Initialization

Code

```
Quit
```

```
Print[$MachineName ]
Print[$Version]
```

```
pcl0517
```

```
10.0 for Linux x86 (64-bit) (December  4, 2014)
```

```
<< RBSFA`
Print[$RBSFAversion]
Print[$RBSFAcommit  ]
```

```
RB-SFA v2.1.3, Tue 28 Feb 2017 11:48:14
```

```
commit   3bf93b827ba584ae1cc1b7f265f38a5f0a32ea04
Author: Emilio  Pisanty <emilio .pisanty@icfo.eu>
Date:   Tue Feb 28 11:49:15 2017 +0100
    Improved  testing for previous
     versions of ReIm  to avoid error messages  on package reload.
```

```
$HistoryLength= (*1*) 0;
LaunchKernels[8 - $KernelCount];
ParallelEvaluate[{$MachineName , $RBSFAversion}] // Tally
```

```
{{{pcl0517, RB-SFA v2.1.3, Tue 28 Feb 2017 11:48:14}, 8}}
```

```
$HistoryLength= (*1*) 0;
LaunchKernels[4 - $KernelCount];
ParallelEvaluate[{$MachineName , $RBSFAversion}] // Tally
```

```
{{{pcl0517, RB-SFA v2.1.3, Tue 28 Feb 2017 11:48:14}, 4}}
```

quick data import

```
<< (NotebookDirectory[] <> "data - saddle points on the laboratory frame .txt");
<< (NotebookDirectory[] <> "data - SPA dipole data on the laboratory frame .txt");
<< (NotebookDirectory[] <>
        "data - SPA dipole data for a single ionizationburst on the
            laboratory frame .txt");
<< (NotebookDirectory[] <> "data - SPA dipole data for a single burst, with
            no intrinsicphase, on the laboratory frame .txt");
saddlePoints // Dimensions
SPAdipoleData // Dimensions
singleSPAdipoleData // Dimensions
singleSPAdipoleDataNoIntrinsicPhase // Dimensions
```

```
{3, 1633}

{4, 1633}

{4, 1633}

{4, 1633}
```

Formatting niceties

```
<< MaTeX`
SetOptions[MaTeX, "Preamble " → {"
\\usepackage{amssymb ,upref}
\\usepackage{fourier}
\\usepackage{tgheros}
\\usepackage[T1]{fontenc}
\\usepackage{textcomp }
\\usepackage{microtype }
\\usepackage{siunitx}
\\usepackage{MnSymbol }
"}];
```

Formatting: label, tick and inset font sizes

```
lfs = 7;
tfs = 6;
ifs = 9;
ilfs = 5;
```

```
$OutputDirectory = FileNameJoin [{NotebookDirectory[], "..", "Figures"}];
$MainDirectory =
    StringReplace[FileNameJoin [{NotebookDirectory[], ".."}], {" " → "\\ "}];
pdflatex[] := Which[
    $OperatingSystem == "Unix", Run["cd " <> $MainDirectory <>
          " && pdflatex --output-directory=build Manuscript.tex"] /.
      {0 → "pdflatex successful"},
    $System == "Microsoft Windows (64-bit)", Run["cd " <> NotebookDirectory[] <>
          " && cd .. && pdflatex --output-directory=build Manuscript.tex"] /.
      {0 → "pdflatex successful"},
    True, Print["pdflatex[] has not been tested on Mac OS."];]
```

Definitions

Fields

```
bicircularA[t_] = F/ω {Cos[ω t], Sin[ω t], 0} + F/(2 ω) {Cos[2 ω t], -Sin[2 ω t], 0};
bicircularF[t_] = F {Sin[ω t], -Cos[ω t], 0} + F {Sin[2 ω t], Cos[2 ω t], 0};

(*bicircularArot[t_]=F/ω{3/2 Cos[3/2 ω t],1/2 Sin[3/2 ω t],0};
bicircularFrot[t_]=2F{Sin[3/2 ω t],0,0};*)

(*{{Cos[α t], -Sin[α t], 0}, {Sin[α t], Cos[α t], 0}, {0, 0, 1}}.({Cos[ω t], Sin[ω t], 0} + 1/2 {Cos[2ω t], -Sin[2ω t], 0})/.{α→ω/2}//FullSimplify
  {{Cos[α t], -Sin[α t], 0}, {Sin[α t], Cos[α t], 0}, {0, 0, 1}}.({Sin[ω t], -Cos[ω t], 0} + {Sin[2ω t], Cos[2ω t], 0})/.{α→ω/2}//FullSimplify*)

(*Rzα[t_]={{Cos[ω/2 t], Sin[ω/2 t], 0}, {-Sin[ω/2 t], Cos[ω/2 t], 0}, {0, 0, 1}};*)

pi[{px_, py_, pz_}, t_, tt_] := {px, py, pz} + bicircularA[t]
pi[{px_, py_, pz_}, t_] := pi[{px, py, pz}, t, 0]
```

Building the action

Saddle-point momentum

```
ps[t_, tt_] = -1/(t - tt) FullSimplify[(*Rzα[-tt].*) Integrate[bicircularA[τ], {τ, tt, t}]]
```

$$\left\{-\frac{1}{4\,(t-tt)\,\omega^2}F\,(4\,\mathrm{Sin}[t\,\omega]+\mathrm{Sin}[2\,t\,\omega]-4\,\mathrm{Sin}[tt\,\omega]-\mathrm{Sin}[2\,tt\,\omega]),\; -\frac{1}{4\,(t-tt)\,\omega^2}F\,(-4\,\mathrm{Cos}[t\,\omega]+\mathrm{Cos}[2\,t\,\omega]+4\,\mathrm{Cos}[tt\,\omega]-\mathrm{Cos}[2\,tt\,\omega]),\; 0\right\}$$

Action

```
(*S[t_,tt_]=(Ip-m ω/2) (t-tt) + ((#/.{τ→t}) - (#/.{τ→tt}))&@Integrate[
     Total[1/2 (Rzα[tt-τ].ps[t,tt]+bicircularArot[τ])^2]
     ,τ]*)
```

```
S[t_, tt_] = Ip (t - tt) + ((# /. {τ → t}) - (# /. {τ → tt})) &@Integrate[
    Total[1/2 (ps[t, tt] + bicircularA[τ])^2]
    , τ]
```

$$
\begin{aligned}
&\mathrm{Ip}\,(t-tt)+\frac{1}{48\,(t-tt)^2\,\omega^4}\\
&\quad F^2\big(51\,tt+30\,t^3\,\omega^2-60\,t^2\,tt\,\omega^2+30\,t\,tt^2\,\omega^2+12\,t\,\mathrm{Cos}[3\,t\,\omega]-24\,tt\,\mathrm{Cos}[3\,t\,\omega]-48\,t\,\mathrm{Cos}[(t-tt)\,\omega]-\\
&\qquad 3\,t\,\mathrm{Cos}[2\,(t-tt)\,\omega]-12\,t\,\mathrm{Cos}[3\,tt\,\omega]+48\,t\,\mathrm{Cos}[(-t+tt)\,\omega]-48\,tt\,\mathrm{Cos}[(-t+tt)\,\omega]+\\
&\qquad 3\,t\,\mathrm{Cos}[2\,(-t+tt)\,\omega]-3\,tt\,\mathrm{Cos}[2\,(-t+tt)\,\omega]+12\,tt\,\mathrm{Cos}[(2\,t+tt)\,\omega]+\\
&\qquad 12\,tt\,\mathrm{Cos}[(t+2\,tt)\,\omega]+8\,t^2\,\omega\,\mathrm{Sin}[3\,t\,\omega]-16\,t\,tt\,\omega\,\mathrm{Sin}[3\,t\,\omega]+8\,tt^2\,\omega\,\mathrm{Sin}[3\,t\,\omega]\big)-\\
&\quad \frac{1}{48\,(t-tt)^2\,\omega^4}F^2\big(51\,t+30\,t^2\,tt\,\omega^2-60\,t\,tt^2\,\omega^2+30\,tt^3\,\omega^2-12\,tt\,\mathrm{Cos}[3\,t\,\omega]-48\,t\,\mathrm{Cos}[(t-tt)\,\omega]-\\
&\qquad 3\,t\,\mathrm{Cos}[2\,(t-tt)\,\omega]-24\,t\,\mathrm{Cos}[3\,tt\,\omega]+12\,tt\,\mathrm{Cos}[3\,tt\,\omega]+12\,t\,\mathrm{Cos}[(2\,t+tt)\,\omega]+\\
&\qquad 12\,t\,\mathrm{Cos}[(t+2\,tt)\,\omega]+8\,t^2\,\omega\,\mathrm{Sin}[3\,tt\,\omega]-16\,t\,tt\,\omega\,\mathrm{Sin}[3\,tt\,\omega]+8\,tt^2\,\omega\,\mathrm{Sin}[3\,tt\,\omega]\big)
\end{aligned}
$$

