Feynman's Formula for the Ising Model on Surfaces

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1 Preliminaries and Goals

1.1 Graph definitions and notation

We begin by making a few basic definitions and introducing notation for graphs.

Definition 1.1 (Directed graph)**.** Denote by G a finite connected graph, and by $V(G)$ and $\vec{E}(G) \subseteq V(G) \times V(G)$ the sets of vertices and directed edges of G respectively. Each directed edge $e \in \vec{E}(G)$ is of the form $(o(e), t(e))$, where we call $o(e) \in V(G)$ and $t(e) \in V(G)$ the *origin* and *terminal* vertices of e respectively. We define $\bar{e} := (t(e), o(e))$ (so $\bar{e}$ has opposite orientation), and we require $\bar{e} \in \vec{E}(G)$ (symmetry). Finally, define the set $E(G)$ of undirected edges, which can be thought of as the quotient of $\vec{E}(G)$ by the equivalence relation $\sim$ under which $(o(e), t(e)) \sim (t(e), o(e))$.

We further associate to each edge an abstract variable, which we call the *weight* of that edge:

Definition 1.2 (Weighted graph)**.** Given graph G, define the weight function $x = (x_e)_{e \in \vec{E}(G)}$ for variables x_e, requiring $x_e = x_{\bar{e}}$ for each $e \in \vec{E}(G)$. The latter requirement lets us equally x as indexed by undirected edges $e \in E(G)$. Define the weight of a subset $E \subseteq E(G)$ of edges as $x(E) := \prod_{e \in E} x_e$. Finally, denote the weighted graph by (G, x).

For graph (G, x), selecting appropriate subsets $V(G') \subseteq V(G)$ and $\vec{E}(G') \subseteq E(G)$ (where each edge in $\vec{E}(G')$ of course joins vertices in $V(G')$) amounts to choosing a (weighted) *subgraph* (G', x'), where x' is simply the restriction of x to $\vec{E}(G')$. Of particular importance to our study are *even* subgraphs. We define those now, along with dimer configurations.

Definition 1.3 (Even subgraphs)**.** A given graph G is called *even* if each vertex $v \in V(G)$ has even *degree* - that is, an even number of directed edges have v as origin vertex. Denote by $\mathcal{E}(G)$ the set of even subgraphs of G.

Definition 1.4 (Dimer configurations)**.** A dimer configuration on a graph G is a set of edges $E \subseteq E(G)$ such that every vertex $v \in V(G)$ is adjacent to exactly one edge in E. Denote by $\mathcal{D}(G)$ the set of dimer configurations on G.

Finally, we recall the definition of a graph embedding, which in our context we require to be *non-crossing* (to be consistent with the broader topological definition of an embedding).

Definition 1.5 (Graph embedding)**.** An embedding of a graph G into a surface (connected 2-manifold) Σ consists of functions from $V(G) \to \Sigma$ and $E(G) \to \Sigma$, such that

- Each edge is mapped to a simple arc (homeomorphic image of $[0,1]$) in Σ.
- Each vertex is mapped to a single point in Σ; the endpoints of the arc to which an edge e is mapped are the images of the vertices joined by e.
- At no point interior to any arc does it meet another arc or vertex image. Requiring that no arcs intersect at interior points makes the embedding "non-crossing".

1.2 Introduction of the main results

The main results forming the focus of this investigation, namely the *Kac-Ward formula* and its generalisation, are equations which give the so-called *high-temperature polynomial* (in physics, such a polynomial arises when considering the *partition function* of the Ising model for a ferromagnet at high temperature) of a graph (G, x), defined below, in terms of geometric quantities computed from a particular choice of embedding of G.

Definition 1.6 (High-temperature polynomial)**.** The high-temperature polynomial, $\mathcal{Z}_{\text{high}}$, for a weighted graph (G, x) is defined

$$\mathcal{Z}_{\text{high}} \coloneqq \sum_{P \in \mathcal{E}(G)} x(P).$$

The simpler of the two results is the case where the graph (G, x) in question is planar. In this case, the Kac-Ward formula gives the high-temperature polynomial simply as the squared determinant of the *Kac-Ward matrix*, which we introduce in section 2:

Theorem 1.1 (Kac-Ward formula (planar case))**.** *For a planar weighted graph* (G, x)*,*

$$\mathcal{Z}_{high} = \pm\sqrt{\det W}.$$

The full significance of this result will become more apparent once we have defined the Kac-Ward matrix W (in section 2), but for now suffice to say that the components of W are calculated entirely from the *geometry* of the lines that represent edges of G for a *particular choice of embedding* of G in the plane. In contrast, it is clear that the definition of the polynomial $\mathcal{Z}_{\text{high}}$ (definition 1.6 above) makes no reference at all to any embedding of (G, x). We think of $\mathcal{Z}_{\text{high}}$ as being a topological quantity of the abstract graph itself, and hence it is surprising that it can be given entirely geometrically, for any arbitrary choice of embedding.

One might then ask whether an analogous result holds in cases where (G, x) is not planar. Indeed, there is such a result, which forms the main focus of this essay. Although there is much work to be done for even the formulation of this generalisation to become transparent, we state it early for comparison with theorem 1.1.

Theorem 1.2 (Generalised Kac-Ward formula)**.** *Let* (G, x) *be an arbitrary weighted graph, and* Σ *be any (compact) connected, oriented, genus-g surface into which* G *has a 2-cell embedding. Then*

$$\mathcal{Z}_{high} = \frac{1}{2g} \sum_{\lambda \in \mathcal{S}(\Sigma)} (-1)^{\text{Arf}(\lambda)} (\det W_\lambda)^{1/2},$$

where $(\det W_\lambda)^{1/2}$ *is the square root with sign chosen to have constant coefficient equal to* $+1$.

Although the form of the generalisation is significantly more complicated than that of theorem 1.1, the remark which proceeds the aforementioned theorem remains true: the left-hand-side in theorem 1.2 depends only on the topological structure of the abstract graph (G, x), while the right-hand-side is a calculated from a particular choice of embedding for this graph (indeed, also a particular choice of *surface* into which (G, x) is to be embedded).

To appreciate the full generality of theorem 1.2, it is also worth remarking there always *exists* some (compact connected orientable) surface Σ into which any given graph (G, x) embeds. The details of this construction were first given by Heffter, and can be summarised by what is now usually known as the *Heffter-Edmonds principle*[13]. This principle states that cellular, orientable embeddings of a graph G are in bijection with so-called *rotation systems* - that is, assignments to each vertex $v \in V(G)$ a cyclic permutation, or "rotation", of the adjacent edges.

We omit the full details of this construction, (though Youngs goes into more detail [13]) but loosely speaking, given a rotation system, it is possible to recover the full structure of the associated graph G. Then at each vertex $v \in V(G)$, we consider the collection of half-edges, or *darts*, adjacent to v, and construct for each an equilateral triangle, as in figure 1 below. We then (topologically) glue such triangles together in the obvious way. It is possible to orient the resulting surface such that a clockwise ordering of (the arcs representing) edges around any vertex matches the clockwise ordering given by the rotation system.

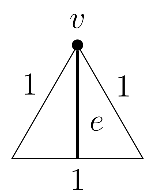

Figure 1: Equilateral triangle for a dart forming half of edge e

2 Kac-Ward Matrix

2.1 Kac-Ward matrix for planar case

We introduce the Kac-Ward matrix first in the context of a planar graph. We will later generalise this definition for the case of an arbitrary graph embedded in some surface, but, as mentioned in the previous section, this generalisation requires some preliminaries which we will only cover in sections 4 and 5. However, once these preliminary tools have been established, the definitions given in this section generally extend in a very natural way.

Definition 2.1 (Kac-Ward matrix (planar case))**.** Given a planar weighted graph (G, x) for which we have chosen a particular embedding in the plane, we define the $\left|\vec{E}(G)\right| \times \left|\vec{E}(G)\right|$ *Kac-Ward matrix* as $W := \mathbb{I} - T$, where

$$T_{e,e'} := \begin{cases} (x_e x_{e'})^{1/2} e^{iw(e,e')/2} & \text{if } t(e) = o(e'), e' \neq \bar{e} \\ 0 & \text{else} \end{cases}.$$

Here $w(e, e')$ is an oriented angle obtained from the chosen embedding, as indicated in figure 2 below.

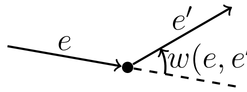

Figure 2: Definition of the oriented angle $w(e, e')$

2.2 Terminal graph and Pfaffian

In this section and the next, we work towards a convenient formula, eventually given by theorem 2.3, which expresses the determinant $\det W$ in terms of a sum over dimer configurations on the so-called *terminal graph* associated with (G, x). This formula will play a significant role in the proofs of both of the main results, theorems 1.1 and 1.2 (in the latter case, for a slightly generalised definition of the Kac-Ward matrix W), so some time is spent arriving to theorem 2.3.

