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INVERSION SEQUENCES AVOIDING QUADRUPLE LENGTH-3 PATTERNS

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Abstract

An *inversion sequence* of length n is a sequence of integers $e = e_0 \cdots e_n$ which satisfies $0 \leq e_i \leq i$, for all $i = 0, 1, \ldots, n$. For a set of patterns B, let $\mathbf{I}_n(B)$ be the set of inversion sequences of length n that avoid all the patterns from B. We say that two sets of patterns B and C are *I-Wilf-equivalent* if $|\mathbf{I}_n(B)| = |\mathbf{I}_n(C)|$, for all $n \geq 0$. In this paper, we show that the number of I-Wilf-equivalences among quadruples of length-3 patterns is at least 212 and at most 215, where three open cases remain.

1. Introduction

Any word $e = e_0 \cdots e_n$ such that $0 \leq e_i \leq i$ for all $i = 0, 1, \ldots, n$ is called an *inversion sequence* [7, 12] of length n. We denote the set of inversion sequences of length n by $\mathbf{I}_n$.

We say that a word $u = u_1 \cdots u_n$ is *order-isomorphic* to a word $v = v_1 \cdots v_n$ if for every pair of indices $i, j \in [n]$, we have $u_i < u_j$ ($u_i = u_j$) if and only if $v_i < v_j$ ($v_i = v_j$). We say that the word w *contains* a word $p = p_1 \cdots p_k$ if w contains a subsequence of length k which is order-isomorphic to p. Otherwise, we say that w *avoids* p. We denote the set of all inversion sequences of length n that avoid a pattern p by $\mathbf{I}_n(p)$. For a set of patterns P, define $\mathbf{I}_n(P) = \cap_{p\in P}\mathbf{I}_n(p)$, for all $n \geq 0$. We say that two sets of patterns P and Q are *I-Wilf-equivalent*, and we write $P \overset{\mathbf{I}}{\sim} Q$, if $|\mathbf{I}_n(P)| = |\mathbf{I}_n(Q)|$, for all $n \neq 0$.

Pattern avoidance in inversion sequences was initiated in [7, 12]. Subsequently, several researchers studied the number of I-Wilf-equivalences for single, pairs of, and triples of length-3 patterns:

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- Martinez and Savage [14] generalized and extended the notion of pattern-avoidance for inversion sequences to triples of binary relations that led to new conjectures and open problems. In particular, some of these results are related to quadruples of length-3 patterns as mentioned in Table 1 (also, see [6]). Most conjectures in this work have been solved in [1,6,9,10], where both generating functions and bijections were employed.

- The results of [2,5,14,16] determined all the I-Wilf-equivalence classes of pairs of length-3 patterns. So, it showed that there are 48 Wilf classes among 78 pairs of length-3 patterns (see [16]); for a complete list of the classes with open cases in terms of enumeration, see Tables 1 and 2 in [16]. Kotsireas, Mansour, and Yıdırım [8] enumerated some of these open cases (see Remark 1).

- Callan, Jelínek, and Mansour [3] showed that the number of I-Wilf-equivalence classes among triples of length-3 patterns is 137, 138 or 139. In particular, it remains to prove $\{101, 102, 110\} \overset{\mathbf{I}}{\sim} \{021, 100, 101\}$ and $\{100, 110, 201\} \overset{\mathbf{I}}{\sim} \{100, 120, 210\}$.

We refer the reader to [3] and references therein. The aim of this paper is to prove the following result.

Theorem 1. *The number of I-Wilf-equivalence classes among quadruple length-3 patterns is at least* 212 *and at most* 215*.*

Based on numerical results (see Table 2) and Theorem 1, we conjecture the following.

Conjecture 1. We conjecture

- Class 152: $\{010, 100, 102, 210\} \overset{\mathbf{I}}{\sim} \{011, 201, 210\}$;

- Class 166: $\{010, 100, 110, 201\} \overset{\mathbf{I}}{\sim} \{010, 101, 120, 201\}$;

- Class 207: $\{100, 101, 110, 201\} \overset{\mathbf{I}}{\sim} \{101, 110, 120, 210\}$.

Note that we checked the conjecture up to $n = 13$ and we commented on these three cases in Table 1 as “still open”. So if we assume that the conjecture is true, then Theorem 1 shows that there are exactly 212 I-Wilf-equivalence classes among quadruple length-3 patterns.

2. The Strategy to Prove Theorem 1

Define $P = \{000, 001, 010, 011, 012, 021, 100, 101, 102, 110, 120, 201, 210\}$ to be the set of all length-3 patterns. The goal of this paper (which we don’t quite achieve) is to prove there are exactly 212 I-Wilf-equivalences among $B \subset P$ with $|B| = 4$. Since the number of subsets B of P with $|B| = 4$ is 715, it seems impossible to reach our goal by constructing explicit bijections between classes of inversion sequences. The way out is to combine several steps as follows.

`First step`: We find all the sequences $\{|\mathbf{I}_n(B)|\}_{n=0}^{9}$, for all $B \subset P$ with $|B| = 4$. Table 2 in the Appendix below divides the 715 4-subsets of P into 212 classes, where the first column of this table assigns the number of the class. Theorem 1 is equivalent to proving that the classes in Table 2 are exactly the I-Wilf-equivalences among quadruples of length-3 patterns.

`Second step`: Let C be any class in Table 2. We say that C is *trivial* if C contains exactly one subset. Otherwise, C is *nontrivial*. Since each trivial class in Table 2 is an I-Wilf-equivalence, we need to consider only the nontrivial classes in Table 2. There are exactly 120 trivial classes, denoted by T in the first column in Table 2. Thus, it remains to consider $212 - 120 = 92$ nontrivial classes. Table 1 below contains only the 92 nontrivial classes, whereas, in its first column, we retain the class number from Table 2.

`Third step`: Let B be any set of patterns in P. We say that B is *reducible* if there exists $C \subsetneq B$ such that $\mathbf{I}_n(B) = \mathbf{I}_n(C)$, for all $n \geq 0$. In this context, we write $C \overset{r}{\sim} B$. Clearly, $C \overset{r}{\sim} B$ implies $C \overset{\mathbf{I}}{\sim} B$.

Theorem 2. *We have*

$$\begin{array}{lll} \{001\} \overset{r}{\sim} \{001, 101\}, & \{001\} \overset{r}{\sim} \{001, 102\}, & \{001\} \overset{r}{\sim} \{001, 201\}, \\ \{011\} \overset{r}{\sim} \{011, 101\}, & \{011\} \overset{r}{\sim} \{011, 110\}, & \{012\} \overset{r}{\sim} \{012, 102\}, \\ \{012\} \overset{r}{\sim} \{012, 120\}, & \{021\} \overset{r}{\sim} \{021, 201\}, & \{021\} \overset{r}{\sim} \{021, 210\}. \end{array}$$

Proof. Since the proofs are similar, we show only the equivalence $\{001\} \overset{r}{\sim} \{001, 101\}$. Clearly, $\mathbf{I}_n(\{001, 101\}) \subseteq \mathbf{I}_n(\{001\})$. So it remains to show that $\mathbf{I}_n(\{001\}) \subseteq \mathbf{I}_n(\{001, 101\})$. Let $\pi = \pi_0\pi_1 \cdots \pi_n \in \mathbf{I}_n(001)$ with $n \geq 3$ (clearly, the statement holds for $n \leq 2$) and assume that π contains 101. Thus, there exists $0 \leq i < j < k \leq n$ such that $\pi_i = \pi_k > \pi_j$. By induction we prove that $\pi_j \geq m$ and $i \geq m+1$, for any $m = 0, 1, \ldots, n-3$.

Clearly, the claim holds for $m = 0$. Since $\pi_0 = 0$, we have $i \geq 1$. If $\pi_j = 0$ then π contains 001 (as $\pi_0\pi_j\pi_k$), so $\pi_i > \pi_j \geq 1$ and then $i \geq 2$. So the claim holds for $m = 1$. Assume that the claim holds for m and let us prove it for $m+1$. Suppose, for a contradiction, that $\pi \in \mathbf{I}_n(\{001\})$ is such that there exists $0 \leq i < j < k \leq n$ with $\pi_i = \pi_k > \pi_j \geq m$ and $i \geq m+1$. Since π avoids 001, all the letters left of π_i are different, so $\pi_s = s$ for all $s = 0, 1, \ldots, m$. If $\pi_j = m$, then π contains $\pi_m\pi_j\pi_k = mm\pi_k$, that is, it contains 001. Thus, $\pi_j \geq m+1$. Since π is an inversion sequence and $\pi_i > \pi_j$, we have $i \geq m+2$. Hence, by induction on m, we have that π satisfies the claim for $m = n - 3$, thus $\pi = 01 \cdots n - 3\pi_{n-2}\pi_{n-1}\pi_n$ and $\pi_n > \pi_{n-1} \geq n-3$. Since π avoids 001, we have that $\pi_{n-1} \geq n-2$, so $\pi_{n-2} > n-2$, which contradicts the fact that $\pi_s \leq s$ for all $s = 0, 1, \ldots, n$ ($\pi \in \mathbf{I}_n$). Thus, π avoids 101, which completes the proof. □

In the next theorem, we describe all the subsets of three length-3 patterns in P reducible to subsets of two length-3 patterns in P that are not obtained from Theorem 2. Here, we omit the proof.

Theorem 3. *We have*

- $\{000, 001\} \overset{r}{\sim} \{000, 001, 100\}$;
- $\{000, 011\} \overset{r}{\sim} \{000, 011, \tau\}$, *for any* $\tau = 100, 201, 210$;
- $\{000, 012\} \overset{r}{\sim} \{000, 012, \tau\}$, *for any* $\tau = 201, 210$;
- $\{000, 021\} \overset{r}{\sim} \{000, 021, 100\}$;
- $\{001, 010\} \overset{r}{\sim} \{001, 010, \tau\}$, *for any* $\tau = 021, 100, 110, 120, 210$;
- $\{001, 011\} \overset{r}{\sim} \{001, 011, 021\}$;
- $\{001, 012\} \overset{r}{\sim} \{001, 012, 021\}$;
- $\{001, 110\} \overset{r}{\sim} \{001, 110, 210\}$;
- $\{001, 120\} \overset{r}{\sim} \{001, 120, 210\}$;
- $\{010, 011\} \overset{r}{\sim} \{010, 011, 100\}$;
- $\{010, 012\} \overset{r}{\sim} \{010, 012, \tau\}$, *for any* $\tau = 101, 201$;
- $\{010, 021\} \overset{r}{\sim} \{010, 021, \tau\}$, *for any* $\tau = 100, 101, 102, 110, 120$.

In the next theorem, we describe all the reducible subsets of four length-3 patterns in P to subsets of three length-3 patterns in P that are not obtained from Theorems 2 and 3. Again, we omit the proof.

Theorem 4. *We have*

- $\{001, 011, 012\} \overset{r}{\sim} \{001, 011, 012, 210\}$;
- $\{001, 011, 100\} \overset{r}{\sim} \{001, 011, 100, 210\}$;
- $\{001, 012, 100\} \overset{r}{\sim} \{001, 012, 100, 210\}$.

