Granular Knowledge and Rational Approximation in General Rough Sets-I

A Mani

MIU, Indian Statistical Institute, Kolkata
203, B. T. Road, Kolkata-700108, India
Homepage: https://www.logicamani.in
Orcid: https://orcid.org/0000-0002-0880-1035

ARTICLE HISTORY
Compiled October 9, 2023

ABSTRACT
Rough sets are used in numerous knowledge representation contexts, and are then empowered with varied ontologies. These may be intrinsically associated with ideas of rationality under certain conditions. In recent papers, specific granular generalizations of graded and variable precision rough sets are investigated by the present author from the perspective of rationality of approximations (and the associated semantics of rationality in approximate reasoning). The studies are extended to ideal-based approximations (sometimes referred to as subsethood-based approximations). It is additionally shown that co-granular or point-wise approximations defined by σ-ideals/filters (for an arbitrary relation σ) fit easily into the entire scheme. Concepts of the rationality of objects (vague or crisp) and their types are introduced, and are shown to be applicable to most general rough sets by the present author. Surprising results on these are proved on these by her in this part of the research paper. The present paper is the first of a three part study on the topic.

KEYWORDS
Rough sets; rational approximations; ideals and filters; rationality of objects; rationality types; co-granular rough sets; substantial parthood; mereology

1. Introduction

A number of granular and nongranular semantics of rough sets are known in the literature. Concepts of knowledge can be associated with these from multiple perspectives such as those based on classical rough ideas (Pawlak & Skowron, 2007; Pawlak, 1991; Orłowska & Pawlak, 1984), mereological axiomatic granular perspectives (Mani, 2020a, 2018a, 2018b), classical granular computing (Yao, 2001; Zadeh, 1997), interpretations of modal logic (Pagliani & Chakraborty, 2008; Mani, Düntsch, & Cattaneo, 2018; Yao & Lin, 1996), constructive logic (Pagliani, 2018; Järvinen, Pagliani, & Radeleczki, 2012), concept analysis (Yao, 2016; Ciucci, Dubois, & Prade, 2012; Mani, 2018b), evidence theory, and machine learning. Importantly, ideas of knowledge are not well-developed in every generalization of rough sets. Researchers also argue about the borders of rough sets from features of knowledge. This has, for example, motivated recent work on ideas of rational knowledge (with applications to validity in clustering) in the context of generalizations (and hybridization) of variable precision rough sets (VPRS) and granular

CONTACT A Mani. Email: *a.mani.cms@gmail.com amani.rough@isical.ac.in*

graded rough sets by the present author (Mani, 2022a; Mani & Mitra, 2022). It may be noted that the literature on VPRS and generalizations, has been mostly focused on practical computations, and not on semantics towards addressing the problems. Therefore, the issues at stake are about possible algebraic semantics, general frameworks, and the very idea of rationality. The last-mentioned concept depends crucially on that of *substantial parthood* ((Burkhardt, Seibt, Imaguire, & Gerogiorgakis, 2017; Mani, 2018b; Vieu & Aurnague, 2007)) as opposed to parthood alone (specifically, VPRS and generalizations are all about a certain specific sense of being *a substantial part of*).

The concept of rational discourse in a context is typically determined by subjective, normative, and rule based constraints on associated concepts. Formal approaches to rationality are known in non-monotonic logic, evidence theory, general rough sets, and epistemology. An overview of older theories of rationality, and rational inference across can be found in the article (Kolodny & Brunero, 2020). A specific version of rationality, called bounded rationality in which possible inferences are bounded or limited has found much application in applied logic. Bounded rationality is additionally relevant in the context of rough set applications. However, the boundedness aspect need not happen always. While deterministic rationality may be realized through well-constructed or formally-verified computer programs, it is possible to write totally unreliable random code (and compile it). People are likely to think rationally provided they have learnt reliable meta-level models or rules, and their application. *Somewhat analogously, general rough approximation operators require additional layers of being well-constructed to qualify as rational ones (though not necessarily in a deterministic point of view).* Importantly, the operators may be capable of handling uncertainty and vagueness at multiple stages of their construction.

From a purely theoretical perspective, this multipart study is about features of the defining conditions of approximations that potentially lead to new classifications of general rough sets and connections with related areas. In relation to these, four broad classes of rough sets (graded, VPRS, ideal based and classical) are investigated, and two more remain to be done. From an application point of view, reasonable (or rational) approximations in the context of general rough approximations are essential for applications to contexts that require high quality approximations, or predictions (especially when robustness cannot be expected). This is very relevant in applications to human learning, and automated evaluation frameworks in education as shown by the present author in (Mani, 2020b). In the second part of this multipart research, the general rough approach to cluster validation introduced in (Mani, 2021) is actually shown to be related to rationality, and learning frameworks. Apart from these a number of potential application areas like intelligent robust image segmentation with unlearning, epidemiology (where feature selection fails badly), unbalanced class problem, and medical diagnostics can be indicated – these are considered separately.

In (Mani, 2022a; Mani & Mitra, 2022), distinct concepts of *rational approximations*, and *rationally constructed objects* are investigated in a mereological perspective. Substantial parthoods of various types are common in application contexts, however the trend towards simplistic solutions ignoring model complexity has led to solutions of poor quality, and reliance on excessive computation. It is therefore essential to build the foundations with such predicates, and additionally explore related possibilities. In this part, new connections of graded rough sets with ideal-based (or subsethood) approximations are introduced, the nature of substantial parthood investigated in ideal-based rough sets in relation to general framework for rational approximations. Concepts of rationality types that correspond to the nature of existence of objects in relation to their representation in terms of granules are introduced, and it is shown to have classificatory value that applies to all types of rough sets.

In the next part of this multipart research, a number of new results on granular VPRS, related distance functions, connections with ideas of rationality in evidence theory and belief functions, and applications to soft cluster validation are investigated. Most other types of rough sets that can coerced within the theoretical framework are covered in the third part.

This research paper is organized as follows: A guide to the necessary background is provided in the next section. In the third section, extensions of earlier results on graded rough sets (Mani, 2022a) to a type based scenario are introduced. Ideal-based co-granular rough sets and the nature of rational approximations is investigated in the following section. Rationality types are introduced and explored in detail in the fifth section. Further, directions are additionally provided in the sixth.

2. Background

Quantifiers are uniformly enclosed in braces for easier reading. So, $\forall a \exists b \Phi(a,b)$ is the same as $(\forall a \exists b)\, \Phi(a,b)$.

Conditional implications of the form *for every x whenever $\Phi(x)$ holds, then $\Psi(x)$ holds as well*, are replaced (whenever possible) by

$$(\forall x)(\Phi(x) \longrightarrow \Psi(x))$$

Definition 2.1. A collection of subsets $\mathcal{L} = \{L_j : j \in J\}$ is a τ_k-covering of a set S if and only if all the following hold:

- $\mathcal{L}$ is a set of pairwise incomparable subsets relative to the usual inclusion order,
- $\mathcal{L}$ is a cover for S (that is, its union is S), and
- if A is a subset of S which is not included in any L_j, then there exist k elements $\{a_i\}_1^k$ of A which are not included in any L_j.

A τ_2 covering is also referred to as a *normal cover* (Chajda, Niederle, & Zelinka, 1976).

A *lattice ideal* K of a lattice $L = (L, \vee, \wedge)$ is a subset (possibly empty) of L that satisfies the following ($\leq$ being the definable lattice order on L):

$$(\forall a \in L)(\forall b \in K)(a \leq b \longrightarrow a \in K) \qquad \text{(o-Ideal)}$$

$$(\forall a, b \in K)\, a \vee b \in K \qquad \text{(Join Closure)}$$

An ideal P in a lattice L is *prime* if and only if $(\forall a, b)(a \wedge b \in P \longrightarrow a \in P$ or $b \in P)$. $Spec(L)$ shall denote the set of all prime ideals. A lattice filters is the dual of an ideal. Maximal lattice filters are the same as ultrafilters. In Boolean algebras, any filter F that satisfies $(\forall a) a \in F$ or $a^c \in F$ is an ultra filter. *Chains* are subsets of a poset in which any two elements are comparable, while *antichains* are subsets of a poset in which no two distinct elements are comparable. Singletons are both chains and antichains.

For basics of partial algebras, the reader is referred to (Burmeister, 1986, 2002; Ljapin, 1996).

A *partial algebra* P is denoted by a tuple of the form

$$\langle \underline{P},\, f_1,\, f_2,\, \ldots,\, f_n, (r_1,\, \ldots,\, r_n)\rangle$$

with $\underline{P}$ being a set, f_i's being partial function symbols of arity r_i. The interpretation of f_i on the set $\underline{P}$ should be denoted by $f_i^{\underline{P}}$; however, the superscript will be dropped in this paper as the application contexts are simple enough. If predicate symbols enter into the signature, then P is termed a *partial algebraic system*.

For two terms s, t, $s \stackrel{\omega}{=} t$ shall mean that, whenever both the terms are defined (after interpretation) then they are equal. That is $(\forall a, b)(s^{\underline{P}} = a \,\&\, t^{\underline{P}} = b \longrightarrow a = b)$. $\stackrel{\omega}{=}$ is the same as the existence equality (sometimes written as $\stackrel{e}{=}$) in the present paper.

2.1. Rationality and Semantic Domains

Semantic domains are important in any logical or mathematical approach that focuses on meaning and models. It suffices to specify these without completely formalizing them within specific domains of knowledge. One way of identifying semantic domains in contexts involving approximate reasoning or rough sets is through the type of objects involved in the discourse. This can be specified in a number of ways such as through direct observation of all attributes, observation of parts of an object's attributes, approximations of objects, operations permitted (or admissible predicates) on objects.

For example, a teacher's understanding of a lesson is very different from that of a student, and each may be approximating a certain knowledge structure (see (Ball, Thames, & Phelps, 2008; Mani, 2022b) for a more detailed discussion). In the context, more than the described six semantic domains may be constructed for the purpose of modeling from different perspectives simply because multiple agents and knowledge hierarchies are involved.

Consider the sentences',

S-A The golden mountain is golden and a mountain.
S-B The tallest mountain is tall and a mountain.

It is common knowledge that S-A is an unverifiable assertion, while S-B is a verifiable fact. However, relative to a knowledge base in which the concept of *being tall* is missing, both S-A and S-B would be unverifiable. Related semantic domains can be respectively formalized in different ways.

Informally, an information table is a usual table with each row corresponding to a crisp entity, and columns corresponding to attributes. Cell entries are the attribute values for the entity. However, the main concerns are about objects defined by subsets of attributes and related valuations (for details see (Mani, 2018a, 2020a)). In more abstract settings, one directly specifies conditions on approximation operators on sets or partial algebraic systems, and with no reference to information tables.

Formally, an *information table* $\mathcal{I}$, is a tuple of the form

$$\mathcal{I} = \langle \mathfrak{O}, \mathbb{A}, \{V_a : a \in \mathbb{A}\}, \{f_a : a \in \mathbb{A}\}\rangle$$

with $\mathfrak{O}$, $\mathbb{A}$ and V_a being respectively sets of *Objects*, *Attributes* and *Values* respectively. $f_a : \mathfrak{O} \longmapsto \wp(V_a)$ being the valuation map associated with attribute $a \in \mathbb{A}$. Values may also be denoted by the binary function $\nu : \mathbb{A} \times \mathfrak{O} \longmapsto \wp(V)$ defined by for any $a \in \mathbb{A}$ and $x \in \mathfrak{O}$, $\nu(a, x) = f_a(x)$.