Definition 2.2 (Terminal graph)**.** Given weighted graph (G, x), we construct the (weighted) terminal graph (G^K, x^K) by the replacing each vertex $v \in V(G)$ of degree $d(v)$ with a *clique* $K_{d(v)}$ - that is a graph of $d(v)$ vertices wherein every pair of vertices are joined by an undirected edge - as shown in figure 3 below.

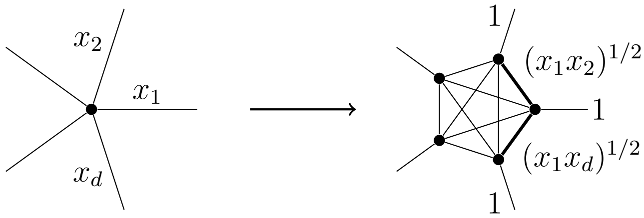

Figure 3: Construction of terminal graph (G^K, x^K)

We call those edges in (G^K, x^K) which belong to one of the cliques $K_{d(v)}$ that we inserted *short*, and the others edges *long*. Note first the bijection $\vec{E}(G) \cong V(G^K)$, obtained by mapping each directed edge $e \in \vec{E}(G)$ to the origin vertex of the corresponding long edge in G^K. The weights x^K are assigned to the terminal graph as shown in the figure above. Each long edge is given a weight of 1, and a short edge joining the vertices in G^K which correspond to edges $e, e' \in \vec{E}(G)$ is weighted by $x^K_{e,e'} := (x_e x_{e'})^{1/2}$.

It is the bijection $\vec{E}(G) \cong V(G^K)$ mentioned above which makes the terminal graph useful for computing $\det W$. To notice this connection, we must first express the desired determinant in terms of the *Pfaffian*, which we define below.

Definition 2.3 (Pfaffian)**.** The Pfaffian $\operatorname{Pf} M$ of a $2n \times 2n$ skew-symmetric matrix M is defined

$$\operatorname{Pf} M := \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} M_{\sigma(2i-1),\sigma(2i)},$$

where S_{2n} is the permutation group of order $(2n)!$, and $\operatorname{sgn}\sigma$ is the signature of permutation $\sigma \in S_{2n}$, and $M_{j,k}$ is the j,k-th component of M.

We note, but do not prove, that the square of the Pfaffian is simply the determinant, $(\operatorname{Pf} M)^2 = \det M$, as originally shown by Cayley[4]. To relate the Pfaffian to a sum over dimer configurations, we simply make use of the skew-symmetry of M to reduce the sum in the definition of the Pfaffian to run only over all possible pairings of indices (which for $\operatorname{Pf} W$ will correspond to dimer configurations on x^K):

Lemma 2.1. *Let $\mathcal{D}$ be the set of all partitions of $\{1, 2, \ldots, 2n\}$ into pairs. Write any such partition $D \in \mathcal{D}$ as $D = \{\{d_1, d_2\}, \{d_3, d_4\}, \ldots, \{d_{2n-1}, d_{2n}\}\}$, and let $\sigma_D \in S_{2n}$ be the permutation mapping each $i \in \{1, 2, \ldots, 2n\}$ to d_i. Then*

$$\operatorname{Pf} M = \sum_{D \in \mathcal{D}} \operatorname{sgn}(\sigma_D) \prod_{i=1}^{n} M_{\sigma_D(2i-1), \sigma_D(2i)}.$$

Proof. Let $(i, j) \in S_{2n}$ denote a transposition. For any given $\sigma \in S_{2n}$ and $j \in \{1, 2, \ldots, n\}$, let $\sigma' := (\sigma(2j-1), \sigma(2j)) \circ \sigma$. Then we have $\operatorname{sgn}(\sigma') = -\operatorname{sgn}(\sigma)$, and $M_{\sigma'(2j-1), \sigma'(2j)} = M_{\sigma(2j), \sigma(2j-1)} = -M_{\sigma(2j-1), \sigma(2j)}$, by skew-symmetry of M. Therefore, restricting the sum in the definition of $\operatorname{Pf} M$ to (arbitrarily) fix the order of each pair $(\sigma(2i-1), \sigma(2i))$ simply incurs an overall factor of 2^n. The sum now runs over all pairings $D \in \mathcal{D}$, as well as all permutations of the pairs $\{d_i, d_{i+1}\} \in D$. Finally, since each permutation of the pairs in $D \in \mathcal{D}$ contributes equally to the sum (swapping two adjacent pairs involves 4 transpositions, so does not change the permutation signature), we may restrict the sum further to (arbitrarily) fix the order of the pairs, incurring an overall factor of $|S_n| = n!$. This gives the form in the lemma statement. ■

2.3 Kac-Ward determinant for planar case

Since the Pfaffian is only defined for skew-symmetric matrices, we finally concern ourselves with finding a skew-symmetric matrix $\hat{W}$ with $\det W = \det \hat{W}$. Such a matrix $\hat{W}$ is given, for the planar graph case, by the following definition. Once we have introduced spin structures in section 5, this transformation, as with the original definition of W, will naturally generalise, as will the results of the subsequent lemma, lemma 2.2. This justifies once more spending some time to understand these results in the simpler context.

Definition 2.4 (Skew-symmetric Kac-Ward matrix (planar case))**.** Given the Kac-Ward matrix W for a planar weighted graph (G, x) (which we consider as embedded in the complex plane), define $\hat{W}$ by

$$\hat{W} := iU^* \left(\delta_{\bar{e}, e'}\right)_{e, e'} WU,$$

where U is the diagonal matrix containing entries $\eta_e \in \mathbb{C}$, with each η_e is chosen so that $(\eta_e^*)^2$ gives the direction of $e \in \vec{E}(G)$ (that is, the direction of the vector representing e in the complex plane).

While the choice of square root is arbitrary, and hence not canonical, we can make it consistent by taking the direction of each edge $e \in \vec{E}(G)$ to be the Arg of the complex number representing it in the plane, forcing it to be in the range $(-\pi, \pi]$. We can then consistently and unambiguously define $\eta_e := e^{i \operatorname{Arg}(e)/2}$.

In the next lemma, we prove that $\hat{W}$ has the desired properties, and we give an explicit form for its entries.

Lemma 2.2. *The matrix $\hat{W}$ defined above*

1. *satisfies* $\det W = \det \hat{W} = \left(\operatorname{Pf} \hat{W}\right)^2$
2. *is skew-symmetric (with real entries)*

3. *has explicit entries* $\hat{W}_{e,e'} = \varepsilon_{e,e'} \cdot x^K_{e,e'}$*, considering* $e, e' \in \vec{E}(G)$ *as adjacent vertices in* G^K*, where*

$$\varepsilon_{e,e'} = \pm 1 = \begin{cases} i\eta_e^* \eta_{e'} & \text{if a long edge links } e \text{ and } e' \\ -i\eta_e^* \eta_{e'} e^{iw(\bar{e},e')/2} & \text{if a short edge links } e \text{ and } e' \end{cases}, \quad \text{so} \quad \varepsilon_{e,e'} \in \{-1, 0, 1\}.$$

Proof. Let J be the $\left|\vec{E}(G)\right| \times \left|\vec{E}(G)\right|$ matrix with entries $J_{e,e'} := \delta_{\bar{e},e'}$.

1. We first prove that $\det J = (-1)^{|E(G)|}$. We expand the following standard definition of the determinant:

$$\det J = \sum_{\sigma \in S_{|\vec{E}(G)|}} \operatorname{sgn}(\sigma) \prod_{e \in \vec{E}(G)} J_{e,\sigma(e)} = \sum_{\sigma \in S_{|\vec{E}(G)|}} \operatorname{sgn}(\sigma) \prod_{e \in \vec{E}(G)} \delta_{\bar{e},\sigma(e)}.$$

Now the product of Kronecker deltas is clearly only non-zero if σ is the exact permutation mapping each edge $e \in \vec{E}(G)$ to its unique opposite edge $\bar{e}$. Hence the sum collapses to just the signature of that single permutation, which factorises into transpositions as such:

$$\begin{aligned} \det J &= \operatorname{sgn}\left(\begin{bmatrix} e_1 & e_2 & \cdots & e_{|\vec{E}(G)|} \\ \bar{e_1} & \bar{e_2} & \cdots & \bar{e}_{|\vec{E}(G)|} \end{bmatrix}\right) \\ &= \operatorname{sgn}\left((e_1, \bar{e_1}) \circ \cdots \circ \left(e_{|\vec{E}(G)|}, \bar{e}_{|\vec{E}(G)|}\right)\right) = (-1)^{|E(G)|}. \end{aligned}$$

It is now clear (since, manifestly, $\det(U^*)\det(U) = \det \mathbb{I} = 1$) that

$$\det \hat{W} = i^{\left|\vec{E}(G)\right|}(-1)^{|E(G)|} \det W = \left((-1)^{|E(G)|}\right)^2 \det W = \det W.$$

3. We now prove the given explicit form for $\hat{W}_{e,e'}$. First expand the product,

$$\hat{W}_{e,e'} = i \sum_{f,g,h \in \vec{E}(G)} (U^*)_{e,f} J_{f,g} W_{g,h} U_{h,e'} = i\eta_e^* W_{\bar{e},e'} \eta_{e'}.$$

Now a long edge links e and e' in G^K if and only if $e' = \bar{e}$. In this case, $W_{\bar{e},e'} = W_{\bar{e},\bar{e}} = \mathbb{I}_{\bar{e},\bar{e}} = 1 = x^K_{e,e'}$. A short edge links e and e' in G^K if and only if $o(e) = o(e')$, but $e \neq e'$. In this case, $W_{\bar{e},e'} = -(x_{\bar{e}} x_{e'})^{1/2} e^{iw(\bar{e},e')/2} = -x^K_{e,e'} e^{iw(\bar{e},e')/2}$. When e and e' are not adjacent vertices in G^K, it is clear that $W_{\bar{e},e'} = 0$, so the given entries are the only ones.