Building the prefactor

Ground-state quantities for a short-range potential

```
shortRangeΨ[m_, κ_, {kx_, ky_, kz_}] /; (m^2 == 1) :=
  (2 i)/Sqrt[2 π κ] SolidHarmonicS[1, m, kx, ky, kz] 1/(kx^2 + ky^2 + kz^2)
  (κ/(kx^2 + ky^2 + kz^2 + κ^2) - ArcTan[Sqrt[kx^2 + ky^2 + kz^2]/κ]/Sqrt[kx^2 + ky^2 + kz^2])

shortRangeϒ[m_, κ_, {kx_, ky_, kz_}] /; (m^2 == 1) :=
  (2 i)/Sqrt[2 π κ] SolidHarmonicS[1, m, kx, ky, kz] 1/(kx^2 + ky^2 + kz^2)
  (κ - (kx^2 + ky^2 + kz^2 + κ^2)/Sqrt[kx^2 + ky^2 + kz^2] ArcTan[Sqrt[kx^2 + ky^2 + kz^2]/κ])
(*equal to (kx^2+ky^2+kz^2+κ^2) times shortRangeΨ.*)

reducedϒ[m_, κ_, {kx_, ky_, kz_}] /; (m^2 == 1) := SolidHarmonicS[1, m, kx, ky, kz]

doubleReducedϒ[m_, κ_, {kx_, ky_, kz_}] /; (m^2 == 1) := SolidHarmonicS[1, m, kx, ky, kz]/Sqrt[kx^2 + ky^2 + kz^2]

shortRangeΨ[0, κ_, {kx_, ky_, kz_}] := (1/κ)/(Sqrt[2] π) 1/(kx^2 + ky^2 + kz^2 + κ^2)
```

```
shortRangeϒ[0, κ_, {kx_, ky_, kz_}] :=
  (1/κ)/(Sqrt[2] π) 1 (*equal to (kx²+ky²+kz²+κ²) times shortRangeΨ.*)


RadialF[0, 1, κ_, {kx_, ky_, kz_}] := 2/(kx² + ky² + kz² + κ²)²
RadialF[1, 1, κ_, {kx_, ky_, kz_}] := (2/κ)/(kx² + ky² + kz² + κ²)²
RadialF[2, 1, κ_, {kx_, ky_, kz_}] :=
  1/(kx² + ky² + kz²)² (-(5 (kx² + ky² + kz²) + 3 κ²)/(kx² + ky² + kz² + κ²)² + 3/(κ Sqrt[kx² + ky² + kz²]) ArcTan[Sqrt[kx² + ky² + kz²]/κ])

ConstantN[l_, ll_, m_ , q_] := 2^(3/2) ⅈ^ll (-1)^q Sqrt[((2 l + 1) (2 ll + 1))/(4 π)]
    ThreeJSymbol [{l, 0}, {ll, 0}, {1, 0}] ThreeJSymbol [{l, -m }, {ll, m -q}, {1, q}]
UnitE[0] := {0, 0, 1}
UnitE[q_ /; (q² == 1)] := (-q)/Sqrt[2] {1, q ⅈ, 0}

Table[
    shortRangeDipoleConj[m ][{px_, py_, pz_}, κ_] = (0
          +UnitE[+m ]*ConstantN[1, 0, m , m ]
            SolidHarmonicS [0, 0, {px, py, pz}] RadialF[0, 1, κ, {px, py, pz}]
          +UnitE[+m ]*ConstantN[1, 2, m , m ] SolidHarmonicS [2, 0, {px, py, pz}]
            RadialF[2, 1, κ, {px, py, pz}]
          +UnitE[-m ]*ConstantN[1, 2, m , -m ] SolidHarmonicS [2, -2 m , {px, py, pz}]
            RadialF[2, 1, κ, {px, py, pz}]
          +UnitE[+0]*ConstantN[1, 2, m , 0] SolidHarmonicS [2, -m , {px, py, pz}]
            RadialF[2, 1, κ, {px, py, pz}]
       ) /. {(px² + py² + pz²)^(5/2) → (px² + py² + pz²)² Sqrt[px² + py² + pz²]}
    , {m , {1, -1}}];

shortRangeDipoleConj[0][{px_, py_, pz_}, κ_] := (ⅈ Sqrt[2])/(π κ) {px, py, pz}/(px² + py² + pz² + κ²)²;
```

Building the dipole

Note that the harmonic-dipole functions encode into the fictional $m = 2$ orbital the full-2p response, as it simplifies a bunch of data-handling structures.

```
(*rotatingFrameHarmonicDipole [S_,mm_ ,Ω_,saddleAssociation_]:=
  Block[{t,τ,t0,τ0,m =mm },
    {t,τ}=saddleAssociation[m ];
    {t0,τ0}=saddleAssociation[0];

    i (2π/(i τ))^(3/2) HessianRoot[S,t,τ]×
      shortRangeDipoleConj[m ][pi[Rzα[-τ].ps[t,t-τ],t,t-τ],κ]×
      shortRangeΥ[m ,κ,pi[ps[t0,t0-τ0],t0-τ0,t0-τ0]]×Exp[-i S[t,t-τ]+i Ω t]
  ]*)

(*rotatingFrameHarmonicDipoleNoIntrinsicPhase [S_,mm_ ,Ω_,saddleAssociation_]:=
  Block[{t,τ,t0,τ0,m =mm },
    {t,τ}=saddleAssociation[m ];
    {t0,τ0}=saddleAssociation[0];

    i (2π/(i τ))^(3/2) HessianRoot[S,t,τ]×
      shortRangeDipoleConj[m ][pi[Rzα[-τ].ps[t,t-τ],t,t-τ],κ]×
      shortRangeΥ[m ,κ,pi[ps[t0,t0-τ0],t0-τ0,t0-τ0]]×
      Abs[Exp[-i S[t,t-τ]+i Ω t]]
  ]*)

laboratoryFrameHarmonicDipole [S_, mm_ , Ω_, saddleAssociation_] :=
  Block[{t, τ, t0, τ0, m = mm },
    {t, τ} = saddleAssociation[m ];
    {t0, τ0} = saddleAssociation[0];

    i (2 π/(i τ))^(3/2) HessianRoot[S, t, τ] × shortRangeDipoleConj[m ][pi[ps[t, t - τ], t, t - τ], κ] ×
      shortRangeΥ[m , κ, pi[ps[t0, t0 - τ0], t0 - τ0, t0 - τ0]] × Exp[-i S[t, t - τ] + i Ω t]
  ]

laboratoryFrameHarmonicDipoleNoIntrinsicPhase [S_, mm_ , Ω_, saddleAssociation_] :=
  Block[{t, τ, t0, τ0, m = mm },
    {t, τ} = saddleAssociation[m ];
    {t0, τ0} = saddleAssociation[0];

    i (2 π/(i τ))^(3/2) HessianRoot[S, t, τ] × shortRangeDipoleConj[m ][pi[ps[t, t - τ], t, t - τ], κ] ×
      shortRangeΥ[m , κ, pi[ps[t0, t0 - τ0], t0 - τ0, t0 - τ0]] × Abs[Exp[-i S[t, t - τ] + i Ω t]]
  ]
```

Functions for deeper analysis

```
(*recombinationDipole[S_,mm_   ,Ω_,saddleAssociation_]:=Block[{t,τ,t0,τ0,m =mm   },
    {t,τ}=saddleAssociation[m ];
    {t0,τ0}=saddleAssociation[0];

    shortRangeDipoleConj[m ][pi[Rzα[-τ].ps[t,t-τ],t,t-τ],κ]
  ]*)

recombinationDipole[S_, mm_   , Ω_, saddleAssociation_] :=
  Block[{t, τ, t0, τ0, m  = mm   },
    {t, τ} = saddleAssociation[m ];
    {t0, τ0} = saddleAssociation[0];

    shortRangeDipoleConj[m ][pi[ps[t, t - τ], t, t - τ], κ]
  ]

ionizationFactor[S_, mm_   , Ω_, saddleAssociation_] := Block[{t, τ, t0, τ0, m  = mm   },
    {t, τ} = saddleAssociation[m ];
    {t0, τ0} = saddleAssociation[0];

    shortRangeϒ[m  , κ, pi[ps[t0, t0 - τ0], t0 - τ0, t0 - τ0]]
  ]

ionizationFactorTotal[S_, mm_   , Ω_, saddleAssociation_] :=
  Block[{t, τ, t0, τ0, m  = mm   },
    {t, τ} = saddleAssociation[m ];
    {t0, τ0} = saddleAssociation[0];

    shortRangeϒ[m  , κ, pi[ps[t0, t0 - τ0], t0 - τ0, t0 - τ0]] Abs[Exp[-ⅈ S[t, t - τ] + ⅈ Ω t]]
  ]

(*actionWithoutIpModification[S_,mm_   ,Ω_,saddleAssociation_]:=
  Block[{t,τ,t0,τ0,m =0},
    {t,τ}=saddleAssociation[m ];
    Abs[Exp[-ⅈ S[t,t-τ]+ⅈ Ω t]]
  ]*)

(*actionWithIpModification[S_,mm_   ,Ω_,saddleAssociation_]:=
  Block[{t,τ,t0,τ0,m =mm   },
    {t,τ}=saddleAssociation[m ];
    Abs[Exp[-ⅈ S[t,t-τ]+ⅈ Ω t]]
  ]*)

actionFactor[S_, mm_   , Ω_, saddleAssociation_] := Block[{t, τ, t0, τ0, m  = mm   },
    {t, τ} = saddleAssociation[m ];
    Abs[Exp[-ⅈ S[t, t - τ] + ⅈ Ω t]]
  ]
```

Trajectory functions

```
(*rotatingFrameTrajectory [ttr_,{t_,tt_}]=
    Rzα[-(ttr-tt)].Integrate[ps[t,tt]+Rzα[-tt].bicircularA[td],{td,tt,ttr}];*)

(*rotatingFrameVelocity [ttr_,{t_,tt_}]:=ps[t,tt]+bicircularArot[ttr]*)
```

```
laboratoryFrameTrajectory [ttr_, {t_, tt_}] =
    Integrate[ps[t, tt] + bicircularA[td], {td, tt, ttr}];

laboratoryFrameVelocity [ttr_, {t_, tt_}] := ps[t, tt] + bicircularA[ttr]
```

Functions for analysis

For more details, see https://physics.stackexchange.com/questions/308009/how-can-i-get-the-axes-of-the-polarization-ellipse-from-the-jones-vector-of-the

```
getEllipticity::nnzez = "getEllipticity called with nonzero third argument `1`.";
getEllipticity[{Ex_, Ey_, Ez_}] := (If[N[Ez] ≠ 0., Message[getEllipticity::nnzez, Ez]];
    getEllipticity[{Ex, Ey}])
getEllipticity[{Ex_, Ey_}] := If[
    Total[{Ex, Ey}^2] == 0,
    Sign[Cross[Re[#], Im [#]].{0, 0, 1}] &[{Ex, Ey, 0}]
    ,
    (Sign[Cross[Re[#], Im [#]].{0, 0, 1}] Norm [Im [#]]/Norm [Re[#]] &[
        Sqrt[Total[{Ex, Ey}^*2]]/Abs[Sqrt[Total[{Ex, Ey}^*2]]] {Ex, Ey, 0}])
  ]


getAngle::nnzez = "getAngle called with nonzero third argument `1`.";
getAngle[{Ex_, Ey_, Ez_}] := (If[N[Ez] ≠ 0., Message[getAngle::nnzez, Ez]];
    getAngle[{Ex, Ey}])
getAngle[{Ex_, Ey_}] := Mod[(ArcTan@@Re[#] &[Sqrt[Total[{Ex, Ey}^*2]]/Abs[Sqrt[Total[{Ex, Ey}^*2]]] {Ex, Ey}]), π]


getPhaseGamma ::nnzez =
    "getPhaseGamma called with nonzero third argument `1`.";
getPhaseGamma [{Ex_, Ey_, Ez_}] :=
  (If[N[Ez] ≠ 0., Message[getPhaseGamma ::nnzez, Ez]];
    getPhaseGamma [{Ex, Ey}])
getPhaseGamma [{Ex_, Ey_}] := If[
    Total[{Ex, Ey}^2] == 0,
    Arg[UnitE[getEllipticity[{Ex, Ey}]]^*.{Ex, Ey, 0}]
    ,
    1/2 Arg[Total[{Ex, Ey}^2]]
  ]
```