Note that when we have a reducible subset of four patterns in P, we can consider the references [3, 14, 16]. These references considered the I-Wilf-equivalences and enumerations of $|\mathbf{I}_n(B)|$ whenever B is any pair or triple of length-3 patterns from P. In Table 1, for a given B in a class C, if the computations for $\mathbf{I}_n(B)$ are not simple, we write $B = B'$ and cite Theorems 2-4, where B is reducible to B'.

`Fourth step`: First, following [8], we define the generating tree (see [15]) $\mathcal{T}(B)$. Set $\mathbf{I}_B = \cup_{n=0}^{\infty}\mathbf{I}_n(B)$. The tree $\mathcal{T}(B)$ is understood to be empty if there is no inversion sequence of arbitrary length avoiding the set B, that is, $0 \in B$. Otherwise, the root can always be taken as 0. Starting with this root which stays at level 0, we construct the remainder of the nodes of the tree $\mathcal{T}(B)$ as follows: the children of $e_0e_1\cdots e_{n-1} \in \mathbf{I}_{n-1}(B)$ are obtained from the set $\{e_0e_1\cdots e_{n-1}e_n \mid e_n = 0, 1, \ldots, n\}$ by obeying the pattern-avoiding restrictions of the patterns in B.

We define an equivalence relation on nodes of $\mathcal{T}(B)$. Let $\mathcal{T}(B; e)$ be the subtree consisting of the inversion sequence e as the root and its descendants in $\mathcal{T}(B)$. We say that e is *equivalent* to e' if and only if $\mathcal{T}(B; e) \cong \mathcal{T}(B; e')$ (in the sense of plain trees). Let $\mathcal{T}'(B)$ be the same tree $\mathcal{T}(B)$ where we replace each node e by the first node $e' \in \mathcal{T}(B)$ from top to bottom and from left to right in $\mathcal{T}(B)$ such that $\mathcal{T}(B; e) \cong \mathcal{T}(B; e')$. From now on, we identify $\mathcal{T}'(B)$ with $\mathcal{T}(B)$.

We are now ready to describe the details of the fourth step which is based on the algorithm of [8]. Let C be any nontrivial class in Table 2. For each subset $B \in C$, we run the main algorithm of [8], call it Algorithm KMY, for guessing and proving (if possible) the rules of the generating tree $\mathcal{T}(B)$. Then, we translate these rules into a system of equations and we solve for $F_B(x) = \sum_{n\geq 0} |\mathbf{I}_n(B)|x^{n+1}$. For examples, we refer the reader to [3, 8, 13]. See Table 1 for all the generating trees $\mathcal{T}(B)$ that we obtained and all the corresponding generating functions $F_B(x) = \sum_{n\geq 0} |\mathbf{I}_n(B)|x^{n+1}$.

Remark 1. In [8], Kotsireas, Mansour, and Yıdırım suggested an algorithm for guessing and proving (if possible) the rules of the generating tree $\mathcal{T}(B)$. In particular, they solved six open cases (see Tables 1 and 2 in [16]) for such pattern classes: $\mathbf{I}_n(000, 021)$, $\mathbf{I}_n(102, 021)$, $\mathbf{I}_n(100, 012)$, $\mathbf{I}_n(120, 210)$, Wilf-equivalent $\mathbf{I}_n(011, 201)$ and $\mathbf{I}_n(011, 210)$, and Wilf-equivalent $\mathbf{I}_n(100, 021)$ and $\mathbf{I}_n(110, 021)$. Moreover, they extended the algorithm to the case of restricted growth sequences (for pattern avoidance on restricted growth sequences, see [11]) and presented an explicit formula for the generating function for the number of restricted growth sequences of length n that avoid either $\{12313,12323\}$, $\{12313, 12323, 12333\}$, or $\{123\cdots\ell 1\}$.

We end this section with Table 1 which presents the 92 nontrivial classes (see the second column). In the third and fourth columns of the table, we present the rules of the generating tree $\mathcal{T}(B)$ (the root is always 0) and the corresponding generating function $F_B(x)$, whenever B is any set in the first column of the table.

Table 1: Succession rules for the generating trees $\mathcal{T}'(B)$ and generating functions $F_B(x)$ for pattern set $B \in C$, where C is a class in Table 2. Note that we distinguished those pattern sets B with same generating tree by dashed lines and we denote the pattern set B at line d in class C by $C(d)$.

Beginning of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
1	{000,001,010,012} {000,001,011,012}	$0 \rightsquigarrow 00, 01$; $01 \rightsquigarrow 00$	$x + 2x^2 + x^3$
2	{000,001,010,011} {001,010,011,012}	$0 \rightsquigarrow 00, 01$; $01 \rightsquigarrow 01$	$x + 2x^2 + \frac{x^3}{1-x}$
4	{000,001,012,021} {000,001,012,100} {000,001,012,101} {000,001,012,102} {000,001,012,120} {000,001,012,201} {000,001,012,210}	$0 \rightsquigarrow 00, 01$; $01 \rightsquigarrow 00, 011$; $011 \rightsquigarrow 00$	
	{000,010,011,012}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 01, 002$; $002 \rightsquigarrow 01$	$x + 2x^2 + 2x^3 + x^4$
5	{000,001,011,120}	$0 \rightsquigarrow 00, 01$; $01 \rightsquigarrow 00, 012$; $012 \rightsquigarrow 012$	
	{001,011,012,100}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 010$	$x + 2x^2 + 2x^3 + \frac{x^4}{1-x}$
6	{000,001,010,021} {000,001,010,100} {000,001,010,101} {000,001,010,102} {000,001,010,110} {000,001,010,120} {000,001,010,201} {000,001,010,210} {000,001,011,021} {000,001,011,100} {000,001,011,101} {000,001,011,102} {000,001,011,110} {000,001,011,201} {000,001,011,210}	$0 \rightsquigarrow 00, 0$	
	{001,010,011,021} {001,010,011,100}		

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{001,010,011,101}		
	{001,010,011,102}		
	{001,010,011,110}		
	{001,010,011,120}		
	{001,010,011,201}		
	{001,010,011,210}		
	{001,010,012,021}		
	{001,010,012,100}		
	{001,010,012,101}		
	{001,010,012,102}		
	{001,010,012,110}		
	{001,010,012,120}		
	{001,010,012,201}		
	{001,010,012,210}		
	{001,011,012,021}		
	{001,011,012,101}		
	{001,011,012,102}		
	{001,011,012,110}		
	{001,011,012,120}		
	{001,011,012,201}		
	{001,011,012,210}	$0 \rightsquigarrow (00)^2$; $00 \rightsquigarrow 00$	$x + \frac{2x^2}{1-x}$
8	{000,011,012,100}		
	{000,011,012,101}		
	{000,011,012,102}		
	{000,011,012,110}		
	{000,011,012,120}		
	{000,011,012,201}		
	{000,011,012,210}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 01$; $01 \rightsquigarrow 001$	$x + 2x^2 + 3x^3 + x^4$
10	{001,011,100,120}		
	{001,012,100,110}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 010, 00$	$x + 2x^2 + 3x^3 + \frac{2x^4}{1-x}$
11	{000,010,012,100}		
	{000,010,012,110}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 01, 002$; $01 \rightsquigarrow 011$; $002 \rightsquigarrow 01, 011$	$x + 2x^2 + 3x^3 + 3x^4 + x^5$
12	{000,010,012,101}		
	{000,010,012,102}		
	{000,010,012,120}		
	{000,010,012,201}		

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{000,010,012,210}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 01, 002$; $01 \rightsquigarrow 011$; $002 \rightsquigarrow 01, 0022$; $0022 \rightsquigarrow 01$	$x + 2x^2 + 3x^3 + 3x^4 + 2x^5 + x^6$
14	{000,001,021,110} {000,001,110,120}	$0 \rightsquigarrow 00, 01$; $01 \rightsquigarrow (00)^2, 01$	
	{001,011,021,120} {001,011,101,120} {001,011,102,120} {001,011,110,120} {001,011,120,201} {001,011,120,210} {001,012,021,110}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow (00)^2$	
	{001,011,021,100} {001,011,100,101} {001,011,100,102} {001,011,100,110} {001,011,100,201} {001,011,100,210} {001,012,021,100} {001,012,100,101} {001,012,100,102} {001,012,100,120} {001,012,100,201} {001,012,100,210}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 010, 01$	
	{001,012,101,110} {001,012,102,110} {001,012,110,120} {001,012,110,201} {001,012,110,210}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow (00)^2$	$x + 2x^2 + \frac{3x^3}{1-x}$
15	{000,001,021,100} {000,001,021,101} {000,001,021,102} {000,001,021,201} {000,001,021,210} {000,001,100,120} {000,001,101,120}		

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{000,001,102,120} {000,001,120,201} {000,001,120,210}	$0 \rightsquigarrow 00, 01;$ $01 \rightsquigarrow 00, 011, 01;$ $011 \rightsquigarrow 00$	$x + 2x^2 + 3x^3 + \frac{4x^4}{1-x}$
16	{000,001,100,110} {000,001,101,110} {000,001,102,110} {000,001,110,201} {000,001,110,210}	$a_m \rightsquigarrow (00)^{m+1}, a_{m+1},$ where $a_m = 01 \cdots m$	
	{000,010,011,021}	$0 \rightsquigarrow 0, 01;\ 01 \rightsquigarrow 01$	
	{001,010,021,100} {001,010,021,101} {001,010,021,102} {001,010,021,110} {001,010,021,120} {001,010,021,201} {001,010,021,210} {001,010,100,101} {001,010,100,102} {001,010,100,110} {001,010,100,120} {001,010,100,201} {001,010,100,210} {001,010,101,102} {001,010,101,110} {001,010,101,120} {001,010,101,201} {001,010,101,210} {001,010,102,110} {001,010,102,120} {001,010,102,201} {001,010,102,210} {001,010,110,120} {001,010,110,201} {001,010,110,210} {001,010,120,201} {001,010,120,210} {001,010,201,210} {001,011,021,101}		

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{001,011,021,102}		
	{001,011,021,110}		
	{001,011,021,201}		
	{001,011,021,210}		
	{001,011,101,102}		
	{001,011,101,110}		
	{001,011,101,201}		
	{001,011,101,210}		
	{001,011,102,110}		
	{001,011,102,201}		
	{001,011,102,210}		
	{001,011,110,201}		
	{001,011,110,210}		
	{001,011,201,210}		
	{001,012,021,101}		
	{001,012,021,102}		
	{001,012,021,120}		
	{001,012,021,201}		
	{001,012,021,210}		
	{001,012,101,102}		
	{001,012,101,120}		
	{001,012,101,201}		
	{001,012,101,210}		
	{001,012,102,120}		
	{001,012,102,201}		
	{001,012,102,210}		
	{001,012,120,201}		
	{001,012,120,210}		
	{001,012,201,210}	$0 \rightsquigarrow 00, 0$; $00 \rightsquigarrow 00$	
	{010,011,012,021}	$a_m \rightsquigarrow a_{m+1}, (01)^m$, where $a_m = 0^m$	$\frac{x}{(1-x)^2}$
17	{000,001,100,210}		
	{000,001,101,210}		
	{000,001,102,210}		
	{000,001,201,210}	$a_0 \rightsquigarrow b_0, a_1$; $a_m \rightsquigarrow b_0^m, b_m, a_{m+1}$; $b_m \rightsquigarrow b_0^m$, where $a_m = 01 \cdots m$, $b_m = a_m m$	
	{000,010,011,102}	$0 \rightsquigarrow 00, 01$, $00 \rightsquigarrow 00, 002$, $01 \rightsquigarrow 01$, $002 \rightsquigarrow 0021, 002$	$\frac{x(1+x^3)}{(1-x)^2}$