Relations may be derived from information tables by way of conditions of the following form: For $x, w \in \mathfrak{O}$ and $B \subseteq \mathbb{A}$, $(x, w) \in \sigma$ if and only if $(\mathbf{Q}a, b \in B)\, \Phi(\nu(a, x), \nu(b, w),)$ for some quantifier $\mathbf{Q}$ and formula Φ. The relational system $S = \langle \underline{S}, \sigma\rangle$ (with $\underline{S} = \mathbb{A}$) is said to be a *general approximation space*. In particular if σ

is an equivalence relation then S is referred to as an *approximation space*.

In the general rough context, the concepts of definite and rough (or vague) objects (these objects are in the rough semantic domain as opposed to the crisp entities of an information table in the classical semantic domain) are specified through one or more approximation operators and additional constraints. Concepts of crisp objects are additionally specified in the same way or may be designated as such. In the case of Pawlakian (or classical) rough sets, if the lower approximation of an object X coincides with itself, then X is said to be crisp or definite. Objects that are not definite are *rough*. There are several ways of representing rough objects in terms of crisp objects such as a pair of definite objects (A, B) with $A \subset B$. It may additionally be reasonable to think of sets of objects that properly contain common maximal crisp objects as a rough object, or take orthopairs (Cattaneo & Ciucci, 2018) as the primary objects of interest. For a longer list see (Mani, 2018a). Naturally, these lead to different rough semantic domains.

Classical rough sets (see the book (Pawlak, 1991)) starts from approximation spaces (derived from information tables) of the form $\langle S, R\rangle$, where R is an equivalence on the set S. The Boolean algebra on the power set $\wp(S)$ with lower and upper approximation operations forms a model that does not describe the rough objects alone. This extends to arbitrary general approximation spaces where R is permitted to be any relation or even to those based on covers. The *classical semantic domain* associated with such classes of models may be understood in terms of the collection of restrictions on possible objects, predicates, constants, functions and low level operations.

The problem of defining rough objects that permit reasoning about both intensional and extensional aspects posed in (Chakraborty, 2016) corresponds to identification of suitable semantic domains. Explicit perspectives, as for example in (Yao, 2015; Cattaneo & Ciucci, 2018), correspond similarly. Other semantic domains, including hybrid semantic domains, can be built from more complicated objects such as maximal antichains of mutually discernible objects (Mani, 2017b, 2015) (mentioned earlier), or even over the power set of the set of possible order-compatible partitions of the set of roughly equivalent elements (Mani, 2009, 2018a). In fact, any general rough or soft reasoning context may be associated with a number of semantic domains (Mani, 2012, 2017b, 2016, 2018a, 2020a, 2014). These are sometimes vaguely referred to as meta levels in the artificial intelligence and machine learning (AIML) literature. For example, type-I and type-II fuzzy sets are read in terms of descriptions of functions from different meta levels. Thus, the concept of *semantic domains* in abstract model theory (Mundici, 1984) is analogous to the usage here (though formalization of domains can be objected to because the additional work involved may not justify the coverage of issues afforded). *Contamination* (Mani, 2012, 2018a, 2020a), is the unjustified use of concepts from one semantic domain in another. For example, in a domain intended for reasoning about rough objects alone, presuming absolute knowledge of all objects would amount to contamination. Data intrusion, as in introducing poorly justified stochastic assumptions into a rough domain are additionally a form of contamination. The problem of consistently avoiding such assumptions is the *contamination problem*.

Even when *rough sets are formalized as well-formed formulas* in a fixed language they do not refer to the same domain of discourse. For example, (Banerjee & Khan, 2007; Banerjee & Chakraborty, 2004; Mani, 2005; Düntsch & Orłowska, 2011) refer to semantics of classical rough sets from different perspectives with that of (Mani, 2005) being a higher order semantics of the ability of objects to approximate – this is not expressible in the others. To see this, note that the algebraic model of (Mani, 2005) is built over blocks of a tolerance relation on the set of roughly equivalent objects, while the model of (Banerjee & Chakraborty, 2004) (for example) is essentially about the set of roughly equivalent objects.

For semantic domains in the context of fuzzy sets, the reader is referred to (Turksen, 2005). On page 46, the author essentially points out that possible solutions of the following problem (of subjective probability arguably) depend on the semantic domain used:

> A box contains ten balls of various sizes. Several of these are large, and a few are small. What is the probability that a ball drawn at random is neither large nor small?

Implicit in the problem is that subjective perceptions determine the being of objects. From a rough perspective, information about attributes of the objects would be necessary.

From the above scenario, it can be expected that possible concepts of rationality of approximations are additional impositions on the domain of discourse. Current practices in rough sets do not explicitly require approximations to be rational (in the sense that the approximations are well-constructed from its constituents) in most cases; however, many conditions on possible decisions can be imposed. This lack of universality across domains further makes it natural to explore concepts of rationality at the object level. Apparently, the most convenient way is to define reasonable concepts through ideas of *being a substantial part of* that are related to rough approximations and rationality.

Rationality is additionally approached through formalizations of principles in the context of modal or nonmonotonic logics with epistemological concerns. Ideal-based rough sets can potentially be related to such theories, and this is an open problem. The relatively weaker assumptions on approximation operators can be a cause for concern.

Human-reasoning is often not about truth of statements or grades of truth of statements. Valuation of statements with grades of truth in $\{F, T\}$ or hypercubes generated by $[0, 1]$ are typically meaningless and never intended. They are however used in soft decision-making with no proper validation or opaque expert approved mechanisms as in the LSP (Logic Scoring Preference) method (Dujmovic, 2018). Related assumptions do not provide a good basis for rationality.

2.2. Mereology

Mereology (Burkhardt et al., 2017) consists of a number of theoretical and philosophical approaches to relations of parthood (or *is a part of* predicates) and relatable ones such as those of *being connected to, being apart from, and being disconnected from.* Such relations can be found everywhere, and they relate to ontological features of any body of soft or hard knowledge (and their representation). In the second part of this paper, the relation between other mereological predicates, and the core assumptions about the part-of relations used in this paper are discussed in more detail. It builds on earlier work of the present author (Mani, 2012, 2018b, 2018a, 2020a) and others.

Many types of mereologies (Burkhardt et al., 2017; Cotnoir & Varzi, 2021; Mani, 2012; Lewis, 1991; Janicki & Le, 2007) are known in the literature. The differences can be about axioms (when a common formal language is possible) or domains of discourse. (Cotnoir & Varzi, 2021) imposes a common formal language on a number of approaches. Such reductionist reasoning is additionally evident in rough mereology (Polkowski, 2004; Polkowski & Polkowska, 2008), inspired by the ontology due to Lesniewski, where even the perspective of (Grzegorczyk, 1955) (that *theorems of ontology are those that are true in every model for atomic Boolean algebras without a null element*) is accepted. This makes the resulting model unsuitable for modeling human reasoning, though it appears to work for simpler robotic tasks, and such. Note that the relation of *being roughly* *included to a degree r* is not transitive. Reasoning about vagueness in mereological perspectives requires one to add additional predicates almost always, and therefore associated axiomatics is involved (Polkowski, 2011; Mani, 2012, 2018b) and may need to be enhanced with ontologies. Another mereology (Maffezioli & Varzi, 2021) that assumes many properties of the mereological predicates is however formalized in an intuitionist perspective. The extensionality of parthood and overlap considered in the work do not hold in rough reasoning.

Apart from the motivations and reasons explained in (Mani, 2012, 2018b, 2018a) by the present author, the need for formalizing rationality and flexible languages for teaching and educational research provides additional motivations. The latter is explained in (Mani, 2022b). Basically, in classroom teaching of a subject like mathematics (and especially from a student-centric perspective (Mani, 2020b; Jacobs, Renandya, & Power, 2016)), it is useful to use a language that is easy to understand, and at the same time be formalizable (possibly in a number of ways). In (Mani, 2022b), the use of mereology is suggested as a universal solution to the problem. It may be noted that partial formalizations of negative valuations in multisets (Felisiak, Qin, & Li, 2020) are related.

2.3. Granules and Granulations

A granule may be vaguely defined as some concrete or abstract realization of relatively simpler (or crisper) objects through the use of which more complex concerns may be solved. They exist relative to the issues being solved in question, and can be specified in different non-equivalent ways. For example, they can be specified by the internal attributes of objects, precision levels of possible solutions, or precision levels attained by objects. An axiomatic approach to granular computing is proposed by the present author in (Mani, 2012). Further improvements are in her later works (Mani, 2020a, 2018a, 2013-15). Differences between primitive granular computing and classical granular computing (typically involving numeric precision values) (T. Y. Lin, 2009; Zadeh, 1979; T. Lin & Liu, 1994; Liu, 2006; Yao, 1996, 2001, 2007) are additionally explained. In (Mani, 2020a, 2018a), it is actually argued that the latter can be traced to algorithms in ancient mathematics.

The axiomatic frameworks (AGCP) (Mani, 2018a, 2012) do not refer to numeric precision for defining granules, and the problem of defining or rather extracting concepts that qualify require much work in the specification of semantic domains and process abstraction. A granulation is a collection of all granules and is denoted by $\mathcal{G}$. For a granulation to be *admissible*, it is required that every approximation is term-representable by granules, that every granule in $\mathcal{G}$ coincides with its lower approximation (granules are lower definite), and that all pairs of distinct granules are part of definite objects (those that coincide with their own lower and upper approximations).

By a neighborhood granulation $\mathcal{G}$ on a set W will be meant a subset of the power set $\wp(W)$ for which there exists a map $\lambda : W \longmapsto \mathcal{G}$ such that

$$(\forall B \in \mathcal{G})(\exists x \in W)\, \lambda(x) = B \qquad \text{(Surjectivity)}$$

$$\bigcup_{x \in W} \lambda(x) = S \qquad \text{(Cover)}$$

λ is referred to as a neighborhood map. Given a collection of granules, it may often be possible to generate newer types of interesting granules and granulations using set-theoretical constraints. (Atef, Khalil, Li, Azzam, & Atik, 2020; Al-shami & Ciucci, 2022; Allam, Bakeir, & Abo-Tabl, 2006; Mani, 2017a) use special cases of neighborhood granules to investigate subset neighborhoods.

Let $L, U : \wp(W) \longmapsto \wp(W)$ be operations that satisfy $\forall A \in \wp(W) L(A) \subseteq A \subseteq U(A)$. L will be referred to as a *pointwise lower approximation* if $(\forall A \in \wp(W))L(A) = \{x : x \in W \,\&\Phi(x, A)\}$ for some formula Φ (possibly involving λ). *Pointwise upper approximations* are defined analogously. An example of a pointwise lower approximation is the one defined by $(\forall A \in \wp(W))L(A) = \{x : \lambda(x) \subseteq A\}$, and of a pointwise upper approximation is $(\forall A \in \wp(W))L(A) = \{x : \lambda(x) \cap A \neq \emptyset\}$. The definition is necessarily tied to the language.

In general rough contexts involving any number of approximation operators, it is possible to speak of definite and rough objects in multiple senses that arise from preferred concepts of approximate equality and approximations. Even situations in which the goal is to look for possible explanations that fit approximations specified by agents without explanation, can be handled by the general frameworks of granular operator spaces/partial algebras introduced by the present author. For detailed explanations of the assumptions underlying the frameworks of this subsection, the reader is referred to the papers (Mani, 2020a; Mani & Mitra, 2022) and related references. The *high* adjective is used as an abbreviation for higher order. Key definitions are mentioned below for convenience.

