On a matrix decomposition and its applications

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Abstract

We show the uniqueness and construction (of the Z matrix in Theorem 1, to be exact) of a matrix decomposition and give an affirmative answer to a question proposed in [J. Math. Anal. Appl. 407 (2013) 436-442].

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1 Introduction

Several recent papers [2, 3, 4, 5, 10] are devoted to the study of matrices with numerical range in a sector of the complex plane. In particular, this includes the study of accretive-dissipative matrices and positive definite matrices as special cases. A matrix decomposition plays a fundamental role in these works. The aim of this paper is twofold: show the uniqueness along with other properties of the key matrix in the decomposition and give an affirmative answer to a question raised in [12].

As usual, the set of $n \times n$ complex matrices is denoted by $\mathbb{M}_n$. For $A \in \mathbb{M}_n$, the singular values and eigenvalues of A are denoted by $\sigma_i(A)$ and $\lambda_i(A)$, respectively, $i = 1, \ldots, n$. The singular values are always arranged in nonincreasing order: $\sigma_1(A) \geq \cdots \geq \sigma_n(A)$. If A is Hermitian, then all eigenvalues of A are real and ordered as $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. Note that $\sigma_j(A) = \lambda_j(|A|)$, where $|A|$ is the modulus of A, i.e., $|A| = (A^*A)^{1/2}$ with A^* for the conjugate transpose of A. We denote $\sigma(A) = (\sigma_1(A), \ldots, \sigma_n(A))$ and $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))$.

For a square complex matrix A, recall the Cartesian (or Toeplitz) decomposition (see, e.g., [1, p. 6] and [7, p. 7]) $A = \Re A + i\Im A$, where

$$\Re A = \frac{1}{2}(A + A^*), \quad \Im A = \frac{1}{2i}(A - A^*).$$

There are many interesting properties for such a decomposition. For instance, $\Re(R^*AR) = R^*(\Re A)R$ for any $A \in \mathbb{M}_n$ and any $n \times m$ matrix R. A celebrated result due to Fan and Hoffman (see, e.g., [1, p. 73]) sates that

$$\lambda_j(\Re A) \le \sigma_j(A), \quad j = 1, \dots, n. \tag{1}$$

For $A \in \mathbb{M}_n$, the numerical range of A is the set in the complex plane

$$W(A) = \{x^*Ax \mid x \in \mathbb{C}^n, \|x\| = 1\}.$$

For $\alpha \in [0, \frac{\pi}{2})$, let S_α be the sector in the complex plane given by

$$S_\alpha = \{z \in \mathbb{C} \mid \Re z > 0, |\Im z| \le \Re z \tan \alpha\} = \{re^{i\theta} \mid r > 0, |\theta| \le \alpha\}.$$

Apparently, if the numerical range $W(A)$ is contained in a sector S_α for some $\alpha \in [0, \frac{\pi}{2})$, then A is nonsingular and $\Re A$ is positive definite. Moreover, $W(A) \subseteq S_\alpha$ implies $W(R^*AR) \subseteq S_\alpha$ for any nonzero $n \times m$ matrix R.

If $W(A)$ is contained in the first quadrant of the complex plane, then $\Re A$ and $\Im A$ are positive semidefinite. We call such a matrix A accretive-dissipative. Note that if A is accretive-dissipative and nonsingular, then $W(A) \subseteq e^{i\pi/4}S_{\pi/4}$, i.e., $W(e^{-i\pi/4}A) \subseteq S_{\pi/4}$. With continuity argument, we assume that the accretive-dissipative matrices to be considered in this paper are nonsingular.

We write $A \ge 0$ if A is positive semidefinite and $A > 0$ if A is positive definite. For two Hermitian matrices A and B of the same size, we denote $A \ge B$ if $A - B \ge 0$. Note that $A \ge B$ implies $\lambda_j(A) \ge \lambda_j(B)$ for all j.

In Section 2, we provide a detailed analysis of the so-called sectoral decomposition and show some important properties. In section 3, we use the decomposition and majorization as a tool to obtain some norm inequalities; a question raised in [12] is answered.