Calculation of the saddle points

Parameters

```
parameters = {getIonizationPotential["Neon", 0], Sqrt[1.88] 0.053, 0.057}; (*{Ip,F,ω}*)
```

Neon, 1.88×10^{14} W/cm ^{2}, 800 nm

Getting the saddle points

Calculation

```
Block[{Ip, F, ω, γ},
  {Ip, F, ω} = parameters ;
  γ = Sqrt[2 Ip ω]/F;
  ΩRange = Range[7 ω, 75 ω, (1/24.) ω];

  Print[DateString[]];
  AbsoluteTiming [
    saddlePoints= Association[Table[
          m  → GetSaddlePoints[
              ΩRange, S, Table[
                {{(0 - ⅈ 2 γ)/ω, (200 ° + ⅈ 2 γ)/ω}, {(55. ° + 0.35 ⅈ)/ω + (120 ° k)/ω, (100 ° + 1.1 ⅈ)/ω + (120 ° k)/ω}}
                , {k, 0, 5}]
              , IndependentVariables→ {"τ", "tt"}
              , Tolerance → 10^-5 /ω, Seeds → 75
              , Jacobian→ FiniteDifference
            ]
          , {m  , -1, 1}]]
  ]
]
DateString[]
```

```
Fri 5 May 2017 18:24:34
```

```
{1067.742754,
  <|-1 → <|0.399 → {{103.258 + 2.832 ⅈ, 5.81996 - 11.6052 ⅈ}, {66.5138 + 2.832 ⅈ, 5.81996 -
        11.6052 ⅈ}, {29.7701 + 2.832 ⅈ, 5.81996 - 11.6052 ⅈ}, ⋯6⋯,
      {46.7384 - 11.2229 ⅈ, 20.8151 - 23.9868 ⅈ}, {230.457 - 11.2229 ⅈ,
        20.8151 - 23.9868 ⅈ}, {156.97 - 11.2229 ⅈ, 20.8151 - 23.9868 ⅈ}},
    ⋯1631⋯, 4.275 → {{⋯1⋯}, ⋯16⋯, ⋯1⋯}|>,
   0 → ⋯1⋯, 1 → <|⋯1⋯|>|>}

large output | show less | show more | show all | set size limit ...
```

```
Fri 5 May 2017 18:42:25
```

Data handling

In-notebook save:

```
With[{data = Compress [saddlePoints]},
  Button["Restore saddle points", Set[saddlePoints, Uncompress [data]];
    saddlePoints;]
]
```

Restore saddle points

External export:

```
Save[NotebookDirectory[] <>
    "data - saddle points on the laboratory frame .txt", saddlePoints]
```

Import from external export:

```
<< (NotebookDirectory[] <> "data - saddle points on the laboratory frame .txt");
```

Check the import worked correctly:

```
saddlePoints // Dimensions
```

```
{3, 1633}
```

Initial map of the saddle points

```
Block[{Ip, F, ω, γ, saddles},
  {Ip, F, ω} = parameters ;
  γ = Sqrt[2 Ip ω]/F;
  Row[Table[
      saddles = saddlePoints[m ];

      Column [Table[
          Show[
            Graphics[
              ParallelTable[
                Map[
                  Apply[Function[{t, τ},
                      Tooltip[Point[ReIm [ω (time /. {"tt" → t - τ, "t" → t, "τ" → τ})]],
                        {Ω/ω, ω{t - τ, t, τ}, Floor[ωRe[t - τ], 2π/3]/(2π/3)}]
                    ]], saddles[Ω][[All]]]
                , {Ω, Keys[saddles][[1 ;; ;; 10]]}]
            ]
            , Frame → True, Axes → True
            , ImageSize → 750
            , FrameLabel → {"Re(ω" <> time <> ")", "Im (ω" <> time <> ")"}
          ]
          , {time , {"tt", "t", "τ"}}]]
      , {m , {1, -1}}]]
]
```

Classifying the saddle points

Note that the saddle-point classification involves the manual fixing of the parameters $\omega\, t_{\mathrm{min}} = 0.95$ and $\omega\, t_{\mathrm{min}} = 2.68$, chosen so that a vertical strip in τ with those limits will include only the short trajectories. If the parameters change, these might need to be adjusted.

Naive sort

```
Block[{Ip, F, ω, γ, saddles, classifierFunction, sortingFunction, selection},
  {Ip, F, ω} = parameters ;
  γ = Sqrt[2 Ip ω]/F;
  classifierFunction = Function[{t, τ, Ω}, Which@@Flatten[{Table[
          {And[0.95 < ω Re[τ] < 2.68, Floor[ω Re[t - τ], 2 π/3]/(2 π/3) == k - 1], "A" <> ToString[k]
          , {k, 1, 6}], {True, "Discard"}}]];
  sortingFunction =
    Function[list, SortBy[list, Function[Re[ω #[[1]] - Floor[ω Re[#[[1]] - #[[2]]], 2 π]]]]];

  Show[
    Table[
      selection = ClassifyQuantumOrbits [saddlePoints[m ],
          classifierFunction, sortingFunction, DiscardedLabels → {"Discard"}];
      Graphics[{
          RGBColor[(1 + m)/2, 0, (1 - m)/2],
          Table[ParallelTable[
              Point[
                ReIm [ω #2 &@@ selection[index, Ω, 1]]
              ]
              , {Ω, Keys [selection[index]][[1 ;; ;; 1]]}], {index, Keys [selection][[{1}]]}]
        }]
      , {m , -1, 1}]
    , Frame  → True, Axes → True
    , ImageSize  → 600
    , FrameLabel  → {"Re(ωτ)", "Im  (ωτ)"}
  ]

]
```

Refactored sort

This puts the orbital number m as the innermost index so that the $m = 0$ saddles can be used in the

```
Block[{Ip, F, ω, κ, γ, saddles, classifierFunction, sortingFunction, selection},
  {Ip, F, ω} = parameters ; κ = Sqrt[2 Ip] ; γ = κω/F;
  classifierFunction = Function[{t, τ, Ω}, Which@@Flatten[{Table[
          {And[0.95 < ωRe[τ] < 2.68, Floor[ωRe[t-τ], 2π/3]/(2π/3) == k-1], "A" <> ToString[k]
          , {k, 1, 6}], {True, "Discard"}}]];
  sortingFunction =
    Function[list, SortBy[list, Function[Re[ω#[[1]] - Floor[ωRe[#[[1]] - #[[2]]], 2π]]]]];
  selection = Query[Transpose] /@Query[Transpose] [Association[ParallelTable[
        m → KeySort[ClassifyQuantumOrbits [saddlePoints[m ], classifierFunction,
             sortingFunction, DiscardedLabels → {"Discard"}]]
        , {m , -1, 1}]]];

  Column [Table[
      Show[
        Graphics[
          Table[
            Table[{
                Blend[{Hue[ToExpression[StringTake[index, {2}]]/6],
                    GrayLevel[(1+m)/2]}, 0.5],
                Tooltip[
                  Line[
                    Table[
                      Function[{t, τ},
                          ReIm [ω (time /. {"tt" → t-τ, "t" → t, "τ" → τ})]
                        ]@@selection[index, Ω, m , 1]
                      , {Ω, Keys[selection[index]]}]
                  ]
                  , {index, m }]
              }, {m , -1, 1}]
            , {index, Keys[selection]}]
        ]
        , Frame → True, Axes → True
        , ImageSize → 900
        , FrameLabel → {"Re(ω" <> time <> ")", "Im (ω" <> time <> ")"}
      ]


      , {time , {"tt", "t", "τ"}}]]

]
```

Harmonic dipoles and spectra

Calculation of dipole data

Bare calculation

In the dipole data, $m = 2$ corresponds to the full-2p response for simpler data handling.

```
Block[{Ip, F, ω, κ, γ, classifierFunction, sortingFunction},
  {Ip, F, ω} = parameters ; κ = Sqrt[2 Ip] ; γ = κ ω / F;
  classifierFunction = Function[{t, τ, Ω}, Which@@Flatten[{Table[
              {And[0.95 < ω Re[τ] < 2.68, Floor[ω Re[t - τ], 2 π/3] / (2 π/3) == k - 1], "A" <> ToString[k]}
              , {k, 1, 6}], {True, "Discard"}}]];
  sortingFunction =
    Function[list, SortBy[list, Function[Re[ω #[[1]] - Floor[ω Re[#[[1]] - #[[2]]], 2 π]]]]];

  Print[AbsoluteTiming[
      selection = Query[Transpose] /@ Query[Transpose][Association[ParallelTable[
                  m → KeySort[ClassifyQuantumOrbits[saddlePoints[m ],
                          classifierFunction, sortingFunction,
                          DiscardedLabels → {"Discard"}]][[All, All, 1]]
                  , {m , -1, 1}]]];
    ]];

  Print[DateString[]];
  Print[AbsoluteTiming[
      Dimensions /@ {
          SPAdipoleData = Map[Total,
              Query[Transpose] /@ Association[Table[
                    m → Association[ParallelTable[
                          index → Association[Table[
                                Ω → laboratoryFrameHarmonicDipole[
                                      S, m , Ω, selection[index, Ω]]
                                , {Ω, Keys[selection[index]][[1 ;; -1]]}]]
                          , {index, Keys[selection[[1 ;; -1]]]}]]
                    , {m , -1, 1}]]
              , {2}],
          Print["Done with data 1 at ", DateString[]];
          singleSPAdipoleData = Map[Total,
              Query[Transpose] /@ Association[Table[
                    m → Association[ParallelTable[
                          index → Association[Table[
                                Ω → laboratoryFrameHarmonicDipole[
                                      S, m , Ω, selection[index, Ω]]
                                , {Ω, Keys[selection[index]][[1 ;; -1]]}]]
                          , {index, Keys[selection[[1 ;; 1]]]}]]
```

```
                    , {m , -1, 1}]]
                , {2}],
            Print["Done with data 2 at ", DateString[]];
            singleSPAdipoleDataNoIntrinsicPhase = Map[Total,
                Query[Transpose] /@Association[Table[
                    m → Association[ParallelTable[
                        index→ Association[Table[
                            Ω → laboratoryFrameHarmonicDipoleNoIntrinsicPhase [
                                S, m , Ω, selection[index, Ω]]
                            , {Ω, Keys[selection[index]][[1 ;; -1]]}]]
                        , {index, Keys[selection[[1 ;; 1]]]}]]
                    , {m , -1, 1}]]
                , {2}]
        }
    ]];
  Print["Done with data 3 at ", DateString[]];
  SPAdipoleData[2] = SPAdipoleData[1] + SPAdipoleData[-1];
  singleSPAdipoleData[2] = singleSPAdipoleData[1] + singleSPAdipoleData[-1];
  singleSPAdipoleDataNoIntrinsicPhase[2] =
    singleSPAdipoleDataNoIntrinsicPhase[1] + singleSPAdipoleDataNoIntrinsicPhase[-1];

]
Print[DateString[]];
```

```
{4.253887, Null}

Fri 5 May 2017 18:53:13

Done with data 1 at Fri 5 May 2017 18:57:10

Done with data 2 at Fri 5 May 2017 18:59:37

{530.104031, {{3, 1633}, {3, 1633}, {3, 1633}}}

Done with data 3 at Fri 5 May 2017 19:02:03

Fri 5 May 2017 19:02:03
```

```
{4.013075, Null}

Fri 21 Apr 2017 22:26:22

Done with data 1 at Fri 21 Apr 2017 22:30:17

Done with data 2 at Fri 21 Apr 2017 22:32:42

{522.983914, {{3, 1633}, {3, 1633}, {3, 1633}}}

Done with data 3 at Fri 21 Apr 2017 22:35:05

Fri 21 Apr 2017 22:35:05
```