Definition 2.2. An *high mereological approximation Space* (mash) $\mathbb{S}$ is a partial algebraic system of the form $\mathbb{S} = \langle \underline{\mathbb{S}}, l, u, \mathbf{P}, \leq, \vee, \wedge, \bot, \top \rangle$ where $\underline{\mathbb{S}}$ is a set, and l, u are operators : $\underline{\mathbb{S}} \longmapsto \underline{\mathbb{S}}$ that satisfy the following ($\underline{\mathbb{S}}$ is replaced with $\mathbb{S}$ if clear from the context. $\vee$ and $\wedge$ are idempotent partial operations and $\mathbf{P}$ is a binary predicate.):

$$(\forall x)\mathbf{P}xx \tag{PT1}$$

$$(\forall x, b)(\mathbf{P}xb \,\&\, \mathbf{P}bx \longrightarrow x = b) \tag{PT2}$$

$$(\forall a, b) a \vee b \stackrel{\omega}{=} b \vee a;\ (\forall a, b) a \wedge b \stackrel{\omega}{=} b \wedge a \tag{G1}$$

$$(\forall a, b)(a \vee b) \wedge a \stackrel{\omega}{=} a;\ (\forall a, b)(a \wedge b) \vee a \stackrel{\omega}{=} a \tag{G2}$$

$$(\forall a, b, c)(a \wedge b) \vee c \stackrel{\omega}{=} (a \vee c) \wedge (b \vee c) \tag{G3}$$

$$(\forall a, b, c)(a \vee b) \wedge c \stackrel{\omega}{=} (a \wedge c) \vee (b \wedge c) \tag{G4}$$

$$(\forall a, b)(a \leq b \leftrightarrow a \vee b = b \leftrightarrow a \wedge b = a) \tag{G5}$$

$$(\forall a \in \mathbb{S})\, \mathbf{P}a^l a \,\&\, a^{ll} = a^l \,\&\, \mathbf{P}a^u a^{uu} \tag{UL1}$$

$$(\forall a, b \in \mathbb{S})(\mathbf{P}ab \longrightarrow \mathbf{P}a^l b^l \,\&\, \mathbf{P}a^u b^u) \tag{UL2}$$

$$\bot^l = \bot \,\&\, \bot^u = \bot \,\&\, \mathbf{P}\top^l \top \,\&\, \mathbf{P}\top^u \top \tag{UL3}$$

$$(\forall a \in \mathbb{S})\, \mathbf{P}\bot a \,\&\, \mathbf{P}a\top \tag{TB}$$

In a *high general granular operator space* (GGS), defined below, aggregation and co-aggregation operations ($\vee$, $\wedge$) are conceptually separated from the binary parthood ($\mathbf{P}$), and a basic partial order relation ($\leq$). Parthood is assumed to be reflexive and antisymmetric. It may satisfy additional generalized transitivity conditions in many contexts. Real-life information processing often involves many non-evaluated instances of aggregations (fusions), commonalities (conjunctions) and implications because of laziness or supporting metadata or for other reasons – this justifies the use of partial operations. Specific versions of a GGS and granular operator spaces have been studied in the research paper (Mani, 2018a). Partial operations in GGS permit easier handling of adaptive granules (Skowron, Jankowski, & Dutta, 2016) through morphisms. The universe $\underline{\mathbb{S}}$ may be a set of collections of attributes, labeled or unlabeled objects among other things.

Definition 2.3. A *High General Granular Operator Space* (GGS) $\mathbb{S}$ is a partial algebraic system of the form $\mathbb{S} = \langle \underline{\mathbb{S}}, \gamma, l, u, \mathbf{P}, \leq, \vee, \wedge, \bot, \top \rangle$ where $\mathbb{S} = \langle \underline{\mathbb{S}}, l, u, \mathbf{P}, \leq, \vee, \wedge, \bot, \top \rangle$ is a mash, and γ is a unary predicate that determines $\mathcal{G}$ (by the condition γx if and only if $x \in \mathcal{G}$) an *admissible granulation*(defined below) for $\mathbb{S}$. Further, γx will be replaced by $x \in \mathcal{G}$ for convenience.): Let $\mathbb{P}$ stand for proper parthood, defined via $\mathbb{P}ab$ if and only if $\mathbf{P}ab \,\&\, \neg \mathbf{P}ba$). A granulation is said to be admissible if there exists a term operation t formed from the weak lattice operations such that the following three conditions hold:

$$(\forall x \exists x_1, \ldots x_r \in \mathcal{G})\, t(x_1,\, x_2, \ldots\, x_r) = x^l$$
$$\text{and } (\forall x\, \exists x_1,\, \ldots\, x_r \in \mathcal{G})\, t(x_1,\, x_2, \ldots\, x_r) = x^u, \qquad \text{(Weak RA, WRA)}$$
$$(\forall a \in \mathcal{G})(\forall x \in \underline{\mathbb{S}}))\, (\mathbf{P}ax \longrightarrow \mathbf{P}ax^l), \qquad \text{(Lower Stability, LS)}$$
$$(\forall x,\, a \in \mathcal{G} \exists z \in \underline{\mathbb{S}})\, \mathbb{P}xz,\, \&\, \mathbb{P}az \,\&\, z^l \,=\, z^u \,=\, z. \qquad \text{(Full Underlap, FU)}$$

If a granulation is admissible, then the approximations l and u are *granular*.

Definition 2.4.
- In the above definition, if the anti-symmetry condition PT2 is dropped, then the resulting system will be referred to as a *Pre-GGS*. If the restriction $\mathbf{P}a^l a$ is removed from UL1 of a pre-GGS, then it will be referred to as a *Pre*-GGS*.
- In a GGS (resp Pre*-GGS), if the parthood is defined by $\mathbf{P}ab$ if and only if $a \leq b$ then the GGS is said to be a *high granular operator space* GS (resp. Pre*-GS).
- A *higher granular operator space* (HGOS) (resp Pre*-HGOS) $\mathbb{S}$ is a GS (resp Pre*-GS) in which the lattice operations are total.
- In a higher granular operator space, if the lattice operations are set theoretic union and intersection, then the HGOS (resp. Pre*-HGOS) will be said to be a *set HGOS* (resp. *set Pre*-HGOS*). In this case, $\underline{\mathbb{S}}$ is a subset of a power set, and the partial algebraic system reduces to $\mathbb{S} = \langle \underline{\mathbb{S}}, \gamma, l, u, \subseteq, \cup, \cap, \bot, \top \rangle$ with $\underline{\mathbb{S}}$ being a set, γ being a unary predicate that determines $\mathcal{G}$ (by the condition γx if and only if $x \in \mathcal{G}$). Closure under complementation is not guaranteed in it.

In general rough sets, approximations may be granular (in the axiomatic sense), or pointwise, or abstract or co-granular. Abstract approximations are those operators that are merely required to satisfy a few universal conditions. Classical rough approximations can be formalized in all four perspectives.

2.4. Frameworks for Rationality

In earlier papers (Mani, 2022a; Mani & Mitra, 2022), the concept of a *pre-general pre-substantial granular space* (pGpsGS) and stronger forms of partial algebraic systems were proposed to describe a framework for comparing multiple types of rough sets. This assumes that approximations are granular in the sense of the present author (Mani, 2012, 2020a, 2022a) because all pGpsGS are assumed to be pre*-GGS. To accommodate non-granular approximations, it may be necessary to weaken or remove the granularity axioms of weak representability (WRA), lower stability (LS) and full underlap (FU). A full discussion of the connections are done in the next part of this paper, as the approximations are not always granular in this paper. For convenience, the main definition is *The reader is referred to the mentioned papers for relevant explanations and missing* *concepts.*

A *Pre-General Pre-Substantial Granular Space* (pGpsGS) $\mathbb{S}$ is a partial algebraic system of the form

$$\mathbb{S} = \langle \underline{\mathbb{S}}, \gamma, l, u, \mathbf{P}, \mathbf{P}_s, \leq, \vee, \wedge, \bot, \top \rangle$$

where $\langle \underline{\mathbb{S}}, \gamma, l, u, \mathbf{P}, \leq, \vee, \wedge, \bot, \top \rangle$ is a Pre*-GGS and satisfies sub3-sub4 ($\approx$ is intended as any definable rough equality):

$$(\forall a, b)\, (\mathbf{P}_s ab \& \mathbf{P}_s ba \longrightarrow a \approx b) \tag{sub3}$$

$$(\forall a, b, e)\, (\mathbf{P}_s ae \& \mathbf{P}_s ab \longrightarrow \mathbf{P}_s a(b \vee e)) \tag{sub4}$$

If in addition, sub1, sub2, sub5, UL1 and sub6 are satisfied, then $\mathbb{S}$ will be said to be a *Pre-General Substantial Granular Space* (pGsGS).

$$(\forall a)\, \mathbf{P}_s aa \tag{sub1}$$

$$(\forall a, b)\, (\mathbf{P}_s ab \longrightarrow \mathbf{P}ab) \tag{sub2}$$

$$(\forall a, b, e)\, (\mathbf{P}_s ba \& \mathbf{P}_s be \& \mathbf{P}ae \longrightarrow \mathbf{P}_s ae) \tag{sub5}$$

$$(\forall a, b, e)\, (\mathbf{P}_s ab \& \mathbf{P}_s eb \& \mathbf{P}ae \longrightarrow \mathbf{P}_s ae) \tag{sub6}$$

If a pGsGS satisfies the condition PT1, then it will be referred to as a *General Substantial Granular Space.*

2.5. Representation of Concept Maturation

A practical example concerning evaluation contexts (see (Mani, 2020b; Sands, Parker, Hedgeland, Jordan, & Galloway, 2018)) is constructed to demonstrate aspects of the framework in this section. Suppose that labeled or partially labeled data about the understanding of concepts in a mixed class (consisting of students aged 5-11 years, say) is available in the form (Student_Id, Concept). This can be used to generate a pGpsGS $\mathbb{S}$ under the following interpretation: $\underline{\mathbb{S}}$ is a set of sets of crisp and vague concepts, $\vee$: union, $\wedge$: intersection, $\top$: a universal set of collections of concepts, $\bot$ is the set containing the empty set of concepts or whatever that should be regarded as empty, l: the greatest set of crisp concepts that are part of the given element, u: the smallest set of crisp concepts of which the given element is part of, and the granulation predicate γ is determined by a suitable subset of sets of concepts. A parthood relation $\mathbf{P}$ can be defined on the set of concepts in a perspective as follows:

$$\mathbf{P}ab \leftrightarrow b \text{ is an accessible super concept from } a$$

Transitivity of $\mathbf{P}$ can fail possibly because the agency of accessibility may require more. In many cases it may be transitive. Antisymmetry cannot be expected to be true without an additional layer of equivalencing because different concepts that mutually explain each other are always possible. For example, straight lines can be modeled by different types of algebraic equations.

Proposition 2.5. $\mathbf{P}$ *is a reflexive relation that is not antisymmetric on the set of concepts, that in turn induces a reflexive non-antisymmetric relation on the granulation* $\mathcal{G}$.

Irrespective of whether $\mathbf{P}$ is a quasi-order or not, a *filter* of $\mathbf{P}$ is any subset $F \subseteq \mathbb{S}$ that satisfies

$$(\forall x \in F)(\forall z \in \mathbb{S})(\mathbf{P}xz \Rightarrow z \in F).$$

Filters, in this sense, may be read as a weak generalization of sets closed under the consequence afforded by $\mathbf{P}$, or as sets that consist of elements that are not part of elements beyond those in the set.

Such a filter can be used to define a *substantial part of* relation as follows: $\mathbf{P}_s ab$ if and only if $a \in F$ and $\mathbf{P}ab$. Concrete definitions of rational lower approximation relative to such a substantial parthood can then be formulated as:

$$(\forall x) x^{l_s} = \vee\{a : a^l = a \,\&\, \mathbf{P}_s ax\}$$

The lower approximation l_s is rational because it consists of concepts that are not unbounded in the sense afforded by $\mathbf{P}$. Rational upper approximations on the other hand would need to be computed by its definition.

