F(\epsilon w_1, \epsilon w_2w_3) \cdot v \\ &= \delta \cdot vw_1 + \delta \cdot vw_2w_3 + F(vw_1, vw_2w_3) \\ &= \delta \cdot (vw_1 + vw_2w_3) \\ &= \delta \cdot vk\ , \end{aligned}$$

and hence $vk = 0$, for all $v \in V$. So $k = 0$. □

Remark 8.42. It will follow from the classification that the statements in Lemma 8.40 and Lemma 8.41 actually hold in a much broader generality, for all reduced quadrangular systems. More precisely, we have that

(i) $F(vk_1, vk_2) = F(\epsilon k_1, \epsilon k_2)v$;
(ii) $vk + vk^\sigma = vF(\epsilon k, \epsilon)$;
(iii) $vkk^\sigma = v \cdot \delta(\epsilon k)$;
(iv) if $\epsilon k = 0$, then $k = 0$;

for all $v \in V$ and all $k, k_1, k_2 \in K$. However, we are not aware of a simple proof of these facts at this step of the classification.

By definition, $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is normal if and only if $\epsilon M = \epsilon W$. To complete the proof of Theorem 8.6, it will thus suffice to prove the following theorem:

THEOREM 8.43. *If $V \neq E$ and $F \not\equiv 0$, then $\epsilon M = \epsilon W$. Furthermore, $vm = vm^\sigma$ and $\overline{vm} = \overline{v}m$ for all $v \in V \setminus E$ and all $m \in M$.*

PROOF. We start by showing that $\overline{vw} = \overline{v}w$ for all $v \in V \setminus E$ and all $w \in W$. So let $v \in V \setminus E$ and $w \in W$ be arbitrary. By 3.20, $\overline{\overline{v}w} = v \cdot \delta v^{-1} \cdot w\overline{v}$, and hence $vw - \overline{\overline{v}w} = vk$ for $k = w - \delta v^{-1} \bullet w\overline{v}$. On the other hand,

$$\begin{aligned} vw - \overline{\overline{v}w} &= vw - \overline{\epsilon F(v,\epsilon)w - vw} \\ &= vw - \epsilon F(\epsilon F(v,\epsilon)w - vw, \epsilon) + \epsilon F(v,\epsilon)w - vw \\ &= \epsilon F(v,\epsilon)w - \epsilon F(\epsilon F(v,\epsilon)w - vw, \epsilon) \\ &\in \epsilon K \ , \end{aligned}$$

hence $vk \in \epsilon K$. Suppose that $vk \neq 0$. Then it would follow from Lemma 8.41 that $\epsilon k \neq 0$ and that $vkk^\sigma \cdot (\delta(\epsilon k))^{-1} = v$. Hence $v \in \epsilon K \cdot k^\sigma \cdot (\delta(\epsilon k))^{-1} \subseteq \epsilon K = E$, which contradicts the choice of v. So we must have $vk = 0$, and thus $\overline{\overline{v}w} = vw$, which shows that $\overline{vw} = \overline{v}w$.

We will now show by induction on n that $\overline{vw_1 \dots w_n} = \overline{v}w_1 \dots w_n$ for all $v \in V \setminus E$ and all $w_1, \dots, w_n \in W$. We have already shown this for $n = 1$, so suppose that it holds for $n-1$. We may assume that $w_i \neq 0$ for all $i \in \{1, \dots, n\}$. If $v \in V \setminus E$, then also $vw_1 \dots w_{n-1} \in V \setminus E$, since $v = vw_1 \dots w_{n-1}w_{n-1}^{-1} \dots w_1^{-1}$. Hence we can substitute $vw_1 \dots w_{n-1}$ for v in the result of the previous paragraph, and we get that

$$\begin{aligned} \overline{vw_1 \dots w_n} &= \overline{vw_1 \dots w_{n-1} \cdot w_n} \\ &= \overline{vw_1 \dots w_{n-1}} \cdot w_n \\ &= \overline{v}w_1 \dots w_n \ ; \end{aligned}$$

the statement thus holds for n as well. This shows that $\overline{vm} = \overline{v}m$ for all $v \in V \setminus E$ and all $m \in M$.

We will now prove by induction on n that $\overline{\overline{vw_1 \dots w_n}} = \overline{v}w_n \dots w_1$ for all $v \in V$ and all $w_1, \dots, w_n \in W$. Again, we have already shown this for $n = 1$, so suppose that it holds for $n-1$. Then, by Lemma 8.17,

$$\begin{aligned} \overline{vw_1 \dots w_n} &= \overline{\overline{\overline{v}w_{n-1} \dots w_1}w_n} \\ &= \overline{vw_n}w_{n-1} \dots w_1 \\ &= \overline{v}w_n \dots w_1 \ , \end{aligned}$$

so it holds for n as well. We have thus proved that $\overline{vm} = \overline{v}m^\sigma$ for all $m \in M$.

Since $v \in V \setminus E$ if and only if $\overline{v} \in V \setminus E$, it follows from the previous two paragraphs that $vm = vm^\sigma$ for all $v \in V \setminus E$ and all $m \in M$.

Now, we first assume that $\epsilon \notin \mathrm{Rad}(F)$. Since E is a proper subgroup of V, V is generated by $V \setminus E$. Since $\epsilon \notin \mathrm{Rad}(F)$, this implies that $F(\epsilon, V \setminus E) \neq 0$, so there exists an element $v \in V \setminus E$ such that $F(\epsilon, v) \neq 0$. Let m be an arbitrary element of M, and let $m_2 := F(\epsilon, v)^{-1} \bullet m \in M$. Then it follows from $\overline{vm_2} = \overline{v}m_2$ that

$$\epsilon F(\epsilon, vm_2) - vm_2 = \epsilon F(\epsilon, v)m_2 - vm_2$$

and hence

$$\epsilon F(\epsilon, vF(\epsilon, v)^{-1}m) = \epsilon F(\epsilon, v)F(\epsilon, v)^{-1}m$$

from which it follows that

$$\epsilon m = \epsilon F(\epsilon, vF(\epsilon, v)^{-1}m) \in \epsilon W \ ,$$

for all $m \in M$. So we have shown that $\epsilon M = \epsilon W$ in this case.

Now assume that $\epsilon \in \mathrm{Rad}(F)$. By Lemma 8.11, all elements of V and W have order at most 2, and $\overline{v} = v$ for all $v \in V$.

Since $F \not\equiv 0$, there exists an element $\eta \in V \setminus \mathrm{Rad}(F)$. By Lemma 8.22, $\mathrm{Rad}(F) \cdot K = \mathrm{Rad}(F)$, so $E = \epsilon K \subseteq \mathrm{Rad}(F)$, and hence $\eta \in V \setminus E$. It follows that $\eta m = \eta m^\sigma$ for all $m \in M$, which implies, by Lemma 8.36, that $\pi_\eta(\eta m) = \eta m^\sigma = \eta m$. By the definition of π_η, this implies that $\eta^{-1} F(\eta, \eta m) = 0$, and hence $F(\eta, \eta m) = 0$ for all $m \in M$. Since F is additive, this in turn implies that $F(\eta, \eta K) = 0$. Since we chose $\eta \notin \mathrm{Rad}(F)$, we conclude that $V \neq \eta K$.

We now show that $\pi_\eta(vw) = \pi_\eta(v)w$ for all $v \in V \setminus \eta K$ and all $w \in W$. So let $v \in V \setminus \eta K$ and $w \in W$ be arbitrary. If we substitute η for v and $v(\delta\eta)^{-1}$ for c in ($\mathbf{Q}_{25}$), then we get that $\pi_\eta(v)w = \pi_\eta(v(\delta\eta)^{-1}(w\eta))$, and hence $\pi_\eta(\pi_\eta(v)w) = v(\delta\eta)^{-1}(w\eta)$. Hence $vw + \pi_\eta(\pi_\eta(v)w) = vk$ where $k = w + (\delta\eta)^{-1} \bullet (w\eta) \in K$. On the other hand,

$$\begin{aligned} vw + \pi_\eta(\pi_\eta(v)w) &= vw + \pi_\eta(vw + \eta F(\eta^{-1}, v)w) \\ &= vw + vw + \eta F(\eta^{-1}, v)w + \eta F(\eta^{-1}, vw + \eta F(\eta^{-1}, v)w) \\ &= \eta F(\eta^{-1}, v)w + \eta F(\eta^{-1}, vw + \eta F(\eta^{-1}, v)w) \\ &\in \eta K, \end{aligned}$$

hence $vk \in \eta K$. In a similar way as in the first paragraph, it would follow from $vk \neq 0$ that $v \in \eta K$, which would contradict the choice of v. Hence $vk = vw + \pi_\eta(\pi_\eta(v)w) = 0$, and thus $\pi_\eta(vw) = \pi_\eta(v)w$.

Again, it follows by induction on n that $\pi_\eta(vw_1 \dots w_n) = \pi_\eta(v)w_1 \dots w_n$ for all $v \in V \setminus \eta K$ and all $w_1, \dots w_n \in W$, that is, $\pi_\eta(vm) = \pi_\eta(v)m$ for all $v \in V \setminus \eta K$ and all $m \in M$.

Since ηK is a proper subgroup of V, V is generated by $V \setminus \eta K$. Since $\eta \notin \mathrm{Rad}(F)$, this implies that $F(\eta, V \setminus \eta K) \neq 0$, so there exists an element $v \in V \setminus \eta K$ such that $F(\eta, v) \neq 0$. It follows from ($\mathbf{Q}_{17}$) that $F(\eta^{-1}, v) = F(\eta, v)\eta^{-1} \neq 0$ as well.

Let m be an arbitrary element of M, and let $m_2 := F(\eta^{-1}, v)^{-1} \bullet m \in M$. Then it follows from $\pi_\eta(vm_2) = \pi_\eta(v)m_2$ that

$$vm_2 + \eta F(\eta^{-1}, vm_2) = vm_2 + \eta F(\eta^{-1}, v)m_2$$

and hence

$$\eta F(\eta^{-1}, vF(\eta^{-1}, v)^{-1}m) = \eta F(\eta^{-1}, v)F(\eta^{-1}, v)^{-1}m ,$$

from which it follows that

$$\eta m = \eta F(\eta^{-1}, vF(\eta^{-1}, v)^{-1}m) ,$$

for all $m \in M$. So we have shown that $\eta M = \eta W$.

Since $\eta \in V \setminus \mathrm{Rad}(F)$ and $\epsilon \in \mathrm{Rad}(F)$, we have that $\eta + \epsilon \in V \setminus \mathrm{Rad}(F)$ as well. So the conclusion of the previous paragraph is also valid for $\eta + \epsilon$, that is, $(\eta + \epsilon)M = (\eta + \epsilon)W$. Now let m be an arbitrary element of M. Then $\eta m = \eta w_1$ and $(\eta + \epsilon)m = (\eta + \epsilon)w_2$ for some $w_1, w_2 \in W$. It follows that

$$\begin{aligned} \epsilon m &= \eta m + (\eta + \epsilon)m \\ &= \eta w_1 + (\eta + \epsilon)w_2 \\ &= \eta(w_1 + w_2) + \epsilon w_2 . \end{aligned}$$

If $w_1 + w_2 \neq 0$, then it would follow from this that

$$\eta = (\epsilon m + \epsilon w_2) \cdot (w_1 + w_2)^{-1} \in \epsilon K = E \subseteq \mathrm{Rad}(F) ,$$

which contradicts the choice of η. Hence we must have $w_1 + w_2 = 0$, and it follows that $\epsilon m = \epsilon w_2$. Since m was arbitrary, we have shown that $\epsilon M = \epsilon K$ also in this case.

This completes the proof of this theorem, and thereby also the proof of Theorem 8.6. □

8.2. Quadrangular Systems of Quadratic Form Type

Our goal in this section is to classify the quadrangular systems which are normal.

Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system which is normal. In particular, Ω is reduced, so $H \equiv 0$.

Lemma 8.44. *K is abelian, i.e. $vk_1k_2 = vk_2k_1$ for all $v \in V$ and all $k_1, k_2 \in K$. Equivalently, $\sigma = 1$.*

Proof. Note that by the definition of σ, K is abelian if and only if $\sigma = 1$.

It suffices to show that $vw_1w_2 = vw_2w_1$ for all $v \in V$ and all $w_1, w_2 \in W$. If $v \in V \setminus E$, then this follows by substituting $w_1 \bullet w_2$ for m in Theorem 8.43. If $v = \epsilon$, then $\epsilon w_1 w_2 = \epsilon w_3$ for some $w_3 \in W$ since Ω is normal. Hence, by Theorem 8.19, $\epsilon w_1 w_2 = \epsilon w_3 = \overline{\epsilon w_3} = \overline{\epsilon w_1 w_2} = \epsilon w_2 w_1$. Finally, assume that $v = \epsilon k$ for some $k \in K$. Then $vw_1w_2 = \epsilon k w_1 w_2 = \overline{\epsilon w_2 w_1 k^\sigma} = \overline{\epsilon w_1 w_2 k^\sigma} = \epsilon k w_2 w_1 = vw_2w_1$, again by Theorem 8.19. This shows the lemma in all possible cases. □

Lemma 8.45. *For all $v \in V$, we have that $vK = vW$.*

Proof. It suffices to show that $vM = vW$.

First, assume that $v \in E$, say $v = \epsilon k$ for some $k \in K$. Let $m \in M$ be arbitrary. Then $\epsilon m = \epsilon w$ for some $w \in W$, since Ω is normal. Since K is abelian by Lemma 8.44, it follows that $vm = \epsilon k m = \epsilon m k = \epsilon w k = \epsilon k w = vw$, which shows that $vM = vW$ in this case.

Now, assume that $v \in V \backslash E$. Note that it is sufficient to show that $vw_1w_2 \in vW$ for all $w_1, w_2 \in W$; it then follows by induction that $vM = vW$. Choose two arbitrary elements $w_1, w_2 \in W$. By Lemma 8.40(ii) and Lemma 8.44, we have that $vw_1w_2w_3 + vw_1w_2w_3 = vF(\epsilon w_1, \epsilon w_2 w_3)$, or equivalently, $vw_1w_2w_3(\delta + \delta) = vF(\epsilon w_1, \epsilon w_2 w_3)$, for all $w_3 \in W$.

We now distinguish two cases. First, assume that $\delta + \delta = 0$. It then follows, by $(\mathbf{Q}_{12})$, that all elements of V and W have order at most 2. Since Ω is normal, there exists a $w \in W$ such that $\epsilon w_1 w_2 = \epsilon w$. By the previous paragraph, $F(\epsilon w_1, \epsilon w_2 w) = 0$, and hence, by Lemma 8.22 and Lemma 8.40(i), $F(vw_1w_2, vw) = F(vw_1, vw_2w) = F(\epsilon w_1, \epsilon w_2 w)v = 0$ as well. By Lemma 8.15, it follows that

$$\begin{aligned}
\delta(vw_1w_2 + vw) &= \delta \cdot vw_1w_2 + \delta \cdot vw + F(vw_1w_2, vw) \\
&= \delta \cdot \epsilon w_1 w_2 \cdot v + \delta \cdot \epsilon w \cdot v \\
&= \delta \cdot \epsilon w \cdot v + \delta \cdot \epsilon w \cdot v \\
&= 0 ,
\end{aligned}$$

which implies that $vw_1w_2 + vw = 0$, hence $vw_1w_2 = vw \in vW$, which is what we had to show.

Now, assume that $\delta + \delta \neq 0$. Then we set $w_3 = (\delta + \delta)^{-1}$ in the identity $vw_1w_2w_3(\delta + \delta) = vF(\epsilon w_1, \epsilon w_2 w_3)$, which yields

$$vw_1w_2 = vF(\epsilon w_1, \epsilon w_2(\delta + \delta)^{-1}) \in vW \ ,$$

which proves the lemma in this case as well. □

LEMMA 8.46. *Let $w \in W$ and $k \in K$ be such that $vk = vw$ for some $v \in V^*$. Then $k = w$.*

PROOF. We will show that $uk = uw$ for all $u \in V$. We distinguish two cases.

First, assume that $u \in vK$. Since $vK = vW$ by Lemma 8.45, there exists a $z \in W$ such that $u = vz$. Then $uk = vzk = vkz = vwz = vzw = uw$, since K is abelian.

Now, assume that $u \notin vK$. By Lemma 8.45, there exists a $w_2 \in W$ such that $uk = uw_2$, and there exists a $w_3 \in W$ such that $(v + u)k = (v + u)w_3$. We have to show that $w = w_2$. We have that $vw_3 + uw_3 = (v + u)w_3 = (v + u)k = vk + uk = vw + uw_2$, from which it follows that $u(w_3 - w_2) = v(w - w_3)$. Since $u \notin vK$, this can only occur if $w_3 - w_2 = 0$, and then $w - w_3 = 0$ as well. Hence $w = w_3 = w_2$, which is what we had to show. □

REMARK 8.47. It follows from this lemma that if $k_1, k_2 \in K$ are such that $vk_1 = vk_2$ for some $v \in V^*$, then $k_1 = k_2$, since, by Lemma 8.45, there exists a $w \in W$ such that $vk_1 = vw = vk_2$.

THEOREM 8.48. *$K_{+,\bullet}$ is a commutative field.*

PROOF. We have already shown in Lemma 8.44 that K is a commutative ring. It only remains to show that every element of K^* is invertible. Let k be an arbitrary non-zero element of K. Since Ω is normal, $\epsilon k = \epsilon w$ for some $w \in W$; hence by Lemma 8.46, $k = w$. It follows that k is invertible with inverse $k^{-1} = w^{-1}$, since $w \bullet w^{-1} = w^{-1} \bullet w = \delta$. □

By Lemma 8.46, $(K, +) \cong W$ as additive groups. We will denote the isomorphism by square brackets, that is, for every $t \in K$, we will denote the corresponding element of W by $[t]$. Since K is a commutative field, V is a (left) vector space over K, with scalar multiplication given by $tv := v[t]$, for all $t \in K$ and all $v \in V$. From now on, we will denote the multiplicative identity of K by 1 in place of δ. Then $\delta = [1] \in W$. If there is no danger of confusion, we will also write st in place of $s \bullet t$ for $s, t \in K$, and t^2 in place of $t \bullet t$ for $t \in K$.

DEFINITION 8.49. We define a map q from V to K by setting $[q(v)] = \delta v = [1]v$ for all $v \in V$. Furthermore, we define a map f from $V \times V$ to K by setting $[f(v_1, v_2)] = F(v_1, v_2)$ for all $v_1, v_2 \in V$.

LEMMA 8.50. *For all $v \in V$, all $w \in W$ and all $t \in K$, we have that*

(i) $\overline{vw} = \overline{v}w$;
(ii) $\overline{tv} = t\overline{v}$.

PROOF. We first show (i). If $v \in \epsilon W$, then $vw \in \epsilon W$ as well, and it follows from Theorem 8.19 and Lemma 8.44 that $\overline{vw} = vw = \overline{v}w$ (remember that $\sigma = 1$). If $v \notin \epsilon W = \epsilon K$, then we have already shown this in Theorem 8.43.

Identity (ii) now follows by substituting $[t]$ for w in (i). □

LEMMA 8.51. *For all $v \in V$ and all $t \in K$, we have that $[t]v = [tq(v)]$.*

PROOF. Let $w := [t] \in W$. We have to show that $wv = w \bullet \delta v$. By ($\mathbf{Q}_{16}$) and Lemma 8.50(i), $v^{-1} \cdot wv = \overline{vw} = \overline{v}w$. On the other hand, $v^{-1} \cdot (w \bullet \delta v) = v^{-1} \cdot \delta v \cdot w = \overline{v}w$, again by ($\mathbf{Q}_{16}$). Hence $v^{-1} \cdot wv = v^{-1} \cdot (w \bullet \delta v)$, which implies by Lemma 8.46 that $wv = w \bullet \delta v$. □

LEMMA 8.52. *For all $u, v \in V$ and all $t \in K$, we have that $\pi_u(tv) = t\pi_u(v)$.*

PROOF. Let $w := [t] \in W$. If we substitute u for v and $\overline{v}(\delta u)^{-1}$ for c in ($\mathbf{Q}_{25}$), then we get that $\pi_u(v)w = \pi_u\big(\overline{\overline{v}(\delta u)^{-1}(wu)}\big)$, and hence, by Lemma 8.50(i) and Lemma 8.51, that

$$\begin{aligned} t\pi_u(v) &= \pi_u(v)w \\ &= \pi_u(v(\delta u)^{-1}(wu)) \\ &= \pi_u(v \cdot [q(u)^{-1}] \cdot [tq(u)]) \\ &= \pi_u(tq(u)q(u)^{-1}v) \\ &= \pi_u(tv)\ , \end{aligned}$$

which is what we had to show. □

THEOREM 8.53. *q is an anisotropic quadratic form from V to K with corresponding bilinear form f.*

PROOF. Let $v \in V$ and $t \in K$ be arbitrary, and let $w := [t] \in W$. Then, by ($\mathbf{Q}_{26}$),

$$[q(tv)] = [q(vw)] = \delta \cdot vw = \delta \cdot \epsilon w \cdot v = \delta v \cdot \epsilon w\ .$$

By Lemma 8.46, it follows from Lemma 8.38 that $w \bullet w \bullet z = z \cdot \epsilon w$ for all $z \in W$. Hence

$$[q(tv)] = \delta v \cdot \epsilon w = w \bullet w \bullet \delta v = [t] \bullet [t] \bullet [q(v)] = [t^2 q(v)]\ .$$

Next, it follows from ($\mathbf{Q}_{11}$) that for all $u, v \in V$, $[q(u+v)] = \delta(u+v) = \delta u + \delta v + F(u,v) = [q(u)] + [q(v)] + [f(u,v)] = [q(u) + q(v) + f(u,v)]$. We now show that f is bilinear over K. Let $u, v \in V^*$ and $t \in K$ be arbitrary. By Lemma 8.52, we have that $\pi_u(tv) = t\pi_u(v)$. By the definition of π_u, this yields

$$tv - \overline{u^{-1}F(u,tv)} = tv - t\overline{u^{-1}F(u,v)}\ .$$

By Lemma 8.50(ii), it follows that $u^{-1}F(u,tv) = tu^{-1}F(u,v)$, hence

$$u^{-1} \cdot [f(u,tv)] = u^{-1} \cdot [f(u,v)] \cdot [t] = u^{-1} \cdot [tf(u,v)]\ .$$

By Lemma 8.46, this implies that $f(u,tv) = tf(u,v)$. Since f is symmetric, it follows from this that f is bilinear over K.