Data handling

Data handling

In-notebook save:

```
With[{data = Compress [SPAdipoleData]},
  Button["Restore SPA dipole data", Set[SPAdipoleData, Uncompress [data]];
    SPAdipoleData; ]
]
With[{data = Compress [singleSPAdipoleData]},
  Button["Restore single SPA dipole data",
    Set[singleSPAdipoleData, Uncompress [data]];
    singleSPAdipoleData; ]
]
With[{data = Compress [singleSPAdipoleDataNoIntrinsicPhase]},
  Button["Restore single SPA dipole data without intrinsicphase",
    Set[singleSPAdipoleDataNoIntrinsicPhase, Uncompress [data]];
    singleSPAdipoleDataNoIntrinsicPhase]
]
```

Restore SPA dipole data

Restore single SPA dipole data

Restore single SPA dipole data without intrinsic phase

External export:

```
Save[NotebookDirectory[] <> "data - SPA dipole data on the laboratory frame .txt",
  SPAdipoleData]
Save[NotebookDirectory[] <>
    "data - SPA dipole data for a single ionizationburst on the
        laboratory frame .txt", singleSPAdipoleData]
Save[NotebookDirectory[] <> "data - SPA dipole data for a single burst,
        with no intrinsicphase, on the laboratory frame .txt",
  singleSPAdipoleDataNoIntrinsicPhase]
```

Import from external export:

```
<< (NotebookDirectory[] <> "data - SPA dipole data on the laboratory frame .txt");
<< (NotebookDirectory[] <>
        "data - SPA dipole data for a single ionizationburst on the
            laboratory frame .txt");
<< (NotebookDirectory[] <> "data - SPA dipole data for a single burst, with
            no intrinsicphase, on the laboratory frame .txt");
```

Check the import worked correctly:

```
SPAdipoleData // Dimensions
singleSPAdipoleData // Dimensions
singleSPAdipoleDataNoIntrinsicPhase // Dimensions
```

```
{4, 1633}

{4, 1633}

{4, 1633}
```

Spectra

```
sizeI = 170;

Block[{Ip, F, ω, κ, γ, saddles, classifierFunction, sortingFunction},
  {Ip, F, ω} = parameters ; κ = Sqrt[2 Ip] ; γ = κ ω / F;
  classifierFunction = Function[{t, τ, Ω}, Which @@ Flatten[{Table[
        {And[0.95 < ω Re[τ] < 2.68, Floor[ω Re[t - τ], 2 π / 3] / (2 π / 3) == k - 1], "A" <> ToString[k]
        , {k, 1, 6}], {True, "Discard"}}]];
  sortingFunction =
    Function[list, SortBy[list, Function[Re[ω #[[1]] - Floor[ω Re[#[[1]] - #[[2]]], 2 π]]]]];
  (*selection = Query[Transpose] /@ Query[Transpose] [Association[ParallelTable[
        m →
          KeySort[ClassifyQuantumOrbits [saddlePoints[m ], classifierFunction,
                sortingFunction, DiscardedLabels → {"Discard"}]] [[All, All, 1]]
        , {m , -1, 1}]]];*)


  Column @AbsoluteTiming @Column [Table[Column [{
          figureI[m ] = Show[
              Join[{},
                Transpose[Table[{
                    Graphics[{
                        RGBColor[s, 0, 1 - s],
                        Thickness[0.0025],
                        Tooltip[
                          Line[
                            DeleteCases[
                              ParallelTable[
                                {Ω / ω, Log10[Abs[
                                  {1, -s i, 0} / Sqrt[2] . SPAdipoleData[m , Ω]
                                ]^2]}
                                , {Ω, Select[Keys[SPAdipoleData[m ]][[
```

```
                    1 ;; ;; 1]], 12 ω ≤ # ≤ 60 ω &]}]
                  , {_, _? (# < -28 &)}]
                ]
                , {m , s}]
            }]
          ,
          Graphics[{
              Opacity[0.7],
              RGBColor[s, 0, 1 - s],
              Thickness[0.003],
              Tooltip[
                Line[
                  ParallelTable[
                    {Ω/ω, Log10[Abs[
                      {1, -s i, 0}/√2 . (6 singleSPAdipoleData[m , Ω])
                      ]^2]}
                    , {Ω, Select[Keys[singleSPAdipoleData[m ]][[
                      1 ;; ;; 1]], 12 ω ≤ # ≤ 60 ω &]}]
                  ]
                  , {m , s}]
            }]
        }
        , {s, {1, -1}}]], {
      Graphics[{Black, Thickness[0.001],
          Line[{{Ip/ω, -10}, {Ip/ω, -5.6}}]}],
      Graphics[{White, FilledCurve[{{Line[ImageScaled /@
                  {{0, -0.1}, {1, -0.1}, {1, 1}, {0, 1}}]},
              {Line[Scaled/@({{0, 0}, {1, 0}, {1, 1}, {0, 1}})]}}]}]
  }]
, ImageSize → (Automatic → {sizeI, Automatic })
, AspectRatio→ 1/1.6
, Frame → True
, PlotRange → {{12, 60}, {-9.2, -5.6}}
, PlotRangeClipping→ False
, FrameTicks → {{If[m == -1 || m == 0, ## &[], None], Join[
          {#, If[m == -1 || m == 0, MaTeX["10^{" <> ToString[# + 6] <>
                      "}" /. {"10^{0}" → "1"}, FontSize→ tfs],
                  ""], {0.01, 0}} & /@Range[-10, -6],
          {#, "", {0.005, 0}} & /@Flatten[Outer[Plus,
                Log10[Range[2., 9.]], Range[-11, -5]]]
        ], If[m == -1 || m == 0, None, ## &[]]}, {Join[
          {#, ""(*MaTeX[ToString[#],FontSize→tfs]*),
                {0.016, 0}} & /@Range[0, 60, 3],
```

```
            {#, "", {0.009, 0}} &/@Range[1, 60, 1]
          ], Join[{#, "", {0.016, 0}} &/@Range[0, 60, 3],
            {#, "", {0.009, 0}} &/@Range[1, 60, 1]]}}
  , FrameLabel → {None, If[m == -1 || m == 0,
          MaTeX["|\\mathbf  D(\\Omega )|^2\\ \\mathrm  {(arb.u.)}",
            FontSize→ lfs], None]}
  , GridLines→ {Range[3/2, 90, 3], Join[
          Range[-12, -6],
          Flatten[
            Outer[Plus, Log10[Range[2., 9.]], Range[-11, -5]]]
        ]}
  , GridLinesStyle→ Directive[GrayLevel[0.8]]
  , ImagePadding → {
        {If[m == -1 || m == 0, Scaled[0.085], Scaled[0.006]],
          Scaled[0.006]},
        {Scaled[0.001], Scaled[0.001]}
      }
  , Epilog→ {
        Inset[MaTeX["m =" <> ToString[m ] /. {"m =2" → "2p"}, FontSize→ if
          Scaled[{0.5, 0.98}], Scaled[{0.5, 1}]],
        Inset[MaTeX["\\text{(" <> {"a", "c", "b", "d"}[[m +2]] <> ")}",
            FontSize→ ifs], Scaled[{0.04, 0.98}], Scaled[{0, 1}]],
        If[m == 1, Inset[Grid[{
                {Graphics[{Red, Thickness[0.05], Line[{{0, 0}, {1, 0}}]}
                    PlotRange → {{0, 1}, 0.1 {-1, 1}},
                    PlotRangePadding→ None, ImageSize → 10],
                  MaTeX["\\rcirclearrowleft", FontSize→ 5]},
                {Graphics[{Blue, Thickness[0.05], Line[{{0, 0},
                        {1, 0}}]}, PlotRange → {{0, 1}, 0.1 {-1, 1}},
                    PlotRangePadding→ None, ImageSize → 10],
                  MaTeX["\\lcirclearrowright", FontSize→ 5]}
              }, Background → White, Spacings→ {0.5, 0},
              Frame → True, FrameStyle → Directive[
                  Gray, Thickness[0.00001]]]
            , Scaled[{0.99, 0.99}], Scaled[{1, 1}]], ## &[]]
      }
  ]

,

figureIellipticity[m ] = Show[{
      Graphics[{Black, Thickness[0.001], Line[
            {{Ip / ω, -1.2}, {Ip / ω, 1.2}}], Line[{{12, 0}, {60, 0}}]}],
      Graphics[{
          Black,
          RGBColor[0.0, 0., 0.8],
          RGBColor[0.025, 0.01, 0.01],
          Thickness[0.003],
          Line[
            ParallelTable[
```

```
                {Ω/ω, getEllipticity[
                    singleSPAdipoleData[m , Ω][[{1, 2}]]]}
                , {Ω, Select[Keys[singleSPAdipoleData[m ]][[
                        1 ;; ;; 1]], 12 ω ≤ # ≤ 60 ω &]}]
            ]
        }]
    }
    , ImageSize → (Automatic → {sizeI, Automatic })
    , AspectRatio→1/5
    , Frame →True
    , PlotRange→{{12, 60}, {-1.2, 1.2}}
    , PlotRangeClipping→True
    , PlotRangePadding→{{None, None}, {Scaled[0.005], Scaled[0.005]}}
    , GridLines→{Range[3/2, 90, 3], Range[-1, 1, 0.2]}
    , GridLinesStyle→Directive[GrayLevel[0.8]]
    , Axes→False
    , Axes→True
    , Method→{"AxesInFront"→False}
    , AxesOrigin→{0, 0}
    , FrameLabel →{If[m ⩵0||m ⩵2, MaTeX["\\Omega /\\omega ",
            FontSize→lfs], None], If[m ⩵-1||m ⩵0,
          MaTeX["\\varepsilon", FontSize→lfs], None]}
    , FrameTicks →{
        {If[m ⩵-1||m ⩵0, ## &[], None],
          Join[{#, If[m ⩵-1||m ⩵0, MaTeX[#, FontSize→ilfs], ""],
                {0.01, 0}} &/@Range[-1, 1],
            {#, "", {0.005, 0}} &/@Range[-1, 1, 0.2]],
          If[m ⩵1||m ⩵2, ## &[], None]
        },
        {Join[
            {#, MaTeX[ToString[#], FontSize→tfs],
                {0.016, 0}} &/@Range[0, 60, 3],
            {#, "", {0.009, 0}} &/@Range[1, 60, 1]
          ], Join[{#, "", {0.016, 0}} &/@Range[0, 60, 3],
            {#, "", {0.009, 0}} &/@Range[1, 60, 1]]}}

    , ImagePadding →{
        (*{If[m ==-1||m ==0,37,3],3},*)
        {If[m ⩵-1||m ⩵0,
            Scaled[0.085], Scaled[0.006]], Scaled[0.006]},
        (*{Scaled[0.001],Scaled[0.001]}*)
        {If[m ⩵0||m ⩵2,
            Scaled[0.06], Scaled[0.001]], Scaled[0.001]}
        (*{If[m ==0||m ==2,200,200],5}*)
      }
    ]
  }], {m , -1, 2}]]

]
Table[{
```

```
    FileByteCount[
      Export[FileNameJoin[{$OutputDirectory, "figureI" <> {"a", "c", "b", "d"}[[m + 2]] <>
              "-spectra-" <> {"p-", "s", "p+", "2p"}[[m + 2]] <> ".pdf"}], figureI[m ]]],
    FileByteCount[Export[FileNameJoin[{$OutputDirectory,
            "figureI" <> {"a", "c", "b", "d"}[[m + 2]] <> "-ellipticity" <>
              {"p-", "s", "p+", "2p"}[[m + 2]] <> ".pdf"}], figureIellipticity[m ]]]
  }, {m , -1, 2}]
pdflatex[]
```