3. Summary and Extensions: Granular Graded Rough Sets

In graded rough sets, approximations are constructed relative to integral grades that are related to the cardinality of sets. Neighborhood granulations of points are used to construct point-wise approximations in (Yao & Lin, 1996), while equivalent granular approximations generated by equivalences are studied in (Zhang, Mo, Xiong, & Cheng, 2012). These are generalized to arbitrary granulations and explored by the present author in a recent work (Mani, 2022a). While such granulations can be related to variable precision rough sets, they can be suggestive of the use of mutually inconsistent procedures in their construction. A few incorrect semantic claims in (Zhang et al., 2012) are also rectified in the research (Mani, 2022a). Additionally, many examples and applications are part of the paper. Semantics of graded modal logics (see (Chen, van Ditmarsch, Greco, & Tzimoulis, 2021; de Rijke, 2000) and related references), and their limitations are applicable to the non-granular context of the paper (Yao & Lin, 1996).

A *grade* k is a fixed positive integer that refers to the cardinality of granules or sets used. Let S be a collection of subsets of a universe H, and $\mathcal{G}$ a subset of S. If x is a set in the collection S, then the *k-lower* and *k-upper* approximations, and related regions are ($\setminus$ is used for the set difference operation)

$$x^{u_k} = \bigcup\{h : \, h \in \mathcal{G} \,\&\, \#(h \cap x) > k\} \quad \text{(k-upper)}$$

$$x^{l_k} = \bigcup\{h : \, h \in \mathcal{G} \,\&\, \#(h) - \#(h \cap x) \leq k\} \quad \text{(k-lower)}$$

$$Pos_k(x) = x^{u_k} \cap x^{l_k} \quad \text{(k-positive region)}$$

$$Neg_k(x) = H \setminus (x^{l_k} \cup x^{u_k}) = H \cap (x^{l_k} \cup x^{u_k})^c \quad \text{(k-negative region)}$$

$$Bnd_k^u(x) = x^{u_k} \setminus x^{l_k} \quad \text{(upper k-boundary)}$$

$$Bnd_k^l(x) = x^{l_k} \setminus x^{u_k} \quad \text{(lower k-boundary)}$$

The lower approximation can result in strange values, and the following regularized version comes naturally:

$$x^{l_k^r} = \bigcup\{h : h \in \mathcal{G} \,\&\, h \subseteq x \,\&\#(h) - \#(h \cap x) \leq k\} \tag{k-reg.lower}$$

Proposition 3.1. *The range of the operations u_k and l_k on $\wp(H)$ need not be equal even when $\mathcal{G}$ is a partition of H, and $S = \wp(H)$.*

In the context of granular graded rough sets, the following substantial parthood predicates are defined (Mani, 2022a): For any $a, b \in S$, and a fixed positive integer k

$$\mathbf{P}_s^1 ab \text{ if and only if } a^{l_k} \subseteq b^{l_k} \tag{s1}$$

$$\mathbf{P}_s^2 ab \text{ if and only if } a \subseteq b \& \#(a) > k \tag{s2}$$

$$\mathbf{P}_s^* ab \text{ if and only if } \#(a \cap b) > k \& b \not\subset a \tag{s*}$$

$$\mathbf{P}_s^3 ab \text{ if and only if } \#(a \cap b) > k \& a \subseteq b \tag{s3}$$

The compatibility of these with defining conditions in the pGpsGS framework is tabulated below (**Tr, Sy** and **Asy** respectively abbreviate transitivity, symmetry and antisymmetry respectively. $*$ indicates the property is satisfied, and $-$ that it is not necessarily satisfied):

Rel	sub1	**Tr**	**Sy**	**Asy**	sub2	sub3	sub4	sub5	sub6	sub9
$\mathbf{P}_s^1$	$*$	$*$	$-$	$-$	$-$	$*$	$*$	$*$	$*$	$*$
$\mathbf{P}_s^2$	$-$	$-$	$-$	$*$	$*$	$*$	$*$	$*$	$-$	$-$
$\mathbf{P}_s^*$	$-$	$-$	$-$	$-$	$-$	$-$	$*$	$*$	$*$	$-$
$\mathbf{P}_s^3$	$-$	$*$	$-$	$*$	$*$	$*$	$*$	$*$	$-$	$-$

Table 1. Summary k-Grade Rough Sets

This shows that the introduced framework may be sufficient for representing substantial parthood in the context of granular k-grade rough sets. Additional variants are also motivated.

3.1. Extensions by Derived Neighborhoods

In a few recent papers (Atef et al., 2020; Al-shami & Ciucci, 2022; Allam et al., 2006), a number of neighborhoods derived from more basic ones generated in a general approximation space, are studied from a mathematical perspective. Some of them approximate better than approximations defined by successor neighborhoods (in the point-wise perspective). However, the parthoods defining the approximations may fail to be substantial. It is of natural interest to investigate graded variants of the same. The main ideas are abstracted and granular graded variants are proposed below. A more thorough investigation of these cases will appear in the second part of this paper. They can serve as examples of point-wise co-granular graded rough sets – the co-granularity being in the sense of the present author (Mani, 2017a). The substantial parthoods of co-granular rough sets do not depend on arbitrary assumptions about grade or precision degrees, and they carry over to the neighborhoods of this section too.

On a general approximation space $S = \langle \underline{S}, R \rangle$, the basic neighborhoods of a point $x \in S$ are (the $\mathcal{N}$ notation is used in (Atef et al., 2020; Al-shami & Ciucci, 2022))

$$[x] := \{a;\, Rax\} = \mathcal{N}_l(x) \quad \text{(Successor)}$$
$$[x]_i := \{a;\, Rxa\} = \mathcal{N}_r(x) \quad \text{(Predecessor)}$$
$$[x]_o := \{a;\, Rax \,\&\, Rxa\} = \mathcal{N}_i(x) \quad \text{(Multiplicative)}$$
$$[x]_\vee := \{a;\, Rax \vee Rxa\} = \mathcal{N}_u(x) \quad \text{(Additive)}$$

From these other neighborhoods such as ()

$$\mathcal{N}_{<l>}(x) = \begin{cases} \bigcap_{x \in \mathcal{N}_l(v)} \mathcal{N}_l(v) & \text{exists } v \text{ s.t. } x \in \mathcal{N}_l(v) \\ \emptyset & \text{otherwise} \end{cases}$$

$$\mathcal{N}_{<r>}(x) = \begin{cases} \bigcap_{x \in \mathcal{N}_r(v)} \mathcal{N}_r(v) & \text{exists } v \text{ s.t. } x \in \mathcal{N}_r(v) \\ \emptyset & \text{otherwise} \end{cases}$$

$$\mathcal{N}_{<i>}(x) = \mathcal{N}_{<r>}(x) \cap \mathcal{N}_{<l>}(x)$$

$$\mathcal{N}_{<u>}(x) = \mathcal{N}_{<r>}(x) \cup \mathcal{N}_{<l>}(x)$$

These eight neighborhoods are generally denoted by $\mathcal{N}_\rho(x)$ (with ρ being a member of $\{l, r, < l >, < r >, i, u, < i >, < u >\}$). Relations between these neighborhoods are used to generate other classes ($\mathcal{E}$, $\mathcal{P}$, $\mathcal{C}$, $\mathcal{S}$) of neighborhoods in the same spirit. For example, the last mentioned class of neighborhoods are

$$\begin{aligned}
\mathcal{S}_l(x) &= \{v : \mathcal{N}_l(x) \subseteq \mathcal{N}_l(v)\} \\
\mathcal{S}_r(x) &= \{v : \mathcal{N}_r(x) \subseteq \mathcal{N}_r(v)\} \\
\mathcal{S}_i(x) &= \mathcal{S}_l(x) \cap \mathcal{S}_r(x) \\
\mathcal{S}_u(x) &= \mathcal{S}_l(x) \cup \mathcal{S}_r(x) \\
\mathcal{S}_{<l>}(x) &= \{v : \mathcal{N}_{<l>}(x) \subseteq \mathcal{N}_{<l>}(v)\} \\
\mathcal{S}_{<r>}(x) &= \{v : \mathcal{N}_{<r>}(x) \subseteq \mathcal{N}_{<r>}(v)\} \\
\mathcal{S}_{<i>}(x) &= \mathcal{S}_{<l>}(x) \cap \mathcal{S}_{<r>}(x) \\
\mathcal{S}_{<u>}(x) &= \mathcal{S}_{<l>}(x) \cup \mathcal{S}_{<r>}(x)
\end{aligned}$$

The standard point-wise approximations for any given neighborhood operation n :

$S \longmapsto \wp(S)$ the n-lower and n-upper approximation is defined as follows for any $B \subseteq S$

$$\mathfrak{A}_n(B) = \{z : n(z) \subseteq B\} \tag{n-lower}$$
$$\mathfrak{A}^n(B) = \{z : n(z) \cap B \neq \emptyset\} \tag{n-upper}$$

The logical meaning of these classes of neighborhoods and associated approximations are not explained in detail in the literature. While it is easy to obtain alternative granulations (finer, under additional conditions on the original relation), the neighborhoods represent the interpretation of patterned connections in the model. For example, v is in the $\mathcal{S}_l$ neighborhood of x provided anything R-related to x is also R-related to v. So the neighborhood is essentially a higher order representation of a derived relation.

Let $\mathcal{N}_l(S) = \{\mathcal{N}_l(x) : x \in S\}$. This collection is partially ordered by set inclusion, and it can be proved that

Proposition 3.2. *For each $x \in S$, the principal filter generated by $\mathcal{N}_l(x)$ in $\mathcal{N}_l(S)$ coincides with the set of $\mathcal{N}_l$- neighborhoods of the elements of $\mathcal{S}_l(x)$.*

In fact, the connection of filters or ideals generated similarly is a central motivation for the co-granular approximations of co-granular approximations. This eventually means that a type-theoretic generalization as opposed to the granular graded approach to rationality is also possible. The following example is used to illustrate the difference

Example 3.3. Two Generalizations

Let $S = \{a, b, c, e\}$, and

$$R = \{(a, a), (c, c), (e, a), (a, c), (b, c), (c, b), (e, c), (e, b)\}.$$

The table of neighborhoods of R are in Table 2

x	$\mathcal{N}_l(x)$	$\mathcal{N}_r(x)$	$\mathcal{N}_i(x)$	$\mathcal{S}_l(x)$	$\mathcal{S}_r(x)$	$\mathcal{S}_{<l>}(x)$
a	$\{a, e\}$	$\{a, c\}$	$\{a\}$	$\{a, c\}$	$\{a, e\}$	$\{a, b\}$
b	$\{c, e\}$	$\{c\}$	$\{c\}$	$\{b.c\}$	S	$\{b\}$
c	S	$\{b, c\}$	$\{b, c\}$	$\{c\}$	$\{c, e\}$	$\{b, c\}$
e	$\emptyset$	$\{a, b, c\}$	$\emptyset$	$\{e\}$	$\{e\}$	S

Table 2. A Few Neighborhoods

These neighborhoods can also be regarded as granulations, and relative to the granulation $\mathcal{G} = \{\mathcal{S}_l(x) : x \in S\}$, the 2-graded approximations of the subset $F = \{a, b, e\}$ are

$$F^{l_2} = S \tag{L2}$$
$$F^{u_2} = S \tag{U2}$$

By contrast, the $\mathfrak{A}_{\mathcal{S}_l}$ and $\mathfrak{A}^{\mathcal{S}^l}$ approximations of F are respectively $\{e\}$ and F. In addition, the $\mathfrak{A}_{\mathcal{S}_{<l>}}$ and $\mathfrak{A}^{\mathcal{S}^{<l>}}$ approximations of F are both equal to F. The graded lower approximations are necessarily proper.