Finally, q is anisotropic, since $q(v) = 0$ implies that $\delta v = 0$ and hence $v = 0$. □

LEMMA 8.54. *For all $u, v \in V$, we have that $q(\overline{v}) = q(v)$ and $f(\overline{u}, \overline{v}) = f(u,v)$.*

PROOF. We have that

$$\begin{aligned} q(\overline{v}) &= q(f(\epsilon,v)\epsilon - v) \\ &= q(f(\epsilon,v)\epsilon) + q(v) - f(f(\epsilon,v)\epsilon, v) \\ &= f(\epsilon,v)^2 q(\epsilon) + q(v) - f(\epsilon,v)f(\epsilon,v) \\ &= q(v)\ , \end{aligned}$$

and hence

$$\begin{aligned} f(\overline{u},\overline{v}) &= q(\overline{u}+\overline{v}) - q(\overline{u}) - q(\overline{v}) \\ &= q(u+v) - q(u) - q(v) \\ &= f(u,v) \end{aligned}$$

as well. □

THEOREM 8.55. $(V, W, \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_Q(K, V, q)$.

PROOF. Observe that $q(\epsilon) = 1$, since $[q(\epsilon)] = \delta\epsilon = \delta = [1]$.

Let ϕ be the isomorphism from $[V]$ to V which maps $[v]$ to v for all $v \in V$, and let ψ be the isomorphism from $[K]$ to W which maps $[t]$ to $[t]$ for all $t \in W$. Then $\phi([\epsilon]) = \epsilon$ and $\psi([1]) = [1] = \delta$. Furthermore,

$$\begin{aligned} &\phi([v][t]) = \phi([tv]) = tv = v[t] = \phi([v])\psi([t]) \ , \text{ and} \\ &\psi([t][v]) = \psi([tq(v)]) = [tq(v)] = [t]v = \psi([t])\phi([v]) \ , \end{aligned}$$

for all $t \in K$ and all $v \in V$. Hence (ϕ, ψ) is an isomorphism from $\Omega_Q(K, V, q)$ to $(V, W, \tau_V, \tau_W, \epsilon, \delta)$. □

This completes the proof of Theorem 8.7.

8.3. Quadrangular Systems of Indifferent Type

Our goal in this section is to classify the quadrangular systems which are indifferent.

Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a quadrangular system which is indifferent. Then $F \equiv 0$ and $H \equiv 0$. By Lemma 8.11, all elements of V and W have order 1 or 2, and for all $v \in V$, we have $\overline{v} = v$. Furthermore, we have $\pi_v(c) = c$ for all $v, c \in V$.

LEMMA 8.56. *K is abelian, i.e. $vk_1k_2 = vk_2k_1$ for all $v \in V$ and all $k_1, k_2 \in K$.*

PROOF. By Lemma 8.36, we have that $\pi_v(vk) = vk^\sigma$ and hence $vk = vk^\sigma$ for all $v \in V$ and all $k \in K$. It follows that $vk_1k_2 = v(k_1 \bullet k_2) = v(k_1 \bullet k_2)^\sigma = v(k_2^\sigma \bullet k_1^\sigma) = vk_2^\sigma k_1^\sigma = vk_2k_1$ for all $v \in V$ and all $k_1, k_2 \in K$. □

LEMMA 8.57. *For all $v \in V$ and all $k \in K$, we have that $vkk = v \cdot \delta(\epsilon k)$.*

PROOF. It already follows from Lemma 8.39 that $vmm = v \cdot \delta(\epsilon m)$ for all $m \in M$. Now suppose that $vk_1k_1 = v \cdot \delta(\epsilon k_1)$ and $vk_2k_2 = v \cdot \delta(\epsilon k_2)$ for some $k_1, k_2 \in K$. We will then show that $v(k_1+k_2)(k_1+k_2) = v \cdot \delta(\epsilon(k_1+k_2))$, which will prove the lemma.

By $(\mathbf{Q}_{11})$, $(\mathbf{Q}_{12})$ and Lemma 8.56, we have that

$$\begin{aligned} v(k_1+k_2)(k_1+k_2) &= vk_1k_1 + vk_2k_2 + vk_1k_2 + vk_2k_1 \\ &= v \cdot \delta(\epsilon k_1) + v \cdot \delta(\epsilon k_2) + vk_1k_2 + vk_1k_2 \\ &= v \cdot (\delta(\epsilon k_1) + \delta(\epsilon k_2)) \\ &= v \cdot \delta(\epsilon k_1 + \epsilon k_2) \\ &= v \cdot \delta(\epsilon(k_1+k_2)) \ , \end{aligned}$$

and we are done. □

LEMMA 8.58. *For all $v \in V$ and all $k \in K$, we have that $\delta \cdot vk = \delta \cdot \epsilon k \cdot v$.*

PROOF. In Lemma 8.15, we have already shown this for all $k \in M$. Now suppose that $\delta \cdot vk_1 = \delta \cdot \epsilon k_1 \cdot v$ and $\delta \cdot vk_2 = \delta \cdot \epsilon k_2 \cdot v$ for some $k_1, k_2 \in K$. We will then show that $\delta \cdot v(k_1 + k_2) = \delta \cdot \epsilon(k_1 + k_2) \cdot v$, which will prove the lemma.

By ($\mathbf{Q}_{11}$), we have that

$$\begin{aligned} \delta \cdot v(k_1 + k_2) &= \delta \cdot (vk_1 + vk_2) \\ &= \delta \cdot vk_1 + \delta \cdot vk_2 \\ &= \delta \cdot \epsilon k_1 \cdot v + \delta \cdot \epsilon k_2 \cdot v \\ &= \delta \cdot (\epsilon k_1 + \epsilon k_2) \cdot v \\ &= \delta \cdot \epsilon(k_1 + k_2) \cdot v \ , \end{aligned}$$

and we are done. □

THEOREM 8.59. *$K_{+,\bullet}$ is a commutative field of characteristic 2 with multiplicative identity δ.*

PROOF. We have already shown in Lemma 8.56 that $K_{+,\bullet}$ is a commutative ring. Let $k \in K$ be arbitrary. If $\delta(\epsilon k) = 0$, then it would follow from Lemma 8.58 that $vk = 0$ for all $v \in V$ and thus $k = 0$. Hence $\delta(\epsilon k)$ is invertible for all $k \neq 0$, and it then follows from Lemma 8.57 that $vkk(\delta(\epsilon k))^{-1} = v$ for all $v \in V$. This implies that k is invertible with inverse $k^{-1} := k \bullet (\delta(\epsilon k))^{-1}$.

Furthermore, for all $v \in V$ and all $k \in K$, we have that $v(k+k) = vk + vk = 0$, hence $k + k = 0$, so $\mathrm{char}(K) = 2$. □

LEMMA 8.60. *If $vk_1 = vk_2$ for some $v \in V^*$ and some $k_1, k_2 \in K$, then $k_1 = k_2$.*

PROOF. If $vk_1 = vk_2$ for some $v \in V^*$ and some $k_1, k_2 \in K$, then $v(k_1+k_2) = 0$. If we would have that $k_1 \neq k_2$, then $k_1 + k_2$ would be invertible, and it would then follow that $v = v(k_1+k_2)(k_1+k_2)^{-1} = 0$, a clear contradiction. Hence $k_1 = k_2$. □

THEOREM 8.61. *$(K, W, \delta V)$ is an indifferent set. Moreover, $\delta v \bullet w = wv$ and $w \bullet w \bullet \delta v = \delta \cdot vw$ for all $v \in V$ and all $w \in W$.*

PROOF. It is obvious that W is a subgroup of $(K, +)$. Since $\delta v_1 + \delta v_2 = \delta(v_1 + v_2)$ by ($\mathbf{Q}_{11}$), δV is a subgroup of $(K, +)$ as well. Furthermore, both W and δV contain the multiplicative identity δ.

By ($\mathbf{Q}_{25}$), $\epsilon \cdot \delta v \cdot w = \epsilon \cdot wv$ for all $v \in V$ and all $w \in W$. It follows by Lemma 8.60 that $\delta v \bullet w = wv$, and hence $\delta V \bullet W \subseteq W$.

By ($\mathbf{Q}_{26}$), we have that $\delta \cdot vw = \delta \cdot \epsilon w \cdot v = \delta v \cdot \epsilon w$, for all $v \in V$ and all $w \in W$. By Lemma 8.60, it follows from Lemma 8.38 that $w \bullet w \bullet z = z \cdot \epsilon w$ for all $z \in W$. Hence $\delta v \cdot \epsilon w = w \bullet w \bullet \delta v$. It follows that $w \bullet w \bullet \delta v = \delta \cdot vw$, and hence $W^2 \bullet \delta V \subseteq \delta V$.

Finally, it follows from the definition of K that K is generated by W as a ring. This shows that $(K, W, \delta V)$ is an indifferent set. □

THEOREM 8.62. *$(V, W, \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_D(K, W, \delta V)$.*

PROOF. First, we observe that v is uniquely determined by δv, since $\delta(v_1 + v_2) = \delta v_1 + \delta v_2$ by ($\mathbf{Q}_{11}$).

Let ϕ be the isomorphism from $[\delta V]$ to V which maps $[\delta v]$ to v for all $v \in V$, and let ψ be the isomorphism from $[W]$ to W which maps $[w]$ to w for all $w \in W$.

Then $\phi([\delta]) = \phi([\delta\epsilon]) = \epsilon$ and $\psi([\delta]) = \delta$. Furthermore, it follows from Theorem 8.61 that

$$\phi([\delta v][w]) = \phi([w^2 \bullet \delta v]) = \phi([\delta \cdot vw]) = vw = \phi([\delta v])\psi([w]) \ , \text{ and}$$
$$\psi([w][\delta v]) = \psi([\delta v \bullet w]) = \psi([wv]) = wv = \psi([w])\phi([\delta v]) \ ,$$

for all $v \in V$ and all $w \in W$. Hence (ϕ, ψ) is an isomorphism from $\Omega_D(K, W, \delta V)$ to $(V, W, \tau_V, \tau_W, \epsilon, \delta)$. □

This completes the proof of Theorem 8.8, and thereby the classification of all *reduced* quadrangular systems.

8.4. Quadrangular Systems of Pseudo-quadratic Form Type, I

Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a wide quadrangular system which is the extension of a reduced quadrangular system Λ of proper involutory type; more precisely, let $\Lambda = (V, \mathrm{Rad}(H), \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_I(K, K_0, \sigma)$ with $K = \langle K_0 \rangle$ and $\sigma \neq 1$, where τ_V and τ_W are as in Remark 8.3. In particular, $V = [K]$.

DEFINITION 8.63. Let $Y := \mathrm{Rad}(H)$. Note that Y is a normal subgroup of W since $Y \subseteq Z(W)$ by $(\mathbf{Q}_8)$; let $\widetilde{W} := W/Y$. Let ι be the canonical surjection from W to $\widetilde{W}$. We will also write $\tilde{w}$ in place of $\iota(w)$, for all $w \in W$. Then $\tilde{w}_1 = \tilde{w}_2$ if and only if $w_1 \boxminus w_2 \in Y$.

By $(\mathbf{Q}_8)$ and $(\mathbf{Q}_7)$, $[w_1, w_2] \in \mathrm{Im}(F) \subseteq Y$, hence $\widetilde{W}$ is abelian; we will use the additive notations $+$ and $-$ for $\widetilde{W}$. We can define a map $\tilde{\tau}_W$ from $\widetilde{W} \times V$ to $\widetilde{W}$, which we will denote by $\cdot$ or by juxtaposition, by setting

$$\tilde{\tau}_W(\tilde{w}, v) := \tilde{w} \cdot v := \widetilde{wv}$$

for all $v \in V$ and all $w \in W$. This is well defined: let $\tilde{w}_1 = \tilde{w}_2$, then $w_1 \boxminus w_2 \in Y$, and hence $w_1 v \boxminus w_2 v = (w_1 \boxminus w_2)v \in Y$ since $Y \cdot V = Y$; it then follows that $\widetilde{w_1 v} = \widetilde{w_2 v}$.

REMARK 8.64. If $s \in K_0$, then the notation $[s]$ is ambiguous. If we want to make clear whether we mean $[s] \in V$ or $[s] \in W$, we will write $[s]_V$ and $[s]_W$, respectively. Note that $[s]_W \in Y$ for all $s \in K_0$, and that $\epsilon[s]_W = [s]_V$ for all $s \in K_0$.

THEOREM 8.65. *$\widetilde{W}$ is a right vector space over K, with scalar multiplication given by $\tilde{w}t := \tilde{w} \cdot [t]$, for all $t \in K$ and all $\tilde{w} \in \widetilde{W}$.*

PROOF. We have that

$$\begin{aligned}(\tilde{w}_1 + \tilde{w}_2)t &= (\tilde{w}_1 + \tilde{w}_2) \cdot [t] = \iota\big((w_1 + w_2) \cdot [t]\big) \\ &= \iota(w_1 \cdot [t]) + \iota(w_2 \cdot [t]) = \tilde{w}_1 \cdot [t] + \tilde{w}_2 \cdot [t] = \tilde{w}_1 t + \tilde{w}_2 t\end{aligned}$$

for all $t \in K$ and all $w_1, w_2 \in W$. By $(\mathbf{Q}_{11})$ and $(\mathbf{Q}_7)$,

$$\begin{aligned}\tilde{w}(t_1 + t_2) &= \tilde{w} \cdot [t_1 + t_2] = \tilde{w} \cdot ([t_1] + [t_2]) = \iota(w \cdot ([t_1] + [t_2])) \\ &= \iota(w \cdot [t_1] \boxplus w \cdot [t_2] \boxplus F([t_2]w, [t_1])) \\ &= \iota(w \cdot [t_1]) + \iota(w \cdot [t_2]) = \tilde{w} \cdot [t_1] + \tilde{w} \cdot [t_2] = \tilde{w}t_1 + \tilde{w}t_2\end{aligned}$$

for all $t_1, t_2 \in K$ and all $w \in W$.

It only remains to show that $\tilde{w}(t_1 t_2) = (\tilde{w}t_1)t_2$ for all $t_1, t_2 \in K$ and all $w \in W$. (The other axioms for a vector space are obvious.) We thus have to check that $\iota(w \cdot [t_1 t_2]) = \iota(w \cdot [t_1] \cdot [t_2])$. Since $K = \langle K_0 \rangle$, it suffices to show this for $t_1 \in K_0$;

the result for $t_1 = s_1 \dots s_n$ with $s_1, \dots, s_n \in K_0$ will then follow by induction on n, and since we have already shown that $\tilde{w} \cdot [t_3 + t_4] = \tilde{w} \cdot [t_3] + \tilde{w} \cdot [t_4]$ for all $t_3, t_4 \in K$, the result then follows for all $t_1 \in K$.

By Remark 8.64, $(\mathbf{Q}_{26})$ and the definition of $\Omega_I(K, K_0, \sigma)$, we have that

$$\begin{aligned} w \cdot [s]_V \cdot [t] &= w \cdot \epsilon [s]_W \cdot [t] \\ &= w \cdot [t][s]_W \\ &= w \cdot [st]_V \end{aligned}$$

and hence $\iota(w \cdot [s]_V \cdot [t]) = \iota(w \cdot [st]_V)$ for all $s \in K_0$ and all $t \in K$, which is what we had to show. ☐

DEFINITION 8.66. Let π be the map from $\widetilde{W}$ to $V/[K_0]_V$ which maps $\tilde{w}$ to $\epsilon w \pmod{[K_0]_V}$. This map is well defined: let $w_1, w_2 \in W$ be such that $\tilde{w}_1 = \tilde{w}_2$. Then $w_1 \boxminus w_2 \in Y$, hence

$$\begin{aligned} \epsilon w_1 - \epsilon w_2 &= \epsilon((w_1 \boxminus w_2) \boxplus w_2) - \epsilon w_2 \\ &= \epsilon(w_1 \boxminus w_2) + \epsilon w_2 - \epsilon w_2 \\ &= \epsilon(w_1 \boxminus w_2) \in \epsilon Y = \epsilon [K_0]_W = [K_0]_V \end{aligned}$$

by $(\mathbf{Q}_{12})$.

LEMMA 8.67. *For all $\tilde{w} \in \widetilde{W}$, we have that $\pi(\tilde{w}) = 0$ if and only if $\tilde{w} = 0$.*

PROOF. Let $w \in W$ be such that $\pi(\tilde{w}) = 0$. Then $\epsilon w \in [K_0]_V$, say $\epsilon w = [s]_V$ with $s \in K_0$. By 3.13(ii), $\epsilon(\boxminus [s]_W) = -\epsilon [s]_W$ since $[s]_W \in Y$. It follows that $\epsilon(w \boxminus [s]_W) = \epsilon w + \epsilon(\boxminus [s]_W) = \epsilon w - \epsilon [s]_W = \epsilon w - [s]_V = 0$, and hence $w = [s]_W \in Y$. It follows that $\tilde{w} = 0$. ☐

DEFINITION 8.68. Let h be the map from $\widetilde{W} \times \widetilde{W}$ to K, defined by the identity $[h(\tilde{w}_1, \tilde{w}_2)] := H(w_1, w_2)$ for all $\tilde{w}_1, \tilde{w}_2 \in \widetilde{W}$. Since $\widetilde{W} = W/\operatorname{Rad}(H)$, the map h is well defined.

LEMMA 8.69. *For all $v \in V$, all $w \in W$ and all $y_1, \dots, y_n \in Y$, we have that $\overline{\overline{v}w}y_1 \dots y_n = \overline{\overline{vy_1 \dots y_n}w}$.*

PROOF. We will first prove the lemma for $n = 1$. Let $v \in V$, $w \in W$ and $y \in Y$. We may assume that $w \neq 0$. Note that $\Pi_w(y) = y$ since $y \in \operatorname{Rad}(H)$. It follows from $(\mathbf{Q}_{23})$ that $\overline{vy} = -\overline{\overline{v}(\boxminus w)}y\kappa(w)$, and hence, by $(\mathbf{Q}_{18})$, $\overline{\overline{vy}(\boxminus w)} = \overline{\overline{v}(\boxminus w)}y$. Substituting $\boxminus w$ for w now yields the result for $n = 1$.

We advance to general n by induction. Let $v \in V$, $w \in W$ and $y_1, \dots, y_n \in Y$, and suppose that $\overline{\overline{v}w}y_1 \dots y_{n-1} = \overline{\overline{vy_1 \dots y_{n-1}}w}$. Then

$$\begin{aligned} \overline{\overline{v}w}y_1 \dots y_n &= \overline{\overline{vy_1 \dots y_{n-1}}w} \cdot y_n \\ &= \overline{\overline{vy_1 \dots y_{n-1}y_n}w} \ , \end{aligned}$$

where we have used the lemma for $n = 1$ in the last equality. This completes the proof of this lemma. ☐

LEMMA 8.70. *For all $w \in W$ and all $y_1, \dots, y_n \in Y$, we have that $\overline{\epsilon w}y_1 \dots y_n = \overline{\epsilon y_n \dots y_1 w}$. In particular, for all $w \in W$ and all $s \in K_0$, we have that $\overline{[s]_V w} = \overline{\epsilon w}[s]_W$.*

Proof. By Theorem 8.19, $\overline{\epsilon y_1 \dots y_n} = \epsilon y_n \dots y_1$. The first result follows by substituting ϵ for v in Lemma 8.69.

In the particular case $n = 1$ and $y_1 = [s]_W$, we get that $\overline{\epsilon w}[s]_W = \overline{\epsilon[s]_W w} = \overline{[s]_V w}$. □

Theorem 8.71. *The map h is a skew-hermitian form over K with respect to σ.*

Proof. Since $\overline{H(w_1, w_2)} = -H(w_2, w_1)$ for all $w_1, w_2 \in W$ by $(\mathbf{Q}_{22})$, and since $\overline{[t]} = [t^\sigma]$ for all $t \in K$, it follows that $h(\tilde{w}_1, \tilde{w}_2) = -h(\tilde{w}_2, \tilde{w}_1)^\sigma$, for all $\tilde{w}_1, \tilde{w}_2 \in \widetilde{W}$.