2. We finally prove that $\hat{W}$ is real and skew-symmetric. To show that it is real, we simply need to confirm that indeed $\varepsilon_{e,e'} = \pm 1$. Consider again the two cases explored in 3. If, as in the first case, $e' = \bar{e}$, then we have the forms $\eta_e^* = e^{-i(\theta_1/2 + n\pi)}$ and $\eta_{e'} = e^{i(\theta_2/2 + m\pi)}$, say, for $m, n \in \mathbb{Z}$, and where $\theta_1 - \theta_2 = \pm\pi$. Then $i\eta_e^*\eta_{e'} = ie^{i(\pm\pi/2 + (n-m)\pi)} = i(\pm i) = \pm 1$. It is quite clear also from these expressions that $\varepsilon_{e',e} = -\varepsilon_{e,e'}$ in this case. The other case is entirely analogous, albeit slightly more wordy, so we omit it here.

■

Finally, we can state the formula for $\operatorname{Pf} \hat{W}$ to which we have alluded several times. We note that this relationship between the Pfaffian of certain matrices and dimer configurations associated graphs is explored in more general contexts by Cimasoni[6].

Theorem 2.3. *Let W be the Kac-Ward matrix for a planar weighted graph (G, x). Then*

$$\operatorname{Pf} \hat{W} = \sum_{D \in \mathcal{D}(G^K)} \epsilon(D) x^K(D) = \pm\sqrt{\det W},$$

where the signs $\epsilon(D) = \pm 1$ are defined

$$\epsilon(D) := \operatorname{sgn}(\sigma_D) \prod_{i=1}^{|E(G)|} \varepsilon_{\sigma_D(2i-1),\sigma_D(2i)},$$

where $\sigma_D \in S_{|V(G^K)|}$ is some permutation representing the dimer configuration $D \in \mathcal{D}$ (so that if we enumerate the vertices in $V(G^K)$, the dimers are the edges joining vertices $\sigma(2i-1)$ and $\sigma(2i)$ for each $i \in \{1, 2, \ldots, |V(G^K)|\}$), and where $\varepsilon_{i,j}$ is as given in lemma 2.2, where i, j represent vertices in $V(G^K)$ (alternatively, edges in $\vec{E}(G)$, as in the notation of lemma 2.2).

Proof. The result is immediate from applying lemma 2.1 on the explicit form of $\hat{W}$ given in lemma 2.2, noting that the sum over pair-partitions used in lemma 2.1 reduces to a sum over dimer configurations in this case, since $\hat{W}_{e,e'} = 0$ if the vertices $e, e' \in V(G^K)$ are not adjacent (joined by a short or long edge), so such contributions to the sum may be excluded. ■

Of course, as alluded to in the proof of lemma 2.1, upon which the above result is based, the signs $\epsilon(D)$ do not depend on the specific choice of permutation σ_D used to represent each dimer configuration $D \in \mathcal{D}$.

3 Kac-Ward Formula in Planar Case

Recall the planar case of the main result, the Kac-Ward formula, as was given in theorem 1.1, which stated that $\mathcal{Z}_{\text{high}} = \pm \operatorname{Pf} \hat{W}$. With the expansion of the Pfaffian given in theorem 2.3, we are almost ready to prove this result in the planar case. We require in particular the following crucial lemma, which by itself is already remarkable. As with most of the results and constructions used in the proof of the planar case, we will later encounter a generalisation of lemma 3.1 in the more general context.

Lemma 3.1 (Whitney's lemma)**.** *Given an oriented, piecewise-smooth, closed curve $C \subset \mathbb{C}$, which has with $t(C)$ transverse self-intersections, we have*

$$(-1)^{t(C)+1} = \exp\left[i\pi \operatorname{wind}(C)\right],$$

where $\operatorname{wind}(C)$ *is the (oriented) winding number of C.*

Proof. The proof, taken from Chelkak *et al*[5], relies simply on converting C to a union C' of *simple* closed curves, by locally removing each of the self-intersection points, as shown in figure 4 below.

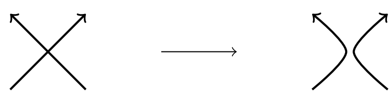

Figure 4: Operation for removing self-intersection points

Now since the winding number of a simple closed curve is ± 1, we have that $\exp\left[i\pi \operatorname{wind}(C')\right] = e^{\pm i\pi m} = (-1)^m$, where m is the number of (connected curves which form) components of

C'. Also note that $\exp\left[i\pi\,\text{wind}(C)\right] = \exp\left[i\pi\,\text{wind}(C')\right]$, since each smoothing operation as described above leaves the winding number unchanged. Hence we just need that $t(C)+1 \equiv m \pmod 2$, but this is clear since C starts with one component, and each smoothing operation changes m by ± 1, so that the parity of m is always opposite to that of $t(C)$. ■

As mentioned above, Whiteney's lemma is already remarkable, as it gives a connection between a topological property of a planar curve - that is, the number of transverse self-intersections - and a property of the geometry of the curve in the plane - that is, its winding number. It is no surprise then that this result will be central to our proof of theorem 1.1, which also gives such a connection between topological and geometric properties.

We pause briefly to illustrate Whitney's lemma with a small example. Consider the curve in figure 5 given below.

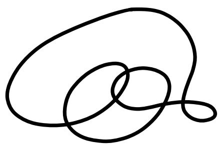

Figure 5: Example closed planar curve

Taking positive orientation, this curve has a winding number of 2, and self-intersects at 6 points. Hence, for this curve, we have $e^{i\pi\,\text{wind}(C)} = e^{2\pi i} = 1 = (-1)^6 = (-1)^{t(C)+1}$, so indeed the lemma seems works in this case.

We state one further lemma which is used in the proof of theorem 1.1. This result, like most encountered thus far, has a simple generalisation in the surface case which we treat later. Note that for $D \in \mathcal{D}(G)$, the self-intersection number $t(D)$ is simply the number of points at which one edge of D crosses another. Note that only short edges can induce such crossings (since G is definitionally non-crossing), so such self-intersections happen only in the cliques of G^K.

Lemma 3.2. *For planar weighted graph* (G, x)*,*

$$\mathcal{Z}_{high} = \sum_{D \in \mathcal{D}(G^K)} (-1)^{t(D)} x^K(D).$$

Proof. Take some $D \in \mathcal{D}(G^K)$, and let G_D be the subgraph of G containing only edges which correspond to long edges in D. Then we note that $G \setminus G_D \in \mathcal{E}(G)$. This is because if any vertex $v \in V(G \setminus G_D)$ has odd degree $d(v)$, then we would require each of the $d(v)$ vertices in the clique $K_{d(v)} \subseteq G^K$ corresponding to v to be adjacent to exactly one *short* edge which is in D. But each such short edge which is chosen to be in D is adjacent to two vertices in $K_{d(v)}$, so such a dimer configuration is impossible for odd $d(v)$.

Hence, we have map $\rho\colon \mathcal{D}(G^K) \to \mathcal{E}(G)\colon D \mapsto G \setminus G_D$, and we note that a given $P \in \mathcal{E}(G)$ is mapped to by $D \in \mathcal{D}(G^K)$ precisely when the short edges of D are those joining the long edges which are (in correspondence with edges) in P. Since only short edges are weighted in G^K, this immediately gives us that $x^K(D) = x(\rho(D))$ for each $D \in \mathcal{D}(G^K)$. We also have a bijection then between $\rho^{-1}(P)$ and the set $\prod_{v \in V(P)} \mathcal{D}(K_{d(v,P)})$, where $K_{d(v,P)}$ is the clique (complete graph) formed at the vertex v of P with $d(v,P)$ points, where $d(v,P)$ is the degree of v *in the even graph* P. To indicate that each such vertex is of even degree, we also write $d(v,P) = 2n_v$. By these observations,

$$\sum_{E\in\rho^{-1}(P)}(-1)^{t(D)}x^K(D)=\left[\prod_{v\in V(P)}\sum_{D_v\in\mathcal{D}(K_{2n_v})}(-1)^{t(D_v)}\right]x(P).$$

We can also prove easily, though we omit the details (see Chelkak, *et al.*[5]), the simple combinatorial fact that for each $n\geq 1$ we have

$$\sum_{D\in\mathcal{D}(K_{2n})}(-1)^{t(D)}=1.$$

Therefore, we obtain, for each $P\in\mathcal{E}(G)$,

$$x(P)=\sum_{D\in\rho^{-1}}(-1)^{t(D)}x^K(D),$$

and hence, summing over $P\in\mathcal{E}(G)$,

$$\mathcal{Z}_{\text{high}}=\sum_{P\in\mathcal{E}(G)}x(P)=\sum_{P\in\mathcal{E}(G)}\sum_{D\in\rho^{-1}}(-1)^{t(D)}x^K(D)=\sum_{D\in\mathcal{D}(G^K)}(-1)^{t(D)}x^K(D).$$

■

We are now finally ready to give the proof of theorem 1.1, which we also re-state below. This proof is an expanded, more thorough version of the proof given by Chelkak *et al*[5] - elements of it will also generalise for the non-planar case, so we opt for a careful and thorough treatment in this simpler case, so the later extension is more natural and requires fewer trivial steps.

