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A STUDY ON GEOMETRIC INEQUALITIES

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Abstract:
In this paper, we present several inequalities among various elements of a triangle.

Key Words: Nesbitt's Inequality & Cauchy – Inequality

Introduction:

Many of the inequalities we have studied and the techniques we have learnt have their direct implications in a class of inequalities known as geometric inequalities. These inequalities explore relations among various geometric elements. When we consider a triangle, we can associate many things with it: angles, sides, area, medians altitudes, circum-radius, in-radius, ex-radii and so on. The classic example is Euler's inequality: $R \geq 2r$. Where R is the circum-radius and r is the in-radius.

We use the following standard notations for a triangle ABC:

- ✓ $a = |BC|$, $b = |CA|$, $c = |AB|$.
- ✓ S is the semi – perimeter of ABC : $S = (a+b+c)/2$
- ✓ R is the circum – radius.

We prove the following theorems:

Theorem 1: $abc > 8(s-a)(s-b)(s-c)$.

Proof: We have $a^2 - (b-c)^2 \leq a^2$ and equality holds if and only if b= c.

Similar inequalities hold : $b^2 - (c-a)^2 \leq b^2$, $c^2 - (a-b)^2 \leq c^2$.

$$\text{Hence, } abc \geq \sqrt{a^2-(b-c)^2}\ \sqrt{b^2-(c-a)^2}\ \sqrt{c^2-(a-b)^2}$$
$$= (a+b-c)(b+c-a)(c+a-b)$$
$$= 8(s-a)(s-b)(s-c).$$

Equality holds if and only if $a = b = c$.

Theorem 2: $abc < \Sigma a^2 (s-a) \leq 3/2\ abc$.

Proof: We have $2\sum a^2 (s-a) = a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c)$

$$= \Sigma a^2 b + \Sigma ab^2 - \Sigma a^3$$

On the other hand, we also see that

$$(b+c-a)(c+a-b)(a+b-c) = (c^2 - a^2 - b^2 + 2ab)(a+b-c) = a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 - a^3 - b^3 - c^3 - 2abc$$

Thus, we obtain $2\Sigma a^2 (s-a) = (b+c-a)(c+a-b)(a+b-c) + 2abc$.

Since a, b, c are the sides of a triangle, we know that $b+c-a > 0$, $c+a-b > 0$ and $a+b-c > 0$.

$$\text{Hence, } abc < \Sigma a^2 (s-a).$$

Now using (1), we get $2\Sigma a2(s-a) \leq abc + 2abc = 3abc$,

Which proves the right hand side inequality.

Again, we may use stolarsky's theorem.

Considering $P(x, y, z) = \sum_{cyclic} x^2(y+z-x) - 2xyz$,

We see that it is a homogeneous polynomial of degree 3 and

$P(1, 1, 1) = 1$, $P(1, 1, 0) = 0$, $P(2, 1, 1) = 0$.

Hence, $P(a, b, c) > 0$, giving the left – side inequality. On the other hand, the polynomial

$Q(x, y, z) = 3xyz - \sum_{cyclic} x^2(y+z-x)$,

Gives $Q(1, 1, 1) = 0$, $Q(1, 1, 0) = 0$ and $Q(2, 1, 1) = 2$.

Thus $Q(a, b, c) > 0$, and we get the right – side inequality.

Theorem 3: $\frac{3}{2} \leq \sum \frac{a}{b+c} < 2$. Equality holds on the left if and only if $a=b=c$.

Proof: The first part of the above inequality is equivalent to

$$\frac{9}{2} \leq \frac{a}{b+c} + 1 + \frac{b}{c+a} + 1 + \frac{c}{a+b} + 1$$
$$= (a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right)$$

If we introduce $a+b = x$, $b+c = y$ and $c+a = z$, this reduces to $9 \leq (x+y+z)\left(\frac{1}{y} + \frac{1}{z} + \frac{1}{x}\right)$

Which is a consequence of the AM – GM inequality. Suppose c is the largest among a, b, c.

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By the symmetry, we may assume $a \le b \le c$. In this case

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \le \frac{a}{a+c} + \frac{c}{c+a} + \frac{c}{a+b}$$
$$= 1 + \frac{c}{a+b}$$
$$\Sigma \frac{a}{b+c} < 2.$$

Since $c < a + b$ by the triangle inequality.

Proof: The left hand side of the above inequality is generally known as Nesbitt's inequality. There are a variety of ways of proving this. We give two such proofs.

(i) Using the Cauchy – Schwarz inequality, we have

$$(a+b+c)^2 = (\Sigma a)^2$$
$$= \left(\sum \sqrt{\frac{a}{b+c}}\sqrt{a(b+c)}\right)^2$$
$$(a+b+c)^2 \le \left(\sum \frac{a}{b+c}\right) \sum a(b+c)$$

This gives

$$\sum \frac{a}{b+c} \ge \frac{(a+b+c)^2}{\sum a(b+c)}$$
$$= \frac{(a+b+c)^2}{2(ab+bc+ca)}$$
$$\sum \frac{a}{b+c} \ge \frac{3(ab+bc+ca)}{2(ab+bc+ca)} = \frac{3}{2}$$

since $(a+b+c)^2 \ge 3(ab+bc+ca)$.

(ii) we may assume $a \le b \le c$, since the inequality is symmetric in a, b, c.

This implies that

$$\frac{1}{b+c} \le \frac{1}{c+a} \le \frac{1}{a+b}$$

Using rearrangement inequality, we obtain

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{a}{c+a} + \frac{b}{a+b} + \frac{c}{b+c},$$
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}$$

Adding these two, we obtain

$$2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \ge 3.$$

This gives the desired inequality.

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