By $(\mathbf{Q}_{22})$, $(\mathbf{Q}_{12})$ and Lemma 8.70,

$$\begin{aligned} H(w_1 \cdot [s]_V, w_2) &= -\overline{H(w_2, w_1 \cdot [s]_V)} \\ &= -\overline{[s]_V \cdot (w_1 \boxplus w_2)} + \overline{[s]_V \cdot w_1} + \overline{[s]_V \cdot w_2} \\ &= -\overline{\epsilon(w_1 \boxplus w_2)}[s]_W + \overline{\epsilon w_1}[s]_W + \overline{\epsilon w_2}[s]_W \\ &= -\overline{H(w_2, w_1)} \cdot [s]_W \\ &= H(w_1, w_2) \cdot [s]_W \ , \end{aligned}$$

for all $w_1, w_2 \in W$ and all $s \in K_0$. Hence

$$[h(\tilde{w}_1 s, \tilde{w}_2)] = [h(\tilde{w}_1, \tilde{w}_2)] \cdot [s]_W = [sh(\tilde{w}_1, \tilde{w}_2)] \ ,$$

from which it follows that $h(\tilde{w}_1 s, \tilde{w}_2) = sh(\tilde{w}_1, \tilde{w}_2)$ for all $w_1, w_2 \in W$ and all $s \in K_0$. Since $K = \langle K_0 \rangle$ and since h is additive in both variables, it follows from this that $h(\tilde{w}_1 t, \tilde{w}_2) = t^\sigma h(\tilde{w}_1, \tilde{w}_2)$ for all $w_1, w_2 \in W$ and all $t \in K$.

Finally, $h(\tilde{w}_1, \tilde{w}_2 t) = -h(\tilde{w}_2 t, \tilde{w}_1)^\sigma = -(t^\sigma h(\tilde{w}_2, \tilde{w}_1))^\sigma = -h(\tilde{w}_2, \tilde{w}_1)^\sigma t = h(\tilde{w}_1, \tilde{w}_2) t$ for all $w_1, w_2 \in W$ and all $t \in K$. This shows that h is a skew-hermitian form over K with respect to σ. □

Definition 8.72. For all $t \in K$, let k_t be the homomorphism from V to itself which maps $[t']$ to $[tt']$ for all $[t'] \in [K] = V$. We denote the action of k_t by right juxtaposition, i.e. $[t']k_t = [tt']$ for all $t, t' \in K$. In particular, we can identify k_s and $[s]_W$ for all $s \in K_0$. Moreover, we set $k_t^\sigma := k_{t^\sigma}$ for all $t \in K$. Note that the set $\{k_t \mid t \in K\}$ coincides with the set K that we defined in the beginning of section 8.1. In particular, we can apply the lemmas and theorems of that section on the sub-quadrangular system $\Lambda = (V, Y, \tau_V, \tau_W, \epsilon, \delta)$.

Lemma 8.73. *For all $w \in W$ and all $t_1, t_2 \in K$, we have that*

$$\epsilon k_1 w k_2^\sigma + \epsilon k_2 w k_1^\sigma = \epsilon F([t_2]w, [t_1]) + H(w[t_2], w[t_1]) \ ,$$

where $k_1 := k_{t_1}$ and $k_2 := k_{t_2}$.

Proof. By Lemma 8.22 and the definition of the map $v \mapsto \overline{v}$, we have that

$$\begin{aligned} \epsilon F([t_2]w, [t_1]) &= \epsilon F(\epsilon k_2 w, \epsilon k_1) \\ &= \epsilon F(\epsilon k_2 w k_1^\sigma, \epsilon) \\ &= \epsilon k_2 w k_1^\sigma + \overline{\epsilon k_2 w k_1^\sigma} \ , \end{aligned}$$

and by Theorem 8.71 and 3.13(ii), we have that

$$\begin{aligned} H(w[t_2], w[t_1]) &= [h(\tilde{w}t_2, \tilde{w}t_1)] \\ &= [t_2^\sigma h(\tilde{w}, \tilde{w}t_1)] \\ &= [h(\tilde{w}, \tilde{w}t_1)]k_2^\sigma \\ &= H(w, w[t_1])k_2^\sigma \\ &= [t_1](\boxminus w)k_2^\sigma + [t_1]wk_2^\sigma \\ &= \epsilon k_1(\boxminus w)k_2^\sigma + \epsilon k_1 w k_2^\sigma \ . \end{aligned}$$

It only remains to show that $\overline{\epsilon k_2 w k_1^\sigma} = -\epsilon k_1(\boxminus w)k_2^\sigma$. By Lemma 8.70, we have that $\epsilon k_t^\sigma w = \overline{\overline{\epsilon w}k_t}$ for all $w \in W$ and all $t \in K$. By $(\mathbf{Q}_{16})$ with ϵ in place of v and Lemma 8.18 with $\epsilon(\boxminus w)$ in place of v, it follows that

$$\begin{aligned} \overline{\epsilon k_2 w k_1^\sigma} &= \overline{\overline{\overline{\epsilon w}k_2^\sigma}k_1^\sigma} \\ &= -\overline{\overline{\epsilon(\boxminus w)k_2^\sigma}k_1^\sigma} \\ &= -\overline{\overline{\epsilon(\boxminus w)}k_1^\sigma}k_2^\sigma \\ &= -\epsilon k_1(\boxminus w)k_2^\sigma \ , \end{aligned}$$

which completes the proof of this lemma. □

THEOREM 8.74. *For all $w \in W$ and all $t \in K$, we have that $\epsilon \cdot w[t] = \epsilon k_t w k_t^\sigma$.*

PROOF. First assume that $t = s_1 \dots s_n$ with $s_1, \dots, s_n \in K_0$. Let $y_i := [s_i]_W \in Y$ for all $i \in \{1, \dots, n\}$. Then we have to show that $\epsilon \cdot w(\epsilon y_1 \dots y_n) = \epsilon y_1 \dots y_n w y_n \dots y_1$ for all $w \in W$. By $(\mathbf{Q}_{16})$, Lemma 8.27 and Lemma 8.70,

$$\begin{aligned} \epsilon y_1 \dots y_n w &= -\overline{(\epsilon y_1 \dots y_n)^{-1}(\boxminus w \cdot \epsilon y_1 \dots y_n)} \\ &= -\overline{\epsilon y_n^{-1} \dots y_1^{-1}(\boxminus w \cdot \epsilon y_1 \dots y_n)} \\ &= -\overline{\epsilon(\boxminus w \cdot \epsilon y_1 \dots y_n)}y_1^{-1} \dots y_n^{-1} \\ &= \epsilon(w \cdot \epsilon y_1 \dots y_n)y_1^{-1} \dots y_n^{-1} \ , \end{aligned}$$

from which it follows that $\epsilon(w \cdot \epsilon y_1 \dots y_n) = \epsilon y_1 \dots y_n w y_n \dots y_1$.

Now suppose that $\epsilon \cdot w[t_1] = \epsilon k_1 w k_1^\sigma$ and $\epsilon \cdot w[t_2] = \epsilon k_2 w k_2^\sigma$ for some $t_1, t_2 \in K$, where where $k_1 := k_{t_1}$ and $k_2 := k_{t_2}$. We will show that

$$\epsilon \cdot w[t_1 + t_2] = \epsilon(k_1 + k_2)w(k_1 + k_2)^\sigma \ ,$$

which will prove the theorem for all $t \in K$, since $K = \langle K_0 \rangle$.

By $(\mathbf{Q}_{11})$, $(\mathbf{Q}_{12})$ with $v = \epsilon$, $(\mathbf{Q}_7)$ and Lemma 8.73, we have that

$$\begin{aligned} \epsilon \cdot w[t_1 + t_2] &= \epsilon \cdot w([t_1] + [t_2]) \\ &= \epsilon \cdot (w[t_1] \boxplus w[t_2] \boxplus F([t_2]w, t_1)) \\ &= \epsilon \cdot w[t_1] + \epsilon \cdot w[t_2] + \epsilon F([t_2]w, t_1) + H(w[t_2], w[t_1]) \\ &= \epsilon k_1 w k_1^\sigma + \epsilon k_2 w k_2^\sigma + \epsilon k_1 w k_2^\sigma + \epsilon k_2 w k_1^\sigma \\ &= \epsilon(k_1 + k_2)w(k_1 + k_2)^\sigma \ , \end{aligned}$$

which completes the proof of this theorem. □

LEMMA 8.75. *Let $w \in W$ and $t \in K$ be arbitrary. Let $x \in K$ be such that $\epsilon w = [x]$. Then $[t]w = [xt]$, and $\epsilon k_t w k_t^\sigma = [t^\sigma x t]$.*

PROOF. By Lemma 8.70, we have that

$$[t]w = \epsilon k_t w = \overline{\overline{\epsilon w} k_t^\sigma} = \overline{[x^\sigma] k_t^\sigma} = \overline{[t^\sigma x^\sigma]} = [xt] \ ;$$

it then follows that

$$\epsilon k_t w k_t^\sigma = [xt] k_t^\sigma = [t^\sigma xt] \ ,$$

and we are done. □

DEFINITION 8.76. For all $\tilde{w} \in \widetilde{W}$, let $p(\tilde{w})$ be any element $t \in K$ such that $[t]$ is contained in the coset $\pi(\tilde{w}) \in V/[K_0]_V$. Hence p is a map from $\widetilde{W}$ to K such that $[p(\tilde{w})] \equiv \epsilon w \pmod{[K_0]_V}$.

THEOREM 8.77. *$(K, K_0, \sigma, \widetilde{W}, p)$ is an anisotropic pseudo-quadratic space with corresponding skew-hermitian form h.*

PROOF. It only remains to show that p is a pseudo-quadratic form. All equivalences will be modulo $[K_0]_V$. By ($\mathbf{Q}_{12}$), we have that

$$\begin{aligned}
[p(\tilde{w}_1 + \tilde{w}_2)] &\equiv [p(\tilde{w}_2 + \tilde{w}_1)] \\
&\equiv \epsilon(w_2 \boxplus w_1) \\
&\equiv \epsilon w_2 + \epsilon w_1 + H(w_1, w_2) \\
&\equiv [p(\tilde{w}_1)] + [p(\tilde{w}_2)] + [h(\tilde{w}_1, \tilde{w}_2)]
\end{aligned}$$

for all $w_1, w_2 \in W$, which shows the first property.

Next, let w be an arbitrary element of W, and let $x \in K$ be such that $\epsilon w = [x]$. Then $[p(\tilde{w})] \equiv [x]$, and hence $[t^\sigma p(\tilde{w}) t] \equiv [t^\sigma xt]$, since $t^\sigma K_0 t \subseteq K_0$. It follows from Theorem 8.74 and Lemma 8.75 that

$$[p(\tilde{w}t)] \equiv \epsilon \cdot w[t] \equiv \epsilon k_t w k_t^\sigma \equiv [t^\sigma xt] \equiv [t^\sigma p(\tilde{w}) t] \ ,$$

which shows the second property.

Finally, if $[p(\tilde{w})] \equiv 0$ for some $\tilde{w} \in \widetilde{W}$, then $\pi(\tilde{w}) = 0$ and hence $\tilde{w} = 0$ by Lemma 8.67. □

LEMMA 8.78. *Let the group T be as in section 7.4. For each element $(a, x) \in T$, there is a unique element $w \in W$ such that $w \in a$ and $\epsilon w = [x]$. If we denote this element by $\chi(a, x)$, then χ is an isomorphism from T to W.*

PROOF. Let $(a, x) \in T$ be arbitrary. Choose an arbitrary element $z \in a$. Then $a = \tilde{z}$, and $\epsilon z \equiv [x] \pmod{[K_0]_V}$ by the definition of T. Hence $\epsilon z - [x] \in [K_0]_V = \epsilon Y$, say $\epsilon z - [x] = \epsilon y$ with $y \in Y$. Set $w = z \boxminus y$, then $\tilde{w} = \tilde{z} = a$, and $\epsilon w = \epsilon(z \boxminus y) = \epsilon z - \epsilon y = [x]$ by ($\mathbf{Q}_{12}$) and 3.13(ii). This shows the existence of w.

Now suppose that $w_1, w_2 \in W$ are such that $\tilde{w}_1 = \tilde{w}_2$ and $\epsilon w_1 = \epsilon w_2$. Then $w_1 \boxminus w_2 \in Y$, and hence, by ($\mathbf{Q}_{12}$),

$$\begin{aligned}
0 &= \epsilon w_1 - \epsilon w_2 \\
&= \epsilon((w_1 \boxminus w_2) \boxplus w_2) - \epsilon w_2 \\
&= \epsilon(w_1 \boxminus w_2) + \epsilon w_2 - \epsilon w_2 \\
&= \epsilon(w_1 \boxminus w_2) \ ,
\end{aligned}$$

from which it follows that $w_1 = w_2$.

Hence $\chi : T \to W$ is a well defined map, which is bijective, with the inverse map given by $\chi^{-1}(w) = (\tilde{w}, x) \in T$, where $[x] = \epsilon w$. In order to show that χ is an isomorphism, it now suffices to show that $\chi^{-1}(w_1 \boxplus w_2) = \chi^{-1}(w_1) \boxplus \chi^{-1}(w_2)$.

Let $x_1, x_2 \in K$ be such that $[x_1] = \epsilon w_1$ and $[x_2] = \epsilon w_2$. Then $\epsilon(w_1 \boxplus w_2) = \epsilon w_1 + \epsilon w_2 + H(w_2, w_1) = [x_1] + [x_2] + [h(\tilde{w}_2, \tilde{w}_1)]$, and hence

$$\begin{aligned}\chi^{-1}(w_1 \boxplus w_2) &= (\tilde{w}_1 + \tilde{w}_2, x_1 + x_2 + h(\tilde{w}_2, \tilde{w}_1)) \\ &= (\tilde{w}_1, x_1) \boxplus (\tilde{w}_2, x_2) \\ &= \chi^{-1}(w_1) \boxplus \chi^{-1}(w_2) \ ,\end{aligned}$$

which completes the proof of this lemma. □

THEOREM 8.79. $(V, W, \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_P(K, K_0, \sigma, \widetilde{W}, p)$.

PROOF. Let ϕ be the isomorphism from $[K]$ to V which maps $[t]$ to $[t]$ for all $t \in K$, and let ψ be the isomorphism from $[T]$ to W which maps $[a, x]$ to $\chi(a, x)$ for all $(a, x) \in T$. Then $\phi([1]) = [1] = \epsilon$ and $\psi([0, 1]) = \delta$ since $\delta \in Y$ (hence $\tilde{\delta} = 0$) and $\epsilon\delta = [1]$.

Now, let $t \in K$ and $(a, x) \in T$ be arbitrary. Let $w = \psi([a, x]) = \chi(a, x)$, then $a = \tilde{w}$ and $\epsilon w = [x]$. By Lemma 8.75, $[xt] = [t]w$, hence

$$\begin{aligned}&\phi([t][a, x]) = \phi([xt]) = [xt] = [t]w = \phi([t])\psi([a, x]) \ , \text{ and} \\ &\psi([a, x][t]) = \psi([at, t^\sigma xt]) = w[t] = \psi([a, x])\phi([t]) \ ,\end{aligned}$$

since $\widetilde{w[t]} = \tilde{w}t = at$ and $\epsilon \cdot w[t] = [t^\sigma xt]$ by Theorem 8.74 and Lemma 8.75. Hence (ϕ, ψ) is an isomorphism from $\Omega_P(K, K_0, \sigma, \widetilde{W}, p)$ to $(V, W, \tau_V, \tau_W, \epsilon, \delta)$. □

This completes the proof of Theorem 8.9.

8.5. Quadrangular Systems of Type F_4

Let $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ be a wide quadrangular system which is the extension of a reduced quadrangular system Λ of quadratic form type; more precisely, let $\Lambda = (V, \text{Rad}(H), \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_Q(K, V_0, q)$, where τ_V and τ_W are as in Remark 8.3.

Our goal in this section is to classify these quadrangular systems in the case that $\text{Rad}(F) \neq 0$.

So assume that $\text{Rad}(F) \neq 0$. It then follows from $(\mathbf{Q}_{10})$ that $\epsilon \in \text{Rad}(F)$. Note that $\overline{v} = v$ for all $v \in V$ by Lemma 8.11.

REMARK 8.80. We will identify V and V_0 in the sequel if there is no danger of confusion, which will allow us to use notations like tv with $t \in K$ and $v \in V$.

Observe that the axiom system is very symmetrical now. (See section A.3.1 in the appendix.) In particular, every identity will have a "dual identity", which is obtained by switching the roles of V and W.

LEMMA 8.81. *For all $v, v_1, v_2 \in V$ and all $w, w_1, w_2 \in W$, we have that*
(i) $F(v_1 w, v_2) = F(v_1, v_2 w)$ *;*
(ii) $H(w_1 v, w_2) = H(w_1, w_2 v)$ *.*

PROOF. Since both V and W are abelian, it follows from $(\mathbf{Q}_{21})$ and $(\mathbf{Q}_{11})$ that $F(v_1 w, v_2) = w(v_2 + v_1) + wv_2 + wv_1 = w(v_1 + v_2) + wv_1 + wv_2 = F(v_2 w, v_1) = F(v_1, v_2 w)$, which proves (i). Similarly, (ii) follows from $(\mathbf{Q}_{22})$ and $(\mathbf{Q}_{12})$. (Identity (ii) is the "dual" of identity (i).) □

Definition 8.82. Let $R := \mathrm{Rad}(F)$. Then $\epsilon \in R$, and $R \neq V$ since $F \not\equiv 0$. Moreover, let $L := q(R) \subseteq K$.

Lemma 8.83. *$\Sigma := (R, W, \tau_V, \tau_W, \epsilon, \delta)$ is a quadrangular system with $F_\Sigma \equiv 0$ and $H_\Sigma \not\equiv 0$; see Remark 8.3.*

Proof. First of all, we observe that $R = \mathrm{Rad}(F)$ is a subgroup of V, since F is additive in both variables. We have that $\mathrm{Im}(F_\Sigma) = F(R, R) = 0$, and $\mathrm{Im}(H_\Sigma) = H(W, W) = \mathrm{Im}(H) \neq 0$. It now only remains to show that $\tau_V(R \times W) \subseteq R$, $H(W, W) \subseteq R$ and $(R^*)^{-1} \subseteq R$.

If $v \in R$, then $F(v, V) = 0$, hence $F(vw, V) = F(v, Vw) \subseteq F(v, V) = 0$ as well for all $w \in W$, by Lemma 8.81(i). Hence $\tau_V(R \times W) = R \cdot W \subseteq R$. Since W is abelian, it follows from ($\mathbf{Q}_8$) that $H(W, W) \subseteq \mathrm{Rad}(F) = R$. Finally, if $v \in R^*$, then $F(v^{-1}, V) = F(v, V)v^{-1} = 0$ by ($\mathbf{Q}_{17}$), and hence $v^{-1} \in R$. Thus $\Sigma := (R, W, \tau_V, \tau_W, \epsilon, \delta)$ is a quadrangular system. □

By Theorem 8.13, $\Sigma^* = (W, R, \tau_W, \tau_V, \delta, \epsilon)$ is a reduced quadrangular system. Suppose that Σ^* were of involutory type, say $\Sigma^* \cong Q_I(J, J_0, \sigma)$ for some involutory set (J, J_0, σ). Then $\delta \in \mathrm{Rad}(H)$ would imply that $0 = F_{\Sigma^*}([1], [a]) = [a + a^\sigma]$ and hence $a + a^\sigma = 0$ for all $a \in J$, from which it would follow that $F_{\Sigma^*}([a], [b]) = [(a^\sigma b) + (a^\sigma b)^\sigma] = 0$ for all $a, b \in J$. Hence $H_\Sigma \equiv F_{\Sigma^*} = 0$, a contradiction.

It follows that Σ^* must be of quadratic form type. In particular, R has the structure of a field, W is a (right) vector space over R, and the map $p : W \to R : w \mapsto \epsilon w$ is a quadratic form. If we denote the multiplication in R by $\bullet$, then we have that $w(r_1 \bullet r_2) = (wr_1)r_2$ for all $w \in W$ and all $r_1, r_2 \in R$.

Lemma 8.84. *L is a subfield of K, with $K^2 \subseteq L \subseteq K$. Moreover, q is a field isomorphism from R to L.*

Proof. We will first prove that q is an isomorphism (both additive and multiplicative) from R to L. Since $L = q(R)$, q is surjective. For all $r_1, r_2 \in R$, we have that $[q(r_1 + r_2)] = \delta(r_1 + r_2) = \delta r_1 + \delta r_2 = [q(r_1) + q(r_2)]$, by ($\mathbf{Q}_{11}$); hence q is additive. In particular, if $q(r_1) = q(r_2)$, then $q(r_1 + r_2) = 0$ and hence $r_1 = r_2$, since q is anisotropic. Hence q is injective. Furthermore, for all $r_1, r_2 \in R$, we have that $[q(r_1 \bullet r_2)] = \delta(r_1 \bullet r_2) = (\delta r_1)r_2 = [q(r_1)]r_2 = [q(r_1)q(r_2)]$ by Lemma 8.51, hence q is multiplicative.

It follows that $L = q(R)$ is a commutative field which is isomorphic to R. Finally, for all $t \in K$, we have that $q(t\epsilon) = t^2 q(\epsilon) = t^2$, hence $K^2 \subseteq q(R) = L$ since $t\epsilon \in \mathrm{Rad}(F)$. □

Definition 8.85. For all $s \in L$, we let $[s] := q^{-1}(s) \in V$. If we want to make clear whether we mean $[s] \in V$ or $[s] \in W$, we will write $[s]_V$ and $[s]_W$, respectively. By Lemma 8.84, we can consider W as a (left) vector space over L via the scalar multiplication $sw := w[s]$ for all $w \in W$ and all $s \in L$.

Definition 8.86. Let $\hat{q}$ be the map from W to L given by $\hat{q}(w) := q(p(w)) = q(\epsilon w)$ for all $w \in W$, and let $\hat{f}$ be the map from $W \times W$ to L given by $\hat{f}(w_1, w_2) := q(H(w_1, w_2))$ for all $w_1, w_2 \in W$. In particular, $\epsilon w = [\hat{q}(w)]_V$ and $H(w_1, w_2) = [\hat{f}(w_1, w_2)]_V$ for all $w, w_1, w_2 \in W$.