2 A matrix decomposition with a sector

We begin with discussions on a matrix decomposition which we refer to as the *sectoral decomposition*. The existence of the matrix decomposition with numerical range contained in a sector has appeared in [2, Lemma 2.1]. A similar observation was made by London [13] three decades ago (or even earlier by A. Ostrowski and O. Taussky) to prove a number of existing results by the factorization. This decomposition theorem, though simple as it looks, has been heavily used in recent papers [2, 3, 4, 5, 10]. In light of its importance and for completeness and convenience, we restate it here; we then show the uniqueness and give a way of constructing the key matrix Z in the decomposition.

Theorem 1 (Sectoral decomposition) *Let A be an $n \times n$ complex matrix such that $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then there exist an invertible matrix X and a unitary diagonal matrix $Z = \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$ with all $|\theta_j| \le \alpha$ such that $A = XZX^*$. Moreover, such a matrix Z is unique up to permutation.*

Proof. *Existence.* Write $A = M + iN$, where $M = \Re A$ and $N = \Im A$ are Hermitian. Since $W(A) \subseteq S_\alpha$, A is invertible and M is positive definite. By [7, Theorem 7.6.4] or [16, Theorem 7.6], M and N are simultaneously *-congruent and diagonalizable, that is, P^*MP and P^*NP are diagonal for some invertible matrix P. It follows that we can write $A = QDQ^*$ for some diagonal matrix D and invertible matrix Q. Since $W(A) \subseteq S_\alpha$, we have $W(D) \subseteq S_\alpha$. Thus we can write $D = \operatorname{diag}(d_1 e^{i\theta_1}, \ldots, d_n e^{i\theta_n})$, where $d_j > 0$ and $|\theta_j| \leq \alpha$, $j = 1, \ldots, n$. Set $X = Q \operatorname{diag}(\sqrt{d_1}, \ldots, \sqrt{d_n})$ and $Z = \operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$. Then $A = XZX^*$, as desired.

Uniqueness. Suppose that $A = XZ_1X^* = YZ_2Y^*$ are two decompositions of A, where X and Y are nonsingular, Z_1 and Z_2 are unitary and diagonal. We may assume $Y = I$ (otherwise replace X with $Y^{-1}X$). We show that Z_1 and Z_2 have the same main diagonal entries (regardless of order). For this, we show that $\beta \in \mathbb{C}$ is a diagonal entry of Z_1 with multiplicity k if and only if β is a diagonal entry of Z_2 with the same multiplicity. Without loss of generality, we may assume $\beta = 1$ (or multiply both sides by $\bar\beta$ and continue the discussion on $X(\bar\beta Z_1)X^* = \bar\beta Z_2$). Let $Z_1 = C_1 + iS_1$ and $Z_2 = C_2 + iS_2$ be the Cartisian decompositions of Z_1 and Z_2, respectively. Then C_1 and C_2 are positive definite. Since $\beta = 1$ is a diagonal entry of Z_1 with multiplicity k, 1 appears on the diagonal of C_1 k times, so S_1 has k zeros on its diagonal. Thus $\operatorname{rank}(XS_1X^*) = n - k$. As $XS_1X^* = S_2$, we have $\operatorname{rank}(S_2) = n - k$. This implies that C_2 contains k 1's on its diagonal. We conclude that Z_2 is permutation similar to Z_1. ∎

Note that $\cos\alpha$ is decreasing in α on $[0, \frac{\pi}{2})$, the following are immediate.

Corollary 1 *Let A be an $n \times n$ complex matrix such that $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$ and let $A = XZX^*$ be a sectoral decomposition of A, where X is invertible and Z is unitary and diagonal. Then*

(i). $I \leq \sec\alpha\,(\Re Z)$.

(ii). $RR^* \leq \sec\alpha\big(R(\Re Z)R^*\big)$ *for any matrix R.*

(iii). $\sigma_j^2(R) \leq \sec\alpha\,\lambda_j\big(R(\Re Z)R^*\big) \leq \sec\alpha\,\sigma_j\big(RZR^*\big)$ *for any R and j.*

(iv). $\sigma_j^2(X) \leq \sec\alpha\,\lambda_j(\Re A) \leq \sec\alpha\,\sigma_j(A)$ *for all $j = 1, \ldots, n$.*

The following result gives a way of constructing the unique matrix Z.