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
19	{000,001,100,101} {000,001,100,102} {000,001,100,201} {000,001,101,102} {000,001,101,201} {000,001,102,201}	$a_m \rightsquigarrow a_{m+1}, b_0, \dots, b_m$; $b_m \rightsquigarrow b_0, \dots, b_{m-1}$, where $a_m = 01 \cdots m$, $b_m = a_m m$	
	{000,010,011,120}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00, 0$; $01 \rightsquigarrow 01$	
	{010,011,012,100} {010,011,012,101} {010,011,012,102} {010,011,012,110} {010,011,012,120} {010,011,012,201}	$a_m \rightsquigarrow a_{m+1}, b_1, \dots, b_m$; $b_m \rightsquigarrow b_1, \dots, b_{m-1}$, where $a_m = 0^m$, $b_m = a_m m$	$\frac{x(1+x)}{1-x-x^2}$
20	{000,010,011,100} {000,010,011,101} {000,010,011,110} {000,010,011,201} {000,010,011,210}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow (00)^2$; $01 \rightsquigarrow 01$	$\frac{x(1-x-x^2)}{(1-x)(1-2x)}$
30	{000,010,101,120} {000,010,110,120}		See Subsection 4.1
36	{000,010,100,201} {000,010,100,210}		See Subsection 4.2
37	{000,010,101,201} {000,010,101,210}		See Subsection 4.3
39	{000,012,021,101}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow (001)^2$; $01 \rightsquigarrow 010, 001$; $001 \rightsquigarrow 010$	
	{000,012,021,110}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow (001)^2$; $01 \rightsquigarrow 001, 011$; $001 \rightsquigarrow 011$	$x + 2x^2 + 4x^3 + 3x^4$
41	{000,012,021,100} {000,012,021,102}		

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{000,012,021,120} {000,012,021,201} {000,012,021,210}	$0 \rightsquigarrow (00)^2$; $00 \rightsquigarrow (001)^2$; $001 \rightsquigarrow 0011$	$x + 2x^2 + 4x^3 + 4x^4$
	{000,012,100,110}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 002$; $01 \rightsquigarrow 001, 011$; $001 \rightsquigarrow 011$; $002 \rightsquigarrow (011)^2$	
42	{000,012,100,101}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 01$; $01 \rightsquigarrow 001, 010$; $001 \rightsquigarrow 010$	$x + 2x^2 + 4x^3 + 4x^4 + x^5$
	{000,012,102,110} {000,012,110,120} {000,012,110,201} {000,012,110,210}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 01,$ $01 \rightsquigarrow 001, 011$; $001 \rightsquigarrow 011$	
43	{000,012,101,102} {000,012,101,120} {000,012,101,201} {000,012,101,210}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 002$ $01 \rightsquigarrow 010, 001$ $001 \rightsquigarrow 010'$; $002 \rightsquigarrow 001, 0022$; $0022 \rightsquigarrow 001,$	$x + 2x^2 + 4x^3 + 4x^4 +$ $2x^5 + x^6$
44	{000,012,100,102}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 002$; $01 \rightsquigarrow 001, 011$; $001 \rightsquigarrow 011$; $002 \rightsquigarrow (011)^2$	$x + 2x^2 + 4x^3 + 5x^4 + x^5$
	{000,012,100,120} {000,012,100,201} {000,012,100,210}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 001, 002$; $01 \rightsquigarrow (001)^2$; $001 \rightsquigarrow 0011$; $002 \rightsquigarrow 0011, 001$	

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
45	{000,012,102,120}		
	{000,012,102,201}		
	{000,012,102,210}		
	{000,012,120,201}		
	{000,012,120,210}		
	{000,012,201,210}	$0 \rightsquigarrow 00, 01;$ $00 \rightsquigarrow 001, 002;$ $01 \rightsquigarrow (001)^2;$ $001 \rightsquigarrow 0011;$ $002 \rightsquigarrow 001, 0022;$ $0022 \rightsquigarrow 001$	$x + 2x^2 + 4x^3 + 5x^4 + 2x^5 + x^6$
46	{000,011,021,102}	$0 \rightsquigarrow 00, 01;$ $00 \rightsquigarrow 00, 002;$ $01 \rightsquigarrow 010, 01;$ $002 \rightsquigarrow 002$	
	{000,011,102,120}	$0 \rightsquigarrow 0, 01;$ $01 \rightsquigarrow 010, 012;$ $012 \rightsquigarrow 012$	
	{001,021,100,110}	$0 \rightsquigarrow 00, 01;$ $00 \rightsquigarrow 00;$ $01 \rightsquigarrow 010, 00, 01$	
	{001,021,100,120}	$0 \rightsquigarrow 00, 01;$ $00 \rightsquigarrow 00;$ $01 \rightsquigarrow 010, 011, 012;$ $011 \rightsquigarrow 010, 011;$ $012 \rightsquigarrow 00, 012$	
	{001,021,110,120}	$0 \rightsquigarrow 00, 01;$ $00 \rightsquigarrow 00;$ $01 \rightsquigarrow (00)^2, 012;$ $012 \rightsquigarrow 00, 012$	
	{001,100,110,120}	$0 \rightsquigarrow 00, 01;$ $00 \rightsquigarrow 00;$ $01 \rightsquigarrow 010, 00, 01$	$\frac{x(1+x^2-x^3)}{(1-x)^2}$
47	{000,011,021,120}	$0 \rightsquigarrow (00)^2;$ $00 \rightsquigarrow 00, 002;$ $002 \rightsquigarrow 002$	
	{000,011,100,102}		
	{000,011,101,102}		
	{000,011,102,110}		
	{000,011,102,201}		
	{000,011,102,210}	$0 \rightsquigarrow 0, 01;$ $01 \rightsquigarrow 010, 01$	
	{001,021,101,110}		
	{001,021,102,110}		
	{001,021,110,201}		
	{001,021,110,210}	$0 \rightsquigarrow 00, 01;$ $00 \rightsquigarrow 00;$ $01 \rightsquigarrow (00)^2, 01$	

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{001,021,101,120} {001,021,102,120} {001,021,120,201} {001,021,120,210}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 00, (011)^2$; $011 \rightsquigarrow 00, 011$	
	{001,021,100,201} {001,021,100,210} {001,100,101,110} {001,100,102,110} {001,100,110,201} {001,100,110,210}	$a_m \rightsquigarrow$ $(010)^m, 00, a_{m+1}$; $00 \rightsquigarrow 00$, where $a_m = 01 \cdots m$	
	{001,021,100,101} {001,021,100,102} {001,100,101,120} {001,100,102,120} {001,100,120,201} {001,100,120,210}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 010, 011, 01$; $011 \rightsquigarrow 010, 011$	
	{001,101,110,120} {001,102,110,120} {001,110,120,201} {001,110,120,210}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow (00)^2, 01$	
	{011,012,021,100}	$a_m \rightsquigarrow (01)^m, a_{m+1}$; $01 \rightsquigarrow 010$, where $a_m = 0^m$	$\frac{x(1+x^2)}{(1-x)^2}$
48	{000,011,021,100} {000,011,021,101} {000,011,021,110} {000,011,021,201} {000,011,021,210}	$0 \rightsquigarrow 0, 00$; $00 \rightsquigarrow 00, 002$; $002 \rightsquigarrow 002$	
	{001,100,101,210} {001,100,102,210}		

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{001,100,201,210}	$a_m \rightsquigarrow$ $(010)^m, b_m, a_{m+1}$; $b_m \rightsquigarrow (010)^m, b_m$, where $a_m = 01\cdots m$, $b_m = a_m m$	
	{001,101,102,110}		
	{001,101,110,201}		
	{001,101,110,210}		
	{001,102,110,201}		
	{001,102,110,210}		
	{001,110,201,210}	$a_m \rightsquigarrow a_{m+1}, (00)^m$; $00 \rightsquigarrow 00$, where $a_m = 0^m$,	
	{001,021,101,102}		
	{001,021,101,201}		
	{001,021,101,210}		
	{001,021,102,201}		
	{001,021,102,210}		
	{001,021,201,210}		
	{001,101,102,120}		
	{001,101,120,201}		
	{001,101,120,210}		
	{001,102,120,201}		
	{001,102,120,210}		
	{001,120,201,210}	$0 \rightsquigarrow 00, 01$; $00 \rightsquigarrow 00$; $01 \rightsquigarrow 00, 011, 01$; $011 \rightsquigarrow 00, 011$	
	{010,012,021,100}		
	{010,012,021,101}		
	{010,012,021,102}		
	{010,012,021,110}		
	{010,012,021,120}		
	{010,012,021,201}		
	{010,012,021,210}		
	{011,012,021,101}		
	{011,012,021,102}		
	{011,012,021,110}		
	{011,012,021,120}		
	{011,012,021,201}		
	{011,012,021,210}	$a_m \rightsquigarrow a_{m+1}, (01)^m$; $01 \rightsquigarrow 01$, where $a_m = 0^m$	$\frac{x(1-x+x^2)}{(1-x)^3}$

Continuation of Table 1

Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
50	{000,011,100,120} {000,011,101,120} {000,011,110,120} {000,011,120,201} {000,011,120,210}	$0 \rightsquigarrow 0, 01$; $01 \rightsquigarrow 0, 012$; $012 \rightsquigarrow 012$	
	{001,100,101,102} {001,100,101,201} {001,100,102,201}	$a_m \rightsquigarrow$ $a_{m+1}, b_1, \ldots, b_m, c_m$; $b_m \rightsquigarrow b_1, \ldots, b_{m-1}$; $c_m \rightsquigarrow b_1, \ldots, b_m, c_m$, where $a_m = 01 \cdots m$, $b_m = a_m(m-1)$, $c_m = a_m m$	
	{011,012,100,201}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow$ $010, b_1, \ldots, b_{m-1}$, where $a_m = 0^m$, $b_m = a_m m$	$\frac{x}{(1-x)(1-x-x^2)}$
51	{011,012,100,101} {011,012,100,102} {011,012,100,110} {011,012,100,120}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_1 \rightsquigarrow 010$; $b_2 \rightsquigarrow b_1^2$; $b_m \rightsquigarrow c_m, b_1, \ldots, b_{m-1}$; $c_m \rightsquigarrow$ $c_1, b_1, c_3, \ldots, c_{m-1}$, where $a_m = 0^m$, $b_m = a_m m$, $c_m = b_m 0$	$\frac{x(1-x+x^2+x^4)}{(1-x)^3}$
54	{001,101,102,210} {001,101,201,210} {001,102,201,210}	$a_m \rightsquigarrow b_0^m, b_m, a_{m+1}$; $b_m \rightsquigarrow b_0^m, b_m$, where $a_m = 01 \cdots m$, $b_m = a_m m$	
	{010,012,110,210}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow b_1^m$, where $a_m = 0^m$, $b_m = a_m m$	
	{010,012,100,210} {011,012,101,210}		