Lemma 8.87. *$\hat{q}$ is a quadratic form from W to L with corresponding bilinear form $\hat{f}$.*

Proof. Since p is a quadratic form from W to R with corresponding bilinear form H, it follows by Lemma 8.84 that $\hat{q} = q \circ p$ is a quadratic form from W to L with corresponding bilinear form $\hat{f} = q \circ H$. □

Remark 8.88. For $s \in L$ and $t \in K$, we will write $q[s]$ and $\hat{q}[t]$ in place of $q([s])$ and $\hat{q}([t])$, respectively.

Lemma 8.89. *For all $v, v_1, v_2 \in V$ and all $w, w_1, w_2 \in W$, we have that*

(i) $F(v_1, v_2) = 0 \implies wv_1v_2 = wv_2v_1$;
(ii) $H(w_1, w_2) = 0 \implies vw_1w_2 = vw_2w_1$.

Proof. Observe that $(\mathbf{Q}_{23})$ can be rewritten as "$v\Pi_w(z) = vwzw^{-1}$", and that $(\mathbf{Q}_{24})$ can be rewritten as "$w\pi_v(c) = wvcv^{-1}$" since $\pi_v(\epsilon) = \epsilon$.

Let $v_1, v_2 \in V$ be such that $F(v_1, v_2) = 0$, and assume that $v_1 \neq 0$. Then $\pi_{v_1}(v_2) = v_2$. It then follows from $(\mathbf{Q}_{24})$ that $wv_2 = wv_1v_2v_1^{-1}$ for all $w \in W$, hence (i). Identity (ii) is the dual of (i). □

In particular, $s(wv) = wv[s] = w[s]v = (sw)v$ and $t(vw) = vw[t] = v[t]w = (tv)w$ for all $v \in V$, $w \in W$, $s \in L$ and $t \in K$. It follows that the notations swv and tvw are unambiguous.

Lemma 8.90. *For all $v \in V^*$ and all $w \in W^*$, we have that*

(i) $v^{-1} = q(v)^{-1}v$;
(ii) $w^{-1} = \hat{q}(w)^{-1}w$.

Proof. If we substitute δ for w in $(\mathbf{Q}_{16})$, then we get that $v = v^{-1} \cdot \delta v = v^{-1}[q(v)]_W = q(v)v^{-1}$, which proves (i). Similarly for (ii). □

Lemma 8.91. *For all $v \in V^*$, $w \in W^*$, $t \in K$ and $s \in L$, we have that*

(i) $w \cdot tv = \hat{q}[t]wv$;
(ii) $v \cdot sw = q[s]vw$;
(iii) $wv = \hat{q}[q(v)]wv^{-1}$;
(iv) $vw = q[\hat{q}(w)]vw^{-1}$.

Proof. We only prove (i) and (iii). By $(\mathbf{Q}_{26})$,

$$w \cdot tv = w \cdot v[t] = w \cdot \epsilon[t] \cdot v = w \cdot [\hat{q}[t]]_V \cdot v = \hat{q}[t]wv \ ,$$

which proves (i). It follows from Lemma 8.90(i) and (i) that $wv = w \cdot q(v)v^{-1} = \hat{q}[q(v)]wv^{-1}$, which proves (iii). □

Lemma 8.92. *For all $v, c \in V$, $w, z \in W$, we have that*

(i) $wvcv = w(f(v, c)v + q(v)c)$;
(ii) $vwzw = v(\hat{f}(w, z)w + \hat{q}(w)z)$.

Proof. We only prove (i). We may assume that $v \neq 0$. By $(\mathbf{Q}_{24})$ and by Lemma 8.90(i),

$$wvcv^{-1} = w\pi_v(c) = w(c + f(v, c)v^{-1}) = w(c + f(v, c)q(v)^{-1}v) \ .$$

It follows by Lemma 8.91(iii) and Lemma 8.91(i) that

$$wvcv = \hat{q}[q(v)]wvcv^{-1} = \hat{q}[q(v)]w(c + f(v, c)q(v)^{-1}v) = w(q(v)c + f(v, c)v) \ ,$$

which is what we had to show. □

Lemma 8.93. *For all $v, c \in V$, $w, z \in W$, we have that*

(i) $wvc + wcv = \hat{q}[f(v,c)]w + [f(v,c)f(vw,c)]$;
(ii) $vwz + vzw = q[\hat{f}(w,z)]v + [\hat{f}(w,z)\hat{f}(wv,z)]$.

Proof. Again, we only prove (i). We may assume that $v \neq 0$. By $(\mathbf{Q}_{24})$, Lemma 8.91(i) and Lemma 8.90(i),

$$\begin{aligned} wvcv^{-1} &= w\pi_v(c) \\ &= w(c + f(v,c)v^{-1}) \\ &= wc + w \cdot f(v,c)v^{-1} + F(f(v,c)v^{-1}, cw) \\ &= wc + \hat{q}[f(v,c)]wv^{-1} + [f(f(v,c)q(v)^{-1}v, cw)] \ , \end{aligned}$$

and hence, by Lemma 8.51,

$$\begin{aligned} wvc &= wcv + \hat{q}[f(v,c)]w + [f(v,c)q(v)^{-1}f(v,cw)]v \\ &= wcv + \hat{q}[f(v,c)]w + [f(v,c)f(v,cw)] \\ &= wcv + \hat{q}[f(v,c)]w + [f(v,c)f(vw,c)] \ , \end{aligned}$$

which is what we had to show. □

Lemma 8.94. *For all $v \in V$, $c \in V^*$, $w \in W$ and $z \in W^*$, we have that*
(i) $z \cdot vz = \hat{q}(z)zv$;
(ii) $c \cdot wc = q(c)cw$;
(iii) $\hat{f}(z, w \cdot vz)z^{-1} = \hat{f}(zv, w)z$;
(iv) $f(c, v \cdot wc)c^{-1} = f(cw, v)c$.

Proof. We will only prove (i) and (iii). First of all, observe that it follows from $(\mathbf{Q}_{12})$ that $H(w, wv) = 0$ for all $v \in V$ and all $w \in W$. In particular, $\Pi_z(z \cdot \epsilon z) = z \cdot \epsilon z$ and $\Pi_z(z \cdot vz) = z \cdot vz$. It thus follows from $(\mathbf{Q}_{26})$ that $z \cdot vz = z \cdot \epsilon z \cdot v = \hat{q}(z)zv$, which shows (i). By Lemma 8.81(ii) and Lemma 8.90(ii), it now follows that

$$\begin{aligned} \hat{f}(z, w \cdot vz)z^{-1} &= \hat{f}(z \cdot vz, w)\hat{q}(z)^{-1}z \\ &= \hat{f}(\hat{q}(z)zv, w)\hat{q}(z)^{-1}z \\ &= \hat{f}(zv, w)z \ , \end{aligned}$$

which shows (iii). □

Lemma 8.95. *For all $v \in V$ and all $w \in W$, we have that*
(i) $[\hat{q}(wv)] = q(v)[\hat{q}(w)]$;
(ii) $[q(vw)] = \hat{q}(w)[q(v)]$.

Proof. We will only prove (i). We may again assume that $v \neq 0$. Since $\epsilon \in \mathrm{Rad}(F)$, it follows by Lemma 8.81(i) that $F(v, \epsilon c) = F(vc, \epsilon) = 0$ and hence $\pi_v(\epsilon c) = \epsilon c$ for all $c \in V$. If we set $c = \epsilon$ in $(\mathbf{Q}_{25})$, we thus get that $\epsilon \cdot \delta v \cdot w = \epsilon \cdot wv$, and hence $[\hat{q}(wv)] = \epsilon \cdot wv = \epsilon \cdot \delta v \cdot w = \epsilon[q(v)]w = q(v)\epsilon w = q(v)[\hat{q}(w)]$. □

Lemma 8.96. *For all $v, c \in V$ and all $w, z \in W$, we have that*
(i) $w \cdot vz + \hat{q}(z)wv = \hat{f}(w, zv)z + \hat{f}(w,z)zv$;
(ii) $v \cdot wc + q(c)vw = f(v, cw)c + f(v,c)cw$.

Proof. We will only prove (i). We may assume that $z \neq 0$. By $(\mathbf{Q}_{26})$, we have that

$$w \cdot \epsilon z \cdot v + z^{-1}H(z, w \cdot \epsilon z) \cdot v = w \cdot vz + z^{-1}H(z, w \cdot vz) \ ,$$

hence

$$\hat{q}(z)wv + \hat{f}(z, w \cdot \epsilon z)z^{-1}v = w \cdot vz + \hat{f}(z, w \cdot vz)z^{-1} \ ,$$

and it follows from Lemma 8.94(iii) that

$$\hat{q}(z)wv + \hat{f}(z, w)zv = w \cdot vz + \hat{f}(zv, w)z \ ,$$

which is what we had to show. □

At this point, we will break the symmetry. We cannot avoid this, since L is a subfield of K, but not vice versa.

Lemma 8.97. *For all $t \in K$ and all $s \in L$, we have that*

(i) $s[t] = [st]$ *;*
(ii) $t[s] = [t^2 s]$.

Proof. By Lemma 8.51, we have that $s[t] = [t][s] = [tq[s]]$. Since $[s] = q^{-1}(s)$ by definition, it follows that $s[t] = [st]$. On the other hand, $t[s] = tq^{-1}(s) = q^{-1}(t^2 s) = [t^2 s]$. □

Now choose fixed arbitrary elements $\xi \in W \setminus Y$ and $d \in V \setminus R$.

Theorem 8.98. *There exists an element $e \in V$ such that $f(d, e) = 1$ and $f(d, e\xi) = 0$. Moreover, $f(d\xi, e\xi) = \hat{q}(\xi) \in L \setminus K^2$.*

Proof. We will first show the last statement. So let $a, b \in V$ be arbitrary elements such that $f(a, b) = 1$. Then, by Lemma 8.81(i) and Lemma 8.91(iv), we have that $f(a\xi, b\xi) = f(a, b\xi\xi) = f(a, q[\hat{q}(\xi)]b) = q[\hat{q}(\xi)]f(a, b) = q[\hat{q}(\xi)] = \hat{q}(\xi)$ (note that $[s] = q^{-1}(s)$ for all $s \in L$ by definition). Let $\alpha := \hat{q}(\xi)$. Suppose that $\alpha \in K^2$, say $\alpha = t^2$ for some $t \in K$. Then $q(t\epsilon) = t^2 = \alpha = \hat{q}(\xi) = q(\epsilon\xi)$, hence $\epsilon[t] = t\epsilon = \epsilon\xi$. Since $[t] \in Y$, this implies that $\epsilon(\xi + [t]) = \epsilon\xi + \epsilon[t] = 0$, and hence $\xi = [t] \in Y$, which contradicts the choice of ξ. Hence $\alpha \notin K^2$.

Since $d \notin R = \mathrm{Rad}(F)$, there exist an elements $u \in V$ such that $F(d, u) \neq 0$. Let $v := f(d, u)^{-1}u$, then

$$\begin{aligned} f(d, v) &= f(d, f(d, u)^{-1}u) \\ &= f(d, u)^{-1}f(d, u) \\ &= 1 \ . \end{aligned}$$

In particular, $f(d\xi, v\xi) = \hat{q}(\xi) = \alpha$. Since $\alpha \notin K^2$, we have that $\alpha^{-1}f(d, v\xi)^2 \neq 1$. Now let

$$e := \bigl(1 + \alpha^{-1}f(d, v\xi)^2\bigr)^{-1}\bigl(v + \alpha^{-1}f(d, v\xi)v\xi\bigr) \ .$$

Then, by Lemma 8.81(i),

$$\begin{aligned} f(d, e) &= \bigl(1 + \alpha^{-1}f(d, v\xi)^2\bigr)^{-1}f(d, v + \alpha^{-1}f(d, v\xi)v\xi) \\ &= \bigl(1 + \alpha^{-1}f(d, v\xi)^2\bigr)^{-1}\bigl(f(d, v) + \alpha^{-1}f(d, v\xi)f(d, v\xi)\bigr) \\ &= \bigl(1 + \alpha^{-1}f(d, v\xi)^2\bigr)^{-1}\bigl(1 + \alpha^{-1}f(d, v\xi)^2\bigr) \\ &= 1 \ , \end{aligned}$$

and

$$\begin{aligned} f(d, e\xi) &= \big(1 + \alpha^{-1} f(d, v\xi)^2\big)^{-1} f(d, v\xi + \alpha^{-1} f(d, v\xi) v\xi\xi) \\ &= \big(1 + \alpha^{-1} f(d, v\xi)^2\big)^{-1} \big(f(d, v\xi) + \alpha^{-1} f(d, v\xi) f(d, v\xi\xi)\big) \\ &= \big(1 + \alpha^{-1} f(d, v\xi)^2\big)^{-1} \big(f(d, v\xi) + \alpha^{-1} f(d, v\xi)\alpha\big) \\ &= 0\ , \end{aligned}$$

which shows that e fulfills the required properties. □

From now on, let $e \in V$ be as in Theorem 8.98, and let $\alpha := f(d\xi, e\xi) = \hat{q}(\xi)$. By Theorem 8.98, $\alpha \in L \setminus K^2$.

THEOREM 8.99. *Let* $B := \langle d, e, d\xi, e\xi \rangle$. *Then* $\dim_K B = 4$ *and* $B \cap R = 0$.

PROOF. Let $v = t_1 d + t_2 e + t_3 d\xi + t_4 e\xi$ with $t_1, t_2, t_3, t_4 \in K$ be an arbitrary element of B. Suppose that $v \in R = \mathrm{Rad}(F) = \mathrm{Rad}(f)$. Then $f(v, d) = f(v, e) = f(v, d\xi) = f(v, e\xi) = 0$. Note that $f(d, d) = f(d, d\xi) = f(e, e) = f(e, e\xi) = f(d\xi, d\xi) = f(e\xi, e\xi) = 0$ by $(\mathbf{Q}_{11})$, and that $f(d, e\xi) = f(e, d\xi) = 0$ by Theorem 8.98 and Lemma 8.81(i). Moreover, $f(d, e) = 1$ and $f(d\xi, e\xi) = \alpha \neq 0$. It now follows from $f(v, d) = 0$ that $t_2 = 0$, from $f(v, e) = 0$ that $t_1 = 0$, from $f(v, d\xi) = 0$ that $t_4 = 0$ and from $f(v, e\xi) = 0$ that $t_3 = 0$. Hence $v = 0$. This shows that $B \cap R = 0$.

Since $0 \in R$, the previous paragraph also shows that it follows from $v = 0$ that $t_1 = t_2 = t_3 = t_4 = 0$, hence d, e, $d\xi$ and $e\xi$ are linearly independent. It follows that $\dim_K B = 4$. □

THEOREM 8.100. $B^\perp = R$, *where* $B^\perp := \{v \in V \mid f(v, B) = 0\}$.

PROOF. It is obvious that $R \subseteq B^\perp$. So let g be an arbitrary element of $B^\perp$. Then $f(g, d) = f(g, e) = f(g, d\xi) = f(g, e\xi) = 0$. If we substitute ξ for z, ξde for w and g for v in Lemma 8.96(i), then we get that

$$\xi de \cdot g\xi + \hat{q}(\xi)\xi deg = \hat{f}(\xi de, \xi g)\xi + \hat{f}(\xi de, \xi)\xi g\ .$$

Since $f(e, g\xi) = 0$ and $f(d, g\xi) = 0$, it follows from Lemma 8.89(i) that $\xi de \cdot g\xi = \xi d \cdot g\xi \cdot e = \xi \cdot g\xi \cdot d \cdot e$, and hence $\xi de \cdot g\xi = \hat{q}(\xi)\xi gde$ by Lemma 8.94(i). On the other hand, since $f(e, g) = 0$ and $f(d, g) = 0$, it follows from Lemma 8.89(i) that $\hat{q}(\xi)\xi deg = \hat{q}(\xi)\xi dge = \hat{q}(\xi)\xi gde$.

Hence $\xi de \cdot g\xi = \hat{q}(\xi)\xi deg$, and it follows that $\hat{f}(\xi de, \xi g)\xi = \hat{f}(\xi de, \xi)\xi g$. By Lemma 8.81(i) and $(\mathbf{Q}_{11})$, we have that

$$\begin{aligned} [\hat{f}(\xi de, \xi)] &= [\hat{f}(\xi d, \xi e)] \\ &= [\hat{q}(\xi d + \xi e)] + [\hat{q}(\xi d)] + [\hat{q}(\xi e)] \\ &= [\hat{q}(\xi(d + e))] + [\hat{q}(\xi d)] + [\hat{q}(\xi e)] \end{aligned}$$

since $F(d\xi, e) = 0$. It follows from Lemma 8.95(i) that

$$\begin{aligned} [\hat{f}(\xi de, \xi)] &= q(d + e)[\hat{q}(\xi)] + q(d)[\hat{q}(\xi)] + q(e)[\hat{q}(\xi)] \\ &= \big(q(d + e) + q(d) + q(e)\big)[\alpha] \\ &= f(d, e)[\alpha] \\ &= [\alpha]\ , \end{aligned}$$

and hence $\hat{f}(\xi de, \xi) = \alpha \neq 0$. It follows that $\xi g = \alpha^{-1}\hat{f}(\xi de, \xi g)\xi = \xi r$ with $r = [\alpha^{-1}\hat{f}(\xi de, \xi g)] \in R$. Since $\xi(g + r) = \xi g + \xi r = 0$ by $(\mathbf{Q}_{11})$, we conclude that $g = r \in R$, which completes the proof of this theorem. □

Since $\dim_K B = 4$ is finite by Theorem 8.99, we have $V = B + B^\perp$. Since $B \cap R = 0$ by Theorem 8.99 and $B^\perp = R$ by Theorem 8.100, it follows that V has a decomposition $V = B \oplus R$. In particular, every complement of R in V has dimension 4 over K. By symmetry, it also follows that every complement of Y in W has dimension 4 over L.

Let $\beta := q(d)^{-1}$. Then $\beta \in K \setminus L$, since $\beta \in L$ would imply that $q(d) = \beta^{-1} = q[\beta^{-1}]$ and hence $d = [\beta^{-1}] \in [L] = R = \mathrm{Rad}(f)$, which contradicts the fact that $f(d, e) = 1$.

THEOREM 8.101. *Let* $A := \langle \xi, \xi ed^{-1}, \xi d^{-1}, \beta^2\xi e\rangle$. *Then* $W = A \oplus Y$.

PROOF. Let $w = s_1\xi + s_2\xi ed^{-1} + s_3\xi d^{-1} + s_4\beta^2\xi e$ with $s_1, s_2, s_3, s_4 \in L$ be an arbitrary element of A. Suppose that $w \in Y = \mathrm{Rad}(H) = \mathrm{Rad}(\hat{f})$. Observe that $\hat{q}[q(d)] = q(\epsilon[q(d)]) = q(q(d)\epsilon) = q(d)^2 = \beta^{-2}$ and hence, by Lemma 8.91(iii), $\hat{f}(\xi ed^{-1}, \xi) = \hat{f}(\xi e, \xi d^{-1}) = \hat{q}[q(d)]^{-1}\hat{f}(\xi e, \xi d) = \alpha\beta^2 \neq 0$.

By $(\mathbf{Q}_{12})$, $\hat{f}(\xi, \xi) = \hat{f}(\xi d^{-1}, \xi) = \hat{f}(\xi e, \xi) = 0$. It thus follows from $\hat{f}(w, \xi) = 0$ that $\hat{f}(s_2\xi ed^{-1}, \xi) = 0$ and hence $s_2 = 0$. We now have $w = s_1\xi + s_3\xi d^{-1} + s_4\beta^2\xi e$.

Since $\hat{f}(\xi, \xi e) = \hat{f}(\xi e, \xi e) = 0$ and $\hat{f}(\xi d^{-1}, \xi e) = \alpha\beta^2 \neq 0$, it follows from $\hat{f}(w, \xi e) = 0$ that $s_3 = 0$, and hence $w = s_1\xi + s_4\beta^2\xi e$.

Since $\hat{f}(\xi, \xi d^{-1}) = 0$ and $\hat{f}(\xi e, \xi d^{-1}) = \alpha\beta^2 \neq 0$, it follows from $\hat{f}(w, \xi d^{-1}) = 0$ that $s_4 = 0$. Hence $w = s_1\xi$.

Finally, it now follows from $\hat{f}(w, \xi ed^{-1}) = 0$ that $\hat{f}(s_1\xi, \xi ed^{-1}) = s_1\alpha\beta^2 = 0$ and hence $s_1 = 0$.

So we have shown that $w \in Y$ implies $w = 0$, and at the same time, we have shown that ξ, ξed^{-1}, ξd^{-1} and $\beta^2\xi e$ are linearly independent. Hence $\dim_L A = 4$ and $A \cap Y = 0$, from which it follows that A is contained in a complement of Y in W. Since every complement of Y in W is 4-dimensional, this implies that A itself is a complement of Y, i.e. $W = A \oplus Y$. □

Let E be the splitting field of the polynomial $\phi(x) \equiv q(d)x^2 + x + q(e)$ over K.

LEMMA 8.102. *E/K is a separable quadratic extension.*

PROOF. Suppose that $t \in K$ would be a root of ϕ. Then

$$q(td + e) = q(td) + f(td, e) + q(e) = t^2q(d) + tf(d, e) + q(e) = \phi(t) \ ,$$

since $f(d, e) = 1$, hence $q(td + e) = 0$. Since q is anisotropic, this implies that $td + e = 0$, which contradicts the fact that d and e are linearly independent. Hence ϕ has no roots in K, so E/K is a quadratic extension. Since the coefficient of x of ϕ is non-zero, the two roots of ϕ are distinct, hence the extension is separable. □

Let $\omega \in E \setminus K$ be one of the roots of ϕ. Let $D := E^2L = L(\omega^2)$. Then D is the splitting field of the polynomial $\phi'(x) \equiv q(d)^2x^2 + x + q(e)^2$ over L. For both extensions E/K and D/L, we will denote the norm by N and the non-trivial element of the Galois group by $x \mapsto \overline{x}$.