Theorem 1.1 (Kac-Ward formula (planar case))**.** For a planar weighted graph (G,x),

$$\mathcal{Z}_{\text{high}}=\pm\sqrt{\det W}.$$

Proof. Recalling the expansion

$$\operatorname{Pf}\hat{W}=\sum_{D\in\mathcal{D}(G^K)}\epsilon(D)x^K(D)=\pm\sqrt{\det W},$$

of theorem 2.3, as well as that of lemma 3.2,

$$\mathcal{Z}_{\text{high}}=\sum_{D\in\mathcal{D}(G^K)}(-1)^{t(D)}x^K(D),$$

we see immediately that it only remains to prove the equality $(-1)^{t(D)}=\epsilon(D_0)\epsilon(D)$ for each $D\in\mathcal{D}$, where $\epsilon(D_0)$ is just an overall global sign.

To this extent, let $D\in\mathcal{D}(G^K)$ be arbitrary, and define $D_0:=\{e\in E(G^K): e \text{ is a long edge}\}\in\mathcal{D}(G^K)$. Now, the symmetric difference $U:=D\triangle D_0$ is a union of $n\in\mathbb{N}_0$ (vertex-disjoint, in G^K) cycles C_m, each consisting of alternating long and short edges (and hence having an even number of total edges). To see this, suppose there is a path in U which is not a cycle. Then there is some vertex $v\in G^K$ at which a long edge e meets short edges $\{f_1,\ldots,f_n\}$, where: either $e\in U$ but each $f_i\notin U$, or $e\notin U$ but some $f_j\in U$. In the former case, since $e\in D_0$, we must have $e\notin D$, but then v is adjacent to no edges in D, contradicting that D is a dimer configuration. Similarly, in the second case, $e\in D$ and $f_j\in D$, so v is adjacent to two edges in D - again a contradiction.

Now, recall that $\epsilon(D)$, as defined in theorem 2.3, relied on the choice of some permutation $\sigma_D \in S_{|V(G^K)|}$ to represent the dimer configuration D. Choose such representatives σ_D and σ_{D_0} such that $\sigma_D \circ \sigma_{D_0}$ is the permutation which rotates each cycle C_m counterclockwise by one edge. Such a choice of representatives can be made because the only requirement on such a permutation σ_D was that it kept pairs of vertices belonging to the same edge in the dimer D which it represented adjacent.

Now by theorem 2.3 along with lemma 2.2, we have

$$\begin{aligned}\epsilon(D)\epsilon(D_0) &= \operatorname{sgn}(\sigma_D)\operatorname{sgn}(\sigma_{D_0}) \prod_{i=1}^{|E(G)|} \varepsilon_{\sigma_D(2i-1),\sigma_D(2i)}\varepsilon_{\sigma_{D_0}(2i-1),\sigma_{D_0}(2i)} \\ &= \operatorname{sgn}(\sigma_D)\operatorname{sgn}(\sigma_{D_0}) \prod_{i=1}^{|E(G)|} \varepsilon_{\sigma_D(2i-1),\sigma_D(2i)} i\eta^*_{\sigma_{D_0}(2i-1)}\eta_{\sigma_{D_0}(2i)},\end{aligned}$$

where we have recognised that every edge in D_0 is (defined to be) long. We can reduce the product further by noting that any edges in $D \cap D_0$ will contribute two identical factors ε (which square to 1), so we need only consider the edges in U. Once we restrict to edges in U, the alternation of long and short edges results in the factors i and $-i$ from the form of $\varepsilon_{e,e'}$ pairing up to produce and overall factor of 1. Also, the factors η_e all appear along with their conjugates η_e^*, and hence cancel, since C_m is a cycle and since we chose $\sigma_D \circ \sigma_{D_0}$ to be a counterclockwise rotation. Hence the product reduces to simply

$$\epsilon(D)\epsilon(D_0) = \operatorname{sgn}(\sigma_D)\operatorname{sgn}(\sigma_{D_0}) \prod_{m=1}^{n} \omega(C_m),$$

where $\omega(C_m)$ is just the product of coefficients $e^{iw(\bar{e},e')/2}$ along the short edges of C_m. Furthermore, since $\operatorname{sgn}\colon S_{|V(G^K)|} \to \{-1,1\}$ is a group homomorphism, we have that $\operatorname{sgn}(\sigma_D)\operatorname{sgn}(\sigma_{D_0}) = \operatorname{sgn}(\sigma_D)\operatorname{sgn}(\sigma_{D_0}^{-1}) = \operatorname{sgn}(\sigma_D \circ \sigma_{D_0}^{-1})$, and $\sigma_D \circ \sigma_{D_0}^{-1}$ was defined to factorise into n cycles, all of even length (and hence odd parity). Thus, $\operatorname{sgn}(\sigma_D)\operatorname{sgn}(\sigma_{D_0}) = (-1)^n$, so that

$$\epsilon(D)\epsilon(D_0) = \prod_{m=1}^{n} \left(-\omega(C_m)\right).$$

Finally, applying Whitney's lemma (lemma 3.1), we get

$$\epsilon(D)\epsilon(D_0) = \prod_{m=1}^{n} \left(-\omega(C_m)\right) = \prod_{m=1}^{n} \left(-\exp\left[i\pi \operatorname{wind}(C_m)\right]\right) = \prod_{m=1}^{n} (-1)^{t(C_m)} = (-1)^{t(D)},$$

where for the last equality we have noticed (as mentioned above lemma 3.2) that all intersections and self-intersections of the cycles C_m occur within the cliques between short edges belonging to D (and it is clear that unequal cycles intersect at an even number of points). ■

4 $\mathbb{Z}_2$-Homology

Having proven the planar case of the Kac-Ward formula, we now begin to introduce the preliminary concepts which are required to understand the statement of its generalisation. To this end, we introduce homology groups, fibre bundles, spin structures and quadratic forms. We begin in this section by introducing $\mathbb{Z}_2$-homology groups, and stating a few results for the homology groups associated with a graph embedded in a surface.

4.1 Homology and Cohomology Groups

We begin with a slightly more general definition of homology spaces as given by Hatcher[9], which in subsequent sections we make more concrete by applying it to CW complexes and graph embeddings.

Definition 4.1 (Chain complex)**.** Let $C_0, C_1, \dots$ be Abelian groups, and for each n, let $\partial_n \colon C_n \to C_{n-1}$ be group homomorphisms, called *boundary maps*, satisfying $\partial_n \circ \partial_{n+1} = 0$ for all n. Then the sequence

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

of homomorphisms is called a *chain complex*, which we write $(C_k, \partial_k)_{k\in\mathbb{N}}$. Elements of each C_n are called *n-chains*, while elements of $Z_n \coloneqq \ker \partial_n$ and $B_n \coloneqq \operatorname{im} \partial_{n+1}$ are called *n-cycles* and *n-boundaries* respectively.

Since we have for each n that $\partial_n \circ \partial_{n+1} = 0$, we note that B_n is a subgroup of Z_n.

Definition 4.2 (Homology groups)**.** Define the n-th homology group as the quotient group $H_n \coloneqq Z_n/B_n$. Elements of H_n, which are cosets of B_n, are called *homology classes*. Two cycles which fall into the same coset in H_n are said to be *homologous*.

We briefly discuss *cohomology groups*. These groups arise by a process of dualisation in the definition of homology groups. We state this more precisely below.

Definition 4.3 (Cohomology groups)**.** Given a chain complex $(C_k, \partial_k)_{k\in\mathbb{N}}$, where the C_k are free Abelian groups over some field $\mathbb{K}$, construct for each n the dual *cochain group* $C_n^* \coloneqq \operatorname{Hom}(C_n, \mathbb{K})$ and the dual *coboundary map* $\delta_n \coloneqq \partial_n^* \colon C_{n-1}^* \to C_n^* \colon \phi \mapsto \phi \circ \partial_n$. Hereby we obtain a new chain complex,

$$\cdots \longleftarrow C_{n+1}^* \xleftarrow{\delta_{n+1}} C_n^* \xleftarrow{\delta_n} C_{n-1}^* \longleftarrow \cdots$$

from which we form the homology groups in the usual way, only now we refer to them as *cohomology groups*. More precisely, the n-th cohomology group associated with the chain complex and fixed group $\mathbb{F}$ is defined $H^n(\mathbb{F}) \coloneqq \ker \delta_{n+1}/\operatorname{im} \delta_n$.