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{011,012,102,210} {011,012,110,210} {011,012,120,210}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow (0021)^{m-1}, b_m$, where $a_m = 0^m$, $b_m = a_m m$	$\frac{x(1-2x+2x^2)}{(1-x)^4}$
55	{010,012,100,101} {010,012,100,102} {010,012,100,120} {010,012,100,201}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow c_2, \ldots, c_m, b_m$; $c_m \rightsquigarrow c_2, \ldots, c_{m-1}$, where $a_m = 0^m$, $b_m = a_m m$, $c_m = b_m(m-1)$	
	{010,012,101,110} {010,012,102,110} {010,012,110,120} {010,012,110,201} {011,012,101,201} {011,012,102,201} {011,012,110,201} {011,012,120,201}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow b_1, b_1, \ldots, b_{m-1}$, where $a_m = 0^m$, $b_m = a_m m$	$\frac{x(1-3x+4x^2-x^3-x^4+x^5)}{(1-x)^2(1-x-x^2)}$
56	{010,012,101,210} {010,012,102,210} {010,012,120,210} {010,012,201,210}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow b_1^{m-1}, b_m$, where $a_m = 0^m$, $b_m = a_m m$	$\frac{x(1-3x+4x^2-2x^3+x^4)}{(1-x)^5}$
57	{000,011,100,101} {000,011,100,110} {000,011,100,201} {000,011,100,210} {000,011,101,110} {000,011,101,201} {000,011,101,210} {000,011,110,201} {000,011,110,210}		

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{000,011,201,210}	$0 \rightsquigarrow 0, 0$	
	{001,101,102,201}	$a_m \rightsquigarrow a_{m+1}, b_0, \dots, b_m$; $b_m \rightsquigarrow b_0, \dots, b_m$, where $a_m = 01 \cdots m$, $b_m = a_m m$	
	{010,011,021,100} {010,011,021,101} {010,011,021,102} {010,011,021,110} {010,011,021,120} {010,011,021,201} {010,011,021,210}	$a_m \rightsquigarrow a_{m+1}, b_1, \dots, b_m$; $b_m \rightsquigarrow b_1, \dots, b_m$, where $a_m = 0^m$, $b_m = a_m 1$	
	{010,012,101,102} {010,012,101,120} {010,012,101,201} {010,012,102,120} {010,012,102,201} {010,012,120,201} {011,012,101,102} {011,012,101,110} {011,012,101,120} {011,012,102,110} {011,012,102,120} {011,012,110,120}	$a_m \rightsquigarrow a_{m+1}, b_1, \dots b_m$; $b_m \rightsquigarrow b_1, \cdots, b_m$, where $a_m = 0^m$, $b_m = a_m m$	$\frac{x}{1-2x}$
61	{010,011,100,102} {010,011,101,102} {010,011,102,110}	$\overset{r}{\sim} \{010, 011, 102\}$ – Theorems 2-3	See [3]
62	{000,010,021,100} {000,010,021,101} {000,010,021,102} {000,010,021,110} {000,010,021,120} {000,010,021,201} {000,010,021,210}	$\overset{r}{\sim} \{000, 010, 021\}$ – Theorems 2-3	See [3]
63	{010,011,120,201}		

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{010,011,120,210}	Theorem 5	Theorem 5
64	{010,011,100,120} {010,011,101,120} {010,011,110,120}	$\stackrel{r}{\sim} \{010, 011, 120\}$ – Theorems 2-3	See [3]
66	{010,011,100,201} {010,011,100,210} {010,011,101,201} {010,011,101,210} {010,011,110,201} {010,011,110,210}	$\stackrel{r}{\sim} \{010, 011, 201\}$ – Theorems 2-3	See [3]
67	{010,011,100,101} {010,011,100,110} {010,011,101,110}	$\stackrel{r}{\sim} \{010, 011\}$ – Theorems 2-3	See [16]
68	{012,021,100,101}	$a_m \rightsquigarrow a_{m+1}, (01)^m$; $01 \rightsquigarrow 010, 01$, where $a_m = 0^m$	$\frac{x(1-x+2x^2)}{(1-x)^3}$
	{012,021,100,110} {012,021,101,110}	$a_m \rightsquigarrow a_{m+1}, (01)^m$; $01 \rightsquigarrow (010)^2$; $010 \rightsquigarrow 010$, where $a_m = 0^m$	
71	{012,021,100,102} {012,021,100,120} {012,021,100,201} {012,021,100,210}	$\stackrel{r}{\sim} \{012, 021, 100\}$ – Theorems 2-3	See [3]
	{012,021,101,102} {012,021,101,120} {012,021,101,201} {012,021,101,210}	$\stackrel{r}{\sim} \{012, 021, 101\}$ – Theorems 2-3	
	{012,021,102,110} {012,021,110,120} {012,021,110,201} {012,021,110,210}	$\stackrel{r}{\sim} \{012, 021, 110\}$ – Theorems 2-3	
74	{012,100,102,110}		

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{012,100,110,120}	$\overset{r}{\sim} \{012, 100, 110\}$ – Theorems 2-3	See [3]
77	{012,100,101,201}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow (010)^2, b_1, \ldots, b_{m-1}$, where $a_m = 0^m$, $b_m = a_m m$	$\frac{x(1-x+x^2+x^3)}{(1-x)^2(1-x-x^2)}$
	{012,101,110,201}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow c_1, \ldots, c_m, b_m$; $c_m \rightsquigarrow c_1 \ldots c_{m-1}$; $010 \rightsquigarrow 010$, where $a_m = 0^m$, $b_m = a_m m$, $c_m = a_m m(m-1)$	
80	{012,100,101,102} {012,100,101,120}	$\overset{r}{\sim} \{012, 100, 101\}$ – Theorems 2-3	See [3]
81	{011,021,100,102}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow 010, b_1, \ldots, b_m$, where $a_m = 0^m$, $b_m = a_m 1$	$\frac{x(1-x+x^2)}{(1-x)(1-2x)}$
	{012,101,102,110} {012,101,110,120}	$\overset{r}{\sim}= \{012, 101, 110\}$ – Theorems 2-3, $a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow c_m, b_1, \ldots, b_{m-1}c_1$; $c_m \rightsquigarrow c_m, b_1, \ldots, b_{m-1}$, where $a_m = 0^m$, $b_m = a_m m$, $c_m = b_m 0$	
84	{012,100,201,210}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow 010, (0021)^{m-1}, b_m$; $010 \rightsquigarrow 010$, where $a_m = 0^m$, $b_m = a_m m$	$\frac{x(3x^2-2x+1)}{(x-1)^4}$
	{012,110,201,210}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow b_1, (011)^m$; $011 \rightsquigarrow 011$, where $a_m = 0^m$, $b_m = a_m m$	
85	{000,021,100,102}	$\overset{r}{\sim} \{000, 021, 102\}$ – Theorems 2-3	See [3]
	{000,021,102,201}		

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{000,021,102,210}	$a_0 \rightsquigarrow b_0c_1$, $a_m \rightsquigarrow b_m, a_m, \ldots, a_1, 002$; $b_m \rightsquigarrow a_{m+1}, \ldots, a_1, 002$; $c_m \rightsquigarrow 0101, d_m, c_m, \ldots, c_2, 012$; $d_m \rightsquigarrow 0101, c_{m+1}, \ldots, c_2, 012$; $010 \rightsquigarrow 0101$; $012 \rightsquigarrow 0101d_1, 012$, where $a_m = 0^2 \cdots (m-1)^2 m$, $b_m = a_m m$, $c_m = 01^2 \cdots (m-1)^2 m$, $d_m = c_m m$	$(1-x-x^2-x^3-(1+x^2)\sqrt{1-2x-3x^2})/(2x^2)-x^2+x^4$
86	{012,100,102,210} {012,100,120,210}	$\overset{r}{\sim}$ $\{012, 100, 210\}$ – Theorem 2	See [3]
	{012,101,201,210}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow (010)^m, b_m$; $010 \rightsquigarrow 010$, where $a_m = 0^m$, $b_m = a_m m$	$\frac{x(1-3x+5x^2-3x^3+x^4)}{(1-x)^5}$
	{012,102,110,210} {012,110,120,210}	$\overset{r}{\sim}$ $\{012, 110, 210\}$ – Theorem 2	See [3]
87	{012,100,102,201} {012,100,120,201}	$\overset{r}{\sim}$ $\{012, 100, 201\}$ – Theorem 2	
	{012,102,110,201} {012,110,120,201}	$\overset{r}{\sim}$ $\{012, 110, 201\}$ – Theorem 2	See [3]
89	{012,101,102,210} {012,101,120,210}	$\overset{r}{\sim}$ $\{012, 101, 210\}$ – Theorem 2	See [3]
90	{011,021,100,120}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow b_1, \ldots, b_m, 012$; $012 \rightsquigarrow 012$, where $a_m = 0^m$, $b_m = 0^m1$	
	{011,021,101,102}		

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{011,021,102,110} {011,021,102,201} {011,021,102,210}	$\overset{r}{\sim}\{011, 021, 102\}$ – Theorem 2	See [3]
	{011,102,120,201}	Theorem 7	Theorem 7
	{011,102,120,210}	Theorem 7	Theorem 7
	{012,021,102,120} {012,021,102,201} {012,021,102,210} {012,021,120,201} {012,021,120,210} {012,021,201,210}	$\overset{r}{\sim}\{012, 021\}$ – Theorem 2	See [16]
	{012,101,102,201} {012,101,120,201}	$\overset{r}{\sim}\{012, 101, 201\}$ – Theorem 2	See [3]
	{012,102,110,120}	$\overset{r}{\sim}\{012, 110\}$ – Theorem 2	See [16], $\frac{x(1-2x+2x^2)}{(1-x)^2(1-2x)}$
91	{011,101,102,120} {011,102,110,120}	$\overset{r}{\sim}\{011, 102, 120\}$ – Theorem 2	See [3]
96	{000,021,101,120} {000,021,110,120}	$a_0 \rightsquigarrow b_0, 01$; $01 \rightsquigarrow a_1, b_0, 002$; $002 \rightsquigarrow b_0, 002$; $a_m \rightsquigarrow b_m, a_m, \ldots, a_1, 002$; $b_m \rightsquigarrow a_{m+1}, \ldots, a_1, 002$, where $a_m = 0^2 \cdots (m-1)^2 m$, $b_m = a_m m$	$((x-1)(x+2)\sqrt{1-2x-3x^2} + x^3 - 4x^2 - 3x + 2)/(2x^2)$
97	{011,100,101,102} {011,100,102,110}	$\overset{r}{\sim}\{011, 100, 102\}$ – Theorem 2	See [3]
102	{012,102,201,210} {012,120,201,210}	$\overset{r}{\sim}\{012, 201, 210\}$ – Theorem 2	See [3]
104	{000,021,100,120} {000,021,120,201} {000,021,120,210}	$\overset{r}{\sim}\{000, 021, 120\}$ – Theorems 2-3	