4. Ideals and Filters for Rationality

If R is a binary relation on a set Z, then concepts of R-ideals and R-filters are definable. Under additional assumptions, such concepts may be interpreted as subsets closed under generalized consequence. In (Abo-Tabl, 2011; Allam, Bakeir, & Abo-Tabl, 2008; Kandil, Yakout, & Zakaria, 2016) point-wise approximations that involve order-theoretic ideals in their definition are studied. They are generalized set-theoretically to general approximation spaces and R-ideals in (Mani, 2017a) relative to rough approximations by the present author. However, all considerations are restricted to point-wise approximations alone. In this section, the essential definitions and results are reinterpreted, granular variations are introduced, and potential connections with rationality are investigated. In this regard, it should be noted that a substantial part of the connections are already mentioned in (Mani, 2017a). The approximation framework embodies a kind of rationality in the following sense: *if a property has little to do (in a structured way) with what something is not, then that something has the property in an approximate sense.* The idea of *having little to do with* or *to set no value of* relative to operations is intended to be captured by concepts of ideals. Thus, substantial parthood is potentially ensured by discarding the inessential.

Apart from these, the obvious cases of general approximation spaces of the form (Z, R) in which R is a partial or quasi or preference order (see the research chapter (Mani, 2018a) for details) should be usable to define filters and ideals, and induce generalized orders on granulations. These would easily lead to a number of applicable concepts of substantial parthoods and rational approximations. However, in the literature such investigations are not known, and these will be considered in a separate paper.

Set theoretic generalizations of the approach in (Abo-Tabl, 2011; Allam et al., 2008; Kandil et al., 2016) are defined as follows by the present author (Mani, 2017a).

Definition 4.1.
- Let $\langle Z, R\rangle$ be a general approximation space where Z is a set and R a reflexive binary relation on Z,
- Let $\mathcal{Z}$ be an algebra of subsets of Z, $\mathcal{I}(Z)$ the lattice of lattice ideals of $\mathcal{Z}$, and $\mathbb{I} \in \mathcal{I}(Z)$
- Let $< x >= \bigcap_{x\in[b]_i} [b]_i$.
- $(\forall A \in \mathcal{Z})\ A^{l_+} = \{a : a \in A \,\&\, < a > \setminus A \in \mathbb{I}\}$
- $(\forall A \in \mathcal{Z})\ A^{u_+} = \{a : a \in X \,\&\, < a > \cap A \notin \mathbb{I}\} \cup A$

The approximations will be referred to as *Set difference approximations by ideals* (IASD approximations).

This general approach proposed in (Mani, 2017a) is based on concepts of generalized ideals determined by binary relations on a set, and those of ideals of partially ordered sets (Duda & Chajda, 1977; Rudeanu, 2015; Venkataranasimhan, 1971). It is additionally possible to use a binary relation on the power set to construct generalized ideals consisting of some subsets of the set. Essential definitions are recalled below.

Definition 4.2. Let σ be any binary relation on a set H then

- The *Principal Up-set* generated by $a, b \in H$ is the set $U(a, b) = \{x : \sigma ax \,\&\, \sigma bx\}$.
- The *Principal Down-set* generated by $a, b \in H$ is the set $L(a, b) = \{x : \sigma xa \,\&\, \sigma xb\}$.

- $B \subseteq H$ is *U-directed* if and only if $(\forall a, b \in B)\, U(a, b) \cap B \neq \emptyset$.
- $B \subseteq H$ is *L-directed* if and only if $(\forall a, b \in B)\, L(a, b) \cap B \neq \emptyset$. If B is both U- and L-directed, then it is *σ-directed*.
- $K \subset H$ is a *σ-ideal* if and only if K is U-directed, and

$$(\forall x \in H)(\forall a \in K)(\sigma xa \longrightarrow x \in K).$$

- $F \subset H$ is a *σ-filter* if and only if F is L-directed, and

$$(\forall x \in H)(\forall a \in F)(\sigma ax \longrightarrow x \in F).$$

- The set of σ-ideals and σ-filters are respectively be denoted by $\mathcal{I}(H)$ and $\mathcal{F}(H)$ respectively. These are all partially ordered by the set inclusion order. If the intersection of all σ-ideals containing a subset $B \subset H$ is an σ-ideal, then it will be called the σ-ideal generated by B and denoted by $\langle B \rangle$. The collection of all principal σ-ideals is denoted by $\mathcal{I}_\sigma(H)$. If $\langle x \rangle$ exists for every $x \in H$, then H is said to be *σ-principal* (principal for short).

If F is a subset of an U-directed set B, then it is possible that it is not U-directed. For this reason, the intersection of two σ-ideals is not necessarily a σ-ideal.

Definition 4.3. By a *Co-Granular Operator Space By Ideals*, GOSI, is meant a structure of the form $S = \langle \underline{S}, \sigma, \mathcal{G}, l_*, u_* \rangle$ where $\underline{S}$ is a set, σ a binary relation on S, $\mathcal{G}$ a *neighborhood granulation* over S determined by the neighborhood map λ, and l_*, u_* are operators : $\wp(\underline{S}) \longmapsto \wp(\underline{S})$ satisfying the following ($\underline{S}$ is replaced with S if clear from the context):

$$(\forall X \in \wp(S))\, X^{l_*} = \{a : a \in X \,\&\, \lambda(a) \cap X^c \in \mathcal{I}_\sigma(S)\} \qquad \text{(*-Lower)}$$
$$(\forall X \in \wp(S))\, X^{u_*} = \{a : a \in S \,\&\, \lambda(a) \cap X \notin \mathcal{I}_\sigma(S)\} \cup X \qquad \text{(*-Upper)}$$

In general, if a lower or upper rough approximation ($\oplus$) is defined by expressions of the form $X^{\oplus} = \{a : \lambda(a) \odot X^* \in \mathcal{I}\}$ where $\mathcal{G} \subset \wp(S)$, $\lambda : S \longmapsto \mathcal{G}$ is a neighborhood map, $*$ a complementation or identity operation, $\mathcal{I}$ a set of σ-ideals, and $\odot \in \{\cap, \cup\}$, then the approximation is said to be *co-granular*.

Theorem 4.4. *The following hold in a GOSI S:*

$$(\forall A \in \wp(S))\, A^{l_*} \subseteq A \subseteq A^{u_*}) \qquad \text{(Inclusion)}$$
$$(\forall A \in \wp(S))\, A^{l_* l_*} \subseteq A^{l_*} \qquad \text{(l-Weak Idempotency)}$$
$$(\forall A \in \wp(S))\, A^{u_*} \subseteq A^{u_* u_*} \qquad \text{(u-Weak Idempotency)}$$
$$\emptyset^{l_*} = \emptyset = \emptyset^{u_*}.\ \ S^{l_*} = S = S^{u_*}. \qquad \text{(Bottom, Top)}$$

Remark 1. Monotonicity of the approximations need not hold in general, the granulation is not admissible, and the approximations l_*, u_* are not granular. This is because the choice of parthood is not *sufficiently coherent* with σ in general. A sufficient condition is that σ is at least a quasi order.

4.1. Issues in Granular Generalization

This generalization was not considered earlier in (Mani, 2017a) as the focus was on the point-wise approximations. Let σ be a binary relation on a set H, Γ an arbitrary collection of subsets of H, and $[[g]]$ be the intersection of ideals contained in a $g \in \Gamma$. The following four pseudo-approximations of a subset A of H may be said to be *pseudo granular*:

$$A^{l_{pgt}} = \bigcup\{g : [[g]] \setminus A \in \mathcal{I} \& g \in \Gamma\} \quad \text{(twisted pg-lower)}$$

$$A^{u_{pgt}} = \bigcup\{g : [[g]] \cap A \notin \mathcal{I} \& g \in \Gamma\} \quad \text{(twisted pg-upper)}$$

$$A^{l_{pg}} = \bigcup\{g : g \setminus A \in \mathcal{I} \& g \in \Gamma\} \quad \text{(pg-lower)}$$

$$A^{u_{pg}} = \bigcup\{g : g \cap A \notin \mathcal{I} \& g \in \Gamma\} \quad \text{(pg-upper)}$$

While all four are representable in a strong sense relative to the condition WRA of Definition 2.3 (also see (Mani, 2012, 2018a)), the approximations would be granular only if they satisfy the conditions LS and FU. This justifies the terminology. Without additional conditions on σ or Γ, it is easy to see that the four are not even reasonable approximations. So rationality related conditions of the point-wise case cannot be extended, apparently.

4.2. Substantial Parthood

The defining conditions of the approximations by σ-ideals are nicely relatable to part of relations. However, for the point-wise/co-granular approximations to be rational, the relation σ and the neighborhood granulation should be well grounded in the context in the sense that σ and λ determine or express relevant causal relations.

Since the parthood relations are determined by σ-ideals (except for s*, and s3), they are indexed by superscripts like $i1$, $i2$, ...

Definition 4.5. In what follows a and b are subsets of the universe, F a set of nonempty lower (l_+) definite subsets, and ξFa means *there is a* $b \in F$ *such that* $b \subseteq a$. New substantial parthoods relations can be defined as follows (the conditions s* and s3 are repeated from the context of Section 3):

$$\mathbf{P}_s^{i1}ab \text{ if and only if } a \cap b^c \in \mathcal{I}_\sigma(S) \quad \text{(si1)}$$

$$\mathbf{P}_s^{i2}ab \text{ if and only if } a \cap b \notin \mathcal{I}_\sigma(S) \quad \text{(si2)}$$

$$\mathbf{P}_s^{i*}ab \text{ if and only if } \#(a \cap b) > k \& b \not\subset a \quad \text{(s*)}$$

$$\mathbf{P}_s^{i3}ab \text{ if and only if } \#(a \cap b) > k \& a \subseteq b \quad \text{(s3)}$$

$$\mathbf{P}_s^{i5}ab \text{ if and only if } a^{l+} \subseteq b^{l+} \& \xi Fa \quad \text{(si5+)}$$

$$\mathbf{P}_s^{i6}ab \text{ if and only if } a \cap b^c \in \mathcal{I}_\sigma(S) \& \xi Fa \quad \text{(si5)}$$

$$\mathbf{P}_s^{i9}ab \text{ if and only if } a \cap b \notin \mathcal{I}_\sigma(S) \& \xi Fa \quad \text{(si6)}$$

Of these, $\mathbf{P}_s^{i*}$ and $\mathbf{P}_s^{i3}$ are barely relatable to the approximations or properties of σ or σ-ideals, while the last three are better grounded. For other approximations in the class, similar substantial parthoods are definable. The next few theorems concern representative properties of the other four. Additional properties of σ with weaker forms of the conditions in si1, si2, si5, si6, si5+ can define other interesting parthoods.

Theorem 4.6. *If $\mathcal{I}_\sigma(S)$ is closed under set intersection then the following hold for all $a, b, e \in \wp(S)$:*

$$\mathbf{P}_s^{i1}ab \& \mathbf{P}_s^{i1}ae \longrightarrow \mathbf{P}_s^{i1}a(b \cup e) \quad \text{(si1A)}$$

$$\mathbf{P}_s^{i1}ab \& \mathbf{P}_s^{i1}be \longrightarrow \mathbf{P}_s^{i1}(a \cap b)(b \cup e) \quad \text{(si1B)}$$

$$\mathbf{P}_s^{i1}aa \quad \text{(si1C)}$$

$$a \subseteq b \longrightarrow \mathbf{P}_s^{i1}ab \quad \text{(si1E)}$$

$$\mathbf{P}_s^{i1}a^{l_*}a \quad \text{(si1F)}$$

Proof. (1) $\mathcal{I}_\sigma(S)$ is closed under set intersection, therefore $a \cap b^c$, $a \cap e^c$ and $a \cap b^c \cap e^c$ are σ-ideals. So $\mathbf{P}_s^{i1}a(b \cup e)$ and si1A holds.