Internal structure of the harmonic dipole

Function definitions

```
recombinationDipole[S_, mm_ , Ω_, saddleAssociation_] :=
  Block[{t, τ, t0, τ0, m = mm },
    {t, τ} = saddleAssociation[m ];
    {t0, τ0} = saddleAssociation[0];

    shortRangeDipoleConj[m ][pi[(*Rzα[-τ].*)ps[t, t-τ], t, t-τ], κ]
  ]

ionizationFactor[S_, mm_ , Ω_, saddleAssociation_] := Block[{t, τ, t0, τ0, m = mm },
    {t, τ} = saddleAssociation[m ];
    {t0, τ0} = saddleAssociation[0];

    shortRangeϒ[m , κ, pi[ps[t0, t0-τ0], t0-τ0, t0-τ0]]
  ]

ionizationFactorTotal[S_, mm_ , Ω_, saddleAssociation_] :=
  Block[{t, τ, t0, τ0, m = mm },
    {t, τ} = saddleAssociation[m ];
    {t0, τ0} = saddleAssociation[0];

    (1/τ)^(3/2) shortRangeϒ[m , κ, pi[ps[t0, t0-τ0], t0-τ0, t0-τ0]]
      Abs[Exp[-i S[t, t-τ] + i Ω t]]
  ]

ionizationFactorϒExponential[S_, mm_ , Ω_, saddleAssociation_] :=
  Block[{t, τ, t0, τ0, m = mm },
    {t, τ} = saddleAssociation[m ];
    {t0, τ0} = saddleAssociation[0];

    shortRangeϒ[m , κ, pi[ps[t0, t0-τ0], t0-τ0, t0-τ0]] × Abs[Exp[-i m ω/2 (t-τ)]]
  ]
```

```
ionizationFactorAction[S_, mm_ , Ω_, saddleAssociation_] :=
  Block[{t, τ, t0, τ0, m = mm },
    {t, τ} = saddleAssociation[m ];
    {t0, τ0} = saddleAssociation[0];

    (1/τ)^(3/2) Abs[Exp[-ⅈ S[t, t-τ] + ⅈ Ω t]]
  ]
(*rotatingFrameHarmonicDipoleNoIntrinsicPhase [S_,mm_ ,Ω_,saddleAssociation_]:=
  Block[{t,τ,t0,τ0,m =mm },
    {t,τ}=saddleAssociation[m ];
    {t0,τ0}=saddleAssociation[0];

    ⅈ (2π/(ⅈ τ))^(3/2) HessianRoot[S,t,τ]×
      shortRangeDipoleConj[m ][pi[Rzα[-τ].ps[t,t-τ],t,t-τ],κ]×
      shortRangeϒ[m ,κ,pi[ps[t0,t0-τ0],t0-τ0,t0-τ0]]×Abs[Exp[-ⅈ S[t,t-τ]+ⅈ Ω t]]
  ]*)
```

Figure J

Calculations and data handling

```
Block[{Ip, F, ω, κ, ϒ, saddles, classifierFunction, sortingFunction},
  {Ip, F, ω} = parameters ; κ = Sqrt[2 Ip] ; ϒ = κω/F;
  classifierFunction = Function[{t, τ, Ω}, Which@@Flatten[{Table[
               {And[0.95 < ω Re[τ] < 2.68, Floor[ω Re[t-τ], 2π/3]/(2π/3) == k-1], "A" <> ToString[k]
               , {k, 1, 6}], {True, "Discard"}}]];
  sortingFunction =
    Function[list, SortBy[list, Function[Re[ω #〚1〛 - Floor[ω Re[#〚1〛 - #〚2〛], 2π]]]]];
  Print[AbsoluteTiming [
      selection = Query[Transpose] /@Query[Transpose] [Association[ParallelTable[
                  m  → KeySort[ClassifyQuantumOrbits [saddlePoints[m ],
                          classifierFunction, sortingFunction,
                          DiscardedLabels→ {"Discard"}]]〚All, All, 1〛
                  , {m , -1, 1}]]];
    ]];

  Print[DateString[]];

  Print[AbsoluteTiming [Dimensions [
        internalStructureData = Association[Table[
              Print["Starting ", function, " at ", DateString[]];
              function→Map[Total,
                  Query[Transpose] /@Association[Table[
                        m  →Association[ParallelTable[
```

```
                                index→Association[Table[
                                      Ω→function[S, m , Ω, selection[index, Ω]]
                                      , {Ω, Keys[selection[index]][[1 ;; -1 ;; 1]]}]]
                                , {index, Keys[selection[[1 ;; 1]]]}]]
                          , {m , -1, 1}]]
                    , {2}]
                , {function, {recombinationDipole,
                    ionizationFactorϒ, ionizationFactorϒExponential,
                    ionizationFactorTotal, ionizationFactorAction}}]]
        ]]];

    Print[DateString[]];

    Print[AbsoluteTiming[Dimensions[
          figureJlineData= Association[Table[
                Print["Starting ", function, " at ", DateString[]];
                function→Association[Table[
                      m →Association[Table[
                            s→ParallelTable[
                                {Ω/ω, Log10[Abs[
                                        If[function=== recombinationDipole,
                                        ({1, -s i, 0}/Sqrt[2]).
                                        internalStructureData[function, m , Ω],
                                        internalStructureData[function, m , Ω]
                                        ]
                                      ]^2]}

                                , {Ω, Select[Keys[internalStructureData[function,
                                        m ]][[1 ;; ;; 1]], 12 ω ≤ # ≤ 61 ω &]}]
                            , {s, {1, -1}}]]
                      , {m , -1, 1}]]
                , {function, {recombinationDipole,
                    ionizationFactorϒ, ionizationFactorϒExponential,
                    ionizationFactorTotal, ionizationFactorAction}}]]
        ]]];

    Print[DateString[]];

]
{4.094288, Null}
Fri 5 May 2017 19:33:48
Starting recombinationDipole at Fri 5 May 2017 19:33:48
Starting ionizationFactorϒ at Fri 5 May 2017 19:35:58
```

```
Starting ionizationFactorϒExponentialat Fri 5 May 2017 19:38:05

Starting ionizationFactorTotal at Fri 5 May 2017 19:40:13

Starting ionizationFactorActionat Fri 5 May 2017 19:42:22

{642.955058, {5, 3, 1633}}

Fri 5 May 2017 19:44:31

Starting recombinationDipole  at Fri 5 May 2017 19:44:31

Starting ionizationFactorϒ at Fri 5 May 2017 19:44:59

Starting ionizationFactorϒExponentialat Fri 5 May 2017 19:45:26

Starting ionizationFactorTotal at Fri 5 May 2017 19:45:52

Starting ionizationFactorActionat Fri 5 May 2017 19:46:20

{137.325361, {5, 3, 2}}

Fri 5 May 2017 19:46:48
```

```
Save[NotebookDirectory[] <>
    "data - internal structure of the harmonic  dipole - laboratory frame .txt",
  internalStructureData]
Save[NotebookDirectory[] <>
    "data - line data for figure J, internal structure - laboratory frame .txt",
  figureJlineData]
```

```
<< (NotebookDirectory[] <> "data - internal structure of
             the harmonic  dipole - laboratory frame .txt");
<< (NotebookDirectory[] <> "data - line data for figure J, internal
             structure - laboratory frame .txt");
```