Theorem 2 *Let A be an $n \times n$ complex matrix with the Cartesian decomposition $A = M + Ni$, where M is positive definite and N is Hermitian. Then the matrix Z in Theorem 1 is determined by the eigenvalues of $M^{-1}N$. Let μ_j be the eigenvalues of $M^{-1}N$ and let $1 + i\mu_j = |1 + i\mu_j| e^{i\gamma_j}$, $|\gamma_j| < \frac{\pi}{2}$, $j = 1, \ldots, n$. Then $Z = \operatorname{diag}(e^{i\gamma_1}, \ldots, e^{i\gamma_n})$. Let $\gamma(A) = \max_j |\gamma_j|$. Then $W(A) \subseteq S_{\gamma(A)}$.*

Proof. Since $M > 0$, there is an invertible matrix P such that $P^*MP = I$ and $P^*NP = D$ is diagonal (see, e.g., [16, p. 213]). Recall that when X and

Y are both $n\times n$ matrices, XY and YX have the same eigenvalues. We have $\lambda_j(P^*NP)=\lambda_j(PP^*N)=\lambda_j(M^{-1}N)$. It follows that $P^*AP=I+Di$ and D is the diagonal matrix of the eigenvalues μ_j of $M^{-1}N$. Let $1+i\mu_j=|1+i\mu_j|e^{i\gamma_j}$, $|\gamma_j|<\frac{\pi}{2}$, $j=1,\dots,n$. Then $Z=\mathrm{diag}(e^{i\gamma_1},\dots,e^{i\gamma_n})$. With $\gamma(A)=\max_j|\gamma_j|$, we see that $W(Z)$, $W(I+Di)$, and $W(A)$ are all contained in $S_{\gamma(A)}$. ∎

Corollary 2 *Let A be an $n\times n$ complex matrix such that $W(A)\subseteq S_\alpha$ for some $\alpha\in[0,\frac{\pi}{2})$. Then there exist a normal matrix Λ such that $A=(\Re A)^{1/2}\Lambda(\Re A)^{1/2}$. Moreover, $\|\Lambda\|_2\le\sec\alpha$ for the spectral norm $\|\cdot\|_2$ on $\mathbb{M}_n$.*

Proof. Let $A=M+Ni$ with $M=\Re A$ and $N=\Im A$. Take $\Lambda=I+M^{-1/2}NM^{-1/2}i$. Then Λ is normal and $W(\Lambda)\subseteq S_\alpha$. For any unit vector z, $z^*\Lambda z$ is a point in the xy-plane with x-coordinate $x=1$. It follows that the numerical radius of Λ, i.e., $w(\Lambda)=\max\{|z^*\Lambda z|\mid z\in\mathbb{C}^n, z^*z=1\}$, is no more than $\sec\alpha$ (as the hypotonus of the right triangle with the adjacent leg of length 1). Since Λ is normal, all the singular values of Λ are no more than $\sec\alpha$. In particular, for the spectral norm $\|\Lambda\|_2$, we have $\|\Lambda\|_2\le\sec\alpha$. ∎

Let θ_a and θ_b be respectively the largest and smallest values of the θ_j's in Theorems 1 and 2. For the Z in the decomposition, $W(Z)$ is the region formed by the portion of the unit circle from $e^{i\theta_a}$ to $e^{i\theta_b}$ and the line segment from $e^{i\theta_a}$ to $e^{i\theta_b}$. For the Λ in the corollary, $W(\Lambda)$ is the vertical line segment $x=1$ from the point $(1,\tan\theta_a)$ to the point $(1,\tan\theta_b)$. All these figures are contained in S_{θ_c}, where $\theta_c=\max\{|\theta_a|,|\theta_b|\}$, which is nothing but the γ in Theorem 2.

Below is an addition-closure property for the numerical ranges in a sector.