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
107	{012,102,120,201}	$\overset{r}{\sim}\{012,201\}$ – Theorem 2	
	{012,102,120,210}	$\overset{r}{\sim}\{012,210\}$ – Theorem 2	See [3]
108	{011,021,100,101} {011,021,100,110} {011,021,100,201} {011,021,100,210}	$\overset{r}{\sim}\{011,021,100\}$ – Theorems 2	See [3]
	{011,021,101,120} {011,021,110,120} {011,021,120,201} {011,021,120,210}	$\overset{r}{\sim}\{011,021,120\}$ – Theorems 2	See [3]
	{011,101,102,210} {011,102,110,210}	$\overset{r}{\sim}\{011,102,210\}$ – Theorems 2	See [3]
109	{011,101,102,201} {011,102,110,201}	$\overset{r}{\sim}\{011,102,201\}$ – Theorem 2	See [3]
110	{000,021,100,110} {000,021,110,201} {000,021,110,210}	$\overset{r}{\sim}\{000,021,110\}$ – Theorems 2 and 3	See [3]
114	{000,021,100,101} {000,021,101,201} {000,021,101,210}	$\overset{r}{\sim}\{000,021,101\}$ – Theorems 2-3	See [3]
115	{011,100,120,201} {011,100,120,210}	Theorem 8	Theorem 8
117	{011,100,101,120} {011,100,110,120}	$\overset{r}{\sim}\{011,100,120\}$ – Theorem 2	See [3]
124	{000,021,100,201} {000,021,100,210} {000,021,201,210}	$\overset{r}{\sim}\{000,021\}$ – Theorems 2-3	See [16]
127	{011,101,120,201} {011,110,120,201}	$\overset{r}{\sim}\{011,120,201\}$ – Theorem 2	
	{011,101,120,210}		

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{011,110,120,210}	$\overset{r}{\sim}\{011,120,210\}$ – Theorem 2	See [3]
128	{010,021,100,101}		
	{010,021,100,102}		
	{010,021,100,110}		
	{010,021,100,120}		
	{010,021,100,201}		
	{010,021,100,210}		
	{010,021,101,102}		
	{010,021,101,110}		
	{010,021,101,120}		
	{010,021,101,201}		
	{010,021,101,210}		
	{010,021,102,110}		
	{010,021,102,120}		
	{010,021,102,201}		
	{010,021,102,210}		
	{010,021,110,120}		
	{010,021,110,201}		
	{010,021,110,210}		
	{010,021,120,201}		
	{010,021,120,210}		
	{010,021,201,210}	$\overset{r}{\sim}\{010,021\}$ – Theorems 2-3	
	{011,021,101,110}		
	{011,021,101,201}		
	{011,021,101,210}		
	{011,021,110,201}		
	{011,021,110,210}		
	{011,021,201,210}	$\overset{r}{\sim}\{011,021\}$ – Theorems 2-3	See [16]
130	{011,100,101,201}		
	{011,100,110,201}	$\overset{r}{\sim}\{011,100,201\}$ – Theorem 2	
	{011,100,101,210}		
	{011,100,110,210}	$\overset{r}{\sim}\{011,100,210\}$ – Theorem 2	See [3]
133	{000,101,120,201}		
	{000,101,120,210}		See Subsection 4.6
152	{010,100,102,210}		
	{011,101,201,210}		

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{011,110,201,210}	$\overset{r}{\sim} \{011, 201, 210\}$ – Theorem 2	See [3], 152(1) $\overset{\mathbf{I}}{\sim}$ 152(2) still open
156	{010,100,101,120} {010,100,110,120}		 See Subsection 4.5
164	{010,100,120,201} {010,110,120,201}		 See Subsection 4.7
166	{010,100,110,201} {010,100,110,210}		
	{011,101,110,201}	$\overset{r}{\sim} \{011, 210\}$ – Theorem 2	See [16]
	{011,101,110,210}	$\overset{r}{\sim} \{011, 210\}$ – Theorem 2	See [16]
	{010,101,110,201}	$a_m \rightsquigarrow a_{m+1}, a_m,$ $b_{m,2}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow$ $a^2_{m+2-j}, b_{m+3-j,2}, \ldots,$ $b_{m,j-1}, b_{m,j}, \ldots, b_{m,m}$, where $a_m = 0^m$, $b_{m,j} = a_m j$	
	{010,100,120,210} {010,101,120,201} {010,110,120,210}	 $a_m \rightsquigarrow a_{m+1}, a_m,$ $b_{m,2}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow$ $a_{m+2-j}, a_{m+1-j},$ $b_{m+3-j,2}, \ldots, b_{m+1,j},$ $b_{m',2}, \ldots, b_{m',m'}$, where $m' = m + 1 - j$, $a_m = 0^m$, $b_{m,j} = a_m j$	
	{010,101,120,210}	$a_m \rightsquigarrow a_{m+1}, a_m,$ $b_{m,2}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow$ $a_{m+2-j}, a_{m+1-j},$ $c_{m+3-j,2}, \ldots, c_{m,j},$ $b_{m+1,j}, b_{m',2}, \ldots, b_{m',m'}$; $c_{m,j} \rightsquigarrow$ $c_{m+4-j,3}, \ldots, c_{m+1,j},$ $a_{m+3-j}, a_{m+2-j},$ $b_{m'+1,2}, \ldots, b_{m'+1,m'+1}$, where $m' = m + 1 - j$, $a_m = 0^m$, $b_{m,j} = a_m j$	166(7) $\overset{\mathbf{I}}{\sim}$ 166(9) holds by Subsection 4.8, 166(1) $\overset{\mathbf{I}}{\sim}$ 166(6) still open
168	{000,100,101,201} {000,100,110,210}		 Subsection 4.4 [14, 16]
169	{010,100,101,201}		

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{010,100,101,210}	Theorem 9	open
172	{010,100,201,210} {010,101,201,210}	$a_m \rightsquigarrow a_{m+1}, a_m, b_{m,2}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow (a_{m+2-j})^{j-1}, b_{m+1,j}, b_{m,j}, \ldots, b_{m,m}$, where $a_m = 0^m$, $b_{m,j} = a_m j$	Theorem 10
174	{021,100,102,120} {021,101,102,120}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow 010, b_{m+1}, c_1, \ldots, c_m$; $c_m \rightsquigarrow c_1, \ldots, c_{m+1}$; $010 \rightsquigarrow 010$, where $a_m = 0^m$, $b_m = a_m 1$, $c_m = a_m 12$	
	{021,102,110,120}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow 010, c_1, \ldots, c_m, 012$; $c_m \rightsquigarrow c_1, \ldots, c_{m+1}, 012$; $010 \rightsquigarrow 010, 0101$; $0101 \rightsquigarrow 0101$; $012 \rightsquigarrow c_1, 012$, where $a_m = 0^m$, $b_m = a_m 1$, $c_m = a_m 12$	$-\sqrt{1-4x}/(2x) - (2x^5 - 11x^4 + 16x^3 - 14x^2 + 6x - 1)/(2x(1-x)^4)$
178	{021,102,110,201} {021,102,110,210}	$\stackrel{r}{\sim} \{021, 102, 110\}$ – Theorem 2	See [3]
179	{021,102,120,201} {021,102,120,210}	$\stackrel{r}{\sim} \{021, 102, 120\}$ – Theorem 2	See [3]
180	{021,100,102,201} {021,100,102,210}	$\stackrel{r}{\sim} \{021, 100, 102\}$ – Theorem 2	See [3]
181	{021,101,102,201} {021,101,102,210}	$\stackrel{r}{\sim} \{021, 101, 102\}$ – Theorem 2	See [3]

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
183	{021,100,101,120}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow c_m, b_{m+1}, c_1, \ldots, c_m$; $c_m \rightsquigarrow c_1, \ldots, c_{m+1}$, where $a_m = 0^m$, $b_m = a_m 1$, $c_m = a_m 10$	
	{021,100,110,120} {021,101,110,120}	$a_m \rightsquigarrow a_{m+1}, b_1, \ldots, b_m$; $b_m \rightsquigarrow c_m, c_1, \ldots, c_m, 012$; $c_m \rightsquigarrow c_1, \ldots, c_{m+1}, 012$; $012 \rightsquigarrow 012, c_1$, where $a_m = 0^m$, $b_m = a_m 1$, $c_m = a_m 10$	$-(2-3x)\sqrt{1-4x}/(2x(1-x)) + (4x^2-7x+2)/(2x(1-x))$
189	{100,102,120,201}	Theorem 11	
	{102,110,120,201}	Theorem 11	Theorem 11
190	{021,100,110,201} {021,100,110,210}	$\overset{r}{\sim} \{021, 100, 110\}$– Theorem 2	See [3]
191	{021,100,120,201} {021,100,120,210}	$\overset{r}{\sim} \{021, 100, 120\}$ – Theorem 2	
	{021,101,120,201} {021,101,120,210}	$\overset{r}{\sim} \{021, 101, 120\}$ – Theorem 2	
	{021,110,120,201} {021,110,120,210}	$\overset{r}{\sim} \{021, 110, 120\}$ – Theorem 2	
	{100,102,120,210}	$\overset{r}{\sim} \{100, 102, 120\}$ – Theorem 2	
	{101,102,120,201} {101,102,120,210}	$\overset{r}{\sim} \{101, 102, 120\}$ – Theorem 2	
	{102,110,120,210}	$\overset{r}{\sim} \{102, 110, 120\}$ – Theorem 2	See [3]
196	{021,100,101,201} {021,100,101,210}	$\overset{r}{\sim} \{021, 100, 101\}$ – Theorem 2	
	{021,101,110,201}		