We can consider E as a 2-dimensional vector space over K, and D as a 2-dimensional vector space over L. Let $B_0 := E \oplus E$, and let $A_0 := D \oplus D$. Then B_0 is a 4-dimensional vector space over K, and A_0 is a 4-dimensional vector space over L. We can identify B and A with B_0 and A_0, respectively, by the following relations.

$$\begin{aligned} t_1d + t_2e + t_3d\xi + t_4e\xi &\quad\longleftrightarrow\quad (t_1 + t_2\omega, t_3 + t_4\omega) \\ s_1\xi + s_2\xi ed^{-1} + s_3\xi d^{-1} + s_4\beta^2\xi e &\quad\longleftrightarrow\quad (s_1 + s_2\omega^2, s_3 + s_4\omega^2) \end{aligned}$$

Since $R = [L]$ and $Y = [K]$, we have actually identified V and W with $B_0 \oplus L$ and $A_0 \oplus K$, respectively:

$$\begin{aligned} t_1d + t_2e + t_3d\xi + t_4e\xi + [s] &\quad\longleftrightarrow\quad (t_1 + t_2\omega, t_3 + t_4\omega, s) \\ s_1\xi + s_2\xi ed^{-1} + s_3\xi d^{-1} + s_4\beta^2\xi e + [t] &\quad\longleftrightarrow\quad (s_1 + s_2\omega^2, s_3 + s_4\omega^2, t) \end{aligned}$$

For all $(b, s) \in B_0 \oplus L$ and all $(a, t) \in A_0 \oplus K$, we will denote the corresponding elements of V and W by $[b, s]$ and $[a, t]$, respectively.

We can now describe the quadratic forms q and $\hat{q}$ on $B_0 \oplus L$ and $A_0 \oplus K$, respectively, via this identification.

THEOREM 8.103. *For all $u, v \in E$, $s \in L$, $x, y \in D$ and $t \in K$, we have that*

(i) $q[u, v, s] = \beta^{-1}(N(u) + \alpha N(v)) + s$;
(ii) $\hat{q}[x, y, t] = \alpha(N(x) + \beta^2 N(y)) + t^2$.

PROOF. Let $u = t_1 + t_2\omega$ and $v = t_3 + t_4\omega$ be arbitrary elements of E, and let s be an arbitrary element of L. Then we have that

$$\begin{aligned} q[u, v, s] &= q[t_1 + t_2\omega, t_3 + t_4\omega, s] \\ &= q(t_1d + t_2e + t_3d\xi + t_4e\xi + [s]) \\ &= q(t_1d + t_2e) + q(t_3d\xi + t_4e\xi) + q[s] \ , \end{aligned}$$

since $f(t_1d + t_2e, t_3d\xi + t_4e\xi) = 0$ and $[s] \in \mathrm{Rad}(f)$. By Lemma 8.95(ii) and Lemma 8.97(i), $[q(v\xi)] = \alpha[q(v)] = [\alpha q(v)]$, and hence $q(v\xi) = \alpha q(v)$ for all $v \in V$. It follows that

$$\begin{aligned} q[u, v, s] &= q(t_1d + t_2e) + \alpha q(t_3d + t_4e) + q[s] \\ &= q(t_1d) + f(t_1d, t_2e) + q(t_2e) + \alpha(q(t_3d) + f(t_3d, t_4e) + q(t_4e)) + q[s] \\ &= t_1^2q(d) + t_1t_2 + t_2^2q(e) + \alpha(t_3^2q(d) + t_3t_4 + t_4^2q(e)) + s \\ &= q(d)N(t_1 + t_2\omega) + \alpha q(d)N(t_3 + t_4\omega) + s \\ &= \beta^{-1}(N(u) + \alpha N(v)) + s \ , \end{aligned}$$

which proves (i). Similarly, let $x = s_1 + s_2\omega^2$ and $y = s_3 + s_4\omega^2$ be arbitrary elements of D, and let t be an arbitrary element of K. Then we have that

$$\begin{aligned} \hat{q}[x, y, t] &= \hat{q}[s_1 + s_2\omega^2, s_3 + s_4\omega^2, t] \\ &= \hat{q}(s_1\xi + s_2\xi ed^{-1} + s_3\xi d^{-1} + s_4\beta^2\xi e + [t]) \ . \end{aligned}$$

Note that $\hat{f}(\xi ed^{-1}, \xi d^{-1}) = \hat{f}(\xi e, \xi d^{-1}d^{-1}) = \hat{q}[q(d)]^{-1}\hat{f}(\xi e, \xi) = 0$ by Lemma 8.91(iii), hence $\hat{f}(s_1\xi + s_2\xi ed^{-1}, s_3\xi d^{-1} + s_4\beta^2\xi e) = 0$. Since $[t] \in \mathrm{Rad}(\hat{f})$, it thus follows that

$$\begin{aligned}\hat{q}[x,y,t] &= \hat{q}(s_1\xi + s_2\xi e d^{-1}) + \hat{q}(s_3\xi d^{-1} + s_4\beta^2\xi e) + \hat{q}[t] \\ &= \hat{q}(s_1\xi) + \hat{f}(s_1\xi, s_2\xi e d^{-1}) + \hat{q}(s_2\xi e d^{-1}) \\ &\qquad + \hat{q}(s_3\xi d^{-1}) + \hat{f}(s_3\xi d^{-1}, s_4\beta^2\xi e) + \hat{q}(s_4\beta^2\xi e) + \hat{q}[t] \\ &= s_1^2\hat{q}(\xi) + s_1 s_2\hat{f}(\xi, \xi e d^{-1}) + s_2^2\hat{q}(\xi e d^{-1}) \\ &\qquad + s_3^2\hat{q}(\xi d^{-1}) + s_3 s_4\beta^2\hat{f}(\xi, \xi e d^{-1}) + s_4^2\beta^4\hat{q}(\xi e) + \hat{q}[t] \ .\end{aligned}$$

By Lemma 8.95(i) and Lemma 8.97(ii), $[\hat{q}(wv)] = q(v)[\hat{q}(w)] = [q(v)^2\hat{q}(w)]$, and hence $\hat{q}(wv) = q(v)^2\hat{q}(w)$ for all $v \in V$ and all $w \in W$. Remember that $\hat{f}(\xi, \xi e d^{-1}) = \alpha\beta^2$ and that $q(d^{-1}) = q(d)^{-1} = \beta$. Since $\hat{q}[t] = q(\epsilon[t]) = q(t\epsilon) = t^2$, it thus follows that

$$\begin{aligned}\hat{q}[x,y,t] &= s_1^2\alpha + s_1 s_2\alpha\beta^2 + s_2^2 q(e)^2 q(d)^{-2}\alpha \\ &\qquad + s_3^2 q(d)^{-2}\alpha + s_3 s_4\beta^2\alpha\beta^2 + s_4^2\beta^4 q(e)^2\alpha + t^2 \\ &= \alpha(s_1^2 + s_1 s_2 q(d)^{-2} + s_2^2 q(e)^2 q(d)^{-2} \\ &\qquad + \beta^2(s_3^2 + s_3 s_4 q(d)^{-2} + s_4^2 q(e)^2 q(d)^{-2})) + t^2 \\ &= \alpha(N(s_1 + s_2\omega^2) + \beta^2 N(s_3 + s_4\omega^2)) + t^2 \\ &= \alpha(N(x) + \beta^2 N(y)) + t^2 \ ,\end{aligned}$$

which proves (ii). □

For all $a \in A_0$ and all $b \in B_0$, we let $q_1(b) := q[b,0]$ and $q_2(a) := \hat{q}[a,0]$. Denote the corresponding bilinear forms by f_1 and f_2, respectively. We now define maps $\tilde{\Upsilon}$, $\tilde{\nu}$, $\tilde{\Theta}$ and $\tilde{\psi}$ from $A_0 \times B_0$ to A_0, K, B_0 and L, respectively, by setting

$$\begin{aligned}[a,0][b,0] &= [\tilde{\Upsilon}(a,b), \tilde{\nu}(a,b)] \ , \\ [b,0][a,0] &= [\tilde{\Theta}(a,b), \tilde{\psi}(a,b)] \ ,\end{aligned}$$

for all $a \in A_0$ and all $b \in B_0$. We will show that these maps coincide with the maps Υ, ν, Θ and ψ defined on page 44.

Lemma 8.104. $\tilde{\Upsilon} \equiv \Upsilon$.

Proof. All the equivalences in the proof of this lemma are modulo Y. Let

$$\begin{aligned} a_1 &:= \xi \ , & b_1 &:= d \ , \\ a_2 &:= \xi e d^{-1} \ , & b_2 &:= e \ , \\ a_3 &:= \xi d^{-1} \ , & b_3 &:= d\xi \ , \\ a_4 &:= \beta^2\xi e \ , & b_4 &:= e\xi \ ,\end{aligned}$$

and let $a_{ij} := a_i b_j$ for all $i, j \in \{1,2,3,4\}$. We first observe that $\xi de + \xi ed \equiv f(d,e)^2\xi \equiv \xi$ and that $\xi d^{-1}e + \xi e d^{-1} \equiv f(d^{-1}, e)^2\xi \equiv \beta^2\xi$ by Lemma 8.93(i). Then

$$\begin{aligned} a_{11} &\equiv \xi \cdot d \equiv \beta^{-2}a_3 \ ; \\ a_{12} &\equiv \xi \cdot e \equiv \beta^{-2}a_4 \ ; \\ a_{13} &\equiv \xi \cdot d\xi \equiv \alpha\xi d \equiv \alpha\beta^{-2}a_3 \ ; \\ a_{14} &\equiv \xi \cdot e\xi \equiv \alpha\xi e \equiv \alpha\beta^{-2}a_4 \ ;\end{aligned}$$

$a_{21} \equiv \xi ed^{-1} \cdot d \equiv \xi e \equiv \beta^{-2} a_4$;
$a_{22} \equiv \xi ed^{-1} \cdot e \equiv (\xi d^{-1} e + \beta^2 \xi) \cdot e \equiv q(e)^2 a_3 + a_4$;
$a_{23} \equiv \xi ed^{-1} \cdot d\xi \equiv \xi e \cdot d\xi \cdot d^{-1} \equiv \xi \cdot d\xi \cdot ed^{-1} \equiv \alpha \xi ded^{-1} \equiv \alpha(\xi + \xi ed) d^{-1} \equiv \alpha a_3 + \alpha \beta^{-2} a_4$;
$a_{24} \equiv \xi ed^{-1} \cdot e\xi \equiv \xi e \cdot e\xi \cdot d^{-1} \equiv \xi \cdot e\xi \cdot ed^{-1} \equiv \alpha \xi eed^{-1} \equiv \alpha q(e)^2 a_3$;
$a_{31} \equiv \xi d^{-1} \cdot d \equiv a_1$;
$a_{32} \equiv \xi d^{-1} \cdot e \equiv \beta^2 \xi + \xi ed^{-1} \equiv \beta^2 a_1 + a_2$;
$a_{33} \equiv \xi d^{-1} \cdot d\xi \equiv \xi \cdot d\xi \cdot d^{-1} \equiv \alpha \xi dd^{-1} \equiv \alpha a_1$;
$a_{34} \equiv \xi d^{-1} \cdot e\xi \equiv \xi \cdot e\xi \cdot d^{-1} \equiv \alpha \xi ed^{-1} \equiv \alpha a_2$;
$a_{41} \equiv \beta^2 \xi e \cdot d \equiv q(d)^2 \beta^2 \xi ed^{-1} \equiv a_2$;
$a_{42} \equiv \beta^2 \xi e \cdot e \equiv \beta^2 q(e)^2 a_1$;
$a_{43} \equiv \beta^2 \xi e \cdot d\xi \equiv \beta^2 \xi \cdot d\xi \cdot e \equiv \beta^2 \alpha \xi de \equiv \alpha \beta^2 (\xi + \xi ed) \equiv \alpha \beta^2 a_1 + \alpha a_2$;
$a_{44} \equiv \beta^2 \xi e \cdot e\xi \equiv \beta^2 \xi \cdot e\xi \cdot e \equiv \beta^2 \alpha \xi ee \equiv \alpha \beta^2 q(e)^2 a_1$.

Hence

$$\begin{aligned}
&\tilde\Upsilon\big((1,0),(1,0)\big) = (0, \beta^{-2}) \ ; && \tilde\Upsilon\big((0,1),(1,0)\big) = (1,0) \ ; \\
&\tilde\Upsilon\big((1,0),(\omega,0)\big) = (0, \beta^{-2}\omega^2) \ ; && \tilde\Upsilon\big((0,1),(\omega,0)\big) = (\beta^2+\omega^2,0) \ ; \\
&\tilde\Upsilon\big((1,0),(0,1)\big) = (0, \alpha\beta^{-2}) \ ; && \tilde\Upsilon\big((0,1),(0,1)\big) = (\alpha,0) \ ; \\
&\tilde\Upsilon\big((1,0),(0,\omega)\big) = (0, \alpha\beta^{-2}\omega^2) \ ; && \tilde\Upsilon\big((0,1),(0,\omega)\big) = (\alpha\omega^2,0) \ ;
\end{aligned}$$

$$\begin{aligned}
&\tilde\Upsilon\big((\omega^2,0),(1,0)\big) = (0, \beta^{-2}\omega^2) \ ; && \tilde\Upsilon\big((0,\omega^2),(1,0)\big) = (\omega^2,0) \ ; \\
&\tilde\Upsilon\big((\omega^2,0),(\omega,0)\big) = (0, q(e)^2+\omega^2) \ ; && \tilde\Upsilon\big((0,\omega^2),(\omega,0)\big) = (\beta^2 q(e)^2,0) \ ; \\
&\tilde\Upsilon\big((\omega^2,0),(0,1)\big) = (0, \alpha+\alpha\beta^{-2}\omega^2) \ ; && \tilde\Upsilon\big((0,\omega^2),(0,1)\big) = (\alpha\beta^2+\alpha\omega^2,0) \ ; \\
&\tilde\Upsilon\big((\omega^2,0),(0,\omega)\big) = (0, \alpha q(e)^2) \ ; && \tilde\Upsilon\big((0,\omega^2),(0,\omega)\big) = (\alpha\beta^2 q(e)^2,0) \ .
\end{aligned}$$

Since $\omega^2 = \beta\omega + \beta q(e)^2$ and $\overline{\omega} = \omega + \beta$, it is now straightforward to check that $\tilde\Upsilon$ coincides with the map Υ defined on page 44 on the set

$$\{(1,0),(\omega^2,0),(0,1),(0,\omega^2)\} \times \{(1,0),(\omega,0),(0,1),(0,\omega)\} \ .$$

By ($\mathbf{Q}_3$) and ($\mathbf{Q}_{11}$), the map $\tilde\Upsilon$ is additive in both variables. Since $(sw)v = s(wv)$ for all $s \in L$, $v \in V$ and $w \in W$, it follows that $\tilde\Upsilon(sa,b) = s\tilde\Upsilon(a,b)$ for all $s \in L$, $a \in A_0$ and $b \in B_0$. By Lemma 8.91(i), we have that $w(tv) = \hat q[t]wv = t^2wv$ for all $t \in K$, $v \in V$ and $w \in W$, and hence $\tilde\Upsilon(a,tb) = t^2\tilde\Upsilon(a,b)$ for all $t \in K$, $a \in A_0$ and $b \in B_0$. Since the same properties hold for Υ, and since $A_0 = \langle(1,0),(\omega^2,0),(0,1),(0,\omega^2)\rangle$ and $B_0 = \langle(1,0),(\omega,0),(0,1),(0,\omega)\rangle$, it thus follows that $\tilde\Upsilon \equiv \Upsilon$. □

LEMMA 8.105. *For all $a \in A_0$ and all $b, b' \in B_0$, we have that*
(i) $q_2(\tilde\Upsilon(a,b)) = q_1(b)^2 q_2(a) + \tilde\nu(a,b)^2$;
(ii) $\tilde\nu(a,b+b') = \tilde\nu(a,b) + \tilde\nu(a,b') + f_1(\tilde\Theta(a,b),b')$;
(iii) $q_1(\tilde\Theta(a,b)) = q_2(a)q_1(b) + \tilde\psi(a,b)$.

PROOF. By Lemma 8.95(i) and Lemma 8.97(ii), $[\hat q(wv)] = q(v)[\hat q(w)] = [q(v)^2\hat q(w)]$, and hence $\hat q(wv) = q(v)^2\hat q(w)$ for all $v \in V$ and all $w \in W$. If we choose $v = [b,0]$

and $w = [a, 0]$, then we get that

$$\hat{q}[\tilde{\Upsilon}(a,b), \tilde{\nu}(a,b)] = q[b,0]^2 \hat{q}[a,0] \ .$$

Hence $q_2(\tilde{\Upsilon}(a,b)) + \tilde{\nu}(a,b)^2 = q_1(b)^2 q_2(a)$, which proves (i).

Similarly, it follows from Lemma 8.95(ii) and Lemma 8.97(i) that $q(vw) = \hat{q}(w)q(v)$ for all $v \in V$ and all $w \in W$. It follows that

$$q(\tilde{\Theta}(a,b), \tilde{\psi}(a,b)) = \hat{q}[a,0]q[b,0] \ .$$

Hence $q_1(\tilde{\Theta}(a,b)) + \tilde{\psi}(a,b) = q_2(a)q_1(b)$, which proves (iii).

Finally, it follows from $(\mathbf{Q}_{11})$ that

$$[a,0] \cdot [b+b',0] = [a,0] \cdot [b,0] + [a,0] \cdot [b',0] + F([b,0] \cdot [a,0], [b',0]) \ .$$

Projecting this identity on $Y = [0, K]$ yields

$$\begin{aligned} \nu(a, b+b') &= \nu(a,b) + \nu(a,b') + f([\tilde{\Theta}(a,b), \tilde{\psi}(a,b)], [b',0]) \\ &= \nu(a,b) + \nu(a,b') + f_1(\tilde{\Theta}(a,b), b') \ , \end{aligned}$$

which proves (ii). □

Theorem 8.106. *$\tilde{\Upsilon} \equiv \Upsilon$, $\tilde{\nu} \equiv \nu$, $\tilde{\Theta} \equiv \Theta$ and $\tilde{\psi} \equiv \psi$.*

Proof. We have already shown in Lemma 8.104 that $\tilde{\Upsilon} \equiv \Upsilon$. It then follows from Lemma 8.105(i) and Theorem 7.12(xii) that $\tilde{\nu} \equiv \nu$. Hence, by Lemma 8.105(ii) and Theorem 7.12(i), we have that $f_1(\tilde{\Theta}(a,b) - \Theta(a,b), b') = 0$ for all $a \in A_0$ and all $b, b' \in B_0$, from which it follows that $\tilde{\Theta}(a,b) - \Theta(a,b) \in \mathrm{Rad}(f_1)$ for all $a \in A_0$ and all $b \in B_0$. Since $B \cap \mathrm{Rad}(f) = B \cap R = 0$ by Theorem 8.99, we have that $B_0 \cap \mathrm{Rad}(f_1) = 0$ as well, and hence $\tilde{\Theta} \equiv \Theta$. Finally, it then follows from Lemma 8.105(iii) and Theorem 7.12(xi) that $\tilde{\psi} \equiv \psi$. □

Theorem 8.107. *$(V, W, \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_F(K, V_0, q)$.*

Proof. First of all, observe that q is indeed a quadratic form of type F_4, since its regular component q_1 has a norm splitting $q_1(u,v) = \beta^{-1}N(u) + \beta^{-1}\alpha N(v)$, and the product of the coefficients of the norm splitting is $\beta^{-1} \cdot \beta^{-1}\alpha = \beta^{-2}\alpha$, which is an element of L.

Let ϕ be the isomorphism from $[V_0] = [B_0 \oplus L]$ to V which maps $[b, s]$ to $[b, s]$ for all $b \in B_0$ and all $s \in L$, and let ψ be the isomorphism from $[W_0] = [A_0 \oplus K]$ to W which maps $[a, t]$ to $[a, t]$ for all $a \in A_0$ and all $t \in K$. Then $\phi([0, 1]) = [0, 1] = [1]_V = \epsilon$ and $\psi([0, 1]) = [0, 1] = [1]_W = \delta$.

Since ϕ and ψ are identity maps, it now only remains to show that it follows from the relations

$$\begin{aligned} [a,0][b,0] &= [\tilde{\Upsilon}(a,b), \tilde{\nu}(a,b)] \ , \\ [b,0][a,0] &= [\tilde{\Theta}(a,b), \tilde{\psi}(a,b)] \ , \end{aligned}$$

for all $a \in A_0$ and all $b \in B_0$ that

$$\begin{aligned} [a,t][b,s] &= [\tilde{\Upsilon}(a,b) + sa, \tilde{\nu}(a,b) + q[b,s]t] \ , \\ [b,s][a,t] &= [\tilde{\Theta}(a,b) + tb, \tilde{\psi}(a,b) + \hat{q}[a,t]s] \ , \end{aligned}$$

for all $a \in A_0$ and all $b \in B_0$. We will only show the first identity, the second one being completely similar. Since $[0, s] \in \mathrm{Rad}(F)$, it follows from $(\mathbf{Q}_{11})$ and Lemma 8.51 that

$$\begin{aligned}[a,t][b,s] &= [a,0][b,s] + [0,t][b,s] \\ &= [a,0][b,0] + [a,0][0,s] + [0,t][b,s] \\ &= [\tilde{\Upsilon}(a,b), \tilde{\nu}(a,b)] + [sa,0] + [0, tq[b,s]] \\ &= [\tilde{\Upsilon}(a,b) + sa, \tilde{\nu}(a,b) + q[b,s]t] \ .\end{aligned}$$

Since ϕ and ψ are identity maps, it is now obvious that $\phi([b,s][a,t]) = \phi([b,s])\psi([a,t])$ and $\psi([a,t][b,s]) = \psi([a,t])\phi([b,s])$ for all $(a,t) \in W_0$ and all $(b,s) \in V_0$; hence (ϕ, ψ) is an isomorphism from $\Omega_F(K, V_0, q)$ to $(V, W, \tau_V, \tau_W, \epsilon, \delta)$. □

8.6. Quadrangular Systems of Pseudo-quadratic Form Type, II

In this section, we continue to assume that $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a wide quadrangular system which is the extension of a quadrangular system Λ of quadratic form type, i.e. $\Lambda = (V, \mathrm{Rad}(H), \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_Q(K, V_0, q)$, where τ_V and τ_W are as in Remark 8.3.

