Speculation on Maxwell-Boltzmann Distribution From a Microcanonical Power Law

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In (1), a power law distribution is obtained through geometric (using n-sphere and n-1 sphere surface areas) means for n one-dimensional particles with a total energy E, i..e. the condition of the microcanonical distribution is imposed. In the large n limit, this power law distribution becomes the Maxwell-Boltzmann distribution. The geometrical approach is based on the equation: p1p1 + p2p2 + … +pnpn= RR where E=RR/2m (nonrelativistic case), where each pi is like a co-ordinate.

We try to argue that the result of (1), namely f(p) = (1-pp/RR) power((n-3/)/2) may be obtained without using the geometric n dimensional sphere approach of (1). We write: e1+e2+..en = E and argue that ultimately each e should have the same probability and degeneracy. The degeneracy arises from: de= d (pp/2m) = p dp/2m or C1 sqrt(e) dp. In (1), a distribution as a function of p is desired. For a given e and E, the energy which may be distributed is E-e. We argue that for n=3, the probability should be the same for all particles. We further suggest a product of degeneracy values and thus suggest: sqrt(E-e) power(n-3). This is the exact result of (1). In the case of 3-dimensional particles, we apply the degeneracy also to dpx dpy dpz so sqrt(e) becomes sqrt(e)sqrt(e)sqrt(e). This then yields the correct relationship between energy and T (temperature) in the Maxwell-Botlzmann large n limit.

Microcanonical Approach of (1) to Obtaining the Maxwell-Boltzmann Distribution

The geometrical approach of (1) seems to be based on e=pp/2m (nonrelativistic). For 1-dimensional motion one has the microcanonical condition:

P1p1 + p2p2 + … pnpn = RR   where E = RR/2m   ((1))

((1)) is interpreted in (1) as an n-dimensional sphere. For a given pn value, the remaining energy is related to RR-pnpn and (1) argues that the p1 to p (n-1) remaining particles are distributed over an Sn-1 surface i.e. a sphere of n-1 dimensions.

To get a sense of this approach, consider three particles, i.e. n=3. Next consider p1 mapping to x, p2 to y and p3 to z. Consider any point on the 3-space sphere. One may immediately rotate the co-ordinate system so that the point lies on the z axis. The Sn-1 object is then a circle of radius R for any point. The probability for any point then is the same, because one has the same circle. Thus, the formula from (1) should yield a constant for f(p) for n=3 which it does.

The argument of (1) is to write:

RR cos(theta) cos(theta) = RR-pnpn ((2))

In other words one projects onto a lower dimension, i.e. Sn-1. In the 3-sphere case, RR=xx+yy+zz. RR-zz is the remaining number if zz has a certain value. Z is Rsin(90-theta1) if theta is measured from the z axis and the projection is linked with R cos(90-theta1). If one calls 90-theta1=theta, then one sees the emergence of ((2)).

((2)) implies that pn/R = sin(theta) so obtaining d theta from ((2)) yields:

R d theta = dpn / sqrt(1-pnpn/RR) ((3))

A key idea, we argue, is that for curved objects (e.g. a circle) one uses equal intervals along the curve. For a straight line, like an x-axis, one uses constant dx pieces. Here the geometrical picture is one of curved spaces, so the degeneracy factor is not dpn, but rather R d theta which may be written in terms of pn, i.e. ((3)).

Thus probability is given by:

Probability =   Sn-1 (are of an n-1 curved sphere) * degeneracy factor   ((4))

The degeneracy factor is R d theta. Using the result:

Sn-1 proportional to Radius power (n-1) one has:

Probability = Radias power(n-1) dpn/ sqrt(1-pnpn/RR)

The radius of the n-1 sphere is sqrt(1-pnpn/RR) so:

Probability = dpn (1-pnpn/RR) power (n/2 -½ -2/2) = dpn (1-pnpn/RR) power (n/2 -3/2)   ((5))

As argued above, this yields a constant for n=3. In the large n limit one obtains the Maxwell-Boltzmann distribution. Thus (1) shows that the microcanonical approach leads to a power law which in the large n limit yields the MB distribution.

Non-geometric Approach

We suggest that ((5)) may be obtained without using n-space spheres. We use instead:

e1+e2 +.... en = E   ((6))

We argue that in ((6)) all ei values appear on the same footing and should carry the same weight, i.e. degeneracy. This degeneracy is given by:

De = d (pp/2m) or   de= p dp /2m = sqrt(e) C1 dp ((7))

(1) Wishes to write the probability distribution in terms of dp, but we are interested in degeneracy associated with e, so we use the degeneracy factor sqrt(e), except that we apply it to the energy which may be distributed when given en, i.e. E-en.

Thus we consider the degeneracy sqrt(E-en) which applies to n-3 particles if n=3 is to yield 0, i.e. a constant probability. We try to argue that n=3 particles should have a constant probability in the following way.

For a given number e3, one must distribute E-e3 to e1 and e2. One may, however, distribute E-e3 uniformly to e1 as e2 automatically receives the remaining value. Thus, one really performs a uniform distribution of E-e3 to e1. This argument, however, should apply separately to e1, e2 and e3.. This suggests a constant probability distribution in e, we argue. This leads to:

Probability =dp sqrt(E-n) power (n-3)   ((8))

Here we suggest multiplying the degeneracies leaving out factors for 3 particles.

This is the same result as the (1) and leads to the Maxwell-Boltzmann result for high n.. The Maxwell-Boltzmann result is achieved even if one uses n instead of n-3 in the high n limit. One may note that the sqrt in ((8) leads to eave/2 appearing in the MB distribution if E= eave n. For the three dimensional case, we argue that one has dpxdpydpz and may apply the degeneracy sqrt(E-en) to each (just like to a separate particle). This is then consistent with eave = 3/2 n T.

Conclusion

Ref. 1 presens a geometrical n-sphere approach to finding a power law probability associated with n one-dimensionally moving particles based on Sum over i p(i)p(i)=RR. The probability is Sn-1 (i.e. a sphere of one less dimension), but a degeneracy factor must be found which follows from: RR cos(theta)cos(theta) = RR-pnpn. Thus, spherical symmetry is key to the solution of (1).

We argue that one may use e1+e2 +... en = E with no geometrical picture.. We suggest that each ei carries the same degeneracy weight which follows from: e= pp/2m or de = C p dp since one wishes to find a function in p. Thus we use sqrt(e) as the degeneracy weight, or rather sqrt(E-e). We argue that for n=3, i.e. 3 particles, each has a uniform independent distribution, so the probability distribution for n=3 is a constant. Thus we use sqrt(E-e) power (n-3) which is the same result as (1). In the high n limit with E=eave n, one obtains the Maxwell-Boltzmann distribution from the power law one.

References

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