Figure

```
sizeJ = 115;
paddingJ[mfront_ ] := {If[mfront  == -1, 30, 3.5], 3.5}
labelPositionJ = -0.16;
```

```
ClearAll[sizeE, paddingE, labelPositionE]
```

```
Block[{Ip, F, ω, κ, γ, saddles, classifierFunction, sortingFunction},
  {Ip, F, ω} = parameters ; κ = Sqrt[2 Ip] ; γ = κω/F;
  classifierFunction = Function[{t, τ, Ω}, Which@@Flatten[{Table[
              {And[0.95 < ωRe[τ] < 2.68, Floor[ωRe[t-τ], 2π/3]/(2π/3) == k-1], "A" <> ToString[k]
              , {k, 1, 6}], {True, "Discard"}}]];
  sortingFunction =
    Function[list, SortBy[list, Function[Re[ω#[[1]] - Floor[ωRe[#[[1]] - #[[2]]], 2π]]]]];
```

```
Row[Table[
    Column [{

        figureJ1[mfront ] = Show[
            Join[
              Table[
                If[m == 1, Reverse, # &]@Table[
                    {
                      Graphics[{
                          Thickness[0.006],
                          If[m == mfront , RGBColor[
                              s, 0, 1-s], GrayLevel[0.7]],
                          Tooltip[Line[figureJlineData[recombinationDipole,
                                m , s]], {m , s}]
                        }]
                    }
                    , {s, {1, -1}}]
                , {m , Join[DeleteCases[Range[-1, 1], mfront ], {mfront }]}], {
                Graphics[{White, FilledCurve[
                      {{Line[ImageScaled /@{{0, -0.1}, {1, -0.1}, {1, 1}, {0, 1}}]},
                        {Line[Scaled/@({{0, 0}, {1, 0}, {1, 1}, {0, 1}})]}}]}],
                Graphics[{Inset[Rotate[MaTeX["|\\mathbf {d}_\\mathrm {rec}|^2",
                          FontSize→lfs], 90°],
                      Scaled[{labelPositionJ, 0.5}], Scaled[{1, 0.5}]]}],
                Graphics[{Inset[MaTeX["m ="<>ToString[mfront ], FontSize→lfs],
                      Scaled[{0.5, 1.005}], Scaled[{0.5, 0}]]}],
                Graphics[{Inset[MaTeX["(\\mathrm {"<>
                          {"a", "b", "c"}[[mfront +2]]<>"})", FontSize→lfs],
                      Scaled[{0, 1}], Scaled[{-0.1, 1.1}]]}]
              }
            ]
            , ImageSize → (Automatic → sizeJ)
            , AspectRatio→1/1.6
            , Frame →True
            , PlotRange→{{12, 60}, {-4.9, -1.55}}
            , FrameTicks →{{Join[
                    {#, MaTeX["10^{"<>ToString[#]<>"}" /. {"10^{0}"→"1"}, FontSize
                          {0.01, 0}} &/@Range[-6, 0],
                    {#, "", {0.005, 0}} &/@Flatten[Outer[Plus,
                          Log10[Range[2., 9.]], Range[-6, 0]]]
                  ], None}, {Join[
                    {#, "", {0.016, 0}} &/@Range[0, 60, 3],
                    {#, "", {0.009, 0}} &/@Range[1, 60, 1]
                  ], Join[{#, "", {0.016, 0}} &/@Range[0, 60, 3],
                    {#, "", {0.009, 0}} &/@Range[1, 60, 1]]}}
            , GridLines→{Range[3/2, 90, 3], Flatten[Outer[Plus,

                    Log10[Range[1., 9.]], Range[-7, 0]]]}
            , GridLinesStyle→Directive[GrayLevel[0.8]]
            , ImagePadding →{paddingJ[mfront ], {Scaled[0.001], Scaled[0.02]}}
```

```
  ]

,

figureJ2[mfront ] = Show[
    Join[
      Table[
        {
          Graphics[{
              Thickness[0.006],
              If[m == mfront , Black, GrayLevel[0.7]],
              Tooltip[
                Line[figureJlineData[ionizationFactorΥ m , 1]], {m }]
            }]
        }
        , {m , Join[DeleteCases[Range[-1, 1], mfront ], {mfront }]}], {
        Graphics[{White, FilledCurve[
              {{Line[ImageScaled /@{{0, -0.1}, {1, -0.1}, {1, 1}, {0, 1}}]},
                {Line[Scaled/@({{0, 0}, {1, 0}, {1, 1}, {0, 1}})]}}]}],
        Graphics[{Inset[Rotate[MaTeX["|\\Upsilon_\\mathrm  {ion}|^2",
                FontSize→lfs], 90°],
              Scaled[{labelPositionJ, 0.5}], Scaled[{1, 0.5}]]}],
        Graphics[{Inset[MaTeX["(\\mathrm  {"<>
                {"d", "e", "f"}[[mfront +2]]<>"})", FontSize→lfs],
              Scaled[{0, 1}], Scaled[{-0.1, 1.1}]]}]
      }]
    , ImageSize → (Automatic →sizeJ)
    , AspectRatio→1/2
    , Frame →True
    , PlotRange→{{12, 60}, {-1.95, -0.65}}
    , FrameTicks →{{Join[
              {#, "", {0.016, 0}} &/@Range[-2, 1],
              {#, If[EvenQ[Rationalize[100×10^#]],
                      MaTeX[PaddedForm [10^#, {3, 2}],
                        FontSize→tfs], ""], {0.008, 0}} &/@
                Flatten[Outer[Plus, Log10[Range[1., 9.]], Range[-2, 1]]]
            ], None}, {Join[
              {#, MaTeX[ToString[#], FontSize→tfs], {0.016, 0}} &/@
                Range[0, 60, 3],
              {#, "", {0.009, 0}} &/@Range[1, 60, 1]
            ], Join[{#, "", {0.016, 0}} &/@Range[0, 60, 3],
              {#, "", {0.009, 0}} &/@Range[1, 60, 1]]}}
    , GridLines→{Range[3/2, 90, 3], Join[
          Range[-2, 1],
          Flatten[Outer[Plus, Log10[Range[2., 9.]], Range[-2, 1]]]
        ]}
    , GridLinesStyle→Directive[GrayLevel[0.8]]
    , ImagePadding →{paddingJ[mfront ], {Scaled[0.001], Scaled[0.001]}}
  ]
```

```
,

figureJ3[mfront_] = Show[
    Join[
      Table[
        {
          Graphics[{
              Thickness[0.006],
              If[m == mfront, Black, GrayLevel[0.7]],
              Tooltip[Line[
                  figureJlineData[ionizationFactorAction, m, 1]], {m}]
            }]
        }
        , {m, Join[DeleteCases[Range[-1, 1], mfront], {mfront}]}], {
        Graphics[{White, FilledCurve[
              {{Line[ImageScaled /@ {{0, -0.1}, {1, -0.1}, {1, 1}, {0, 1}}]},
                {Line[Scaled /@ ({{0, 0}, {1, 0}, {1, 1}, {0, 1}})]}}]}],
        Graphics[{Inset[Rotate[MaTeX["|e^{-iS}|^2", FontSize → lfs], 90°],
              Scaled[{labelPositionJ, 0.5}], Scaled[{1, 0.5}]]}],
        Graphics[{Inset[MaTeX["(\\mathrm{"<>
                  {"g", "h", "i"}[[mfront + 2]] <> "})", FontSize → lfs],
              Scaled[{0, 1}], Scaled[{-0.1, 1.1}]]}]
      }]
    , ImageSize → (Automatic → sizeJ)
    , AspectRatio → 1/3
    , Frame → True
    , PlotRange → {{12, 60}, {-10.9, -8.9}}
    , FrameTicks → {{Join[
            {#, "", {0.016, 0}} & /@ Range[-11, -9],
            {#, If[MemberQ[{1, 2, 5}, Round[10 MantissaExponent[
                          10^(#+9)][[1]]]], MaTeX[PaddedForm[10^(#+9),
                      {3, 2}], FontSize → tfs], ""], {0.008, 0}} & /@
              Flatten[Outer[Plus, Log10[Range[1., 9.]], Range[-11, -9]]]
          ], None}, {Join[
            {#, MaTeX[ToString[#], FontSize → tfs], {0.016, 0}} & /@
              Range[0, 60, 3],
            {#, "", {0.009, 0}} & /@ Range[1, 60, 1]
          ], Join[{#, "", {0.016, 0}} & /@ Range[0, 60, 3],
            {#, "", {0.009, 0}} & /@ Range[1, 60, 1]]}}
    , GridLines → {Range[3/2, 90, 3], Flatten[Outer[Plus,
            Log10[Range[1., 9.]], Range[-11, -6]]]}
    , GridLinesStyle → Directive[GrayLevel[0.8]]
    , ImagePadding → {paddingJ[mfront], {Scaled[0.001], Scaled[0.001]}}
  ]

,

figureJ4[mfront_] = Show[
    Join[
      Table[
```

```
{
  Graphics[{
      Thickness[0.006],
      If[m == mfront , Black, GrayLevel[0.7]],
      Tooltip[
        Line[figureJlineData[ionizationFactorTotal, m , 1]], {m }]
    }]
}
, {m , Join[DeleteCases[Range[-1, 1], mfront ], {mfront }]}], {
Graphics[{White, FilledCurve[
      {{Line[ImageScaled /@{{0, -0.1}, {1, -0.1}, {1, 1}, {0, 1}}]},
        {Line[Scaled/@ ({{0, 0}, {1, 0}, {1, 1}, {0, 1}})]}}]}],
Graphics[{Inset[MaTeX["\\Omega /\\omega ", FontSize→lfs],
      Scaled[{0.5, -0.30}], Scaled[{0.5, 1}]]}],
Graphics[{Inset[Rotate[MaTeX["|\\Upsilon_\\mathrm {ion}
             e^{-iS}|^2", FontSize→lfs], 90 °],
      Scaled[{labelPositionJ, 0.5}], Scaled[{1, 0.5}]]}],
Graphics[{Inset[MaTeX["(\\mathrm {"<>
           {"j", "k", "l"}[[mfront +2]]<>"})", FontSize→lfs],
      Scaled[{0, 1}], Scaled[{-0.1, 1.1}]]}]
 }]
, ImageSize → (Automatic →sizeJ)
, AspectRatio→1/3
, Frame →True
, PlotRange→{{12, 60}, {-11.45, -10.7}}
, FrameTicks →{{Join[
        {Log10[1.6]+#, "", {0.016, 0}}&/@Range[-12, 1],
        {Log10[1.6]+#, If[MemberQ [{1, 2, 4, 6, 8},
                  Round[10 MantissaExponent[10^(#+11)][[1]]]],
               MaTeX[PaddedForm [10^(#+11), {2, 1}],
                  FontSize→tfs], ""], {0.008, 0}}&/@
          Flatten[Outer[Plus, Log10[Range[1., 9.]], Range[-12, -10]]]
      ], None}, {Join[
        {#, MaTeX[ToString[#], FontSize→tfs], {0.016, 0}}&/@
          Range[0, 60, 3],
        {#, "", {0.009, 0}}&/@Range[1, 60, 1]
      ], Join[{#, "", {0.016, 0}}&/@Range[0, 60, 3],
        {#, "", {0.009, 0}}&/@Range[1, 60, 1]]}}
, GridLines→{Range[3/2, 90, 3], Log10[1.6]+
      Flatten[Outer[Plus, Log10[Range[1., 9.]], Range[-13, -6]]]}
, GridLinesStyle→Directive[GrayLevel[0.8]]
, ImagePadding →{paddingJ[mfront ], {Scaled[0.05], Scaled[0.001]}}
]

 }]
, {mfront , -1, 1}]]

]
Table[FileByteCount[Export[FileNameJoin [{$OutputDirectory,
```

```
        "figureJ"<>{"a", "b", "c"}[[m +2]]<>"-recombination-dipole-"<>
          {"p-", "s", "p+"}[[m +2]]<>".pdf"}], figureJ1[m ]]], {m , -1, 1}]
Table[FileByteCount[Export[FileNameJoin [{$OutputDirectory,
        "figureJ"<>{"d", "e", "f"}[[m +2]]<>"-ionizationfactor-"<>
          {"p-", "s", "p+"}[[m +2]]<>".pdf"}], figureJ2[m ]]], {m , -1, 1}]
Table[FileByteCount[Export[FileNameJoin [{$OutputDirectory,
        "figureJ"<>{"g", "h", "i"}[[m +2]]<>"-action-"<>
          {"p-", "s", "p+"}[[m +2]]<>".pdf"}], figureJ3[m ]]], {m , -1, 1}]
Table[FileByteCount[Export[FileNameJoin [{$OutputDirectory,
        "figureJ"<>{"j", "k", "l"}[[m +2]]<>"-total-"<>
          {"p-", "s", "p+"}[[m +2]]<>".pdf"}], figureJ4[m ]]], {m , -1, 1}]
pdflatex[]
```

```
{43498, 40438, 40568}

{19364, 15286, 15313}

{13694, 9548, 9463}

{32339, 29045, 28795}

pdflatex successful
```