Our goal in this and the next section is to classify these quadrangular systems if $\mathrm{Rad}(F) = 0$. So assume that $\mathrm{Rad}(F) = 0$. We continue to identify V and V_0 if there is no danger of confusion.

LEMMA 8.108. *For all $v \in V$, all $w \in W$ and all $t \in K$, we have that $(tv)w = t(vw)$. It follows that the notation tvw is unambiguous.*

PROOF. If we substitute $[t]$ for z, $\overline{v}$ for v and $\boxminus w$ for w in $(\mathbf{Q}_{23})$, then we get, since $\Pi_w([t]) = [t]$, that

$$t\overline{v} = -\overline{\overline{(t \cdot \overline{vw})}\kappa(\boxminus w)} \ ,$$

and hence, by Lemma 8.50(ii), that

$$\overline{tv} = -\overline{(t \cdot vw)\kappa(\boxminus w)} \ .$$

It follows that

$$tv \cdot w = -(t \cdot vw)\kappa(\boxminus w) \cdot w \ ,$$

and hence, by $(\mathbf{Q}_{18})$, that $tv \cdot w = t \cdot vw$, which is what we had to show. □

DEFINITION 8.109. If $\mathrm{char}(K) \neq 2$, let $\zeta := \epsilon/2$. If $\mathrm{char}(K) = 2$, define $S_1 := \{v \in V \mid F(\epsilon, v) \neq 0\}$ and $S_2 := \{\epsilon w \mid w \in W\}$. If $S_1 \cap S_2 \neq \emptyset$, choose a fixed element $z \in S_1 \cap S_2$; if $S_1 \cap S_2 = \emptyset$, choose a fixed element $z \in S_1$. Observe that $S_1 \neq \emptyset$ since $\epsilon \notin \mathrm{Rad}(F)$. In both cases, we let $\zeta := f(\epsilon, z)^{-1} z$.

It follows that $f(\epsilon, \zeta) = 1$, independent of the characteristic.

REMARK 8.110. This somewhat strange definition will become clear in section 8.7.

DEFINITION 8.111. An element $w \in W$ is called *ζ-orthogonal* if and only if $f(\zeta, \epsilon w) = 0$.

LEMMA 8.112. *Each coset of Y in W contains a unique ζ-orthogonal element.*

Proof. Consider an arbitrary coset $w \boxplus Y$ of Y in W (where $w \in W$). An arbitrary element of this coset, say $w \boxplus [t]$ with $t \in K$, is ζ-orthogonal if and only if $f(\zeta, \epsilon(w \boxplus [t])) = 0$. By ($\mathbf{Q}_{12}$), $f(\zeta, \epsilon(w \boxplus [t])) = f(\zeta, \epsilon w + \epsilon[t]) = f(\zeta, \epsilon w) + f(\zeta, t\epsilon) = f(\zeta, \epsilon w) + t$, hence $w \boxplus [t]$ is ζ-orthogonal if and only if $t = -f(\zeta, \epsilon w)$. □

Since $Y = \mathrm{Rad}(H)$ is a normal subgroup of W, we can define the quotient group $X := W/Y$. Since $[W, W]_{\boxplus} \le Y$, the group X is abelian. We will use the additive notations $+$ and $-$ for X.

Definition 8.113. We define a map $\rho : X \to W$ as follows. For each element $w \boxplus Y \in X$, we let $\rho(w \boxplus Y)$ be the unique element $w \boxplus y \in w \boxplus Y \subseteq W$ which is ζ-orthogonal; see Lemma 8.112. Moreover, for all $x \in X$ and all $t \in K$, we let $(x, t) := \rho(x) \boxplus [t] \in W$. Note that $\rho(x) \in x$ for all $x \in X$, and hence $(x, t) \in x$ for all $x \in X$ and all $t \in K$ as well.

Lemma 8.114. *For all $w \in W$, there exist unique elements $x \in X$ and $t \in K$ such that $w = (x, t)$.*

Proof. Let $w \in W$ be arbitrary. Let $x := w \boxplus Y \in X$, and let

$$y := \boxminus\rho(x) \boxplus w \in \boxminus(w \boxplus Y) \boxplus w = \boxminus Y \boxminus w \boxplus w = Y \ .$$

Hence $y = [t]$ for some $t \in K$, and we thus have that $(x, t) = \rho(x) \boxplus [t] = \rho(x) \boxplus y = w$.

Now suppose that $(x_1, t_1) = (x_2, t_2)$ for some $x_1, x_2 \in X$ and some $t_1, t_2 \in K$. Since $(x_1, t_1) \in x_1$ and $(x_2, t_2) \in x_2$, it follows that the cosets x_1 and x_2 have an element in common, and hence they are equal, i.e. $x_1 = x_2$. It then follows from $(x_1, t_1) = (x_2, t_2)$ that $t_1 = t_2$ as well. □

Definition 8.115. We define a map $G : X \times X \to W$ by setting

$$G(a, b) := \boxminus\rho(a + b) \boxplus \rho(a) \boxplus \rho(b)$$

for all $a, b \in X$. Note that a and b are cosets of Y in W. It follows that $G(a, b) \in -(a + b) + a + b = Y$. Hence we can define a map $g : X \times X \to K$ by setting $G(a, b) = [g(a, b)]$ for all $a, b \in X$.

Lemma 8.116. *$(a, t) \boxplus (b, s) = (a + b, t + s + g(a, b))$ for all $a, b \in X$ and all $t, s \in K$.*

Proof. Since $Y \subseteq Z(W)$ by ($\mathbf{Q}_8$), we have that

$$\begin{aligned}(a, t) \boxplus (b, s) &= \rho(a) \boxplus [t] \boxplus \rho(b) \boxplus [s] \\ &= \rho(a) \boxplus \rho(b) \boxplus [t + s] \\ &= \rho(a + b) \boxplus [g(a, b)] \boxplus [t + s] \\ &= (a + b, g(a, b) + t + s) \ ,\end{aligned}$$

which is what we had to show. □

Definition 8.117. We define a map θ from $X \times V$ to V, a map π from X to V and a map h from $X \times X$ to V by setting

$$\begin{aligned}\theta(a, v) &:= v \cdot (a, 0) \ , \\ \pi(a) &:= \theta(a, \epsilon) = \epsilon \cdot (a, 0) \ , \\ h(a, b) &:= H\big((a, 0), (b, 0)\big) \ ,\end{aligned}$$

for all $a, b \in X$ and all $v \in V$.

By definition, $(a, 0)$ is ζ-orthogonal for all $a \in X$, hence $f(\pi(a), \zeta) = f(\epsilon \cdot (a, 0), \zeta) = 0$ for all $a \in X$. Furthermore, it follows from ($\mathbf{Q}_{12}$) that $v \cdot (a, t) = v \cdot (a, 0) + v \cdot [t] = \theta(a, v) + tv$ for all $v \in V$, all $a \in X$ and all $t \in K$.

Definition 8.118. We define a map $(a, v) \mapsto av$ from $X \times V$ to X and a map φ from $X \times V$ to K by the relation

$$(a, 0) \cdot v = (av, \varphi(a, v)) \ .$$

Since we did not define a multiplication yet between elements of X and elements of V, this will not cause confusion.

Note that it follows from $(a, 0) \cdot \epsilon = (a, 0)$ that $a\epsilon = a$ and $\varphi(a, \epsilon) = 0$ for all $a \in X$. Furthermore, we have that $(a, t) \cdot v = (av, tq(v) + \varphi(a, v))$ by Lemma 8.51, and that $H\big((a, t), (b, s)\big) = h(a, b)$, for all $a, b \in X$, all $v \in V$ and all $t, s \in K$.

Lemma 8.119. *For all $a \in X$, we have that*

$$g(a, -a) = g(-a, a) = f(\pi(a), \epsilon) = f(\pi(-a), \epsilon) \ .$$

Proof. Let $w := (a, 0) \in W$. Then w is ζ-orthogonal. By 3.13(i), we have that $w(-\epsilon) = [f(\epsilon w, \epsilon)] \boxminus w$. By ($\mathbf{Q}_6$), $f(\epsilon \cdot w(-\epsilon), \zeta) = f(\epsilon w, \zeta) = 0$, and hence $w(-\epsilon)$ is ζ-orthogonal as well. It follows that $w(-\epsilon) = (b, 0)$ for some $b \in X$. Since $[f(\epsilon w, \epsilon)] \in Y$, we now have that $b = w(-\epsilon) \boxplus Y = [f(\epsilon w, \epsilon)] \boxminus w \boxplus Y = \boxminus w \boxplus Y = -a$. It follows by Lemma 8.116 that

$$\begin{aligned} [f(\pi(a), \epsilon)] &= [f(\epsilon w, \epsilon)] \\ &= w(-\epsilon) \boxplus w \\ &= (-a, 0) \boxplus (a, 0) \\ &= (-a + a, 0 + 0 + g(-a, a)) \\ &= [g(-a, a)] \ . \end{aligned}$$

Hence $f(\pi(a), \epsilon) = g(-a, a)$, and since it follows from ($\mathbf{Q}_6$) that

$$\begin{aligned} f(\pi(-a), \epsilon) &= f(\epsilon \cdot (-a, 0), \epsilon) \\ &= f(\epsilon \cdot (a, 0)(-\epsilon), \epsilon) \\ &= f(\epsilon \cdot (a, 0), \epsilon) \\ &= f(\pi(a), \epsilon) \ , \end{aligned}$$

we conclude that $g(a, -a) = g(-a, a) = f(\pi(a), \epsilon) = f(\pi(-a), \epsilon)$. □

Definition 8.120. We define a map $(t, a) \mapsto ta$ from $K \times X$ to X by setting $ta := a \cdot t\epsilon$ for all $t \in K$ and all $a \in X$. We will prove later on (see Theorem 8.123) that this makes X into a vector space over K.

Lemma 8.121. *For all $a \in X$ and all $t \in K$, we have that $\varphi(a, t\epsilon) = 0$. Moreover, for all $a \in X$, all $v \in V$ and all $t \in K$, we have that*

(i) $ta \cdot v = a \cdot tv = t \cdot av$;
(ii) $\varphi(ta, v) = \varphi(a, tv) = t^2 \varphi(a, v)$.

Proof. Let $w := (a, 0) \in W$ and let $y := [t] \in Y$. Since $\Pi_{\boxminus y}(z) = z$ for all $z \in W$, it follows from ($\mathbf{Q}_{26}$) that $w \cdot \epsilon y \cdot v = w \cdot vy$, for all $v \in V$. It thus follows from ($\mathbf{Q}_{11}$) that

$$\begin{aligned} F(\epsilon \cdot w(\epsilon y), \zeta) &= \boxminus w \cdot \epsilon y \cdot \epsilon \boxminus w \cdot \epsilon y \cdot \zeta \boxplus w \cdot \epsilon y \cdot (\zeta + \epsilon) \\ &= \boxminus w \cdot \epsilon y \boxminus w \cdot \zeta y \boxplus w \cdot (\zeta + \epsilon) y \\ &= F(\epsilon y \cdot w, \zeta y) \\ &= F(t\epsilon w, t\zeta) \\ &= [t^2 f(\epsilon w, \zeta)] \\ &= 0 \ , \end{aligned}$$

since w is ζ-orthogonal. It follows that $w \cdot \epsilon y$ is ζ-orthogonal as well. Since

$$w \cdot \epsilon y = w \cdot t\epsilon = (a, 0) \cdot t\epsilon = (a \cdot t\epsilon, \varphi(a, t\epsilon)) \ ,$$

it follows that $\varphi(a, t\epsilon) = 0$.

It now follows from $w \cdot \epsilon y \cdot v = w \cdot vy$ that $(a \cdot t\epsilon, 0) \cdot v = (a, 0) \cdot tv$ for all $v \in V$, and hence

$$(ta \cdot v, \varphi(ta, v)) = (ta, 0) \cdot v = (a \cdot t\epsilon, 0) \cdot v = (a, 0) \cdot tv = (a \cdot tv, \varphi(a, tv))$$

for all $v \in V$. This implies that $ta \cdot v = a \cdot tv$ and $\varphi(ta, v) = \varphi(a, tv)$.

Now observe that $\pi_{t\epsilon}(c) = \pi_\epsilon(c) = -\bar{c}$, for all $c \in V$. If we substitute $t\epsilon$ for v and v for c in ($\mathbf{Q}_{24}$), we thus get that $wv = w \cdot t\epsilon \cdot v \cdot (t\epsilon)^{-1}$, and hence

$$(a, 0) \cdot v \cdot t\epsilon = (a, 0) \cdot t\epsilon \cdot v \ .$$

It follows that

$$(av, \varphi(a, v)) \cdot t\epsilon = (ta, 0) \cdot v \ ,$$

and finally, since $q(t\epsilon) = t^2$, that

$$(t \cdot av, t^2 \varphi(a, v)) = (ta \cdot v, \varphi(ta, v)) \ ,$$

and we are done. □

LEMMA 8.122. *The map $(a, v) \mapsto av$ is additive in both variables. Moreover, the following hold for all $a, b \in X$ and all $u, v \in V$:*

(i) $\varphi(a + b, v) + g(a, b)q(v) = \varphi(a, v) + \varphi(b, v) + g(av, bv)$;

(ii) $\varphi(a, u + v) = \varphi(a, u) + \varphi(a, v) + g(av, au) + f(\theta(a, u), v)$.

PROOF. It follows from ($\mathbf{Q}_3$) that

$$\big((a, 0) \boxplus (b, 0)\big)v = (a, 0) \cdot v \boxplus (b, 0) \cdot v \ ,$$

and hence, by Lemma 8.116, that

$$(a + b, g(a, b)) \cdot v = (av, \varphi(a, v)) \boxplus (bv, \varphi(b, v)) \ ,$$

from which it follows that

$$\big((a + b)v, g(a, b)q(v) + \varphi(a + b, v)\big) = \big(av + bv, \varphi(a, v) + \varphi(b, v) + g(av, bv)\big) \ .$$

So we have shown that $(a + b)v = av + bv$ and that (i) holds.

On the other hand, it follows from ($\mathbf{Q}_{11}$) that

$$(a, 0) \cdot (u + v) = (a, 0) \cdot (v + u) = (a, 0) \cdot v \boxplus (a, 0) \cdot u \boxplus F(u \cdot (a, 0), v) \ ,$$

and hence

$$\begin{aligned}\big(a(u+v),\varphi(a,u+v)\big) &= (av,\varphi(a,v)) \boxplus (au,\varphi(a,u)) \boxplus [f(\theta(a,u),v)] \\ &= \big(av+au,\varphi(a,v)+\varphi(a,u)+g(av,au)+f(\theta(a,u),v)\big)\ .\end{aligned}$$

So we have shown that $a(u+v) = au + av$ and that (ii) holds. □

THEOREM 8.123. *X_0 is a vector space over K, with the scalar multiplication given by the map $(t,a) \mapsto ta = a \cdot t\epsilon$.*

PROOF. First of all, we have that $1a = a \cdot \epsilon = a$ for all $a \in X$. By Lemma 8.122, the two distributivity laws hold, since

$$t(a+b) = (a+b) \cdot t\epsilon = a \cdot t\epsilon + b \cdot t\epsilon = ta + tb$$

for all $t \in K$ and all $a, b \in X$, and

$$(s+t)a = a \cdot (s+t)\epsilon = a \cdot (s\epsilon + t\epsilon) = a \cdot s\epsilon + a \cdot t\epsilon = sa + ta$$

for all $s, t \in K$ and all $a \in X$. Finally, it follows from Lemma 8.121(i) that

$$st \cdot a = ts \cdot a = a \cdot (ts)\epsilon = a \cdot t(s\epsilon) = ta \cdot s\epsilon = s \cdot ta$$

for all $s, t \in K$ and all $a \in X$. □

LEMMA 8.124. *For all $a, b \in X$, all $u, v \in V$ and all $t \in K$, we have that*

(i) $\theta(ta,v) = t^2\theta(a,v)$ *;*
(ii) $\theta(a,tv) = t\theta(a,v)$ *;*
(iii) $\theta(a+b,v) + g(a,b)v = \theta(a,v) + \theta(b,v) + h(b,av)$ *;*
(iv) $\theta(a,u+v) = \theta(a,u) + \theta(a,v)$ *.*

PROOF. Let $w := (a,0) \in W$. Note that $\pi_{t\epsilon}(c) = \pi_\epsilon(c) = -\bar{c}$, for all $c \in V$. It thus follows by substituting $t\epsilon$ for v and v for c in $(\mathbf{Q}_{25})$ that $v \cdot \delta(t\epsilon) \cdot w = v \cdot w(t\epsilon)$. Hence

$$\begin{aligned}\theta(ta,v) &= v \cdot (ta,0) = v \cdot w(t\epsilon) = v \cdot \delta(t\epsilon) \cdot w \\ &= v \cdot [q(t\epsilon)] \cdot w = t^2 vw = t^2 v \cdot (a,0) = t^2\theta(a,v)\ ,\end{aligned}$$

which proves (i). Since $t \cdot vw = tv \cdot w$, we have that $t\theta(a,v) = \theta(a,tv)$, which proves (ii).

It follows from $(\mathbf{Q}_{12})$ that

$$\begin{aligned}\theta(a+b,v) + g(a,b)v &= v \cdot (a+b, g(a,b)) \\ &= v \cdot \big((a,0) \boxplus (b,0)\big) \\ &= v \cdot (a,0) + v \cdot (b,0) + H\big((b,0),(a,0) \cdot v\big) \\ &= \theta(a,v) + \theta(b,v) + H\big((b,0),(av,\varphi(a,v))\big) \\ &= \theta(a,v) + \theta(b,v) + h(b,av)\ ,\end{aligned}$$

which shows (iii). Finally, it follows from $(\mathbf{Q}_4)$ that

$$\begin{aligned}\theta(a,u+v) &= (u+v) \cdot (a,0) \\ &= u \cdot (a,0) + v \cdot (a,0) \\ &= \theta(a,u) + \theta(a,v)\ ,\end{aligned}$$

which proves (iv). □

LEMMA 8.125. *For all $a, b \in X$ and all $t \in K$, we have that $h(ta, b) = h(a, tb) = th(a, b)$.*

PROOF. If we substitute $t\epsilon$ for v in Lemma 8.124(iii), then we get, by Lemma 8.124(ii), that

$$\begin{aligned} h(b, a \cdot t\epsilon) &= \theta(a + b, t\epsilon) + g(a, b)t\epsilon - \theta(a, t\epsilon) - \theta(b, t\epsilon) \\ &= t\theta(a + b, \epsilon) + tg(a, b)\epsilon - t\theta(a, \epsilon) - t\theta(b, \epsilon) \\ &= th(b, a) \ , \end{aligned}$$

hence $h(b, ta) = th(b, a)$. It follows by ($\mathbf{Q}_{22}$) and Lemma 8.50(ii) that

$$h(ta, b) = -\overline{h(b, ta)} = -\overline{th(b, a)} = -t\overline{h(b, a)} = th(a, b)$$

as well, and we are done. □

LEMMA 8.126. *For all $a, b \in X$, we have that $f(h(a, b), \epsilon) = g(b, a) - g(a, b)$.*

PROOF. If we set $v = \epsilon$, $w_1 = (b, 0)$ and $w_2 = (a, 0)$ in ($\mathbf{Q}_8$), then we get that

$$\boxminus(b, 0) \boxminus (a, 0) \boxplus (b, 0) \boxplus (a, 0) = [f(h(a, b), \epsilon)] \ .$$

Since

$$\begin{aligned} \boxminus(b, 0) \boxminus (a, 0) \boxplus (b, 0) \boxplus (a, 0) &= \boxminus\big((a, 0) \boxplus (b, 0)\big) \boxplus \big((b, 0) \boxplus (a, 0)\big) \\ &= \boxminus(a + b, g(a, b)) \boxplus (a + b, g(b, a)) \\ &= (0, -g(a, b) + g(b, a)) \ , \end{aligned}$$

it follows that $f(h(a, b), \epsilon) = -g(a, b) + g(b, a)$. □

LEMMA 8.127. *For all $a, b \in X$ and all $v \in V$, we have that*

$$f(h(a, b), v) = f(h(av, b), \epsilon) = f(h(a, b\overline{v}), \epsilon) \ .$$

PROOF. It follows from ($\mathbf{Q}_8$) that

$$F(H(w_2, w_1), v) = [w_1, w_2 v]_{\boxplus} = F(H(w_2 v, w_1), \epsilon)$$

for all $w_1, w_2 \in W$. If we choose $w_2 = (a, 0)$ and $w_1 = (b, 0)$, then we get that $f(h(a, b), v) = f(h(av, b), \epsilon)$. It then follows from Lemma 8.54 that

$$\begin{aligned} f\big(h(a, b), v\big) &= f\big(\overline{h(a, b)}, \overline{v}\big) = -f\big(h(b, a), \overline{v}\big) \\ &= -f\big(h(b\overline{v}, a), \epsilon\big) = -f\big(\overline{h(b\overline{v}, a)}, \overline{\epsilon}\big) \\ &= f\big(h(a, b\overline{v}), \epsilon\big) \end{aligned}$$

as well. □

LEMMA 8.128. *We have that $av\overline{v} = q(v)a$ and $au\overline{v} + av\overline{u} = f(u, v)a$ for all $a \in X$ and all $u, v \in V$.*

PROOF. Let $w := (a, 0) \in W$. It then follows from ($\mathbf{Q}_{15}$) that $avv^{-1} = a$. Since $q(v)v^{-1} = \overline{v}$, it follows from Lemma 8.121(i) that $q(v)a = q(v)avv^{-1} = av \cdot q(v)v^{-1} = av\overline{v}$. It then follows that

$$\begin{aligned} f(u, v)a &= q(u + v)a - q(u)a - q(v)a \\ &= a(u + v)(\overline{u} + \overline{v}) - au\overline{u} - av\overline{v} \\ &= au\overline{v} + av\overline{u} \end{aligned}$$

as well. □

We have now come to a point which is very similar to Chapter 26 in [**20**]. For some of the remaining identities, we will thus simply refer to the appropriate place in [**20**]. Note that [**20**] uses δ where we use ζ.