Continuation of Table 1			
Class	B	Rules of $\mathcal{T}'(B)$	$F_B(x)$
	{021,101,110,210}	$\overset{r}{\sim}\{021,100,110\}$ – Theorem 2	See [3]
203	{021,120,201,210}	$\overset{r}{\sim}\{021,120\}$ – Theorem 2	See [16]
	{101,102,201,210}		$\frac{1-4x+\sqrt{1-8x+20x^2-16x^3}}{2(1-x)(1-4x)}$, see [14, 16]
204	{021,100,201,210}	$\overset{r}{\sim}\{021,100\}$ – Theorem 2	
	{021,110,201,210}	$\overset{r}{\sim}\{021,110\}$ – Theorem 2	See [16]
206	{100,101,120,201} {100,101,120,210}	Theorem 13	
	{101,110,120,201}	Theorem 13	206(3) $\overset{\mathbf{I}}{\sim}$ 206(1), see Subsection 4.9
207	{100,101,110,201} {100,101,110,210}		Subsection 4.10
	{101,110,120,210}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow c_{m,j}, \ldots, c_{m',1}, a_{m'+1}, b_{m',1}, \ldots, b_{m',m'}$; $c_{m,j} \rightsquigarrow c_{m+1,j}, \ldots, c_{m'+1,1}, a_{m'+1}, b_{m',1}, \ldots, b_{m',m'}$, where $m' = m+1-j$, $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m j0$	207(1) $\overset{\mathbf{I}}{\sim}$ 207(3) still open
208	{100,110,120,201} {100,110,120,210}	$a_m \rightsquigarrow a_{m+1}, b_{m,1}, \ldots, b_{m,m}$; $b_{m,j} \rightsquigarrow a^2_{m'+1}, b_{m'+1,1}, \ldots, b_{m,j-1}, b_{m',1}, \ldots, b_{m',m'}$, where $m' = m+1-j$, $a_m = 0^m$, $b_{m,j} = a_m j$	Theorem 12
210	{100,101,201,210} {100,110,201,210} {101,110,201,210}		See [4, 14]
211	{100,120,201,210} {110,120,201,210}		See [14]
End of Table 1			

3. Proof of Theorem 1

Let B be any set of patterns and let $\mathcal{T}(B)$ be the generating tree for the class $\mathbf{I}_B$. The *length* of a node $v \in T(B)$ is defined to be the number of letters in v. For any $k \geq 1$, let $D_k(B)$ be the multiset of all nodes of length k at level $k-1$ in $\mathcal{T}(B)$. For each node $v \in D_k(B)$, we denote the multiset of all children of v at level k in $T(B)$ by $N_k(B;v)$. A generating tree $\mathcal{T}(B)$ is said to be *d-regular* (see [13]) if there exists $k \geq 1$ such that

- the number of different nodes in $D_r(B)$ equals d, for all $r > k$;
- for any $v \in D_r(B)$ and $w \in N_r(B;v)$, the number of occurrences of w in $N_r(B;v)$ does not depend on r, whenever $r > k$.

Clearly, $\mathcal{T}(B)$ is 0-regular if and only if the set of all nodes of the generating tree $\mathcal{T}(B)$ is finite. For instance, the generating tree

$$\begin{aligned} 0^m &\rightsquigarrow (0^{m+1})(01)\cdots(0^m1), \\ 0^m1 &\rightsquigarrow (010)(0021)^{m-1}(0^m1), \\ 010 &\rightsquigarrow 010 \end{aligned}$$

is 2-regular. But the generating tree

$$\begin{aligned} 0^m &\rightsquigarrow (0^{m+1})(0^m1)\cdots(0^m m), \\ 0^m j &\rightsquigarrow (0^{m+1})(0^m j)(0^m(j+1))\cdots(0^m m), \quad 1 \leq j \leq m \end{aligned}$$

and the generating tree

$$\begin{aligned} 0^m &\rightsquigarrow (0^{m+1})(01)\cdots(0^m1), \\ 0^m1 &\rightsquigarrow (0^m1)^m(01)^2 \end{aligned}$$

are not d-regular, for any d. Several examples of d-regular generating trees are presented in [3, 8, 13] and [13] contains a five-step algorithm to obtain an explicit formula for the generating function $F_B(x)$ from a d-regular generating tree $\mathcal{T}(B)$. Accordingly, in this paper, we omit the proof details for the generating function $F_B(x)$ whenever the generating tree $\mathcal{T}(B)$ is d-regular for some d. So, we only consider the classes C in Table 1 such that there exists $B \in C$ where $\mathcal{T}(B)$ is not d-regular, for any $0 \leq d \leq 6$.

Theorem 5 (Class 63). *We have* $\{010, 011, 120, 201\} \stackrel{\mathbf{I}}{\sim} \{010, 011, 120, 210\}$. *Moreover, we have*

$$F_{\{010,011,120,201\}}(x) = \frac{x}{1-x} - \sum_{j\geq 1} \frac{(1-x)(1-2x)x^{2j}\prod_{i=1}^{j}(x^i-2x+1)^2}{(x^{j+2}+1-2x)\prod_{i=2}^{j+1}(x^i+1-2x)\prod_{i=2}^{j+1}(x^i-(1-2x)^2)}.$$

Proof. Let $B = \{010, 011, 120, 201\}$ or $B = \{010, 011, 120, 210\}$. Then, the rules of the generating tree of $\mathcal{T}(B)$ are given by (here we used Algorithm KMY to guess and prove)

$$\begin{aligned} &a_m \rightsquigarrow a_{m+1} b_{m,1} \cdots b_{m,m}, \\ &b_{m,j} \rightsquigarrow b_{m,j-1} \cdots b_{m+2-j,1} b_{m+1-j,1} \cdots b_{m+1-j,m+1-j}, \end{aligned}$$

where $a_m = 0^m$ and $b_{m,j} = 0^m j$ with $j = 1, 2, \ldots, m$. Define $A_m(x)$ (respectively, $B_{m,j}(x)$) to be the generating function for the number of nodes at level $n \geq 1$ for the subtree of $\mathcal{T}(B; a_m)$ (respectively, $\mathcal{T}(B; b_{m,j})$), where its root stays at level 0. Thus,

$$\begin{aligned} A_m(x) &= x + x A_{m+1}(x) + x \sum_{j=1}^{m} B_{m,j}(x), \\ B_{m,j}(x) &= x + x \sum_{i=1}^{j-1} B_{m+i-j+1,i}(x) + x \sum_{i=1}^{m+1-j} B_{m+1-j,i}(x). \end{aligned}$$

Define $B_m(u) = \sum_{j=1}^{m} B_{m,j}(x) u^{m-j}$, $A(v) = \sum_{m \geq 1} A_m(x) v^{m-1}$, and $B(v, u) = \sum_{m \geq 1} B_m(u) v^{m-1}$. Then, the above recurrence can be written as

$$A(v) = \frac{x}{1-v} + \frac{x}{v}(A(v) - A(0)) + x B(v, 1), \quad (1)$$

$$B(v, u) = \frac{x}{(1-v)(1-uv)} + \frac{x}{u(1-v))}(B(v, u) - B(v, 0)) + \frac{x}{1-v} B(vu, 1). \quad (2)$$

By substituting $u = x/(1-v)$ into (2), we have

$$B(v, 0) = \frac{x}{1-v-xv} + \frac{x}{1-v} B(xv/(1-v), 1).$$

Thus, by substituting $u = 1$ into (2) and then solving for $B(v, 1)$, we have

$$B(v, 1) = \frac{x(1-x-v)}{(1-v)(1-v-vx)(1-2x-v)} - \frac{x^2}{(1-v)(1-2x-v)} B(xv/(1-v), 1).$$

By iterating this equation indefinitely, we obtain for $|x| < 1$,

$$B(v, 1) = -\sum_{j \geq 0} \frac{(1-x)(1-x-v) x^{2j+1} \prod_{i=0}^{j} (vx^i - v - x + 1)^2}{\prod_{i=1}^{j+2} (vx^i + 1 - v - x) \prod_{i=1}^{j+1} (vx^i - (1-2x)(1-x-v))}.$$

Hence, by taking $v = x$, (1) gives $A(0) = \frac{x}{1-x} + x B(x, 1)$, which completes the proof. □

Theorem 6 (Class 82). *We have*

$$F_{\{011,100,102,120\}}(x) = \frac{x(1-x-x^2)}{(1+x)(1-2x)}.$$

Proof. Let $B = \{011, 100, 102, 120\}$. Then, the rules of the generating tree of $\mathcal{T}(B)$ are given by

$$\begin{aligned}
&a_m \rightsquigarrow a_{m+1} b_{m,1} \cdots b_{m,m},\\
&b_{m,1} \rightsquigarrow c_{1,0} d_m b_{m-1,1} \cdots b_{m-1,m-1},\\
&b_{m,j} \rightsquigarrow c_{j,0} \cdots c_{j,j-2} c_{j,0} d_{m+1-j} b_{m-j,1} \cdots b_{m-j,m-j}, \quad j = 2, 3, \ldots, m-1,\\
&b_{m,m} \rightsquigarrow c_{m,0} \cdots c_{m,m-2} c_{m,0} d_1,\\
&c_{m,1} \rightsquigarrow c_{m-2,0} \cdots c_{m-2,m-4} c_{m-2,0},\\
&c_{m,j} \rightsquigarrow c_{j,0} \cdots c_{j,j-2} c_{j,0} c_{m-1-j,0} \cdots c_{m-1-j,m-3-j} c_{m-1-j,0}, \quad j = 1, 2, \ldots, m-3,\\
&c_{m,m-2} \rightsquigarrow c_{m-2,0} \cdots c_{m-2,m-4} c_{m-2,0} c_{1,0},\\
&d_m \rightsquigarrow d_m b_{m-1,1} \cdots b_{m-1,m-1},\\
&c_{1,0} \rightsquigarrow c_{1,0},
\end{aligned}$$

where $a_m = 0^m$, $b_{m,j} = 0^m j$ with $j = 1, 2, \ldots, m$, $c_{m,j} = 0^m j0$, and $d_m = 0^m 12$. Define $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_{m,j}(x)$, $D_m(x)$) to be the generating function for the number of nodes at level $n \geq 1$ for the subtree of $\mathcal{T}'(B; a_m)$ (respectively, $\mathcal{T}'(B; b_{m,j})$, $\mathcal{T}'(B; c_{m,j})$, $\mathcal{T}'(B; d_m)$), where its root stays at level 0. Thus,

$$\begin{aligned}
A_m(x) &= x + xA_{m+1}(x) + x \sum_{i=1}^{m} B_{m,i}(x),\\
B_{m,j}(x) &= x + x\left(C_{j,0} + \sum_{i=0}^{j-2} C_{j,i}(x)\right) + xD_{m+1-j}(x) + x \sum_{i=1}^{m-j} B_{m-j,i}(x),\\
C_{m,j}(x) &= x + xC_{1,0}(x)(\delta_{j=1} + \delta_{j=m-2}) + x\left(C_{m-1-j,0}(x) + \sum_{i=0}^{m-3-j} C_{m-1-j,i}(x)\right)\\
&\quad + x\left(C_{j,0}(x) + \sum_{i=0}^{j-2} C_{j,i}(x)\right),\\
D_m(x) &= x + xD_m(x) + x \sum_{i=1}^{m-1} B_{m-1,i}(x).
\end{aligned}$$