We note that in general, for each n, there is a natural homomorphism $h\colon H^n(\mathbb{F}) \to \operatorname{Hom}(H_n, \mathbb{F})$, which we briefly explain. A general element $[\phi] \in H^n(\mathbb{F})$ is represented by homomorphism $(\phi\colon C_n \to \mathbb{F}) \in C_n^*$, and $\phi \in \ker \delta_{n+1}$, so $\delta_{n+1}\phi = \phi \circ \partial_{n+1} = 0$, whence ϕ vanishes on $B_n = \operatorname{im} \partial_{n+1}$. Therefore, by restricting ϕ to Z_n we get a well-defined homomorphism $\tilde{\phi}\colon H_n = Z_n/B_n \to \mathbb{F}$. Furthermore, for $\phi \in \operatorname{im} \delta_n$, say $\phi = \delta_n\psi = \psi \circ \partial_n$, then ϕ is zero on Z_n, so $\tilde{\phi} = 0$. Therefore we get a well-defined map $h\colon H^n(\mathbb{F}) \to \operatorname{Hom}(H_n, \mathbb{F})\colon [\phi] \mapsto \tilde{\phi}$, which is clearly a homomorphism.

Hatcher[9] shows that the homomorphism h described above is surjective, but need not be injective in general. The conditions for the injectivity of h are characterised by the so-called *universal coefficient theorem for cohomology.* Without formulating the theorem in its full generality, we note that one of its consequences (when taken with the *splitting lemma*, also formulated in Hatcher[9]) is that $H^n(\mathbb{F}) \cong \mathrm{Hom}(H_n, \mathbb{F})$ whenever H_{n-1} is free and finitely-generated. In particular, for the spaces $H_1(\Sigma; \mathbb{Z}_2)$ and $H^1(\Sigma; \mathbb{Z}_2)$ introduced in the next section, we will have $H^1(\Sigma; \mathbb{Z}_2) \cong \mathrm{Hom}(H_1(\Sigma; \mathbb{Z}_2), \mathbb{Z}_2)$.

4.2 The Spaces $H_1(\Sigma; \mathbb{Z}_2)$ and $H^1(\Sigma; \mathbb{Z}_2)$

The previous section introduced homology groups entirely algebraically. To demonstrate their applicability to the geometry of an embedded graph, we first introduce the notion of a *CW complex.* Here we follow an similar approach to Hatcher[9], defining a CW complex of a given dimension n inductively by an explicit construction using CW complexes of lower dimension.

Definition 4.4 (CW complex)**.** An n-dimensional CW complex C^n is defined inductively, as follows:

- A *0-dimensional CW complex* is a discrete set C^0 of zero or more points, equipped with the discrete topology
- An *n-dimensional CW complex* C^n is constructed from an $(n-1)$-dimensional one, C^{n-1}, by the following process:
  1. Construct the disjoint union of C^{n-1} with one or more n-balls B^n_α,
  $$U^n := C^{n-1} \sqcup \left(\bigsqcup_\alpha B^n_\alpha \right),$$
  where of course, U^n is equipped with the disjoint union topology.
  2. Define maps $\phi_\alpha \colon \partial B^n_\alpha \to C^{n-1}$ which glue the boundaries of each of the n-balls to the CW complex C^{n-1} used in the definition of U^n.
  3. Form the quotient space under the gluing maps in the usual way:
  $$C^n := U^n / \sim \quad \text{where } x \sim \phi_\alpha(x), \text{ for each } x \in \partial B^n_\alpha,$$
  taking the quotient topology.

The interior of each k-ball added in the k-th step of the construction of C^n is called a *k-cell*, and the union of all such k-cells is called the *k-skeleton* of C^n.

Every graph G can be considered a one-dimensional cell complex with 0-skeleton $V(G)$ and 1-skeleton $E(G)$. This amounts to simply choosing some embedding of G into any topological space. We note, as an aside, that perhaps the simplest such embedding for a finite graph (as we work with throughout this paper) is given by Cohen[7], who shows that any finite graph embeds into $\mathbb{R}^3$.

Similarly, there are some cases in which we may consider a graph G embedded in a surface Σ as a 2-dimensional CW complex. For such embeddings, called *2-cell embeddings*, we require that every *face* - that is every component in the disjoint union of components which comprises

$\Sigma \setminus G$ - be homeomorphic to a topological disk. In such cases, the graph (or rather the collection of points and simple arcs which represents it in Σ) forms the 1-skeleton of a 2-dimensional CW complex, where the 2-cells are the faces.

We are now ready to introduce the $\mathbb{Z}_2$-homology space (our homology group will turn out to be a vector space) for such a 2-cell embedding of our graph G into a surface Σ.

Definition 4.5 (Homology space $H_1(\Sigma; \mathbb{Z}_2)$)**.** Suppose that graph G has a 2-cell embedding in compact connected orientable surface Σ. Then we construct a chain complex (and thereby homology space) as follows:

1. For each $k \in \{1, 2, 3\}$, define the set C_k of k-chains, as the free $\mathbb{Z}_2$-vector space over the set of k-cells (vertices, edges, and faces respectively for $k = 1, 2, 3$) for the 2-dimensional CW complex associated with the 2-cell embedding.
2. Define the group homomorphisms (in this case linear maps)

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

   to be the *boundary operators.* Each ∂_k is the (defined-to-be linear) map sending a k-cell to the $\mathbb{Z}_2$-sum of the $(k-1)$-cells which form its boundary.

It is not difficult to confirm that $\partial_1 \circ \partial_2 = 0$, so that indeed the above definition gives a chain complex. We need only check this on a basis, so consider an arbitrary face. It is mapped under ∂_2 to a sum of edges which form a cycle, so that $\partial_1 \circ \partial_2$ maps it to a sum of vertices with each vertex appearing twice in the sum. Since we work in $\mathbb{Z}_2$, this sum is then 0.

We now also see the advantage of choosing to work with $\mathbb{Z}_2$-homology specifically. Were we working over some other field, finite or otherwise, we would need to be more careful in our definition of the boundary operators ∂_k, to ensure that they still satisfy $\partial_k \circ \partial_{k+1} = 0$ for each k. Each boundary operator would still map a k-cell to a *linear combination* of the $(k-1)$-cells forming its boundary, but now the relative signs in this linear combination, representing orientations of the $(k-1)$-cells, become relevant. Usually this problem is approached as, for instance, Munkres[12] does, by first studying homology of simplices, and subsequently generalising. But for our applications it suffices to avoid the problem altogether by simply working over $\mathbb{Z}_2$.

We note the following result, which tells us that the homology space $H_1(\Sigma; \mathbb{Z}_2)$ is independent of the graph G (as long as we consider a *2-cell embedding* of G into Σ). Hence, $H_1(\Sigma; \mathbb{Z}_2)$ is a property entirely of the surface, with dimension dependent only on the *genus* of Σ. In fact, the latter part of the following theorem is implied by *Euler's theorem*[3], which gives the genus of a surface in terms of its *Euler characteristic*; indeed, this relationship is used as the definition of the *Euler genus*, g of the surface.

Theorem 4.1. *The first homology space $H_1(\Sigma; \mathbb{Z}_2)$, determined by a 2-cell embedding of a graph G into the (compact connected) surface Σ, depends only on Σ (i.e. not on the graph G). It has dimension $2g$ for a closed surface Σ and dimension $2g + b - 1$ if Σ has $b \geq 1$ boundary components.*

As mentioned at the end of the previous section, in this case the cohomology group $H^1(\Sigma;\mathbb{Z}_2)$ is also particularly simple because we have $H^1(\Sigma;\mathbb{Z}_2) \cong \mathrm{Hom}(H_1(\Sigma;\mathbb{Z}_2),\mathbb{Z}_2)$. By definition of the cohomology spaces, we regard $H^1(\Sigma;\mathbb{Z}_2)$ as the space of maps $\psi\colon \vec{E}(G) \to \mathbb{Z}_2$ satisfying $\psi(e) = \psi(\bar{e})$ for each $e \in \vec{E}(G)$, and $\sum_{e\in\partial f} \psi(e) = 0$ for each face f of the 2-cell embedding of G. We would say that $H^1(\Sigma;\mathbb{Z}_2)$ is the space of gauge equivalence classes of $\mathbb{Z}_2$-valued *flat connections*. However the isomorphism $H^1(\Sigma;\mathbb{Z}_2) \cong \mathrm{Hom}(H_1(\Sigma;\mathbb{Z}_2),\mathbb{Z}_2)$ gives us arguably a more straightforward interpretation.

5 Spin Structures

5.1 Definition of spin structure

We begin by defining some basic notions, developing towards that of a *spin structure* on a surface. In our context, it is possible to use the alternative simpler characterisation of spin structures in terms of vector fields that we present in the next section in place of the more general definition. However, we wish to explore the origin of this characterisation, so we begin with the general definition.