Trajectories

SlideView overview over the harmonic plateau

```
Block[{Ip, F, ω, κ, γ, saddles, classifierFunction, sortingFunction, data},
  {Ip, F, ω} = parameters ; κ = Sqrt[2 Ip] ; γ = κω/F;
  classifierFunction= Function[{t, τ, Ω}, Which@@Flatten[{Table[
          {And[0.95 < ωRe[τ] < 2.68, Floor[ωRe[t-τ], 2π/3]/(2π/3) == k-1], "A"<>ToString[k]
          , {k, 1, 6}], {True, "Discard"}}]];
  sortingFunction=
    Function[list, SortBy[list, Function[Re[ω#[[1]] - Floor[ωRe[#[[1]] - #[[2]]], 2π]]]]];
  selection= Query[Transpose] /@Query[Transpose] [Association[ParallelTable[
          m  → KeySort[ClassifyQuantumOrbits [saddlePoints[m ], classifierFunction,
                sortingFunction, DiscardedLabels→ {"Discard"}]][[All, All, 1]]
          , {m , -1, 1}]]];

  Block[{index= First[Keys[selection]],
      (*Ω=First[Nearest[Keys[selection[[1]]],18ω]],*)m  = 0},

    SlideView[Table[
        Column  [{
            Row[{
                ParametricPlot [
                    Evaluate[Flatten[Table[
                          Tooltip[
```

```
{{0, -1}, {1, 0}}.part[
    laboratoryFrameTrajectory [
        Interpolation[
          Function[{t, τ}, {
          (*{{0,t-τ},{1,t}},
          {{0,t-τ},{1/3,Re[t-τ]},
          {2/3,Re[t]},{1,t}},*)
          {{0, t-τ}, {1/2, Re[t-τ]+ⅈIm [t]}, {1, t}}
          }[[j]]]@@selection[index, Ω, m ]
          , InterpolationOrder→1][s]
        , Function[{t, τ},
          {t, t-τ}
          ]@@selection[index, Ω, m ]
      ][[{1, 2}]]
    ]
  , part]
 , {j, 1, 1}, {part, {Im , Re}}], 1]]
, {s, 0, 1}
, PlotPoints→60
, PlotStyle→{(*Lighter[Blue,0.7],Lighter[Red,0.7],
    Lighter[Blue,0.35],Lighter[Red,0.35],*)Blue, Red}
, Frame →True
, Method→{"AxesInFront"→False}
, ImageSize →{{1000}, {370}}
, PlotLabel→"Ω="<>ToString[Ω/ω]<>"ω"
, PlotRange→{{-25, 8}, {-8, 6}}
, PlotRange→{{-8, 6}, {-8, 25}}
] /. {Line[pts_]:→{Arrowheads[0.025], Line[pts], Arrow[pts]}}
,
Show[{
    ParametricPlot [
        Evaluate[Flatten[Table[
              Tooltip[
                {{0, -1}, {1, 0}}.part[
                    laboratoryFrameVelocity [
                      Interpolation[
                      Function[{t, τ}, {
                      (*{{0,t-τ},{1,t}},
                      {{0,t-τ},{1/3,Re[t-τ]},
                      {2/3,Re[t]},{1,t}},*)
                      {{0, t-τ}, {1/2, Re[t-τ]+ⅈIm [t]}, {1, t}}
                      }[[j]]]@@selection[index, Ω, m ]
                      , InterpolationOrder→1][s]
                      , Function[{t, τ},
                      {t, t-τ}
                      ]@@selection[index, Ω, m ]
                      ][[{1, 2}]]
                  ]
                , part]
              , {j, 1, 1}, {part, {Im , Re}}], 1]]
        , {s, 0, 1}
        , PlotPoints→60
        , PlotStyle→{(*Lighter[Blue,0.7],Lighter[Red,0.7],
            Lighter[Blue,0.35],Lighter[Red,0.35],*)Blue, Red}
```

```
                    ] /. {Line[pts_] :> {Arrowheads[0.025],
                            Line[pts], Arrow[pts]}}
                    ,
                    Graphics[{PointSize[0.02],
                        Function[{t, τ}, {
                              Red, Point[Re[ps[t, t-τ][[{1, 2}]]]],
                              Blue, Point[Im [ps[t, t-τ][[{1, 2}]]]]
                            }] @@selection[index, Ω, m ]
                        }]
                  }
                  , Frame  → True
                  , Method → {"AxesInFront" → False}
                  , ImageSize  → {{1000}, {370}}
                  , PlotLabel → "Ω=" <> ToString[Ω / ω] <> "ω"
                  , PlotRange → {{-1.5, 2.5}, {-1.5, 1.5}}
                  , PlotRange → {{-1.5, 1.5}, {-2.5, 1.5}}
                ]
              }]
            ,
            ParametricPlot [
                Evaluate[Table[
                    ReIm [ω×
                        Interpolation[
                            Function[{t, τ}, {
                                    (*{{0,t-τ},{1,t}},
                                    {{0,t-τ},{1/3,Re[t-τ]},
                                      {2/3,Re[t]},{1,t}},*)
                                    {{0, t-τ}, {1/2, Re[t-τ] + ⅈ Im [t]}, {1, t}}
                                }[[j]] @@selection[index, Ω, m ]
                            , InterpolationOrder→ 1] [s]
                    ]
                    , {j, 1, 1}]]
                , {s, 0, 1}
                , PlotPoints→ 60
                , PlotStyle → {Lighter[Blue, 0.7], Lighter[Blue, 0.35], Blue}
                , Frame  → True
                , Method → {"AxesInFront" → False}
                , ImageSize  → 700
                , PlotLabel → "Ω=" <> ToString[Ω / ω] <> "ω"
                , PlotRange → {{1, 4}, {-1, 1}}
              ] /. {Line[pts_] :> {Arrowheads[0.025], Line[pts], Arrow[pts]}}
          }]
        , {Ω, First[Nearest[Keys[selection[[1]]], #]] & /@Range[10 ω, 70 ω, 0.5 ω]}]]

    ]
  ]
```