(2) si1B is an extension of si1A. It is valid because additional constraints are not imposed on the first argument of $\mathbf{P}_s^{i1}$.

(3) The empty set is a trivial σ-ideal. So si1C holds

(4) si1E follows from si1C

(5) si1F follows from the properties of l_* and si1E

□

Theorem 4.7. $\mathbf{P}_s^{i5}$ *satisfies all the following:*

$$\xi Fa \longrightarrow \mathbf{P}_s^{i5}aa \quad \text{(F- reflexivity)}$$

$$\mathbf{P}_s^{i5}ab \& \mathbf{P}_s^{i5}be \longrightarrow \mathbf{P}_s^{i5}ae \quad \text{(Transitivity)}$$

$$\textit{In general, } a \subseteq b \nrightarrow \mathbf{P}_s^{i5}ab \quad \text{(anti set inclusion)}$$

$$\mathbf{P}_s^{i5}ab \& \mathbf{P}_s^{i5}ba \longrightarrow a^{l_+} = b^{l_+} \quad \text{(l-antisymmetry)}$$

$$\textit{Under F9, } \xi Fa \& a \subseteq b \longrightarrow \mathbf{P}_s^{i5}a^{l_+}b^{l_+} \quad \text{(F-substantiality)}$$

F9: F is closed under intersection.

Proof. (1) $a^{l_+} = a^{l_+}$ is trivial, and ξFa ensures F-reflexivity.

(2) $a^{l_+} \subseteq b^{l_+} \subseteq e^{l_+}$ follows from the premise. In conjunction with ξFa, it follows that $\mathbf{P}_s^{i5}ae$.

(3) Because, ξFa is not true for all a, the property of anti set inclusion follows. Counterexamples are easy to construct.

(4) $\mathbf{P}_s^{i5}ab \& \mathbf{P}_s^{i5}ba$ imply that ξFa and ξFb. This does not contradict the consequence $a^{l_+} = b^{l_+}$.

(5) For any subset x, $x^{l_+l_+} = x^{l_+}$. So ξFa implies ξFa^{l_+}. This combined with monotony yields F-substantiality.

□

Theorem 4.8. $\mathbf{P}_s^{i6}$ *has the following properties*

$$a \subseteq b \& \xi Fa \longrightarrow \mathbf{P}_s^{i6}ab \quad \text{(i61)}$$

$$\mathbf{P}_s^{i6}aa \leftrightarrow \xi Fa \quad \text{(i62)}$$

$$\mathbf{P}_s^{i6}aa \& a \subset b \longrightarrow \mathbf{P}_s^{i6}ab \quad \text{(i63)}$$

$$\textit{If F9 holds } \mathbf{P}_s^{i6}ab \& \mathbf{P}_s^{i6}ae \longrightarrow \mathbf{P}_s^{i6}a(b \cup e) \quad \text{(i64)}$$

Proof. (1) If $a \subseteq b$, then $a \cap b^c = \emptyset$, and this is a trivial σ-ideal. ξFa in conjunction with this ensures that $\mathbf{P}_s^{i6}ab$

(2) That $\mathbf{P}_s^{i6}aa$ implies ξFa follows from the definition.

(3) Monotony of approximations with ξFa that follows from $\mathbf{P}_s^{i6}aa$, ensures property i63.

(4) $\mathbf{P}_s^{i6}ab \& \mathbf{P}_s^{i6}ae$ imply that $a \cap b^c$ and $a \cap b^e$ are σ-ideals and by F9 that $a \cap b^c \cap e^c = a \cap (b \cup e)^c$. Combining this with ξFa yields the conclusion $\mathbf{P}_s^{i6}a(b \cup e)$.

□

Theorem 4.9. *(1)* $\mathbf{P}_s^{i2}$ *is a symmetric, partially reflexive relation.*

(2) If $a = \emptyset$*, then for any* b, $\neg \mathbf{P}_s^{i2}ab$

(3) $\mathbf{P}_s^{i9}$ *is a not necessarily symmetric, partially reflexive relation.*

(4) If ξFa *then for any* b, $\neg \mathbf{P}_s^{i2}ab$

(5) If ξFa *and* ξFb *then* $\mathbf{P}_s^{i9}ab$ *implies* $\mathbf{P}_s^{i9}ba$*.*

From the above, it is clear that the last three conditions can serve as nice conditions defining substantial parthood, while s* and s3 are difficult to work with in general. The rest have the potential when the trivial cases are eliminated. This class of approximations is therefore very different from both VPRS and graded rough sets. It is however closer to rough sets with generalized orders on the set of attributes or collections of subsets of attributes. A number of parthoods are possible with additional conditions on the collection of σ-ideals.

4.3. Discussion and Meaning

In the above, conditions of the form $g \setminus A \in \mathcal{I}$ for a neighborhood or granule g essentially refer to the fact that those differences are similar to a generalized zero or an algebraically closed and absorptive subset. In the point-wise perspective, a point that generates such an instance is regarded as a part of the lower approximation. This happens because ideals of an algebra have absorptive properties relative to the algebraic operations, and are closed. Further, in universal algebras with 0 (that are *ideal determined*), ideals are the 0-class of a congruence (Gumm & Ursini, 1984; Mckenzie, McNulty, & Taylor, 1987; Freese, Mckenzie, McNulty, & Taylor, 2022). Additionally, the requirement that $g \setminus A$ is an ideal means the following:

- *Whatever is a substantial part of* g *is also a substantial part of* A *(because the difference is a generalized zero),*
- if g is substantial, then it is a substantial part of A, and $g \setminus A$ is not a substantial part of S.

These assertions are consistent.

Moreover, if K is an ideal in a GOSI S, A a subset of S, and $x \in S$, then

$$\lambda(x) \subseteq A \longrightarrow \lambda(x) \cap A^c = \emptyset \longrightarrow \lambda(x) \cap A^c \subseteq K \tag{1}$$

This extends to the higher order variant of GOSI (GOSIH) (Mani, 2017a) as well. For reference, a *higher co-granular operator space by ideals* GOSIH is a structure of the form $S = \langle \underline{S}, \sigma, \mathcal{G}, l_o, u_o \rangle$ where $\underline{S}$ is a set, σ *a binary relation on the powerset* $\wp(\underline{S})$, $\mathcal{G}$ a *neighborhood granulation* over S and l_o, u_o *o-lower and o-upper approximation operators*

: $\wp(\underline{S}) \longmapsto \wp(\underline{S})$ defined as follows (for any $X \in \wp(S)$, and a fixed $\mathbb{I} \in \mathcal{I}_\sigma(\wp(S))$):

$$X^{l_o} = \{a : a \in X \,\&\, \lambda(a) \cap X^c \in \mathbb{I}\} \tag{o-Lower}$$
$$X^{u_o} = \{a : a \in S \,\&\, \lambda(a) \cap X \notin \mathbb{I}\} \cup X \tag{o-Upper}$$

By contrast, note that the condition $g \cap A \notin \mathcal{I}$ means that $g \cap A$ may be a part of some σ-filters and σ-ideals, but is not a σ-ideal. Therefore, in a nice enough situation (for example, when g is a neighborhood of a point), $g \cap A$ is not an σ-ideal, and possibly closed under a generalized consequence afforded by the model. g may additionally be said to share a common substantial part with A. This is the point that there exists a c such that $\mathbf{P}_s cg$ and $\mathbf{P}_s cA$. A paraconsistent interpretation of the scenario is also justified.

However, as noted in (Mani, 2017a), a *subset of an ideal need not behave like a generalized zero in general.* This statement is dialectically opposed (Mani, 2018b) to (or alternatively, runs counter to) the idea that *the subset is part of a generalized zero.* A higher order approach helps to simplify the interpretation. Partial interpretation of operations may be appropriate because of the nature of objects in a context (for example, attribute values of objects may need to be integers). Such a contamination avoidance procedure can affect the nature of possible models, and in fact, serves as a motivation for introducing the notion of a GOSIH. In the relational approximation contexts of the papers (Abo-Tabl, 2011; Allam et al., 2008; Kandil et al., 2016), subsets of generalized zeros are generalized zeros. This results in wide differences with properties of their generalizations. Therefore, *if a property has little to do (in a structured way) with what something is not, then that something has the property in an approximate sense.* The idea of *little to do with* or *set no value of* relative operations is intended to be captured by concepts of ideals.

If σ-ideals are seen as essentially empty sets, then they have a hierarchy of their own and function like definite entities. The σ-ideals under some weak conditions permit the following association. If A is a subset then it is included in the smallest σ-ideal containing it and a set of maximal σ-ideals contained in it. These may be seen as a representation of rough objects of a parallel universe.

5. Rationality of Objects

The idea of rationality of objects can be read from the perspective of ontology (as in *an object's existence is justified by its constructive definition*) or from a distributive cognition perspective (as in *the object's defining process, and therefore its environment, is a part of the agent's existence*) – the latter involves a broader understanding of objects (see (Werner, 2020; Mani, 2022b)). In common language discourses, across languages, people often use phrases such as *PCIE-5 slots on computer main-boards are insane* (because their bandwidth cannot be saturated by anything that can be inserted there), *this smartphone is not rational, or it has a mind of its own* (to mean that it behaves erratically to input instructions), and *the apples taste sweet* (in reference to a collection of apples). The ability to ontologically represent such information is important for making robust AIML algorithms. Additionally, such representation can help in the construction of algorithms that avoid learning toxic behaviors (from text or speech data that objectifies women, for example (Godoy & Tommasel, 2021)), and understand jargon better when it matters among others.

Rough objects are defined in relation to approximations. However, such objects need not be functionally defined at the object level as can be seen from possible definitions. Their rationality can be expected to be definable in terms of the language of the rough domain. This is because the concept of rationality can only be relative to available reasoning machinery and associated facts. The concepts of substantial parthood used in the earlier papers in the context of granular graded/variable-precision rough sets, and the general frameworks may not be easily relatable as the language is expected to have built-in predicates. In the light of these observations two new concepts of rational and co-rational objects are proposed next.

A *rationally existing object* will be an exact object that approximates as few objects as is possible. This would ensure that it remains distinct from most others from the rough perspective. A dual of this concept that will be referred to as a *co-rationally existing object* will be an exact object that approximates as many objects as is possible. The two formulations can be easily formalized in a number of ways in a rough language that has at least a parthood predicate, and unary lower and upper approximation operator symbols. Scope for expressing granularity is not essential. The following definitions over models may be rewritten to express similar ideas in a language.

5.1. Derivations for Classical Rough Sets

For classical rough sets, sets of reasonable integers can represent the quality, and type of rational existence of objects in relation to its rough semantic domain. This is a surprising new result.

Let $\mathbf{X} = \langle X, R\rangle$ be an approximation space with R being an equivalence relation on the finite set X. Further, let $S = \wp(X)$, the equivalence classes of R be $g_1, g_2, \ldots g_k$, and $Card(X) = n$, $Card(g_i) = n_i$ (for each i). The set of equivalence classes of R is the granulation $\mathcal{G} = \{g_1, g_2, \ldots g_k\}$. An element $x \in S$ (or equivalently, a subset of X) is said to lower (upper) definite if and only if $x^l = x$ ($x^u = x$). As mentioned earlier, it is definite, if it is both lower and upper definite. Denoting the set of all lower (upper) definite elements by $\delta_l(X)$ ($\delta_u(X)$), it is obvious that $\delta_l(X) = \delta_u(X) = \delta(X)$ – the set of definite elements.