Definition 5.1 (Vector bundle)**.** A real vector bundle consists of topological spaces X and E, called the *base space* and *total space* respectively, together with a continuous surjection $\pi\colon E \to X$, called the *bundle projection* map, such that, for each $x \in X$, the *fibre* $\pi^{-1}(\{x\})$ is equipped with the structure of a finite-dimensional real vector space.

Recall that on a smooth manifold Σ we automatically obtain a vector bundle, called the *tangent bundle*, by attaching to each point $x \in \Sigma$ the tangent space at x as the fibre. Recall also that given any vector bundle we can define the so-called *frame-bundle*, $\mathrm{Fr}(X,E)$, where the fibre over each point $x \in X$ consists of all ordered bases, or *frames*, for the vector space $E_x \equiv \pi^{-1}(\{x\})$. In the case of our smooth manifold Σ, the frame bundle, $\mathrm{Fr}(\Sigma)$, (also called the *tangent frame bundle*) is simply that associated with the tangent bundle.

We note also that for a vector bundle (X,E) of rank n (i.e. with frames each consisting of n basis vectors) $\mathrm{SO}(n)$ acts *freely* and *transitively* on each fibre $\mathrm{Fr}(X,E)_x$ of the frame bundle simply via change-of-basis - we say that $\mathrm{Fr}(X,E)_x$ is a *torsor* for $\mathrm{SO}(n)$. This gives us a (non-canonical) bijection $\mathrm{SO}(n) \cong \mathrm{Fr}(X,E)$, where we choose some fixed reference frame, and associate each frame $F \in \mathrm{Fr}(X,E)_x$ with the group element in $\mathrm{SO}(n)$ which acts to transform the reference frame to F.

Definition 5.2 (Spin structure)**.** A spin structure λ on an oriented Riemannian manifold Σ of dimension n is a double cover of the frame bundle, $\mathrm{Fr}(\Sigma)$ which restricts at each $x \in M$ to the non-trivial double cover, $\mathrm{Spin}(n)$, of $\mathrm{SO}(n) \cong \mathrm{Fr}(\Sigma)_x$

We note that in the case of an oriented surface Σ, which is a Riemannian manifold of dimension 2, we have that $\mathrm{SO}(2) \cong S^1$, and so the non-trivial double cover $\mathrm{Spin}(2)$ of $\mathrm{SO}(2)$ is the Möbius double cover. Also, due to the orientation of Σ, each fibre $\mathrm{Fr}(\Sigma)_x$ of the frame bundle is simply the set, $\mathrm{UT}_x(\Sigma)$, of unit tangent vectors at $x \in \Sigma$.

5.2 Spin structures and homology

In preparation for coming results, we take a moment to informally reflect on the definition of a spin structure, and its implications in the case of surfaces. Since a spin structure is a double cover of the frame bundle, it is classified by its *holonomy*, which loosely describes whether the traversing small loops in the frame bundle returns one to the same point in the covering space - the spin structure.

More precisely, we can consider a spin structure as a homomorphism $\lambda\colon \Pi_1 \operatorname{Fr}(\Sigma) \to \mathbb{Z}_2 = \{0, 1\}$, where $\Pi_1 \operatorname{Fr}(\Sigma)$ is the standard *fundamental group* of loops in Σ modulo homotopy. We also require, since the spin structure should restrict to the Möbius double cover at each fibre, that λ assume the value 1 on the identity element of $\Pi_1 \operatorname{Fr} \Sigma$ - that is, the loop which contracts to a point.

At this point, we take note of the *Hurewicz isomorphism*[9], which tells us that, since the frame bundle on Σ is connected, the fundamental group $\Pi_1 \operatorname{Fr}(\Sigma)$ of homotopy classes in the frame bundle coincides with the first homology space $H_1(\operatorname{Fr}(\Sigma); \mathbb{Z})$. This coincides with our intuition, thinking of elements of the homology space as equivalence classes of cycles modulo boundaries. Therefore we can write:

$$\begin{aligned}\operatorname{Hom}(\Pi_1 \operatorname{Fr}(\Sigma), \mathbb{Z}_2) &\cong \operatorname{Hom}(H_1(\operatorname{Fr}(\Sigma); \mathbb{Z}), \mathbb{Z}_2)\\ &\cong \operatorname{Hom}(H_1(\operatorname{Fr}(\Sigma); \mathbb{Z}_2), \mathbb{Z}_2)\\ &\cong H^1(\operatorname{Fr}(\Sigma); \mathbb{Z}_2),\end{aligned}$$

where the last isomorphism came from the *universal coefficient theorem*, similar to in the discussion at the end of section 4.1. We now have the interpretation that a spin structure can be thought of as a cohomology class, with non-trivial value on the trivial loop of frames.

In our context, we can simplify even further, given the interpretation of the frame bundle in terms of the unit tangent bundle. Considering the fibre at each point gives us maps

$$\mathrm{SO}(2) \longrightarrow \mathrm{UT}(\Sigma) \longrightarrow \Sigma$$

which, after applying the homology functor gives a *short exact sequence*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow H_1(\mathrm{UT}(\Sigma); \mathbb{Z}_2) \longrightarrow H_1(\Sigma; \mathbb{Z}_2) \longrightarrow 0$$

which is to say that the image of each map is the kernel of the next. It is well known (see, for instance [9]) that such short exact sequences give rise to (non-canonical) isomorphisms of the form $H_1(\mathrm{UT}(\Sigma); \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus H_1(\Sigma; \mathbb{Z}_2)$.

Hence, with this association, we can describe a spin structure on such a surface as a morphism (linear functional) on this vector space (into $\mathbb{Z}_2$), which restricts to the identity on the $\mathbb{Z}_2$ subspace (this corresponds to spin structures having to assume non-trivial value on the trivial loops of frames, represented here by $1 \in \mathbb{Z}_2$).

At this point, to fully specify a spin structure λ it only remains to define the corresponding linear functional on the remaining vector space in the direct sum, $H_1(\Sigma; \mathbb{Z}_2)$. Our earlier theorem, theorem 4.1, gave that $\dim H_1(\Sigma; \mathbb{Z}_2) = 2g$ for a connected closed surface of genus g, and of course we are free to define the action of a linear functional on each basis element independently. These considerations lead us to the following assertion:

Lemma 5.1. *For genus g closed, compact, connected, Riemannian surface Σ, there are 2^{2g} spin structures on Σ.*

5.3 Spin structures and vector fields

In our context, we have another simple and convenient representation of a spin structure on a surface. We can consider such a spin structure as represented by a vector field with isolated zeros of even index[2]. We formalise this statement in theorem 5.2.

Theorem 5.2. *Each spin structure λ on compact orientable surface Σ can be represented by a vector field X on Σ which has only isolated zeros of even index.*

Though we omit the details in this exploration, we briefly mention that such a vector field gives another description of the holonomy of the double cover of the frame bundle which constitutes the spin structure. At any given point, we can consider the winding number as one traverses around the point in a small loop, modulo 2. Allowing only zeros of even index ensures that the field X always has even winding number for sufficiently small loops considered around zeros of X, making this well-defined.

5.4 Spin structures and quadratic forms

In this section, we will see that every spin structure $\lambda \in \mathcal{S}(\Sigma)$ on a surface Σ is associated with a *quadratic form* on the homology space $H_1(\Sigma; \mathbb{Z}_2)$, which refines the so-called *intersection pairing* - a particular bilinear form which we define below, after recalling the definition of a bilinear form.

Definition 5.3 (Bilinear form)**.** A bilinear form on a vector space V over scalar field $\mathbb{K}$ is a map $b\colon V \times V \to \mathbb{K}$ which is linear in both arguments separately. That is, for all $u, v, w \in V$ and $\mu \in \mathbb{K}$, we have

$$\begin{aligned} b(u+v,w) &= b(u,w) + b(v,w), \quad \text{and} \quad b(\mu v, w) = \mu b(v,w) \\ b(u, v+w) &= b(u,v) + b(u,w), \quad \text{and} \quad b(u, \mu v) = \mu b(u,v). \end{aligned}$$

The bilinear form is said to be *degenerate* if either of the maps from V to its dual space V induced by b, and defined $b_1 \colon V \to V^* \colon v \mapsto (u \mapsto b(v,u))$ and $b_2 \colon V \to V^* \colon v \mapsto (u \mapsto b(u,v))$ are not isomorphisms. For finite-dimensional space V this is equivalent to b having non-trivial kernel - that is, b is degenerate if there is some $0 \neq u \in V$ such that for all $v \in V$ we have $b(u,v) = 0$. Otherwise, b is said to be *non-degenerate*. Additionally, b is said to be *alternating* if $b(u,u) = 0$ for all $u \in V$.

Definition 5.4 (Intersection pairing)**.** For a given graph G embedded in a surface Σ, define the intersection pairing as the bilinear form

$$\odot \colon H_1(\Sigma; \mathbb{Z}_2) \times H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_2,$$

given by $[C] \odot [D] \coloneqq |C \cap D| \mod 2$, where C and D are cycles representing their respective homology classes. We choose representative cycles in *general position*, which intersect at finitely-many points.