Figure K

```
tfsK = 7;
sizeK= 80;

Block[{Ip, F, ω, κ, γ, saddles, classifierFunction, sortingFunction,
    steppingStones, timePath , lengthsList, totalPathLength},
```

```
{Ip, F, ω} = parameters  ; κ = Sqrt[2 Ip] ; γ = κω/F;
classifierFunction= Function[{t, τ, Ω}, Which@@Flatten[{Table[
        {And[0.95 < ωRe[τ] < 2.68, Floor[ωRe[t-τ], 2π/3]/(2π/3) == k-1], "A" <> ToString[k]
        , {k, 1, 6}], {True, "Discard"}}]];
sortingFunction=
  Function[list, SortBy[list, Function[Re[ω#[[1]] - Floor[ωRe[#[[1]] - #[[2]]], 2π]]]]];
(*selection=Query[Transpose]/@Query[Transpose][Association[ParallelTable[
          m →
            KeySort[ClassifyQuantumOrbits[saddlePoints[m ],classifierFunction,
                sortingFunction,DiscardedLabels→{"Discard"}]][[All,All,1]]
          ,{m ,-1,1}]]];*)

Block[{index= First[Keys[selection]],
    m  = 0(*,Ω=First[Nearest[Keys[selection[[1]]],27ω]]*)},

  figureKinsetBackground= Show[{
      Graphics[{White, Opacity[0.85], Rectangle[{1.25, -0.68}, {4.0, 0.77}]}],
      Graphics[{GrayLevel[0.7], Line[{{1.25, 0}, {4.0, 0}}]}],
      Graphics[
        Table[{
            GrayLevel[0.6],
            AbsoluteThickness[0.85],
            Line[Table[
                Function[{t, τ},
                    ReIm [ω (time /. {"tt" → t-τ, "t" → t, "τ" → τ})]
                  ]@@selection[index, Ω, m ]
                , {Ω, Keys[selection[index]][[1 ;; ;; 1]]}]]
          }, {time , {"t", "tt"}}]
        ]
      }
      , ImageSize → 550
      , Frame → True
      , PlotRangePadding→ None
      , FrameTicks → {{
          {#, MaTeX[ToString[PaddedForm [#, {2, 1}]], FontSize→ 9,
                Magnification→ 0.5]} & /@Range[-1, 1, 0.5],
          {#, ""} & /@Range[-1, 1, 0.5]
        }, {
          {#°, MaTeX["\\SI{" <> ToString[#] <> "}{\\degree}", FontSize→ 9,
                Magnification→ 0.5]} & /@Range[0, 360, 30],
          {#°, ""} & /@Range[0, 360, 30]
        }}
    ];

  Table[
    steppingStones=
      Function[{t, τ}, {t-τ, Re[t-τ] + ⅈ Im [t], t}]@@selection[index, Ω, m ];
    lengthsList= Abs[Rest[steppingStones] -Most[steppingStones]];
    totalPathLength = Total[lengthsList];
```

```
timePath = Interpolation[
    Transpose[{Accumulate [Join[{0}, lengthsList]], steppingStones}],
    InterpolationOrder→1];

Row[{

    figureKinset[Round[Ω/ω]] = Show[{
          figureKinsetBackground,
          ParametricPlot [
              ReIm [ωtimePath [s]]
              , {s, 0, totalPathLength}
              , PlotPoints→60
              , PlotStyle→{Directive[Black, AbsoluteThickness[0.85]]}
              , PlotLabel→"Ω="<>ToString[Ω/ω]<>"ω"
            ] /. {Line[pts_] :→ {Arrowheads[{{0.09, 1}, {0.012, 0, Graphics[
                            {Disk[{0, 0}]}]}}], Line[pts], Arrow[pts]}},
          Graphics[{
              AbsolutePointSize[1.7],
              Point[ReIm [ω×timePath [#]] &/@Most[
                      Rest[Range[0, totalPathLength, totalPathLength/20]]]]
            }]
        }
        , ImageSize →87
      ];

    figureKposition[Round[Ω/ω]] = Show[Join[{
            Graphics[{GrayLevel[0.7],
                Line[{{-15.3, 0}, {1.0, 0}}], Line[{{0, -4.6}, {0, 0.8}}]}]
            (*,PlotRange→{{-15.3,1},{-4.6,1.07}}*)(*,
            PlotRange→{{-15.3,1},{-2.85,3.7}}*)
          }, Table[{
              ParametricPlot [
                  Tooltip[
                    {{0, -1}, {1, 0}}.part[
                        laboratoryFrameTrajectory [timePath [s],
                            Apply[Function[{t, τ}, {t, t-τ}],
                              selection[index, Ω, m ]]][[{1, 2}]]
                      ]
                    , part]
                  , {s, 0, totalPathLength}
                  , PlotPoints→60
                  , PlotStyle→{Directive[part /.
                          {Re→Red, Im →Blue}, AbsoluteThickness[0.85]]}
                ] /. {Line[pts_] :→ {Arrowheads[0.035],
                      Line[pts], Arrow[pts]}},
              Graphics[{
                  AbsolutePointSize[2.2],
                  part /. {Re→Red, Im →Blue},
                  Point[
                    {{0, -1}, {1, 0}}.part[
                          laboratoryFrameTrajectory [timePath [#],
                              Apply[Function[{t, τ}, {t, t-τ}],
```

```
                        selection[index, Ω, m ]]][[{1, 2}]]
                    ] &/@Most[Rest[Range[0, totalPathLength,
                      totalPathLength/20]]]
             ]
           }]
      }, {part, {Im , Re}}], {
      Graphics[Inset[figureKinset[Round[Ω/ω]],
          Scaled[{0.06, 0.98}], Scaled[{0, 1}]]],
      Graphics[{
          Inset[MaTeX["y_{\\mathsf {L}}(t)", FontSize→7],
            Scaled[{0.485, -0.12}], Scaled[{0.5, 1}]],
          Inset[Rotate[MaTeX["x_{\\mathsf {L}}(t)", FontSize→7], 90°],
            Scaled[{-0.06, 0.5}], Scaled[{1, 0.5}]]
        }],
      Graphics[Inset[
          MaTeX["(\\mathrm  {"<>{"a", "c", "e", "g"}[[(Round[Ω/ω]-17)/10 + 1]]<>
              "})\\ \\ \\mathbf {r}_\\mathsf {L}(t), \\ \\Omega ="<>
              ToString[Round[Ω/ω]]<>"\\omega ", FontSize→9]
          , Scaled[{0.64, 0.97}], Scaled[{0.5, 1}]
        ]]
    }]
   , ImageSize → (Automatic → {Automatic , sizeK})
   , PlotRange → {{-15.3, 1}, {-4.6, 0.8}}
   , PlotRangePadding → None
   , PlotRangeClipping → False
   , Frame → True
   , Method → {"AxesInFront" → False}
   , AxesOrigin → {0, 0}
   , AxesStyle → GrayLevel[0.7]
   , FrameTicks → {{
        Join[{#, MaTeX[#, FontSize→ tfsK], {0.016, 0}} &/@Range[-5, 4, 1],
          {#, "", {0.008, 0}} &/@Range[-4, 4, 0.5]],
        Join[{#, "", {0.016, 0}} &/@Range[-5, 4, 1],
          {#, "", {0.008, 0}} &/@Range[-4, 4, 0.5]]
      }, {
        Join[{#, MaTeX["+"<>ToString[-#] /. {"+0" → "0"}, FontSize→ tfsK],
              {0.016, 0}} &/@Range[-15, 0, 5],
          {#, "", {0.008, 0}} &/@Range[-20, 5, 1]],
        Join[{#, "", {0.016, 0}} &/@Range[-15, 0, 5],
          {#, "", {0.008, 0}} &/@Range[-20, 5, 1]]
      }}
   , ImagePadding → {{Scaled[0.05], Scaled[0.001]},
      {If[Ω > 40 ω, Scaled[0.045], Scaled[0.0001]], Scaled[0.0010]}}
 ]
,

figureKvelocity[Round[Ω/ω]] = Show[
   Join[
```

```
Table[
  Graphics[{
      Arrowheads[{{0.035, 0.975}}],
      AbsoluteThickness[0.85],
      Blend[{part /. {Re → Red, Im  → Blue}, GrayLevel[0.9]}, 0.6],
      Arrow[{{0, 0},
          {{0, -1}, {1, 0}}.part[
              laboratoryFrameVelocity [timePath [0],
                  Apply[Function[{t, τ}, {t, t - τ}],
                    selection[index, Ω, m ]]][[{1, 2}]]
            ]
        }]
    }]
  , {part, {Re, Im }}],
Table[{
    ParametricPlot [
        Tooltip[
          {{0, -1}, {1, 0}}.part[
              laboratoryFrameVelocity [timePath [s],
                  Apply[Function[{t, τ}, {t, t - τ}],
                    selection[index, Ω, m ]]][[{1, 2}]]
            ]
          , part]
        , {s, 0, totalPathLength}
        , PlotPoints → 60
        , PlotStyle → {Directive[part /.
                {Re → Red, Im  → Blue}, AbsoluteThickness[0.85]]}
      ] /. {Line[pts_] :→ {Arrowheads[0.05],
            Line[pts], Arrow[pts]}},
    Graphics[{
        AbsolutePointSize[2.2],
        part /. {Re → Red, Im  → Blue},
        Point[
          {{0, -1}, {1, 0}}.part[
                  laboratoryFrameVelocity [timePath [#],
                      Apply[Function[{t, τ}, {t, t - τ}],
                      selection[index, Ω, m ]]][[{1, 2}]]
                ] & /@Most[Rest[Range[0, totalPathLength,
                  totalPathLength/20]]]
        ]
      }]
  }, {part, {Im , Re}}], {
  Graphics[{
      Inset[MaTeX["v_{\\mathsf {L},y}(t)", FontSize → 7],
        Scaled[{0.42, -0.12}], Scaled[{0.5, 1}]],
      Inset[Rotate[MaTeX["v_{\\mathsf {L},x}(t)", FontSize → 7],
          90 °], Scaled[{1.11, 0.5}], Scaled[{0, 0.5}]]
    }],
  Graphics[Inset[

      MaTeX["(\\mathrm  {" <> {"b", "d", "f", "h"}[[(Round[Ω/ω] - 17)/10 + 1]] <>
```

```
                    "})\\ \\ \\mathbf {v}_\\mathsf {L}(t), \\ \\Omega ="<>
                    ToString[Round[Ω/ω]]<>"\\omega ", FontSize→9]
                , Scaled[{0.5, 0.97}], Scaled[{0.5, 1}]
              ]],
            Graphics[{
                If[Ω < 20ω,
                  Inset[Grid[{
                        {Graphics[{Red, AbsoluteThickness[0.85], Line[{{0, 0},
                                {1, 0}}]}, PlotRange→{{0, 1}, 0.1{-1, 1}},
                              PlotRangePadding→None, ImageSize →10],
                            MaTeX["\\mathrm  {Re}", FontSize→5.5]},
                        {Graphics[{Blue, AbsoluteThickness[0.85],
                                Line[{{0, 0}, {1, 0}}]},
                              PlotRange→{{0, 1}, 0.1{-1, 1}},
                              PlotRangePadding→None, ImageSize →10],
                            MaTeX["\\mathrm  {Im }", FontSize→5.5]}
                      }, Background→White, Spacings→{0.5, 0},
                      Frame →True, FrameStyle →Directive[
                          Gray, Thickness[0.00001]]]
                    , Scaled[{0.97, 0.03}], Scaled[{1, 0}]]
                  , ## &[]]
              }]
          }]
        , ImageSize →(Automatic →{Automatic , sizeK})
        , PlotRange→{{-1.35, 2.15}, 1.35{-1, 1}}
        , PlotRangePadding→None
        , PlotRangeClipping→False
        , Frame →True
        , Method→{"AxesInFront"→False}
        , Axes→True
        , AxesOrigin→{0, 0}
        , AxesStyle→GrayLevel[0.7]
        , FrameTicks →{{
              Join[{#, "", {0.02, 0}} &/@Range[-1, 1],
                {#, "", {0.01, 0}} &/@Range[-2, 2, 0.2]],
              Join[{#, MaTeX[#, FontSize→tfsK], {0.02, 0}} &/@Range[-1, 1],
                {#, "", {0.01, 0}} &/@Range[-2, 2, 0.2]]
            }, {
              Join[{#, MaTeX[ToString[-#] /. {"1"→"+1"}, FontSize→tfsK],
                      {0.02, 0}} &/@Range[-1, 2],
                {#, "", {0.01, 0}} &/@Range[-2, 3, 0.2]],
              Join[{#, "", {0.02, 0}} &/@Range[-1, 2],
                {#, "", {0.01, 0}} &/@Range[-2, 3, 0.2]]
            }}
        , ImagePadding →{{Scaled[0.001], Scaled[0.047]},
            {If[Ω > 40ω, Scaled[0.045], Scaled[0.0012]], Scaled[0.0001]}}
      ]
  }]

, {Ω, {17ω, 27ω, 37ω, 47ω}}]
```

```
  ]
]
Table[{
    FileByteCount[Export[FileNameJoin [{$OutputDirectory, "figureK" <>
            {"a", "c", "e", "g"}[[(HO-17)/10 + 1]] <> "-position-" <> ToString[HO] <> ".pdf"}],
        figureKposition[HO], Background -> None]],
    FileByteCount[Export[FileNameJoin [{$OutputDirectory, "figureK" <>
            {"b", "d", "f", "h"}[[(HO-17)/10 + 1]] <> "-velocity-" <> ToString[HO] <> ".pdf"}],
        figureKvelocity[HO], Background -> None]]
  }, {HO, {17, 27, 37, 47}}]
pdflatex[]
{{34195, 13171}, {34513, 11937}, {34629, 12119}, {36190, 13886}}

pdflatex successful
```