Define $A(v) = \sum_{m\geq 2} A_m(x) v^{m-1}$, $B(v,u) = \sum_{m\geq 1} \sum_{j=1}^{m} B_{m,j}(x) u^{j-1} v^{m-1}$, $C(v,u) = \sum_{m\geq 2} \sum_{j=0}^{m-2} C_{m,j}(x) u^j v^{m-1}$, and $D(v) = \sum_{m\geq 1} D_m(x) v^{m-1}$. Then, the above recurrence can be written as

$$A(v) = \frac{x}{1-v} + \frac{x}{v}(A(v) - A(0)) + xB(v,1), \tag{3}$$

$$D(v) = \frac{x}{1-v} + xD(v) + xvB(v,1), \tag{4}$$

$$\begin{aligned} B(v,u) &= \frac{x}{(1-v)(1-uv)} + \frac{x}{1-v}(C(uv,1) + C(uv,0)) \\ &\quad + \frac{x}{1-uv}D(v) + \frac{xv}{1-uv}B(v,1), \end{aligned} \tag{5}$$

$$\begin{aligned} C(v,u) &= \frac{xv}{(1-v)(1-uv)} + \frac{x^2v}{1-uv} \\ &\quad + \frac{x^2v^2u}{1-v} + \frac{xv}{1-uv}(C(v,1) + C(v,0)) + \frac{xv^2u}{1-v}(C(uv,1) + C(uv,0)). \end{aligned} \tag{6}$$

By taking either $u = 1$ or $u = 0$ into (6), we have a system of equations with variables $C(v,1)$ and $C(v,0)$. By solving this system, we have

$$C(v,1) = \frac{(x+1)xv}{(1-v)(1-x-2xv)}, \quad C(v,0) = \frac{(1+x-v-2vx)xv}{2v^2x + v^2 - 2vx - 2v + 1}.$$

By solving (4) for $D(v)$, we have

$$D(v) = \frac{x}{(1-v)(1-x)} + \frac{xv}{1-x}B(v,1).$$

By substituting expressions of $C(v,0)$, $C(v,1)$, and $D(v)$ into (5) with $u = 1$, and then solving for $B(v,1)$, we obtain

$$B(v,1) = \frac{x(2vx + x^2 + v - x - 1)}{(2v^2x + v^2 - 2vx - 2v + 1)(v + x - 1)}.$$

Then, by taking $u = x$ into (3) and using the expression of $B(v,1)$, we complete the proof. □

By similar arguments as in the proof of Theorem 6, we obtain the following result.

Theorem 7 (**Class 90**). *We have*

$$F_{\{011,102,120,201\}}(x) = F_{\{011,102,120,210\}}(x) = \frac{x(1-2x+2x^2)}{(1-x)^2(1-2x)}.$$

Moreover, the rules of the generating tree $\{011, 102, 120, 201\}$ *are given by*

$$
\begin{aligned}
&a_m \rightsquigarrow a_{m+1} b_{m,1} \cdots b_{m,m},\\
&b_{m,j} \rightsquigarrow d_1^j c_{m+1-j,2} \cdots c_{m+1-j,c+2-j}, \quad j = 1, 2,\\
&b_{m,j} \rightsquigarrow d_1^2 b_{1,1} d_4 \cdots d_j c_{m+1-j,2} \cdots c_{m+1-j,m+2-j}, \quad j = 3, 4, \ldots, m-1\\
&b_{m,m} \rightsquigarrow d_1^3 b_{1,1} d_4 \cdots d_m,\\
&c_{m,j} \rightsquigarrow e_3 \cdots e_j c_{m+2-j,2} \cdots c_{m+2-j,m+3-j}, \quad j = 2, 3, \ldots, m,\\
&c_{m,m+1} \rightsquigarrow e_3 \cdots e_{m+1} d_1,\\
&d_m \rightsquigarrow d_1^2 b_{1,1} d_4 \cdots b_{m-1},\\
&e_m \rightsquigarrow e_3 \cdots e_{m-1},
\end{aligned}
$$

where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m 1 j$, $d_m = a_m m(m-1)$, *and* $e_m = a_m 1 m(m-1)$. *The rules of the generating tree* $\{011, 102, 120, 210\}$ *are given by*

$$
\begin{aligned}
&a_m \rightsquigarrow a_{m+1} b_{m,1} \cdots b_{m,m},\\
&b_{m,j} \rightsquigarrow d_j e_2 \cdots e_j c_{m+1-j,2} \cdots c_{m+1-j,m+2-j}, \quad j = 1, 2, \ldots, m-1,\\
&b_{m,m} \rightsquigarrow d_m e_2 \cdots e_m d_1,\\
&c_{m,j} \rightsquigarrow e_2 \cdots e_{j-1} c_{m+2-j,2} \cdots c_{m+2-j,m+3-j}, \quad j = 2, 3, \ldots, m,\\
&c_{m,m+1} \rightsquigarrow e_2 \cdots e_m d_1,\\
&d_m \rightsquigarrow d_m e_2 \cdots e_m,\\
&e_m \rightsquigarrow e_2 \cdots e_{m-1},
\end{aligned}
$$

where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m 1 j$, $d_m = a_m m 0$, *and* $e_m = a_m m 1$.

Theorem 8 **(Class 115).** *We have* $\{011, 100, 120, 201\} \overset{\mathbf{I}}{\sim} \{011, 100, 120, 210\}$. *Moreover,*

$$
\begin{aligned}
&F_{\{011,100,120,201\}}(x)\\
&= \frac{x}{1-x} + \sum_{j\geq 0} \frac{x^{j+4}(1+x)^j}{(v_j + 2x - 1)(1 - v_j)^2} \prod_{i=0}^{j-1} \frac{1 - x - v_i}{(1 - v_i)(v_i + 2x - 1)(v_i x + v_i - 1)}
\end{aligned}
$$

with $v_j = \frac{(1-x)x^{j+1}}{1-2x+x^{j+1}}$.

Proof. The rules of the generating trees

$$\mathcal{T}(\{011, 100, 120, 201\}) \text{ and } \mathcal{T}(\{011, 100, 120, 210\})$$

are given by

$$\begin{aligned}
a_m &\rightsquigarrow a_{m+1}b_{m,1}\cdots b_{m,m},\\
b_{m,j} &\rightsquigarrow c_{m+1-j}c_{m-j}b_{m+2-j,1}\cdots b_{m,j-1}b_{m-j,1}\cdots b_{m-j,m-j},\\
b_{m,m} &\rightsquigarrow c_1b_{2,1}\cdots b_{m,m-1}(012),\\
c_m &\rightsquigarrow c_mb_{m,1}\cdots b_{m,m},\\
012 &\rightsquigarrow 012,
\end{aligned}$$

where $a_m = 0^m$, $b_{m,j} = a_mj$ with $j = 1, 2, \ldots, m$, and $c_m = a_m10$. Define $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_m(x)$) to be the generating function for the number of nodes at level $n \geq 1$ for the subtree of $\mathcal{T}(B; a_m)$ (respectively, $\mathcal{T}(B; b_{m,j})$, $\mathcal{T}(B; c_m)$), where its root stays at level 0. Thus,

$$\begin{aligned}
A_m(x) &= x + xA_{m+1} + x\sum_{j=1}^{m} B_{m,j}(x),\\
C_m(x) &= x + xC_m(x) + x\sum_{j=1}^{m} B_{m,j}(x),\\
B_{m,j}(x) &= x + xC_{m+1-j}(x) + xC_{m-j}(x) + x\sum_{i=1}^{m-j} B_{m-j,i}(x) + x\sum_{i=1}^{j-1} B_{m+1-j+i,i}(x),\\
B_{m,m}(x) &= x + xC_1(x) + x\sum_{i=1}^{m-1} B_{i+1,i}(x) + \frac{x^2}{1-x},
\end{aligned}$$

where $1 \leq j \leq m-1$. Define $F(v) = \sum_{m\geq 1} F_m(x)v^{m-1}$ where $F \in \{A, C\}$ and $B(v, u) = \sum_{m\geq 1}\sum_{j=1}^{m} B_{m,j}(x)u^{m-j}v^{m-1}$. Then, the above recurrence can be written as

$$\begin{aligned}
A(v) &= \frac{x}{1-v} + \frac{x}{v}(A(v) - A(0)) + xB(v, 1),\\
C(v) &= \frac{x}{1-v} + xC(v) + xB(v, 1),\\
B(v, u) &= \frac{x}{(1-v)(1-vu)} + \frac{x^2}{(1-x)(1-v)} + \frac{x}{1-v}(C(vu) + vuC(vu) + vuB(vu, 1))\\
&\quad + \frac{x}{u(1-v)}(B(v, u) - B(v, 0)).
\end{aligned}$$

By the third equation with $u = 1$, we obtain

$$B(v, 0) = -\frac{(v^2 + 2xv - 2v + 2x + 1)B(v, 1) - x^2 - x}{x(1-x)(1-v)}.$$

From the second equation, we express the function $C(v)$ in terms of $B(v, 1)$. Hence, the third equation with $u = vx/(1-x)$ gives

$$B(v, 1) = a(v)B(\frac{vx}{1-v}, 1) + b(v), \tag{7}$$

where $a(v) = \frac{x^3}{(v+2x-1)(1-v)^2}$ and $b(v) = -\frac{x(x+1)(x-1+v)}{(1-v)(v+2x-1)(vx+v-1)}$. By iterating (7) an infinite number of times (here we assumed $|x| < 1$), we have

$$B(v,1) = \sum_{j\geq 0} a\left(\frac{vx^j}{1-\frac{v(1-x^j)}{1-x}}\right)\prod_{i=0}^{j-1} b\left(\frac{vx^i}{1-\frac{v(1-x^i)}{1-x}}\right),$$

which, by the equation of $A(v)$ with $v = x$, implies $A(0) = \frac{x}{1-x} + xB(x,1)$, which completes the proof. □

Theorem 9 (**Class 169**)**.** *We have* $\{010, 100, 101, 201\} \overset{\mathbf{I}}{\sim} \{010, 100, 101, 210\}$.