It is not immediately obvious that the formulation of definition 5.4 is even well-defined, but it turns out to be, and, furthermore, Supposing Σ is closed, we additionally have that $\odot$ is non-degenerate.

Definition 5.5 (Quadratic form)**.** Let V be a finite-dimensional $\mathbb{Z}_2$-vector space endowed with an alternating bilinear form $b\colon V \times V \to \mathbb{K}$. Then a quadratic form on V *refining* b (we sometimes instead say a quadratic form on (V, b)) is a map $q\colon H \to \mathbb{Z}_2$ such that

$$q(u+v) - q(u) - q(v) = b(u,v), \quad \text{for all } y, v \in V.$$

A result due to Arf (originally shown in [1]) shows that non-degenerate alternating bilinear forms, b, give rise to quadratic refinements which are characterised by the invariant, $\mathrm{Arf}(b)$, now known as the *Arf invariant* of b. In our context, we can give the following formula for this invariant:

Definition 5.6 (Arf invariant of quadratic form)**.** The Arf invariant, $\mathrm{Arf}(q) \in \mathbb{Z}_2$ of a quadratic form q, which refines an alternating, non-degenerate bilinear form $\odot$ on the $\mathbb{Z}_2$-vector space H is (implicitly) defined by the following formula:

$$(-1)^{\mathrm{Arf}(q)} := \frac{1}{\sqrt{\dim H}} \sum_{\alpha \in H} (-1)^{q(\alpha)} \in \{-1, 1\}.$$

An important property of the Arf invariant, proven for instance by Loebl[11], is the following identity:

$$1 = \frac{1}{\sqrt{\dim H}} \sum_{\substack{\text{quadratic forms} \\ q \text{ on } (V,b)}} (-1)^{\mathrm{Arf}(q)+q(\alpha)}, \quad \text{for any } \alpha \in V. \tag{1}$$

The next result is central to our discussion. It associates to every spin structure on a surface Σ a quadratic form refining the intersection pairing. This lets us define the Arf invariant of a spin structure, which was one of the key components in the formulation of theorem 1.2.

Theorem 5.3 (Quadratic form from spin structure)**.** *Representing any given homology class $\alpha \in H_1(\Sigma; \mathbb{Z}_2)$ by a disjoint union of oriented simple closed curves C_j (as we have done several times now), the following equation:*

$$(-1)^{q_\lambda(\alpha)} = \prod_j \left(-e^{i\pi \,\mathrm{wind}_\lambda(C_j)}\right),$$

(implicitly) gives a well-defined quadratic form q_λ on $H_1(\Sigma; \mathbb{Z}_2)$, refining $\odot$. Here, $\mathrm{wind}_\lambda(C_j)$ *is the winding number around the cycle C_j, as usual measured with respect to the vector field X that represents spin structure λ.*

The form of the above definition also leads to the following equation, which acts as a sort of generalisation of Whitney's lemma, lemma 3.1 from earlier. The proof, which we omit here, also follows the same structure as that of lemma 3.1. We apply operations locally to remove the self-intersection points of C, and then apply the defining relationship of q_λ, as give in the above theorem.

Theorem 5.4 (Generalised Whitney's lemma)**.** *Given spin structure λ and oriented, piecewise-smooth, closed curve $C \subset \Sigma$,*

$$(-1)^{t(C)+q_\lambda(C)+1} = e^{i\pi \,\mathrm{wind}_\lambda(C)},$$

where $\mathrm{wind}_\lambda(C)$ *is the winding number of C with respect to field X representing λ.*

As mentioned previously, theorem 5.3 allows us to define also the Arf invariant of a spin structure. Indeed, for any spin structure on a closed genus-g surface, and for any $\alpha \in H_1(\Sigma; \mathbb{Z}_2)$, we have

$$\frac{1}{2^g} \sum_{\lambda \in \mathcal{S}(\Sigma)} (-1)^{\mathrm{Arf}(\lambda)+q_\lambda(\alpha)} = 1, \tag{2}$$

which follows directly from the property given in equation 1, together with the observation from earlier that $\dim \mathcal{S}(\Sigma) = 2^{2g}$ (lemma 5.1).

As a matter of fact, Johnson[10] showed a result stronger than that mentioned above. Not only do spin structures give rise to quadratic forms refining the intersection pairing, but indeed, spin structures on a surface are in *bijection* with such quadratic forms. Informally speaking, starting from our earlier description of a spin structure as a linear functional $\lambda\colon H_1(\mathrm{UT}(\Sigma); \mathbb{Z}_2) \to \mathbb{Z}_2$ which assumes a value of 1 on the trivial framed loop, Johnson proceeds to prove that there is a map which *lifts* a simple closed curve C in Σ to a (well-defined and unique) similarly simple closed curve $\tilde{C}$ in $\mathrm{UT}(\Sigma)$. We can then associate with every spin structure $\lambda\colon H_1(\mathrm{UT}(\Sigma); \mathbb{Z}_2) \to \mathbb{Z}_2$ the map $q_\lambda\colon H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_2\colon [C] \mapsto \lambda(\tilde{C})$, which turns out to be a well-defined quadratic form. It is this map which gives us the aforementioned bijection. Nevertheless, for our purposes we can simply accept that 5.3 gives us a quadratic form for each spin structure.

6 Generalising the Kac-Ward Formula

Having introduced the requisite preliminaries, in this section we generalise the definitions and results from earlier, and eventually prove the main result, theorem 1.2.

Our construction for the case of an arbitrary weighted graph (G, x) embedded in some surface Σ begins by (arbitrarily) marking one point interior to each edge of (the representation within Σ of) G. These marked points are used in the following definition which generalises definition 2.1.

Definition 6.1 (Kac-Ward matrix)**.** Let (G, x) be an arbitrary weighted graph, and Σ a surface into which G embeds, endowed with a Riemannian metric. Let λ be a spin structure on Σ which is represented by a vector field X that has isolated even-index zeros in $\Sigma \setminus G$. Define the Kac-Ward matrix $W_\lambda := \mathbb{I} - T_\lambda$, with

$$(T_\lambda)_{e,e'} := \begin{cases} (x_e x_{e'})^{1/2} e^{i w_\lambda(e,e')/2} & \text{if } t(e) = o(e'), e' \neq \bar{e} \\ 0 & \text{else} \end{cases},$$

where $w_\lambda(e, e')$ is rotation angle analogous to that of definition 2.1, but now measured by the rotation of the velocity vector as one moves from the marked point in e to that in e', as measured relative to the vector field X at each point along this path.

It is clear why in the above definition we must require that the zeros of X all lie in $\Sigma \setminus G$, since the rotation angle $w_\lambda(e, e')$ would clearly be ill-defined if any of the edges of G crossed such a zero of X.

It will once again be useful to consider the determinant of W_λ instead in terms of a Pfaffian, so we perform the same transformation as in definition 2.4 to obtain a skew-symmetric matrix $\hat{W}_\lambda$ to which we may apply lemma 2.1 to obtain an expansion entirely analogous to that of theorem 2.3.

Definition 6.2 (Skew-symmetric Kac-Ward matrix)**.** Given the Kac-Ward matrix W_λ for arbitrary weighted graph (G, x) (with 2-cell embedding in compact connected oriented surface Σ on which λ is a spin structure represented by vector field X), define $\hat{W}$ by

$$\hat{W} := iU^* \left(\delta_{\bar{e},e'}\right)_{e,e'} WU,$$

where U is the diagonal matrix containing entries $\eta_e \in \mathbb{C}$, with each η_e is chosen so that $(\eta_e^*)^2$ gives the direction of $e \in \vec{E}(G)$, where now this direction is measured relative to the direction of the vector field X at the particular marked point within the edge e.

We can almost verbatim copy the proof of lemma 2.2 to show that $\hat{W}_\lambda$ is indeed skew-symmetric with $\det \hat{W}_\lambda = \det W_\lambda$. We can also define the terminal graph (G^K, x^K) exactly as in the planar case, whence the result of theorem 2.3 (and indeed its proof as well) apply equally in this more general context with the obvious minor modifications (adding subscripts of λ where necessary). That is to say, we still have that

$$\operatorname{Pf} \hat{W}_\lambda = \sum_{D \in \mathcal{D}(G^K)} \epsilon_\lambda(D) x^K(D),$$

where of course we have been explicit (via the subscript) about the fact that the signs ϵ_λ now depend on w_λ (and hence λ) through their dependence on $\varepsilon_{e,e'}$; however, aside from this small modification, ϵ_λ has the same form as given in theorem 2.3.

There is now only one lemma remaining before the proof of theorem 1.2. Indeed, much of the work for the main result will be accomplished in the proof of the following lemma. Take note of the many similarities between the both the statement and proof of the following lemma and those of theorem 1.1.

Lemma 6.1. *Let (G, x) be a weighted graph with a 2-cell embedding in an orientable compact surface Σ. Then for any spin structure $\lambda \in \mathcal{S}(\Sigma)$, we have*

$$\sum_{P \in \mathcal{E}(G)} (-1)^{q_\lambda([P])} x(P) = \pm\sqrt{\det W_\lambda},$$

where $[P] \in H_1(\Sigma; \mathbb{Z}_2)$ is the homology class of P.