LEMMA 8.129. *For all $a, b \in X$, we have that $g(a,b) = f(h(b,a), \zeta)$.*

PROOF. See [**20**, (26.20)]. $\square$

Since h is bilinear over K, it follows from Lemma 8.129 that g is bilinear over K.

LEMMA 8.130. *For all $a, b \in X$ and all $v \in V$, we have that*

$$h(a, bv) - h(b, av) = f(h(a,b), \epsilon)v \ .$$

PROOF. See [**20**, (26.23)]. $\square$

LEMMA 8.131. *If $\mathrm{char}(K) \neq 2$, then $\varphi \equiv 0$, and for all $a \in X$ and all $v \in V$, we have that*

(i) $g(a,a) = 0$;
(ii) $\theta(a,v) = \frac{1}{2}h(a,av)$.

PROOF. See [**20**, (26.24)]. $\square$

Note that it follows from Lemma 8.131(i) and the fact that g is bilinear over K that g is skew-symmetric if $\mathrm{char}(K) \neq 2$.

LEMMA 8.132. *If $\mathrm{char}(K) = 2$, then*

(i) $h(a, av) = g(a,a)v = f(\epsilon, \pi(a))v$;
(ii) $f(\theta(a,v), v) = g(av, av) = g(a,a)q(v) = f(\epsilon, \pi(a))q(v)$;
(iii) $f(\theta(a,u), v) = f(\theta(a,v), u) + f(\epsilon, \pi(a))f(u,v)$;

for all $a \in X$ and all $u, v \in V$.

PROOF. See [**20**, (26.25)]. $\square$

LEMMA 8.133. *For all $a \in X$ and all $u, v \in V$, we have that*

(i) $f(\theta(a,v), v) = f(\epsilon, \pi(a))q(v)$;
(ii) $f(\theta(a,v), u) + f(\theta(a,u), v) = f(\epsilon, \pi(a))f(u,v)$.

PROOF. See [**20**, (26.26)]. $\square$

LEMMA 8.134. *For all $a \in X$, all $u \in V$ and all $v \in V^*$, we have that*

$$\begin{aligned}\theta(av^{-1}, u) + \varphi(a, v^{-1})u = q(v)^{-1}\overline{\theta(a,\overline{u})} - f(u, v')\overline{\theta(a, v^{-1})} \\ - f(\overline{\theta(a,\overline{u})}, v)q(v)^{-1}v' + f(\overline{\theta(a, v^{-1})}, v)f(u, v')v' \ ,\end{aligned}$$

where $v' = \overline{v^{-1}} = q(v)^{-1}v$.

PROOF. Let $w := (a, 0) \in W$, and let $c := q(v)^{-1}\pi_v(u) \in V$. Since $\delta v = [q(v)]$, it follows by substituting u for c in $(\mathbf{Q}_{25})$ and by Lemma 8.52 that

$$\pi_v(\overline{u \cdot wv}) = q(v)\pi_v(\overline{u}) \cdot w \ .$$

Note that $\pi_{v^{-1}}(\overline{v_2}) = \overline{\pi_v(v_2)}$ for all $v_2 \in V$ by Lemma 3.18(i). If we replace v by v^{-1}, then it follows that

$$\overline{\pi_v(u \cdot wv^{-1})} = q(v^{-1})\overline{\pi_v(u)} \cdot w \ ,$$

and hence, since $q(v^{-1}) = q(v)^{-1}$, that

$$u \cdot wv^{-1} = \pi_v(\overline{\overline{c} \cdot w}) \ ,$$

from which it follows that

$$u \cdot (av^{-1}, \varphi(a, v^{-1})) = \pi_v(\overline{\theta(a, \overline{c})}) \ ,$$

and therefore

$$\theta(av^{-1}, u) + \varphi(a, v^{-1})u = \overline{\theta(a, \overline{c})} - f(v, \overline{\theta(a, \overline{c})})v' \ .$$

Since $\overline{c} = q(v)^{-1}\overline{\pi_v(u)} = q(v)^{-1}\overline{u} - q(v)^{-1}f(v, u)v^{-1} = q(v)^{-1}\overline{u} - f(u, v')v^{-1}$, it follows by Lemma 8.124 that

$$\begin{aligned} \theta(av^{-1}, u) + \varphi(a, v^{-1})u = q(v)^{-1}\overline{\theta(a, \overline{u})} - f(u, v')\overline{\theta(a, v^{-1})} \\ - q(v)^{-1}f(v, \overline{\theta(a, \overline{u})})v' + f(v, \overline{\theta(a, v^{-1})})f(u, v')v' \ , \end{aligned}$$

which is what we had to show. □

We now define

$$v^* := \begin{cases} 0 & \text{if } \operatorname{char}(K) \neq 2 \\ f(v, \zeta)\epsilon + f(v, \epsilon)\zeta + v & \text{if } \operatorname{char}(K) = 2 \end{cases} \ ,$$

for all $v \in V$.

LEMMA 8.135. *If* $\operatorname{char}(K) = 2$, *then*

(i) $\varphi(a, v) = f(\theta(a, v^*), v)$
$\qquad = f(\pi(a), v)f(\zeta, v) + f(\theta(a, \zeta), v)f(\epsilon, v) + f(\epsilon, \pi(a))q(v)$;

(ii) *If* $f(\epsilon, v) = f(\zeta, v) = 0$, *then* $\pi(av) = \pi(a)q(v) + f(\pi(a), v)v$;

(iii) $\pi(a\zeta) = \overline{\pi(a)}q(\zeta) + \theta(a, \zeta) + f(\epsilon, \pi(a))\zeta$;

(iv) $\theta(av, u) = q(v)\overline{\theta(a, \overline{u})} + f(u, \overline{v})\overline{\theta(a, v)} + f(\theta(a, v), \overline{u})\overline{v} + \varphi(a, v)u$;

for all $a \in X$ *and all* $u, v \in V$.

PROOF. By Lemma 8.134, this follows from the proof of [**20**, (26.30)]. □

LEMMA 8.136. *For all* $v \in V$, *all* $w \in W$ *and all* $a \in X$, *we have that*

(i) $q(vw) = q(v)q(\epsilon w)$;

(ii) $q(\theta(a, v)) = q(v)q(\pi(a))$.

PROOF. Since $\delta \cdot V \subseteq \operatorname{Rad}(H)$, it follows by substituting δ for w and w for z in ($\mathbf{Q}_{26}$) that $\delta \cdot \epsilon w \cdot v = \delta \cdot vw$, hence $[q(\epsilon w)] \cdot v = [q(vw)]$. By Lemma 8.51, it follows that $[q(v)q(\epsilon w)] = [q(vw)]$, which proves (i). Substituting $(a, 0)$ for w in (i) now yields (ii). □

LEMMA 8.137. *For all* $a \in X$, *we have that* $\varphi(a, \pi(a)) = 0$.

PROOF. By Lemma 8.131, we may assume that $\operatorname{char}(K) = 2$. Since $f(\epsilon, \zeta) = 1 = q(\epsilon)$, we have that $q(\epsilon + \zeta) = q(\zeta)$. It then follows, by Lemma 8.124(iv) and Lemma 8.136(ii), that

$$\begin{aligned} q(\pi(a) + \theta(a, \zeta)) &= q(\theta(a, \epsilon + \zeta)) \\ &= q(\epsilon + \zeta)q(\pi(a)) \\ &= q(\zeta)q(\pi(a)) \\ &= q(\theta(a, \zeta)) \ , \end{aligned}$$

and hence $q(\pi(a)) = f(\pi(a), \theta(a,\zeta))$. It thus follows from Lemma 8.135(i) that

$$\begin{aligned}\varphi(a,\pi(a)) &= f(\pi(a),\pi(a))f(\zeta,\pi(a)) + f(\theta(a,\zeta),\pi(a))f(\epsilon,\pi(a)) + f(\epsilon,\pi(a))q(\pi(a))\\ &= 0 + q(\pi(a))f(\epsilon,\pi(a)) + f(\epsilon,\pi(a))q(\pi(a))\\ &= 0\ ,\end{aligned}$$

which is what we had to prove. □

LEMMA 8.138. *For all $v \in V$ and all $w \in W^*$, we have that*

$$w \cdot q(\epsilon w)^{-1}\epsilon \cdot \overline{\epsilon w} \cdot vw = wv\ .$$

PROOF. If we substitute $\lambda(w(-\epsilon))$ for w in ($\mathbf{Q}_{19}$), then we get that

$$\lambda(w(-\epsilon)) \cdot vw = wv\ .$$

If we set $v = \epsilon$ in this identity, then we get that $\lambda(w(-\epsilon))\cdot\epsilon w = w$, hence $\lambda(w(-\epsilon)) = w \cdot (\epsilon w)^{-1}$, and therefore

$$w \cdot (\epsilon w)^{-1} \cdot vw = wv\ .$$

Note that it follows by substituting $[t]$ for z in ($\mathbf{Q}_{26}$), with $t \in K$, that $w{\cdot}t\epsilon{\cdot}v = w{\cdot}tv$. Since $(\epsilon w)^{-1} = q(\epsilon w)^{-1}\overline{\epsilon w}$, it follows from this identity that

$$w \cdot q(\epsilon w)^{-1}\epsilon \cdot \overline{\epsilon w} \cdot vw = wv\ ,$$

which is what we had to show. □

LEMMA 8.139. *For all $a \in X$ and all $v \in V$, we have that*

(i) $q(\pi(a))av = a\overline{\pi(a)}\theta(a,v)$ *;*
(ii) $a\pi(a)v = a\theta(a,v)$ *.*

PROOF. We may assume that $a \neq 0$. First, we substitute $(a,0)$ for w in Lemma 8.138, and we get that

$$(a,0) \cdot q(\epsilon(a,0))^{-1}\epsilon \cdot \overline{\epsilon(a,0)} \cdot v(a,0) = (a,0)v,$$

hence

$$(q(\pi(a))^{-1}a, 0) \cdot \overline{\pi(a)} \cdot \theta(a,v) = (a,0)v.$$

If we calculate the X-component of both sides, then we get that

$$q(\pi(a))^{-1}a\overline{\pi(a)}\theta(a,v) = av,$$

which shows (i).

On the other hand, if we substitute $(a,1)$ for w in Lemma 8.138, then we get that

$$(a,1) \cdot q(\epsilon(a,1))^{-1}\epsilon \cdot \overline{\epsilon(a,1)} \cdot v(a,1) = (a,1)v,$$

hence

$$(q(\pi(a)+\epsilon)^{-1}a, q(\pi(a)+\epsilon)^{-2}) \cdot (\overline{\pi(a)+\epsilon}) \cdot (\theta(a,v)+v) = (a,1)v.$$

Again, we calculate the X-component of both sides, and we get that

$$\big(q(\pi(a)) + f(\pi(a),\epsilon) + q(\epsilon)\big)^{-1}a(\overline{\pi(a)}+\epsilon)(\theta(a,v)+v) = av\ ,$$

from which it follows that

$$a\overline{\pi(a)}\theta(a,v) + a\theta(a,v) + a\overline{\pi(a)}v + av = q(\pi(a))av + f(\pi(a),\epsilon)av + av\ .$$

Since $f(\pi(a),\epsilon)av = a\pi(a)v + a\overline{\pi(a)}v$ by Lemma 8.128, it follows by (i) that

$$a\theta(a,v) = a\pi(a)v\ ,$$

which proves (ii). $\square$

LEMMA 8.140. *If* $|K| > 2$, *then*

$$c\theta(a,v) - c\pi(a)v = ah(a,c)v - ah(a,cv)$$

for all $a, c \in X$ *and all* $v \in V$.

PROOF. See [**20**, (26.36)]. $\square$

LEMMA 8.141. *For all* $a \in X$ *and all* $v \in V$, *we have that*

$$\theta(a, \theta(a,v)) = \theta(a,v) f(\epsilon, \pi(a)) - q(\pi(a))v \ .$$

PROOF. See [**20**, (26.33)]. $\square$

We can rephrase this identity in terms of W in place of X, which results in a nice identity.

LEMMA 8.142. *For all* $v \in V$ *and all* $w \in W$, *we have that* $vw(\boxminus w) = -q(\epsilon w)v$.

PROOF. Let $w = (a,t)$, with $a \in X$ and $t \in K$. Let $Q(a) := g(a,-a) = f(\pi(a), \epsilon)$, and observe that $2Q(a) = 0$; see Lemma 8.119 and Lemma 8.131. Note that it follows from Lemma 8.116 that $\boxminus w = (-a, -t + Q(a))$. Hence, by Lemma 8.124 and Lemma 8.141,

$$\begin{aligned}
vw(\boxminus w) &= v(a,t)(-a, -t + Q(a)) \\
&= (\theta(a,v) + tv) \cdot (-a, -t + Q(a)) \\
&= \theta(-a, \theta(a,v) + tv) + (-t + Q(a))(\theta(a,v) + tv) \\
&= \theta(a, \theta(a,v)) + t\theta(a,v) - t\theta(a,v) + Q(a)\theta(a,v) - t^2 v + Q(a)tv \\
&= \theta(a,v)Q(a) - q(\pi(a))v + Q(a)\theta(a,v) - t^2 v + Q(a)tv \\
&= -\big(q(\pi(a)) + Q(a)t + t^2\big)v \\
&= -q\big(\pi(a) + t\epsilon\big)v \\
&= -q(\epsilon w)v \ ,
\end{aligned}$$

and we are done. $\square$

LEMMA 8.143. *For all* $v \in V$ *and all* $w \in W$, *we have that*

$$vww = f(\epsilon, \epsilon w)vw - q(\epsilon w)v \ .$$

PROOF. By 3.13(i), we have that $w(-\epsilon) = F(\epsilon w, \epsilon) \boxminus w$, and hence, by ($\mathbf{Q}_6$), ($\mathbf{Q}_{12}$) and Lemma 8.142,

$$\begin{aligned}
vww &= vw \cdot w(-\epsilon) \\
&= vw \cdot (F(\epsilon w, \epsilon) \boxminus w) \\
&= vw \cdot [f(\epsilon w, \epsilon)] + vw(\boxminus w) \\
&= f(\epsilon w, \epsilon)vw - q(\epsilon w)v \ ,
\end{aligned}$$

which is what we had to show. $\square$

DEFINITION 8.144. For all $v \in V^*$ and all $w \in W^*$, we let $[v]_w := \langle v, vw \rangle$ be the subspace of V (over K) generated by v and vw. Note that $[v]_w$ is 2-dimensional if and only if $w \in W \setminus Y$.

LEMMA 8.145. *For all $v \in V^*$ and all $w \in W \setminus Y$, we have that $[v]_w \cdot w = [v]_w$, i.e. $[v]_w$ is a 2-dimensional subspace of V which is irreducible under the action of w.*

PROOF. It follows from Lemma 8.143 that $[v]_w \cdot w = \langle v, vw \rangle \cdot w = \langle vw, vww \rangle = \langle vw, f(\epsilon, \epsilon w)vw - q(\epsilon w)v \rangle = \langle vw, v \rangle$ since $q(\epsilon w) \neq 0$. □

DEFINITION 8.146. Let $u, v \in V^*$ and $w \in W \setminus Y$. Then u and v are called *w-orthogonal* if and only if $f([u]_w, [v]_w) = 0$.

REMARK 8.147. It is clear that the definition above of $[v]_w$ and the notion of w-orthogonality are generalizations of the definition of $[v]_a$ and the notion of a-orthogonality as defined in [**20**]. See [**20**, (26.37) and (26.38)].

THEOREM 8.148. *Let $a \in X^*$, and let $w := (a, 0) \in W^*$. Suppose that $f(\epsilon, \pi(a)) \neq 0$ if $\mathrm{char}(K) = 2$. Let T be the endomorphism of V given by $T(v) := vw$ for all $v \in V$. Then:*

(i) *The endomorphism T is a norm splitting map of the quadratic space (V, K, q);*
(ii) *The minimal polynomial of T is*
$$p(x) = x^2 + f(\epsilon, \pi(a))x + q(\pi(a)) \ .$$
Let E denote the splitting field of p over K, and let $\gamma \in E$ be a root of p. Then E/K is a separable quadratic extension and there is a scalar multiplication from $E \times V$ to V extending the scalar multiplication from $K \times V$ to V, such that $T(v) = \gamma v$ for all $v \in V$;
(iii) *Let S be a finite set of pairwise w-orthogonal elements of V^*. Then the elements of the set $S \cup Sw$ are linearly independent over K; if this set does not span V, then S can be extended to a larger set of non-zero pairwise w-orthogonal vectors;*
(iv) *Let $\psi : E \to [\epsilon]_w$ be given by*
$$\psi(r + t\gamma) := r\epsilon + t\pi(a)$$
for all $r, t \in K$. Then ψ is an isomorphism of vector spaces and X is a (right) vector space over E with scalar multiplication given by $bu := b\psi(u)$ for all $b \in X$ and all $u \in E$. If σ denotes the non-trivial element in $\mathrm{Gal}(E/K)$, then $\psi(u^\sigma) = \overline{\psi(u)}$ for all $u \in E$. If N denotes the norm of the extension E/K, then $N(u) = q(\psi(u))$ for all $u \in E$.

PROOF. See [**20**, (26.39)]. □

LEMMA 8.149. *Let $a \in X^*$ be arbitrary, and let $w := (a, 0) \in W^*$. Let $D := \langle \epsilon, \epsilon w, v, vw \rangle$ for some $v \in V \setminus \langle \epsilon, \epsilon w \rangle$. Then $\dim_K D = 4$, and $aDD \subseteq aD$ (but not necessarily $bDD \subseteq bD$ for other elements $b \in X$).*

PROOF. See [**20**, (26.41)]. □

THEOREM 8.150. *Let $\dim_K V = 4$. Then V can be made into a division ring such that X is a right vector space over V with scalar multiplication given by the map $(a, v) \mapsto av$ for all $a \in X$ and all $v \in V$.*

PROOF. See [**20**, (26.42)]. □

It will be convenient now to set $v^\sigma := \overline{v}$ for all $v \in V$.

THEOREM 8.151. *Suppose that* $\dim_K V \in \{2,4\}$. *Then there is a multiplication on* V *which gives* V *the structure of a division ring with the following properties:*

(i) $\langle\epsilon\rangle$ *is a subfield lying in the center of* V *and the map* $t \mapsto t\epsilon$ *is an isomorphism from* K *to* $\langle\epsilon\rangle$ *;*
(ii) σ *is an involution of* V *;*
(iii) X *is a right vector space over* V *with scalar multiplication given by the map* $(a,v) \mapsto av$ *;*
(iv) $q(v) = vv^\sigma = v^\sigma v \in \langle\epsilon\rangle$, *and* $f(u,v) = uv^\sigma + vu^\sigma = u^\sigma v + v^\sigma u \in \langle\epsilon\rangle$ *for all* $u,v \in V$ *;*
(v) h *is a skew-hermitian form on* X *with respect to* σ *;*
(vi) $(V, \langle\epsilon\rangle, \sigma)$ *is an involutory set;*
(vii) $\theta(a,v) = \pi(a)v$ *for all* $a \in X$ *and all* $v \in V$ *.*

PROOF. See [**20**, (26.43)]. □

THEOREM 8.152. *Suppose that* $\dim_K V \leq 4$. *Then* $\dim_K V \in \{2,4\}$. *Let* V *be given the structure of a division ring as in Theorem 8.151. Then* (V,K,σ,X,π) *is an anisotropic pseudo-quadratic space. Moreover, we have that*

$$\pi(av) = v^\sigma \pi(a) v - \varphi(a,v)\epsilon$$

for all $a \in X$ *and all* $v \in V$.

PROOF. See [**20**, (26.44)]. □

THEOREM 8.153. *Suppose that* $\dim_K V \leq 4$. *Let* (V,K,σ,X,π) *be as in Theorem 8.152. Then* $(V,W,\tau_V,\tau_W,\epsilon,\delta) \cong \Omega_P(V,K,\sigma,X,\pi)$.

PROOF. Let $(T,\boxplus)$ be the group defined in section 7.4 applied on the pseudo-quadratic space (V,K,σ,X,π). By the definition of T, we have that $\pi(a) - v \in \langle\epsilon\rangle$ for all $(a,v) \in T$. Let $\chi(a,v)$ be the unique element $t \in K$ such that $v - \pi(a) = t\epsilon$.