Proposition. *Let $A,B\in\mathbb{M}_n$. If $W(A),W(B)\subseteq S_\alpha$ for some $\alpha\in[0,\frac{\pi}{2})$, then*

$$W(A+B)\subseteq S_\alpha.$$

Proof. Consider the Cartesian decompositions of A and B,

$$A=R_1+iS_1,\qquad B=R_2+iS_2.$$

Since $W(A)$ and $W(B)$ are contained in S_α, we have $R_1+R_2>0$. Note that for $a,b,c,d>0$, $(a+b)/(c+d)\le\max\{a/c,b/d\}$. We compute, for any $x\ne0$,

$$\begin{aligned}\frac{|x^*(S_1+S_2)x|}{x^*(R_1+R_2)x} &\le \frac{|x^*S_1x|+|x^*S_2x|}{x^*(R_1+R_2)x}\\ &\le \frac{x^*|S_1|x+x^*|S_2|x}{x^*R_1x+x^*R_2x}\\ &\le \max\left\{\frac{x^*|S_1|x}{x^*R_1x},\frac{x^*|S_2|x}{x^*R_2x}\right\}\\ &\le \tan\alpha.\end{aligned}$$

This says $|x^*\Im(A+B)x|\le x^*\Re(A+B)x\tan\alpha$. Thus, $W(A+B)\subseteq S_\alpha$. ∎

We note here that fractional roots (powers) of elements in Banach algebras are studied in [9] by means of numerical range sectors.

3 Norm inequalities for partitioned matrices

Recall that a norm $\|\cdot\|$ on $\mathbb{M}_n$ is unitarily invariant if $\|UAV\| = \|A\|$ for any $A \in \mathbb{M}_n$ and all unitary $U, V \in \mathbb{M}_n$. The unitarily invariant norms of matrices are determined by nonzero singular values of the matrices via symmetric gauge functions (see, e.g., [16, Theorems 10.37 and 10.38]). If B is a submatrix of $A \in \mathbb{M}_n$, then $\|B\|$ is understood as the norm of the $n \times n$ augmented matrix B with 0's, and conventionally B has n singular values with the trailing ones 0; that is, $\sigma(B) = (\sigma_1(B), \ldots, \sigma_r(B), 0, \ldots, 0) \in \mathbb{R}^n$, where r is the rank of B. Thus $\sigma(A)$ and $\sigma(B)$ are both in $\mathbb{R}^n$.

Let A be an n-square complex matrix partitioned in the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{where } A_{11} \text{ and } A_{22} \text{ are square.} \tag{2}$$

In [12], the following norm inequalities are proved (in Hilbert space).

LZ1 [12, Theorem 3.3]: *Let $A \in \mathbb{M}_n$ be accretive-dissipative and partitioned as in (2). Then for any unitarily invariant norm $\|\cdot\|$ on $\mathbb{M}_n$,*

$$\max\{\|A_{12}\|^2, \|A_{21}\|^2\} \leq 4\|A_{11}\|\,\|A_{22}\|. \tag{3}$$

LZ2 [12, Theorem 3.11]: *Let $A \in \mathbb{M}_n$ be accretive-dissipative and partitioned as in (2). Then for any unitarily invariant norm $\|\cdot\|$ on $\mathbb{M}_n$,*

$$\|A\| \leq \sqrt{2}\,(\|A_{11}\| + \|A_{22}\|). \tag{4}$$

It is asked in [12] as an open problem whether the factor 4 in (3) and the factor $\sqrt{2}$ in (4) can be improved. Indeed, the factor $\sqrt{2}$ in (4) is optimal. To construct such an accretive-dissipative matrix, we can find a matrix whose numerical range is contained in the sector $S_{\pi/4}$, then rotate it by $+\pi/4$. The normal matrix $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] i = \left[\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right]$ has eigenvalues $1+i$ and $1-i$. So the matrix $A = e^{i\pi/4}B$ is accretive-dissipative. A and B have the same repeated singular value $\sqrt{2}$. Thus, for the trace norm (sum of all singular values),

$$2\sqrt{2} = \|A\| = \sqrt{2}\,(\|A_{11}\| + \|A_{22}\|) = \sqrt{2}\,(1+1).$$

However, the factor 4 in (3) can be improved to 2 (see Corollary 3). In this section, we show some more general results than (3) and (4).

We adopt the following standard notations. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. We denote the componentwise product of x and y by $x \circ y$. i.e., $x \circ y = (x_1y_1, \ldots, x_ny_n)$. We write $x \leq y$ to mean $x_j \leq y_j$ for $j = 1, \ldots, n$. We say that x is weakly majorized by y, written as $x \prec_w y$, if the partial sum of the first k largest components of x is less than or equal to the corresponding partial sum of y for $k = 1, \ldots, n$. We write $x \prec y$ if $x \prec_w y$ and the sum of all components of x is equal to that of y. (See, e.g., [14, p. 12] or [16, p. 326].)