Proof. First we note that the set $\mathcal{E}(G)$ together with the operation of symmetric difference can be associated with the $\mathbb{Z}_2$-vector space of 1-cycles in G. This is since every even subgraph may be expressed as a symmetric difference of cycles - in fact this expansion need only include cycles which belong to the so-called *cycle basis*[8] of G. This allows us to write

$$\mathcal{Z}_\lambda := \sum_{P \in \mathcal{E}(G)} (-1)^{q_\lambda([P])} x(P) = \sum_{\alpha \in H_1(\Sigma;\mathbb{Z}_2)} (-1)^{q_\lambda(\alpha)} \sum_{\substack{\text{1-cycle } P \\ [P]=\alpha}} x(P).$$

Now by much the same arguments as in the proof of lemma 3.2, we obtain that

$$\mathcal{Z}_\lambda = \sum_{D \in \mathcal{D}(G)} (-1)^{q_\lambda([G \setminus G_D]) + t(D)} x^K(D),$$

where, of course, G_D is defined as it was in lemma 3.2.

So, as in the proof of theorem 1.1, comparing the above expansion with that of $\operatorname{Pf} \hat{W}_\lambda$, we find ourselves left with proving an equality involving only the signs:

$$(-1)^{q_\lambda([G \setminus G_D]) + t(D)} = \epsilon_\lambda(D) \epsilon_\lambda(D_0),$$

where $D_0 := \{e \in E(G^K) : e \text{ is a long edge}\} \in \mathcal{D}(G^K)$ as before.

Unsurprisingly, the arguments which follow almost exactly mimic the planar case, only now we make use of the generalised Whitney lemma, theorem 5.4:

$$\epsilon_\lambda(D)\epsilon_\lambda(D_0) = \prod_{m=1}^{n} \left(-\exp\left[i\pi \,\mathrm{wind}_\lambda(C_m)\right]\right) = \prod_{m=1}^{n} (-1)^{q_\lambda(C_m)+t(C_m)},$$

where $D\triangle D_0$ is a disjoint union of the n cycles C_m, as in theorem 1.1.

Now using that q_λ is a quadratic form refining the intersection pairing $\odot$, we have that

$$\begin{aligned}
\sum_{m=1}^{n}(q_\lambda(C_m) + t(C_m)) &= q_\lambda(C_1 + C_2) - C_1 \odot C_2 + \sum_{m=3}^{n} q_\lambda(C_m) + \sum_{m=1}^{n} t(C_m) \\
&= q_\lambda(C_1 + C_2 + C_3) + C_1 \odot C_2 + (C_1 + C_2) \odot C_3 \\
&\quad + \sum_{m=4}^{n} q_\lambda(C_m) + \sum_{m=1}^{n} t(C_m) \\
&= \cdots = q_\lambda(\sum_{m=1}^{n} C_m) + \sum_{\substack{l,m=1 \\ l<m}}^{n} C_l \odot C_m + \sum_{m=1}^{n} t(C_m) \\
&= q_\lambda([D\triangle D_0]) + \sum_{\substack{l,m=1 \\ l<m}}^{n} C_l \odot C_m + \sum_{m=1}^{n} t(C_m) \\
&= q_\lambda([D\triangle D_0]) + t(D\triangle D_0),
\end{aligned}$$

where we have used the fact that we are working in $\mathbb{Z}_2$, so $-1 \equiv 1$, and in the last line we have used that the number of self-intersections of $D\triangle D_0 = \bigsqcup_{m=1}^{n} C_m$ is the sum of the numbers of self-intersections of each cycle C_m and the number of pairwise intersections between cycles (again, we need only care about this quantity modulo 2, since it appears as a power of -1). We also used the fact that, working over $\mathbb{Z}_2$, sums and disjoint unions of cycles coincide.

Finally, we need only observe that homology classes $[D\triangle D_0] = [G\setminus G_D]$, and that $t(D\triangle D_0) = t(D)$. The latter equality is more or less obvious, since self-intersections of D can only occur in cliques, and D_0 consists of only long edges, so $D\triangle D_0$ contains the same set of short edges as D. For the former equality, note that, by definition, $G \setminus G_D$ consists of every (long) edge except those in D, and $D\triangle D_0$ also contains every long edge except those in D (by definition of D_0). Furthermore, the only short edges in $D\triangle D_0$ are those joining its long edges (since it splits into cycles). It is then clear that the process of inserting such intermediate short edges does not change the homology class. ■

We conclude this section by restating and proving the central result, theorem 1.2 - the generalisation of the Kac-Ward formula to arbitrary graphs. The proof is straightforward application of the results which we have developed thus far.

Theorem 1.2 (Generalised Kac-Ward formula)**.** Let (G, x) be an arbitrary weighted graph, and Σ be any (compact) connected, oriented, genus-g surface into which G has a 2-cell embedding. Then

$$\mathcal{Z}_{\text{high}} = \frac{1}{2g} \sum_{\lambda\in\mathcal{S}(\Sigma)} (-1)^{\mathrm{Arf}(\lambda)} (\det W_\lambda)^{1/2},$$

where $(\det W_\lambda)^{1/2}$ is the square root with sign chosen to have constant coefficient equal to $+1$.

Proof. By lemma 6.1 (given that we chose the square root to have positive constant coefficient),

$$(\det W_\lambda)^{1/2} = \sum_{P\in\mathcal{E}(G)} (-1)^{q_\lambda([P])} x(P) = \sum_{\alpha\in H_1(\Sigma;\mathbb{Z}_2)} (-1)^{q_\lambda(\alpha)} \sum_{\substack{\text{1-cycle } P\\ [P]=\alpha}} x(P)$$

Now using equation 2 from the end of section 5, we have

$$\begin{aligned}
\mathcal{Z}_{\text{high}} &= \sum_{\alpha\in H_1(\Sigma;\mathbb{Z}_2)} \sum_{\substack{\text{1-cycle } P\\ [P]=\alpha}} x(P)\\
&= \sum_{\alpha\in H_1(\Sigma;\mathbb{Z}_2)} \left[\left(\frac{1}{2^g} \sum_{\lambda\in\mathcal{S}(\Sigma)} (-1)^{\text{Arf}(\lambda)+q_\lambda(\alpha)} \right) \sum_{\substack{\text{1-cycle } P\\ [P]=\alpha}} x(P) \right].
\end{aligned}$$

Finally we reorder the sums and apply lemma 6.1 once more to get

$$\begin{aligned}
\mathcal{Z}_{\text{high}} &= \frac{1}{2^g} \sum_{\lambda\in\mathcal{S}(\Sigma)} \left[(-1)^{\text{Arf}(\lambda)} \sum_{\alpha\in H_1(\Sigma;\mathbb{Z}_2)} (-1)^{q_\lambda(\alpha)} \sum_{\substack{\text{1-cycle } P\\ [P]=\alpha}} x(P) \right]\\
&= \frac{1}{2^g} \sum_{\lambda\in\mathcal{S}(\Sigma)} (-1)^{\text{Arf}(\lambda)} (\det W_\lambda)^{1/2}.
\end{aligned}$$

■

References

[1] Cahit Arf. Untersuchungen über quadratische formen in körpern der charakteristik 2.(teil i.). *Journal für die reine und angewandte Mathematik*, 1941(183):148–167, 1941.

[2] Michael F Atiyah. Riemann surfaces and spin structures. *Annales scientifiques de l'École Normale Supérieure*, 4(1):47–62, 1971.

[3] Henning Bruhn and Reinhard Diestel. Embedding graphs in surfaces: Maclane's theorem for higher genus. 2008.

[4] Arthur Cayley. Sur les déterminants gauches.(suite du mémoire t. xxxii. p. 119). *Journal für die reine und angewandte Mathematik*, 1849(38):93–96, 1849.

[5] Dmitry Chelkak, David Cimasoni, and Adrien Kassel. Revisiting the combinatorics of the 2d ising model. *Annales de l'Institut Henri Poincaré D*, 4(3):309–385, 2017.

[6] David Cimasoni. The geometry of dimer models. *Winter Braids Lecture Notes*, 1, 2014. talk:2.

[7] Robert F. Cohen, Peter Eades, Tao Lin, and Frank Ruskey. Three-dimensional graph drawing. *Algorithmica*, 17(2):199–208, 1997.

[8] Reinhard Diestel. Graph theory. *Graduate texts in mathematics*, 173:23–28, 2012.

[9] Allen Hatcher. *Algebraic topology.* Cambridge University Press, 2001.

[10] Dennis Johnson. Spin structures and quadratic forms on surfaces. *Journal of the London Mathematical Society*, 2(2):365–373, 1980.

[11] Martin Loebl and Gregor Masbaum. On the optimality of the arf invariant formula for graph polynomials. *arXiv preprint arXiv:0908.2925*, 2009.

[12] James R Munkres. *Elements of algebraic topology.* CRC Press, 2018.

[13] J. Youngs. Minimal imbeddings and the genus of a graph. *Indiana Univ. Math. J.*, 12:303–315, 1963.