Definition 5.1. For any element $x \in \delta(X)$, let

$$\Lambda_l(x) = \{z :\ z^l = x\} \qquad \text{(lower approximates)}$$
$$\mu_l(x) = Card(\Lambda_l(x)) \qquad \text{(lower power)}$$
$$\Lambda_u(x) = \{z :\ z^u = x\} \qquad \text{(upper approximates)}$$
$$\mu_u(x) = Card(\Lambda_u(x)) \qquad \text{(upper power)}$$
$$\Lambda(x) = \{z :\ z^l = x \text{ or } z^u = x\} \qquad \text{(approximates)}$$
$$Card(\Lambda(x)) = \mu(x) \qquad \text{(full power)}$$

Proposition 5.2. *The following hold:*

$$(\forall a, b \in \delta(X))(\Lambda_l(a) \cap \Lambda_l(b) \neq \emptyset \longrightarrow a = b) \tag{L1}$$
$$(\forall a, b \in \delta(X))(a \neq b \longrightarrow \Lambda_l(a) \cap \Lambda_l(b) = \emptyset) \tag{L2}$$
$$(\forall a, b \in \delta(X))(\Lambda_u(a) \cap \Lambda_u(b) \neq \emptyset \longrightarrow a = b) \tag{U1}$$
$$(\forall a, b \in \delta(X))(a \neq b \longrightarrow \Lambda_u(a) \cap \Lambda_u(b) = \emptyset) \tag{U2}$$
$$(\forall a, b \in \delta(X))(a \subset b \longrightarrow \mu_l(b) < \mu_l(a)) \tag{L3}$$
$$(\forall a, b \in \delta(X))(a \subset b \longrightarrow \mu_u(a) < \mu_u(b)) \tag{U3}$$

In other words, μ_l is an antitone valuation, while μ_u is monotone.

Proof. l and u are many-one functions on S. Λ_l and Λ_u are respectively their inverse images. Therefore, if $a \neq b$, then it is necessary that both $\Lambda_l(a) \cap \Lambda_l(b)$ and $\Lambda_u(a) \cap \Lambda_u(b)$ are empty.

If $a \subset b$, then they must differ by the union of some granules. So let

$$a = \bigcup_{i=1}^{r} g_i \text{ and } b = \bigcup_{i=1}^{r+t} g_i$$

Elements of $\Lambda_l(a)$ will have the form $a \cup x$, where

$$x \subset \cup\{\mathcal{G} \setminus \{g_1, g_2, \ldots g_r\}\}$$

and that x does not include any granule. Elements of $\Lambda_l(b)$ will also have the form $b \cup z$, where z is a union of proper subsets of granules. Therefore, it follows that $\mu_l(b) < \mu_l(a)$. The actual formula is derived in the following theorem.

Elements of $\Lambda_u(a)$ will have the form $x \subseteq a$ subject to the condition that at least one element of each of the granules included in a are part of x. Since a is included in b, elements of $\Lambda_u(b)$ would be constructible in more number of ways. This ensures the monotonicity of μ_u. □ □

Theorem 5.3. *For each granule g_i*

$$\mu_l(g_i) = \prod_{j \neq i} \left[\sum_{0 \leq f < n_j} \binom{n_j}{f} \right] \tag{lpgra}$$

For a definite object $a = \bigcup_{i=1}^{r} g_i$,

$$\mu_l(a) = \prod_{j > r} \left[\sum_{0 \leq f < n_j} \binom{n_j}{f} \right] \tag{lpgen}$$

Proof. The essence of the matter is the statement in the last proof that elements of $\Lambda_l(a)$ will have the form $a \cup x$, where x is a subset of $\cup\{\mathcal{G} \setminus \{g_1, g_2, \ldots g_r\}\}$ that does not include any granule. This means that x can include $0, 1, \ldots n_{r+1} - 1$ elements from g_{r+1} and so on. Each of these number of elements can be selected in $\binom{n_j}{f}$ ways with the variable f ranging over $0 \leq f < n_j$) and so the possible combinations is the sum. The product is the result of combining the selections over the allowed granules. □ □

Theorem 5.4. *For each granule g_i*

$$\mu_u(g_i) = \sum_{1 \leq f \leq n_i} \binom{n_i}{f} \tag{upgra}$$

For a definite object $a = \bigcup_{i=1}^{r} g_i$,

$$\mu_l(a) = \prod_{j=1}^{r} \left[\sum_{1 \leq f \leq n_j} \binom{n_j}{f} \right] \tag{upgen}$$

Proof. The essence of the matter is that elements of $\Lambda_u(a)$ will have the form $\cup_{i=1} r x_i$, where x_i is a nonempty subset of g_i. This means that x can include $1, \ldots n_i$ elements from g_i for $i = 1, \ldots, r$. Each of these number of elements can be selected in $\binom{n_j}{f}$ ways with the variable f ranging over $1 \leq f \leq n_j$) and so the possible combinations is the sum. The product is the result of combining the selections over the r granules. □ □

Remark 2. This approach offers a easier higher order approach to classify classical rough set models and instances. Additionally, it takes the ability of definite objects to approximate others into account.

Definition 5.5. The *l-rationality type* of an approximation space X will be the totally ordered multi-set $lrt(X) = \{\mu_l(x) : x \in \delta_l(X)\}$ (in decreasing order). Similarly, the *u-rationality type* of an approximation space X will be the totally ordered multi-set $urt(X) = \{\mu_u(x) : x \in \delta_u(X)\}$ (in increasing order).

The *normalized l-rationality* and *normalized u-rationality types* will be the totally ordered multi-set $nlrt(X) = \{\frac{\mu_l(x)}{2^n} : x \in \delta_l(X)\}$ (in decreasing order) and the totally ordered multi-set $nurt(X) = \{\frac{\mu_u(x)}{2^n} : x \in \delta_u(X)\}$ (in increasing order).

In practice, the identification of the values of μ_l and μ_u corresponding to granules would be essential for using the l-rationality and u-rationality types effectively for comparing different situations from a rough perspective.

Definition 5.6. Let

$$B_l : \delta_l(X) \longmapsto Z_+, \text{ and } B_u : \delta_u(X) \longmapsto Z_+$$

be two functions defined by

$$B_l(x) = \sum_{z \in \Lambda_l(x)} Card(z \setminus x) + 1 \tag{l-boundary count}$$

$$B_u(x) = \sum_{z \in \Lambda_u(x)} Card(x \setminus z) + 1 \tag{u-boundary count}$$

the *local l-rationality* and *local u-rationality types* will be the totally ordered multi-set $llrt(X) = \{\frac{\mu_l(x) * 2^n}{B_l(x)} : x \in \delta_l(X)\}$ (in decreasing order) and the totally ordered multi-set $nurt(X) = \{\frac{\mu_u(x) * 2^n}{B_u(x)} : x \in \delta_u(X)\}$ (in increasing order).

The local types are intended to account for the number of elements in the lower and upper boundary, and therefore can be related to the quality of approximation in the context.

Problem 5.7. Problems on bounds on the size of μ_l and μ_u can possibly be solved from a purely combinatorial perspective

The number of equivalences with exactly k classes on a set X of cardinality n is given by

$$S(n,k) = \frac{1}{k}\sum_{i=1} k(-1)^{k-i}\binom{k}{i} i^n \qquad \text{(Stirling-2)}$$

While the set of all equivalence relations on the set is given by Bell's equation:

$$B(n) = \sum_{k=1}^{n} S(n,k) \qquad \text{(Bell)}$$

The first few Bell numbers in order are

$$1, 2, 5, 15, 52, 203, 877, 4140, 21147, \ldots.$$

If an equivalence relation has k classes, then the cardinality of $\delta(X) = Card(\wp(\mathcal{G}))$ would be 2^k. However, the rationality types are intended to work across different values of n.

Example 5.8. Let $S = \{1,2,3,4,5,6,7,8,9\}$ be a set, and R an equivalence relation on S defined as follows (Δ_S is the diagonal of S):

$$R := \{(1,\ 2),(2,\ 1),(4,\ 5),(5,\ 4),(6,\ 9),(9,\ 6),\\ (6,\ 7),(7,\ 6),(7,\ 9),(9,\ 7)\} \cup \Delta_S.$$

Consider the subsets

$$A = \{2,\ 3\},\ B = \{1,5,6,8,9\},\ \text{and}\ F = \{1,\ 4\}$$

The approximations of the sets are as in Table 4. In the table, sets of the form $\{1,2\}$ have been abbreviated as 12. The neighborhood granules generated by the elements are as in Table 3.

Table 3. Neighborhoods

S	1	2	3	4	5	6	7	8	9
$[x]$	12	12	3	45	45	679	679	8	679

Note that $\mathcal{G} = \{[x] : x \in S\}$, $Card(\mathcal{G}) = 5$, the empty set is a definite element, and not a granule.

Table 4. Approximations

Set	l	u
A	3	123
B	8	12456789
F	$\emptyset$	1245

The number of elements in $\delta(S)$ is $2^5 = 32$. $x = \{1, 2, 3\}$ is an example of a definite element, and

$$\Lambda_l(x) = \{123, 1234, 1235, 1236, 12346, 12347, 12349, 12356, 12357, 12359, 1237, 1239, 123567, 123579, 123569, 123467, 123469, 123479\} \quad (2)$$

$$\Lambda_u(x) = \{123, 13, 23\} \quad (8)$$

Therefore, $\mu_l(x) = 18$ and $\mu_u(x) = 3$. The l-boundary count $B_l(x) = 36$, and the u-boundary count $B_u(x) = 3$

5.2. More General Rough Contexts

Initially, tolerance spaces are explored, and the concepts are partly extended to more general contexts. These carry over to the general frameworks for rational approximation and granularity.

Rough approximations over tolerance spaces (or similarity approximation spaces) can be defined in a number of nonequivalent ways, and these may or may not involve granulations. This helps in illustrating the difficulty of characterizations based on cardinalities of granules alone in the approaches.

Let $\mathbf{X} = \langle X, T \rangle$ be a general approximation space (tolerance space) with T being a tolerance relation on the finite set X. Possible semantics depend on choice of granulation (see (Mani, 2018a) for details). Some choices of granulations in the context are the following:

(1) The collection $\mathcal{B} = \{g_1, g_2, \ldots g_k\}$ of blocks (maximal subsets B of S that satisfy $B^2 \subseteq T$),
(2) The collection of successor $\mathcal{N}$ and predecessor $\mathcal{N}_i$ neighborhoods generated by T and
(3) The collection $\mathcal{T} = \{\cap(\Gamma) : \Gamma \subseteq \mathcal{B}\}$. These will be called the collection of *squeezed blocks*.

$\mathcal{B}$ may appear to be the most natural choice because it is a proper generalization of the concept of a partition associated with an equivalence relation. Related representation theorems can be found in the chapter on duality in (Mani, 2018c). For this case multiple semantic approaches have been developed by the present author (Mani, 2011, 2009, 2017b). The approach in (Mani, 2017b) can possibly be extended to the point-wise contexts as well. The point-wise nongranular approach has been considered in (Skowron & Stepaniuk, 1996; Pomykala, 1993; Cattaneo, 1998; Cattaneo & Ciucci, 2004; Pagliani & Chakraborty, 2008; Järvinen & Radeleczki, 2017) and other papers. Squeezed blocks are studied in (Ślęzak & Wasilewski, 2007; Wasilewski & Ślęzak, 2008; Mani, 2009).