It is well known (see, e.g., [14, p. 368] or [16, p. 375]) that, for $A, B \in \mathbb{M}_n$, $\|A\| \leq \|B\|$ for all unitarily invariant norms $\|\cdot\|$ on $\mathbb{M}_n$ if and only if $\sigma(A) \prec_w$

$\sigma(B)$. So, to some extend, the norm inequalities are essentially the same as the singular value majorization inequalities. The Fan-Hoffman inequalities (1) yield immediately $\|\Re A\| \le \|A\|$ for any $A \in \mathbb{M}_n$ and any unitarily invariant norm $\|\cdot\|$ on $\mathbb{M}_n$. The following is a reversal. Two useful facts are: the singular value majorization of product $\sigma(AB) \prec_w \sigma(A) \circ \sigma(B)$ (see, e.g., [16, p. 363]) and its companion norm inequality $\|AB\|^2 \le \|AA^*\| \, \|B^*B\|$ (see, e.g., [6, p. 212]).

Lemma 1 *Let $A \in \mathbb{M}_n$ have $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then*

$$\sigma(A) \prec_w \sec\alpha \, \lambda(\Re A).$$

Equivalently, for all unitarily invariant norms $\|\cdot\|$ on $\mathbb{M}_n$,

$$\|A\| \le \sec\alpha \|\Re A\|.$$

Proof. Let $A = XZX^*$ be a sectoral decomposition of A, where X is invertible and Z is unitary and diagonal. Then

$$\sigma(A) = \sigma(XZX^*) \prec_w \sigma(X) \circ \sigma(Z) \circ \sigma(X^*) = \sigma^2(X) \le \sec\alpha \, \lambda(\Re A).$$

The last "$\le$" is by Corollary 1 (iv). The norm inequality follows at once. ∎

Theorem 3 *Let $A \in \mathbb{M}_n$ be partitioned as in (2) and assume $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then for any unitarily invariant norm $\|\cdot\|$ on $\mathbb{M}_n$,*

$$\max\{\|A_{12}\|^2, \|A_{21}\|^2\} \le \sec^2\alpha \, \|A_{11}\| \, \|A_{22}\|. \tag{5}$$

Proof. Let A_{11} be $p \times p$. By Theorem 1, let $A = XZX^*$ be a sectoral decomposition of A, where X is invertible and Z is unitary and diagonal. We partition X as $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $X_1 \in \mathbb{M}_{p \times n}$. Then $\Re A_{11} = X_1(\Re Z)X_1^*$, $\Re A_{22} = X_2(\Re Z)X_2^*$, and $A_{12} = X_1 Z X_2^*$. Using Corollary 1 (ii), we have

$$\begin{aligned} \|A_{12}\|^2 &= \|X_1 Z X_2^*\|^2 \le \|X_1 X_1^*\| \, \|X_2 Z^* Z X_2^*\| \\ &\le \sec^2\alpha \|X_1(\Re Z)X_1^*\| \, \|X_2(\Re Z)X_2^*\| \\ &= \sec^2\alpha \|\Re A_{11}\| \, \|\Re A_{22}\| \\ &\le \sec^2\alpha \|A_{11}\| \, \|A_{22}\|. \end{aligned}$$

So (5) is true for A_{12}. The inequality for A_{21} is similarly proven. ∎

If A is a positive definite matrix, then $\alpha = 0$ and $\sec\alpha = 1$ in (5).

Corollary 3 *Let $A \in \mathbb{M}_n$ be accretive-dissipative and partitioned as in (2). Then for any unitarily invariant norm $\|\cdot\|$ on $\mathbb{M}_n$,*

$$\max\{\|A_{12}\|^2, \|A_{21}\|^2\} \le 2 \, \|A_{11}\| \, \|A_{22}\|. \tag{6}$$

Proof. Set $\alpha = \pi/4$ in the theorem. Then $\sec^2 \alpha = 2$. ∎

(6) is stronger than (3). Moreover, the constant factor 2 is best possible for all accretive-dissipative matrices and unitarily invariant norms. Let $B = \left[\begin{smallmatrix} 1 & 1-i \\ 0 & 1 \end{smallmatrix}\right]$. One may check that $\Re B > 0$ and $\Re B \geq \pm \Im B$, which yield $x^*(\Re B)x \geq |x^*(\Im B)x|$ for all $x \in \mathbb{C}^2$. (Note that $\Re B \ngeq |\Im B|$.) So $W(B) \subseteq S_{\pi/4}$ and $A = e^{i\pi/4}B$ is accretive-dissipative. For the trace norm, apparently, $\|A_{12}\|^2 = 2 = 2(1 \cdot 1) = 2\|A_{11}\| \, \|A_{22}\|$. This answers a question raised in [12, p. 442].