Let ϕ be the isomorphism from $[V]$ to V which maps $[v]$ to v for all $v \in V$, and let ψ be the isomorphism from $[T]$ to W which maps $[a,v]$ to $(a,\chi(a,v))$ for all $(a,v) \in T \subseteq X \times V$. Then $\phi([1]) = [1] = \epsilon$ and $\psi([0,1]) = (0,\chi(0,1)) = (0,1) = \delta$ since $1\epsilon - \pi(0) = 1\epsilon$. (Remember that we have identified K with $\langle\epsilon\rangle \subseteq V$ by Theorem 8.151(i).)

Now, let $v \in V$ and $(a,x) \in T$ be arbitrary. By Lemma 8.139(ii) and Lemma 8.121(i), we have that

$$\begin{aligned} a \cdot xv &= a\pi(a)v + axv - a\pi(a)v \\ &= a\theta(a,v) + a(x - \pi(a))v \\ &= a\theta(a,v) + a \cdot \chi(a,x)\epsilon \cdot v \\ &= a\theta(a,v) + \chi(a,x)av \\ &= a(\theta(a,v) + \chi(a,x)v)\ , \end{aligned}$$

and hence, by Theorem 8.151(iii), it follows that

$$xv = \theta(a,v) + \chi(a,x)v = v \cdot (a,\chi(a,x))\ .$$

By Theorem 8.152 and Theorem 8.151(i and iv), we have that

$$\begin{aligned}\chi(av, v^\sigma xv)\epsilon &= v^\sigma xv - \pi(av) \\ &= v^\sigma xv - v^\sigma \pi(a)v + \varphi(a,v)\epsilon \\ &= v^\sigma (x - \pi(a))v + \varphi(a,v)\epsilon \\ &= v^\sigma \cdot \chi(a,x)\epsilon \cdot v + \varphi(a,v)\epsilon \\ &= \big(\chi(a,x)q(v) + \varphi(a,v)\big)\epsilon \ ,\end{aligned}$$

and hence

$$\chi(av, v^\sigma xv) = \chi(a,x)q(v) + \varphi(a,v) \ .$$

It follows that

$$\begin{aligned}&\phi([v][a,x]) = \phi([xv]) = xv = v \cdot (a, \chi(a,x)) = \phi([v])\psi([a,x]) \ , \text{ and} \\ &\psi([a,x][v]) = \psi([av, v^\sigma xv]) = (av, \chi(av, v^\sigma xv)) = (av, \chi(a,x)q(v) + \varphi(a,v)) \\ &\qquad = (a, \chi(a,x)) \cdot v = \psi([a,x])\phi([v]) \ ,\end{aligned}$$

for all $v \in V$ and all $(a,x) \in T$.

Hence (ϕ, ψ) is an isomorphism from $\Omega_P(V, K, \sigma, X, \pi)$ to $(V, W, \tau_V, \tau_W, \epsilon, \delta)$. □

8.7. Quadrangular Systems of Type E_6, E_7 and E_8

In this section, we continue to assume that $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a wide quadrangular system which is the extension of a quadrangular system Λ of quadratic form type, such that $\mathrm{Rad}(F) = 0$. It only remains to consider the case where $\dim_K V > 4$.

Lemma 8.154. *If* $\mathrm{char}(K) = 2$*, then there exists an element* $\xi \in X^*$ *such that* $\pi(\xi) = \alpha\zeta$ *for some* $\alpha \in K^*$.

Proof. Suppose that $g(a,a) = 0$ for all $a \in X$. Since g is bilinear, it would follow that $g(a,b) = g(b,a)$ for all $a, b \in X$, and hence, by Lemma 8.116, that W is abelian. It would then follow by ($\mathbf{Q}_8$) that $\mathrm{Im}(H) \subseteq \mathrm{Rad}(F)$. Since $\mathrm{Rad}(F) = 0$ and $H \not\equiv 0$, this is a contradiction.

Hence there exists an element $a \in X^*$ such that $g(a,a) \neq 0$. Let $w_1 := (a, 0) \in W^*$. By Lemma 8.119, it follows that $f(\epsilon, \epsilon w_1) = f(\epsilon, \pi(a)) = g(a,a) \neq 0$. Hence (see Definition 8.109)

$$S_1 \cap S_2 = \{v \in V \mid F(\epsilon, v) \neq 0\} \cap \{\epsilon w \mid w \in W\} \neq \emptyset \ ,$$

since $\epsilon w_1 \in S_1 \cap S_2$. By the definition of ζ, this implies that $\zeta = f(\epsilon, z)^{-1} z$ for some $z \in S_1 \cap S_2$. Let $z = \epsilon w_2$ for some $w_2 \in W^*$. Since $f(\epsilon w_2, \zeta) = f(z, \zeta) = f(z, f(\epsilon, z)^{-1} z) = 0$, w_2 is ζ-orthogonal, hence $w_2 = (\xi, 0)$ for some $\xi \in X^*$. We conclude that $\pi(\xi) = \epsilon(\xi, 0) = \epsilon w_2 = z = f(\epsilon, z)\zeta = \alpha\zeta$ for $\alpha = f(\epsilon, z) \in K^*$. □

Definition 8.155. If $\mathrm{char}(K) \neq 2$, let ξ be an arbitrary element of X^*. If $\mathrm{char}(K) = 2$, choose $\xi \in X^*$ as in Lemma 8.154.

By Lemma 8.154 and Theorem 8.148, the endomorphism T of V which maps v to $v(\xi, 0)$ is a norm splitting map of q.

We have come to a point which is completely similar to the beginning of Chapter 27 in [**20**], and the rest of the proof could literally be copied from that chapter.

Theorem 8.156. *The quadratic space* (K, V_0, q) *is of type* E_6*,* E_7 *or* E_8*.*

Proof. The proof is exactly as in [**20**, (27.17)], where we have to use Definition 8.117 and 8.118 and Lemmas 8.124, 8.127, 8.128, 8.129, 8.130, 8.131, 8.132, 8.135, 8.139, 8.140, 8.141 and 8.149. □

Theorem 8.157. $(V, W, \tau_V, \tau_W, \epsilon, \delta) \cong \Omega_E(K, V_0, q)$.

Proof. It follows from the proof of [**20**, (27.19)], using Definition 8.109 and 8.118 as well as Lemmas 8.124, 8.128, 8.129, 8.130, 8.131, 8.132, 8.133, 8.135 and 8.139, that the maps h, g, θ and φ are exactly as in section 7.5.

Let ϕ be the map from $[V_0]$ to V which maps $[v]$ to v for all $v \in V$, and let ψ be the map from $[S]$ to W which maps $[a, t]$ to (a, t) for all $(a, t) \in S$. Since we have seen in Definitions 8.117 and 8.118 that $(a, t) \cdot v = (av, tq(v) + \varphi(a, v))$ and $v \cdot (a, t) = \theta(a, v) + tv$, it is now obvious that (ϕ, ψ) is an isomorphism from $\Omega_E(K, V_0, q)$ to $(V, W, \tau_V, \tau_W, \epsilon, \delta)$. □

This completes the proof of Theorem 8.10, and thereby the proof of the classification of quadrangular systems.

APPENDIX A

Abelian Quadrangular Systems

In this appendix, we will describe the quadrangular systems $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ where W is abelian, and we will restate the axiom system for some specific cases.

A quadrangular system $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ will be called *abelian* if and only if W is abelian. One can check that Ω is abelian if and only if it is of quadratic form type, of involutory type, of indifferent type or of type F_4. (Note that, if Ω is of pseudo-quadratic form type with W abelian, then Ω is in fact reduced, and hence of one of these types.) In this case, we simply write $+$ and $-$ in place of $\boxplus$ and $\boxminus$, respectively, and we get the following description.

Consider an abelian group $(V, +)$ and an abelian group $(W, +)$. Suppose that there is a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W, both of which will be denoted by $\cdot$ or simply by juxtaposition, i.e. $\tau_V(v, w) = vw = v \cdot w$ and $\tau_W(w, v) = wv = w \cdot v$ for all $v \in V$ and all $w \in W$. Consider a map F from $V \times V$ to W and a map H from $W \times W$ to V which are additive in both variables. Suppose furthermore that there exists a fixed element $\epsilon \in V^*$ and a fixed element $\delta \in W^*$, and suppose that, for each $v \in V^*$, there exists an element $v^{-1} \in V^*$, and for each $w \in W^*$, there exists an element $w^{-1} \in W^*$, such that, for all $w, w_1, w_2 \in W$ and all $v, v_1, v_2 \in V$, the following axioms are satisfied.

($\mathbf{A}_1$) $w\epsilon = w$.
($\mathbf{A}_2$) $v\delta = v$.
($\mathbf{A}_3$) $(w_1 + w_2)v = w_1 v + w_2 v$.
($\mathbf{A}_4$) $(v_1 + v_2)w = v_1 w + v_2 w$.
($\mathbf{A}_5$) $v(-w) = -vw$.
($\mathbf{A}_6$) $w(-v) = wv$.
($\mathbf{A}_7$) $\mathrm{Im}(F) \subseteq \mathrm{Rad}(H)$.
($\mathbf{A}_8$) $\mathrm{Im}(H) \subseteq \mathrm{Rad}(F)$.
($\mathbf{A}_9$) $\delta \in \mathrm{Rad}(H)$.
($\mathbf{A}_{10}$) If $\mathrm{Rad}(F) \neq 0$, then $\epsilon \in \mathrm{Rad}(F)$.
($\mathbf{A}_{11}$) $w(v_1 + v_2) = wv_1 + wv_2 + F(v_1 w, v_2)$.
($\mathbf{A}_{12}$) $v(w_1 + w_2) = vw_1 + vw_2 + H(w_1 v, w_2)$.
($\mathbf{A}_{13}$) $(v^{-1})^{-1} = v$ (if $v \neq 0$).
($\mathbf{A}_{14}$) $(w^{-1})^{-1} = w$ (if $w \neq 0$).
($\mathbf{A}_{15}$) $wvv^{-1} = w$ (if $v \neq 0$).
($\mathbf{A}_{16}$) $vww^{-1} = v$ (if $w \neq 0$).
($\mathbf{A}_{17}$) $v^{-1}(wv) = \overline{vw}$ (if $v \neq 0$).
($\mathbf{A}_{18}$) $w^{-1}(vw) = wv$ (if $w \neq 0$).
($\mathbf{A}_{19}$) $F(v_1^{-1}, \overline{v_2})v_1 = F(v_1, v_2)$ (if $v_1 \neq 0$).
($\mathbf{A}_{20}$) $H(w_1^{-1}, w_2)w_1 = H(w_1, w_2)$ (if $w_1 \neq 0$).

Then $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ is an abelian quadrangular system.

A.1. Reduced Quadrangular Systems

If $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is *reduced* or *indifferent*, i.e. if $H \equiv 0$, then Ω is abelian, and we get the following description.

Consider an abelian group $(V, +)$ and an abelian group $(W, +)$. Suppose that there is a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W, both of which will be denoted by $\cdot$ or simply by juxtaposition, i.e. $\tau_V(v, w) = vw = v \cdot w$ and $\tau_W(w, v) = wv = w \cdot v$ for all $v \in V$ and all $w \in W$. Consider a map F from $V \times V$ to W which is additive in both variables. Suppose furthermore that there exists a fixed element $\epsilon \in V^*$ and a fixed element $\delta \in W^*$, and suppose that, for each $v \in V^*$, there exists an element $v^{-1} \in V^*$, and for each $w \in W^*$, there exists an element $w^{-1} \in W^*$, such that, for all $w, w_1, w_2 \in W$ and all $v, v_1, v_2 \in V$, the following axioms are satisfied.

($\mathbf{R}_1$) $w\epsilon = w$.
($\mathbf{R}_2$) $v\delta = v$.
($\mathbf{R}_3$) $(w_1 + w_2)v = w_1 v + w_2 v$.
($\mathbf{R}_4$) $(v_1 + v_2)w = v_1 w + v_2 w$.
($\mathbf{R}_5$) $v(-w) = -vw$.
($\mathbf{R}_6$) $w(-v) = wv$.
($\mathbf{R}_7$) If $\mathrm{Rad}(F) \neq 0$, then $\epsilon \in \mathrm{Rad}(F)$.
($\mathbf{R}_8$) $w(v_1 + v_2) = wv_1 + wv_2 + F(v_1 w, v_2)$.
($\mathbf{R}_9$) $v(w_1 + w_2) = vw_1 + vw_2$.
($\mathbf{R}_{10}$) $(v^{-1})^{-1} = v$ (if $v \neq 0$).
($\mathbf{R}_{11}$) $(w^{-1})^{-1} = w$ (if $w \neq 0$).
($\mathbf{R}_{12}$) $wvv^{-1} = w$ (if $v \neq 0$).
($\mathbf{R}_{13}$) $vww^{-1} = v$ (if $w \neq 0$).
($\mathbf{R}_{14}$) $v^{-1}(wv) = \overline{vw}$ (if $v \neq 0$).
($\mathbf{R}_{15}$) $w^{-1}(vw) = wv$ (if $w \neq 0$).
($\mathbf{R}_{16}$) $F(v_1^{-1}, \overline{v_2})v_1 = F(v_1, v_2)$ (if $v_1 \neq 0$).

Then $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a reduced or indifferent quadrangular system (and it is reduced if and only if $F \not\equiv 0$).

Remark A.1. As we explained in Remark 6.4, axiom ($\mathbf{R}_7$) had only been introduced to simplify the classification result of the *wide* quadrangular systems. In particular, it is not needed for the reduced quadrangular systems, and it is in fact often more convenient – and perfectly allowed – to leave it out.

A.2. Indifferent Quadrangular Systems

If $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ is *indifferent*, i.e. if $F \equiv 0$ and $H \equiv 0$, then Ω is abelian, and we get the following description.

Consider an abelian group $(V, +)$ and an abelian group $(W, +)$. Suppose that there is a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W, both of which will be denoted by $\cdot$ or simply by juxtaposition, i.e. $\tau_V(v, w) = vw = v \cdot w$ and $\tau_W(w, v) = wv = w \cdot v$ for all $v \in V$ and all $w \in W$. Suppose furthermore that there exists a fixed element $\epsilon \in V^*$ and a fixed element $\delta \in W^*$, and suppose that, for each $v \in V^*$, there exists an element $v^{-1} \in V^*$, and for each $w \in W^*$, there exists an element $w^{-1} \in W^*$, such that, for all $w, w_1, w_2 \in W$ and all $v, v_1, v_2 \in V$, the following axioms are satisfied.

($\mathbf{D}_1$) $w\epsilon = w$.

($\mathbf{D}_2$) $v\delta = v$.
($\mathbf{D}_3$) $(w_1 + w_2)v = w_1v + w_2v$.
($\mathbf{D}_4$) $(v_1 + v_2)w = v_1w + v_2w$.
($\mathbf{D}_5$) $w(v_1 + v_2) = wv_1 + wv_2$.
($\mathbf{D}_6$) $v(w_1 + w_2) = vw_1 + vw_2$.
($\mathbf{D}_7$) $(v^{-1})^{-1} = v$ (if $v \neq 0$).
($\mathbf{D}_8$) $(w^{-1})^{-1} = w$ (if $w \neq 0$).
($\mathbf{D}_9$) $wvv^{-1} = w$ (if $v \neq 0$).
($\mathbf{D}_{10}$) $vww^{-1} = v$ (if $w \neq 0$).
($\mathbf{D}_{11}$) $v^{-1}(wv) = vw$ (if $v \neq 0$).
($\mathbf{D}_{12}$) $w^{-1}(vw) = wv$ (if $w \neq 0$).

Then $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ is an indifferent quadrangular system. (Note that we do not have to assume *a priori* that all elements of V and W have order at most 2, but that this follows from these axioms.)

A.3. Radical Quadrangular Systems

An abelian quadrangular system $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ will be called *radical* if and only if $\text{Rad}(F) \neq 0$. One can check that Ω is radical if and only if it is of quadratic form type with $\epsilon \in \text{Rad}(f)$ (and hence $\text{char}(K) = 2$), of indifferent type or of type F_4. We will give two different (but equivalent) descriptions. The first one is useful to check whether a certain system is a radical quadrangular system; the second one is more convenient to work with. Note that each of these descriptions is completely symmetrical.

A.3.1. First Description. Consider an abelian group $(V, +)$ and an abelian group $(W, +)$. Suppose that there is a map τ_V from $V \times W$ to V and a map τ_W from $W \times V$ to W, both of which will be denoted by $\cdot$ or simply by juxtaposition, i.e. $\tau_V(v, w) = vw = v \cdot w$ and $\tau_W(w, v) = wv = w \cdot v$ for all $v \in V$ and all $w \in W$. Consider a map F from $V \times V$ to W and a map H from $W \times W$ to V which are additive in both variables. Suppose furthermore that there exists a fixed element $\epsilon \in V^*$ and a fixed element $\delta \in W^*$, and suppose that, for each $v \in V^*$, there exists an element $v^{-1} \in V^*$, and for each $w \in W^*$, there exists an element $w^{-1} \in W^*$, such that, for all $w, w_1, w_2 \in W$ and all $v, v_1, v_2 \in V$, the following axioms are satisfied.

($\mathbf{F}_1$) $w\epsilon = w$.
($\mathbf{F}_2$) $v\delta = v$.
($\mathbf{F}_3$) $(w_1 + w_2)v = w_1v + w_2v$.
($\mathbf{F}_4$) $(v_1 + v_2)w = v_1w + v_2w$.
($\mathbf{F}_5$) $\text{Im}(F) \subseteq \text{Rad}(H)$.
($\mathbf{F}_6$) $\text{Im}(H) \subseteq \text{Rad}(F)$.
($\mathbf{F}_7$) $\delta \in \text{Rad}(H)$.
($\mathbf{F}_8$) $\epsilon \in \text{Rad}(F)$.
($\mathbf{F}_9$) $w(v_1 + v_2) = wv_1 + wv_2 + F(v_1w, v_2)$.
($\mathbf{F}_{10}$) $v(w_1 + w_2) = vw_1 + vw_2 + H(w_1v, w_2)$.
($\mathbf{F}_{11}$) $(v^{-1})^{-1} = v$ (if $v \neq 0$).
($\mathbf{F}_{12}$) $(w^{-1})^{-1} = w$ (if $w \neq 0$).
($\mathbf{F}_{13}$) $wvv^{-1} = w$ (if $v \neq 0$).
($\mathbf{F}_{14}$) $vww^{-1} = v$ (if $w \neq 0$).

(**F**$_{15}$) $v^{-1}(wv) = vw$ (if $v \neq 0$).
(**F**$_{16}$) $w^{-1}(vw) = wv$ (if $w \neq 0$).
(**F**$_{17}$) $F(v_1^{-1}, v_2)v_1 = F(v_1, v_2)$ (if $v_1 \neq 0$).
(**F**$_{18}$) $H(w_1^{-1}, w_2)w_1 = H(w_1, w_2)$ (if $w_1 \neq 0$).

Then $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a radical quadrangular system. It is of type F_4 if and only if $F \not\equiv 0$ and $H \not\equiv 0$.

A.3.2. Second Description. Let K and L be two commutative fields with $\mathrm{char}(K) = \mathrm{char}(L) = 2$, such that K is a vector space over L and that L is a vector space over K. If t is an element of the field K, then we will denote the corresponding element of the vector space K by $[t]$; if s is an element of the field L, then we will denote the corresponding element of the vector space L by $[s]$. Let V be a vector space over K containing $[L]$ as a subspace, and let W be a vector space over L containing $[K]$ as a subspace.

Suppose that q is an anisotropic quadratic form from V to K, with corresponding bilinear form f, and that $\hat{q}$ is an anisotropic quadratic form from W to L, with corresponding bilinear form $\hat{f}$, such that $[L] \subseteq \mathrm{Rad}(f)$ and $[K] \subseteq \mathrm{Rad}(\hat{f})$. Let $\epsilon := [1] \in [L] \subseteq V$ and $\delta := [1] \in [K] \subseteq W$. Finally, suppose that there is a map τ_V from $V \times W$ to V which is K-linear on V, and a map τ_W from $W \times V$ to W which is L-linear on W, both of which will be denoted by $\cdot$ or simply by juxtaposition, i.e. $\tau_V(v, w) = vw = v \cdot w$ and $\tau_W(w, v) = wv = w \cdot v$ for all $v \in V$ and all $w \in W$. Moreover, suppose that the following axioms hold, for all $v \in V$, $w \in W$, $t \in K$ and $s \in L$.

(**C**$_1$) $v[t] = tv$.
(**C**$_2$) $w[s] = sw$.
(**C**$_3$) $v \cdot sw = vw \cdot s\delta$.
(**C**$_4$) $w \cdot tv = wv \cdot t\epsilon$.
(**C**$_5$) $[t]v = [tq(v)]$.
(**C**$_6$) $[s]w = [s\hat{q}(w)]$.
(**C**$_7$) $vww = v \cdot \hat{q}(w)\delta$.
(**C**$_8$) $wvv = w \cdot q(v)\epsilon$.
(**C**$_9$) $v \cdot wv = q(v)vw$.
(**C**$_{10}$) $w \cdot vw = \hat{q}(w)wv$.
(**C**$_{11}$) $v(w_1 + w_2) = vw_1 + vw_2 + [\hat{f}(w_1 v, w_2)]$.
(**C**$_{12}$) $w(v_1 + v_2) = wv_1 + wv_2 + [f(v_1 w, v_2)]$.

Then $(V, W, \tau_V, \tau_W, \epsilon, \delta)$ is a radical quadrangular system. It is of type F_4 if and only if $f \not\equiv 0$ and $\hat{f} \not\equiv 0$.

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