Consider the following approximations of a subset $A \in S$ ($[x]$ is the successor neighborhood of the point $x \in S$ generated by the tolerance relation T),

$$A^l = \bigcup\{g : g \subseteq A \,\&\, g \in \mathcal{B}\} \qquad \text{(lower)}$$
$$A^u = \bigcup\{A : g \cap A \neq \emptyset \,\&\, g \in \mathcal{B}\} \qquad \text{(upper)}$$
$$A^{l_p} = \{x : [x] \subseteq A\} \qquad \text{(p-lower)}$$
$$A^{u_p} = \{x : [x] \cap A \neq \emptyset\} \qquad \text{(p-upper)}$$
$$A^{l_s} = \bigcup\{g : g \subseteq A \,\&\, g \in \mathcal{T}\} \qquad \text{(sq-lower)}$$
$$A^{u_s} = \bigcup\{A : g \cap A \neq \emptyset \,\&\, g \in \mathcal{T}\} \qquad \text{(sq-upper)}$$
$$A^{u_{sb}} = \bigcup\{g : g \cap A \neq \emptyset \,\&\, g \in \mathcal{T}\} \setminus (X^c)^l \qquad \text{(sqb-upper)}$$

Denoting the above lower and upper approximations by l_* and u_* respectively, the following can be asserted. An element $x \in S$ is said to *-lower (*-upper) definite if and only if $x^{l_*} = x$ ($x^{u_*} = x$). It is said to be *-definite if it both *-lower and *-upper definite. Let the set of *-lower definite, *-upper definite and *-definite elements be $\delta_{l_*}(X)$, $\delta_{u_*}(X)$ and $\delta_*(X)$ respectively.

Definition 5.9. For any $x \in \delta_{l_*}(X)$, $v \in \delta_{u_*}(X)$, and $w \in \delta(X)$, let

$$\Lambda_{l_*}(x) = \{z : z^{l_*} = x\} \qquad \text{(*-lower approximates)}$$
$$\mu_{l_*}(x) = Card(\Lambda_{l_*}(x)) \qquad \text{(*-lower power)}$$
$$\Lambda_{u_*}(v) = \{z : z^{u_*} = v\} \qquad \text{(*-upper approximates)}$$
$$\mu_{u_*}(v) = Card(\Lambda_{u_*}(v)) \qquad \text{(*-upper power)}$$
$$\Lambda_*(w) = \{z : z^{l_*} = w \text{ or } z^{u_*} = w\} \qquad \text{(*-approximates)}$$
$$\mu(w) = Card(\Lambda_*(w)) \qquad \text{(*-full power)}$$

It can be seen that the concepts extend from the classical case to both granular, and point-wise approximations because the entire definition is about the ability to approximate.

Proposition 5.10. *The following hold:*

$$(\forall a, b \in \delta(X))(\Lambda_{l_*}(a) \cap \Lambda_{l_*}(b) \neq \emptyset \longrightarrow a = b) \qquad \text{(*L1)}$$
$$(\forall a, b \in \delta(X))(a \neq b \longrightarrow \Lambda_{l_*}(a) \cap \Lambda_{l_*}(b) = \emptyset) \qquad \text{(*L2)}$$
$$(\forall a, b \in \delta(X))(\Lambda_{u_*}(a) \cap \Lambda_{u_*}(b) \neq \emptyset \longrightarrow a = b) \qquad \text{(*U1)}$$
$$(\forall a, b \in \delta(X))(a \neq b \longrightarrow \Lambda_{u_*}(a) \cap \Lambda_{u_*}(b) = \emptyset) \qquad \text{(*U2)}$$

Proof. l and u are many-one functions on $S = \wp(X)$. Λ_{l_*} and Λ_{u_*} are respectively their inverse images. Therefore, if $a \neq b$, then it is necessary that both $\Lambda_{l_*}(a) \cap \Lambda_{l_*}(b)$ and $\Lambda_{u_*}(a) \cap \Lambda_{u_*}(b)$ be empty. □ □

For any subset x of X, approximate bounds on $\mu_{l_*}(x)$ and $\mu_{u_*}(x)$ can be computed given additional assumptions about the cardinality of $g_i \cap g_j$ for $g_i, g_j \in \mathcal{G}$. The bounds are ensured by the representation results for tolerances by normal covers (see (Chajda et al., 1976)). Rationality types can however be defined by analogy.

Definition 5.11. The *l*-rationality type* of a tolerance space X will be the totally ordered multi-set $l * rt(X) = \{\mu_{l_*}(x) : x \in \delta_{l_*}(X)\}$ (in decreasing order). Similarly, the *u*-rationality type* of X will be the totally ordered multi-set $u * rt(X) = \{\mu_{u_*}(x) : x \in \delta_{u_*}(X)\}$ (in increasing order).

The *normalized l*-rationality* and *normalized u*-rationality* types will be the totally ordered multi-set (in decreasing order)

$$nl * rt(X) = \{\frac{\mu_{l_*}(x)}{2^n} : x \in \delta_{l_*}(X)\},$$

and the totally ordered multi-set (in increasing order)

$$nu * rt(X) = \{\frac{\mu_{u_*}(x)}{2^n} : x \in \delta_{u_*}(X)\}.$$

In practice, the identification of the values of μ_l and μ_u corresponding to granules would be essential for using the l-rationality and u-rationality types effectively for comparing different situations from a rough perspective. Further, the types can be used to classify general rough sets.

Definition 5.12. If $\mathcal{T}$ is the class of finite tolerance spaces, for any $\chi \in \{l * rt, u * rt, nl * rt, nu * rt\}$, the relation τ_χ will be defined by

$$\tau_\chi ab \text{ if and only if } \chi(a) = \chi(b)$$

It is provable that

Theorem 5.13. *τ_χ is a nontrivial equivalence for each $\chi \in \{l*rt, u*rt, nl*rt, nu*rt\}$.*

Definition 5.14. Let $B_{l_*} : \delta_{l_*}(X) \longmapsto Z_+$, and $B_{u_*} : \delta_{u_*}(X) \longmapsto Z_+$ be two functions defined by

$$B_{l_*}(x) = \sum_{z \in \Lambda_{l_*}(x)} Card(z \setminus x) + 1 \qquad \text{(l*-bnd count)}$$

$$B_{u_*}(x) = \sum_{z \in \Lambda_{u_*}(x)} Card(x \setminus z) + 1 \qquad \text{(u*-bnd count)}$$

the *local l*-rationality* and *local u*-rationality types* will be the totally ordered multi-set (in decreasing order)

$$ll * rt(X) = \{\frac{\mu_{l_*}(x) * 2^n}{B_{l_*}(x)} : x \in \delta_{l_*}(X)\},$$

and the totally ordered multi-set (in increasing order)

$$nu * rt(X) = \{\frac{\mu_{u_*}(x) * 2^n}{B_{u_*}(x)} : x \in \delta_{u_*}(X)\}.$$

As in the classical approximation context, the local types are intended to take the number of elements in the lower and upper boundary into account, and therefore can be related to the quality of approximation.

The concepts of rationality type, and boundary counts do not apparently depend on the definition of the approximations. However, it can be seen that types associated with granular approximations have relatively finer bounds due to the block representation theorem for tolerances. Additionally, from the above considerations, it can be seen that definitions 5.9, 5.11, and 5.14 extend to any set equipped with lower and upper approximation operators. Proposition 5.10 can also be extended to the general relational approximation context without any modifications. Pointwise approximations, however lack the ability to impose sharper conditional bounds on types.

5.3. Abstract Examples

Let $S = \{a, b, c, e, f, g\}$, and

$$\sigma = \{(a, c),\ (a, e),\ (b, c),\ (b, e),\ (c, c),\ (c, b), (e, a),\ (f, f)\}.$$

σ is not a symmetric, transitive or reflexive relation. In Table 5, the computed values of the set of upper bounds, lower bounds, and neighborhoods are presented. $*$ in the last two rows refers to any element from the subset $\{a, b, c, e\}$. Values of the form $U(x, x)$ and $L(x, x)$ have been kept in Table 6 because they correspond to values of neighborhoods of σ.

Table 5. Upper and Lower Bounds.

Pair (x, z)	$U(x, z)$	$L(x, z)$
(a, b)	$\{e, c\}$	$\emptyset$
(a, c)	$\{c\}$	$\emptyset$
(a, e)	$\emptyset$	$\emptyset$
(b, c)	$\{c\}$	$\{c\}$
(b, e)	$\emptyset$	$\emptyset$
(c, e)	$\emptyset$	$\{a, b\}$
$(*, f)$	$\emptyset$	$\emptyset$
$(*, g)$	$\emptyset$	$\emptyset$

Table 6. Neighborhoods.

x	$U(x, x)$	$L(x, x)$	$< x >$
a	$\{c, e\}$	$\{e\}$	$\{a\}$
b	$\{c, e\}$	$\{c\}$	$\{b.c\}$
c	$\{b, c\}$	$\{a, b, c\}$	$\{c\}$
e	$\{c\}$	$\{a, b\}$	$\{c, e\}$
f	$\{f\}$	$\{f\}$	$\{f\}$
g	$\emptyset$	$\emptyset$	$\emptyset$

In Table 6, $U(x, x) = [x]_i$ and $L(x, x) = [x]$. Given the above information, it can be deduced that the nontrivial σ- ideals are

$$I_1 = \{a, b, e, c\} \text{ and } I_2 = \{a, b, e, c, f\}$$

If a co-granulation is defined by the neighborhood map $\lambda : S \longmapsto \mathcal{G}$ defined by $\lambda(a) = \{b\}$, $\lambda(b) = \{g\}$, $\lambda(c) = \{c, a\}$, $\lambda(e) = \{e\}$, $\lambda(f) = \{f\}$, and $\lambda(g) = \{g, b, c\}$, then the GOSI approximations of the set $A = \{a, b\}$ can be computed to be $A_*^l = \{b\}$ and $A_*^u = \{a, b, c, g\}$. For distinct lattice ideals many approximations of A by l_i and u_i can be computed. GOSIH related computations of approximations are bound to be cumbersome even for four element sets and so have been omitted.

6. Further, Directions and Conclusions

From the present research, *it is clear that only some generalized rough contexts have a mechanism for controlling the rationality of approximations or objects: graded rough sets, VPRS, probabilistic rough sets, dominance based rough sets, ideal based rough sets, and generalized order based rough sets (including their generalization and hybridization).* Of these, ideal based versions and the rationality of objects are investigated in this first part of a three-part research paper. Generalized order based rough sets will be considered separately as it is a huge topic.

The suitability of the introduced framework for handling substantial parthood is fairly clear from the properties satisfied. Some definitions of substantial parthoods in granular graded and VPRS contexts involve generalized metric condition. This is not the case for ideal-based rough sets (and those defined by subset neighborhoods) that require specifications based on ideals (or other types of sets). *The universal properties of the substantial parthood relations form a basis for deeper explorations.*

Rationality types introduced are shown to be useful for encoding patterns of approximation, and would be useful for comparing different rough contexts of the same theoretical type (like probabilistic, VPRS, and ideal-based), and are by no means restricted by the necessity of substantial parthoods. Once again, they work the best for classical rough sets. These results are new and a bit surprising. In the anti-chain based semantics of (Mani, 2017b, 2015), many operations are studied over antichains constructed in relation to discernibility. *The connections of the rationality types, and related results are of natural interest in the context.* The learning theory proposal is work in progress.

Acknowledgement

This research is supported by Woman Scientist Grant No. WOS-A/PM-22/2019 of the Department of Science and Technology. The present author would like to thank the referees for useful comments that led to improvements in the presentation of the paper.

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