To present next theorem, we need a lemma which is interesting on its own.

Lemma 2 *Let $H = \left[\begin{smallmatrix} H_{11} & * \\ * & H_{22} \end{smallmatrix}\right]$ be an $n \times n$ positive semidefinite matrix, where H_{11} and H_{22} are square submatrices (possibly of different sizes). Then*

$$\lambda(H) \prec \lambda(H_{11}) + \lambda(H_{22}). \tag{7}$$

Consequently, for all unitarily invariant norms $\|\cdot\|$ on $\mathbb{M}_n$,

$$\|H\| \leq \|H_{11}\| + \|H_{22}\|. \tag{8}$$

Proof. Note that a matrix P is positive semidefinite if and only if $P = Q^*Q$ for some matrix Q. Let $H = \left[\begin{smallmatrix} S^* \\ T^* \end{smallmatrix}\right][S, T] = \left[\begin{smallmatrix} S^*S & * \\ * & T^*T \end{smallmatrix}\right]$ with $H_{11} = S^*S$ and $H_{22} = T^*T$. Using the fact that matrices XY and YX have the same nonzero eigenvalues for any $(p \times q)$ matrix X and any $(q \times p)$ matrix Y, we arrive at

$$\begin{aligned} \lambda(H) &= \lambda\Big(\begin{bmatrix} S^* \\ T^* \end{bmatrix} [S, T] \Big) = \lambda\Big([S, T] \begin{bmatrix} S^* \\ T^* \end{bmatrix} \Big) \\ &= \lambda(SS^* + TT^*) \prec \lambda(SS^*) + \lambda(TT^*) \\ &= \lambda(H_{11}) + \lambda(H_{22}). \end{aligned}$$

Here we regard $\lambda(H_{11})$ and $\lambda(H_{22})$ as vectors in $\mathbb{R}^n$ (by adding 0's). ∎

Remark: It is known [14, p. 308] that if $H = \left[\begin{smallmatrix} H_{11} & * \\ * & H_{22} \end{smallmatrix}\right]$ is Hermitian, then

$$\big(\lambda(H_{11}), \lambda(H_{22})\big) = \lambda(H_{11} \oplus H_{22}) \prec \lambda(H).$$

It is also known (see [15] or [11]) that if $H = \left[\begin{smallmatrix} H_{11} & K \\ K^* & H_{22} \end{smallmatrix}\right]$ is positive semidefinite, where K is Hermitian or skew-Hermitian, then

$$\lambda(H) \prec \lambda(H_{11} + H_{22}).$$

We must also point out that (8) has appeared in [6, p. 217] and a more general result is available in [8, Theorem 2.1]. We include our proof here as it is short and elementary, and the most elegant one in author's opinion.

Theorem 4 *Let $A \in \mathbb{M}_n$ be partitioned as in (2) and let $W(A) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. Then for any unitarily invariant norm $\|\cdot\|$ on $\mathbb{M}_n$,*

$$\|A\| \leq \sec \alpha \, (\|A_{11}\| + \|A_{22}\|). \tag{9}$$

Proof. By Lemma 1 and noticing that $\Re A = \begin{bmatrix} \Re A_{11} & * \\ * & \Re A_{22} \end{bmatrix} > 0$, we have

$$\|A\| \le \sec\alpha \|\Re A\| \le \sec\alpha(\|\Re A_{11}\| + \|\Re A_{22}\|).$$

The desired inequality follows at once since $\|\Re X\| \le \|X\|$ for any X. ∎

If A is positive definite, then $\alpha = 0$ and Theorem 4 reduces to (8). If A is accretive-dissipative, then (4) is immediate by setting $\alpha = \pi/4$ in (9).

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