Proof. Let $\mathcal{T}$ be the tree with a root a_1 satisfying the following rules:

$$\begin{aligned}
&a_m \rightsquigarrow a_{m+1}a_m b_{m,2}\cdots b_{m,m},\\
&b_{m,j} \rightsquigarrow c_{m+2-j,2}\cdots c_{m,j}b_{m+1,j}b_{m,j}\cdots b_{m,m}, \quad 2\leq j\leq m,\\
&c_{m,2} \rightsquigarrow a_m b_{m,2}\cdots b_{m,m},\\
&c_{m,j} \rightsquigarrow c_{m+3-j,2}\cdots c_{m,j-1}b_{m,j-1}\cdots b_{m,m}, \quad 3\leq j\leq m.
\end{aligned}$$

Then $\mathcal{T}(\{010, 100, 101, 201\})$ is given by $\mathcal{T}$ with $a_m = 0^m$, $b_{m,j} = a_m j$, and $c_{m,j} = a_m j(j-1)$, and the tree $\mathcal{T}(\{010, 100, 101, 210\})$ is given by $\mathcal{T}$ with $a_m = 0^m$, $b_{m,j} = a_m j$, and $c_{m,j} = a_m j1$. Hence, $\{010, 100, 101, 201\} \overset{\mathbf{I}}{\sim} \{010, 100, 101, 210\}$. □

Remark 2. Let $B = \{010, 100, 101, 201\}$. Define $A_m(x)$ (respectively, $B_{m,j}(x)$) to be the generating function for the number of nodes at level $n \geq 1$ for the subtree of $\mathcal{T}(B; a_m)$ (respectively, $\mathcal{T}(B; b_{m,j})$), where its root stays at level 0. Moreover, let $A(v) = \sum_{m\geq 1} A_m(x)v^{m-1}$, $B(v,u) = \sum_{m\geq 2}\sum_{j=2}^{m} B_{m,j}(x)u^{m-j}v^{m-2}$, and $C(v,u) = \sum_{m\geq 2}\sum_{j=2}^{m} C_{m,j}(x)u^{m-j}v^{m-2}$. Then, the rules of the generating tree $\mathcal{T}(B)$ in the proof of Theorem 9 imply

$$\begin{aligned}
A(v) &= \frac{x}{1-v} + \frac{x}{v}(A(v) - A(0)) + xA(v) + xvB(v,1),\\
B(v,u) &= \frac{x}{(1-v)(1-vu)} + \frac{x}{1-v}C(v,u) + \frac{x}{uv}(B(v,u) - B(v,0))\\
&\quad + \frac{x}{1-u}(B(v,u) - uB(uv,1)),\\
C(v,u) &= \frac{x}{(1-v)(1-vu)} + \frac{x}{uv}(A(uv) - A(0)) + xB(uv,1) - \frac{x}{u}B(v,0)\\
&\quad + \frac{x}{u(1-v)}(C(v,u) - C(v,0)) + \frac{x}{u(1-u)}(B(v,u) - uB(uv,1)).
\end{aligned}$$

Here, we failed to derive from these equations an explicit formula for $A(0)$.

Theorem 10 (**Class 172**)**.** *We have* $\{010, 100, 201, 210\} \overset{\mathbf{I}}{\sim} \{010, 101, 201, 210\}$. *Moreover,*

$$F_{\{010,100,201,210\}}(x) = \frac{K(x)}{1-K(x)},$$

where

$$K(x) = \frac{1}{4}(1+2x-x^2+(1-x)\sqrt{x^2-6x+1}) - \frac{1}{2\sqrt{2}}\sqrt{(x+1)(x^2-4x+1)\sqrt{x^2-6x+1}+x^4-6x^3+4x^2-6x+1}.$$

Proof. By Table 1 (Class 172), we see that the generating tree $\mathcal{T}(\{010, 100, 201, 210\})$ is the same as the generating tree $\mathcal{T}(\{010, 101, 201, 210\})$. Define $A_m(x)$ (respectively, $B_{m,j}(x)$) to be the generating function for the number of nodes at level $n \geq 1$ for the subtree of $\mathcal{T}(B; a_m)$ (respectively, $\mathcal{T}(B; b_{m,j})$), where its root stays at level 0. Thus,

$$A_m(x) = x + xA_{m+1} + xA_m(x) + x\sum_{j=1}^{m} B_{m,j}(x),$$

$$B_{m,j}(x) = x + (j-1)xA_{m+2-j}(x) + xB_{m+1;j}(x) + x\sum_{i=j}^{m} B_{m,i}(x),$$

where $2 \leq j \leq m$.

Define $A(v) = \sum_{m\geq 1} A_m(x)v^{m-1}$ and $B(v,u) = \sum_{m\geq 2}\sum_{j=2}^{m} B_{m,j}(x)u^{m-j}v^{m-2}$. Then, the above recurrence can be written as

$$A(v) = \frac{x}{1-v} + \frac{x}{v}(A(v)-A(0)) + xA(v) + xvB(v,1), \tag{8}$$

$$B(v,u) = \frac{x}{(1-v)(1-vu)} + \frac{x}{vu(1-v)^2}(A(vu)-A(0)) + \frac{x}{uv}(B(v,u)-B(v,0)) + \frac{x}{1-u}(B(v,u)-uB(vu,1)). \tag{9}$$

In order to solve (8)-(9), we **assume** that the generating functions $A(v)$ and $B(v,u)$ satisfy one extra equation

$$B(v,1) = \frac{A(v)}{1-v-\frac{1-x-\sqrt{1-6x+x^2}}{2}}. \tag{10}$$

Note that we guessed (10) by looking at the first terms of the generating function $A(v)/B(v,1)$. Hence, by (8) and (10), we have

$$\left(1-x-\frac{x}{v}-\frac{2xv}{1-2v+x+\sqrt{x^2-6x+1}}\right)A(v) = \frac{x}{1-v} - \frac{x}{v}A(0).$$

By taking $v = v_0 = K(x)$, we obtain $A(0) = \frac{v_0}{1-v_0}$, which, by using same equation, implies

$$A(v,1) = \frac{x(v_0-v)(1-2v+x+\sqrt{x^2-6x+1})}{(1-v)(1-v_0)((xv-v+x)\sqrt{x^2-6x+1}+(v+1)x^2+(1-2v)x+2v^2-v)}.$$

Thus, by (10), we have

$$B(v,1) = \frac{2x(v_0-v)}{(1-v)(1-v_0)((xv-v+x)\sqrt{x^2-6x+1}+(v+1)x^2+(1-2v)x+2v^2-v)}.$$

Now, by (9), we have

$$\left(1-\frac{x}{uv}-\frac{x}{1-u}\right)B(v,u) = \frac{x}{(1-v)(1-vu)}+\frac{x}{vu(1-v)^2}(A(vu)-A(0)) - \frac{x}{uv}B(v,0)-\frac{xu}{1-u}B(vu,1).$$

By taking $u=u_0=\frac{x+v-xv-\sqrt{(v-1)^2x^2-2vx+(1-2x)v^2}}{2v}$, we obtain

$$B(v,0) = \frac{vu_0}{(1-v)(1-vu_0)}+\frac{1}{(1-v)^2}(A(vu_0)-A(0))-\frac{vu_0^2}{1-u_0}B(vu_0,1),$$

which leads to

$$B(v,u) = \frac{\frac{x}{(1-v)(1-vu)}+\frac{x}{vu(1-v)^2}(A(vu)-A(0))-\frac{x}{uv}B(v,0)-\frac{xu}{1-u}B(vu,1)}{1-\frac{x}{uv}-\frac{x}{1-u}},$$

where we do not present the explicit expressions for $B(v,0)$ and $B(v,u)$ because they are too long.

Since $A(v)$ and $B(v,u)$ satisfy (8)-(10), this completes the proof. □

Theorem 11 (Class 189). *We have* $\{100,102,120,201\} \overset{\mathbf{I}}{\sim} \{102,110,120,201\}$. *Moreover,*

$$F_{\{100,102,120,201\}}(x) = \frac{1-3x-x^4-(1-x-2x^3-x^4)\sqrt{1-4x}+1)}{2x^2(2+x)(1-x)^2}.$$

Proof. We proceed by finding the generating function for each class.

(1) The rules of the generating trees $\mathcal{T}(\{100,102,120,201\})$ are given by

$$\begin{aligned}
&a_m \rightsquigarrow a_{m+1}b_{m,1}\cdots b_{m,m},\\
&b_{m,j} \rightsquigarrow c_1\cdots c_j b_{m+1,j}b_{m+1-j,1}\cdots b_{m+1-j,m+1-j}, \quad 1\le j\le m,\\
&c_m \rightsquigarrow d_2\cdots d_{m-1}c_1,\\
&d_m \rightsquigarrow d_2\cdots d_{m-1},
\end{aligned}$$

where $a_m=0^m$, $b_{m,j}=a_mj$ with $j=1,2,\ldots,m$, $c_m=a_mm(m-1)$, and $d_m=a_mm(m-1)(m-2)$. Define $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_m(x)$, $D_m(x)$) to be the generating function for the number of nodes at level $n\ge 1$ for the subtree of

$\mathcal{T}(B;a_m)$ (respectively, $\mathcal{T}(B;b_{m,j})$, $\mathcal{T}(B;c_m)$, $\mathcal{T}(B;d_m)$), where its root stays at level 0. Thus,

$$\begin{aligned}
A_m(x) &= x + xA_{m+1} + x\sum_{j=1}^{m} B_{m,j}(x),\\
B_{m,j}(x) &= x + x\sum_{i=1}^{j} C_i(x) + xB_{m+1,j}(x) + x\sum_{i=1}^{m+1-j} B_{m+1-j,i}(x),\\
C_m(x) &= x + xC_1(x) + x\sum_{i=2}^{m} D_i(x),\\
D_m(x) &= x + x\sum_{i=2}^{m-1} D_i(x),
\end{aligned}$$

where $1 \le j \le m$. Define $F(v) = \sum_{m\ge1} F_m(x)v^{m-1}$ where $F \in \{A, C\}$, $D(v) = \sum_{m\ge3} D_m(x)v^{m-2}$, $B(v,u) = \sum_{m\ge1}\sum_{j=1}^{m} B_{m,j}u^{m-j}v^{m-1}$. Then, by the fact that $C_1(x) = \frac{x}{1-x}$, the above recurrence can be written as

$$\begin{aligned}
A(v) &= \frac{x}{1-v} + \frac{x}{v}(A(v) - A(0)) + xB(v,1),\\
B(v,u) &= \frac{x(1+C(v))}{(1-v)(1-vu)} + \frac{x}{vu}(B(v,u) - B(v,0)) + \frac{x}{1-v}B(vu,1),\\
C(v) &= \frac{x}{(1-x)(1-v)} + \frac{xv}{1-v}D(v),\\
D(v) &= \frac{x}{1-v} + \frac{xv}{1-v}D(v).
\end{aligned}$$

Thus, $D(v) = \frac{x}{1-v-xv}$ and $C(v) = \frac{x(vx^2+v-1)}{(1-x)(1-v)(vx+v-1)}$. The equation of $B(v,u)$ with $u = x/v$ gives

$$B(v,0) = \frac{x(v^2x^2 + vx^3 - vx^2 - v^2 + 2v - 1)}{(1-v)^2(1-x)^2(vx+v-1)} + \frac{x}{1-v}B(x,1).$$

Therefore, by substituting expression of $B(v,0)$ into equation of $B(v,u)$ with $u = 1$ and $v = \frac{1-\sqrt{1-4x}}{2}$, we have

$$B(x,1) = \frac{x(x^2\sqrt{1-4x} - x^2 - 2)}{(x^4 - x^3 - 2x^2 + 3x - 1)\sqrt{1-4x} + x^4 + x^3 - 6x^2 + 5x - 1}.$$

Hence, by taking $v = \frac{1-\sqrt{1-4x}}{2}$ and solving for $A(0)$, we complete the proof.

(2) The rules of the generating trees $\mathcal{T}(\{102, 110, 120, 201\})$ are given by

$$\begin{aligned}
&a_m \rightsquigarrow a_{m+1}b_{m,1}\cdots b_{m,m},\\
&b_{m,j} \rightsquigarrow c_1\cdots c_j a_{m+2-j}b_{m+1-j,1}\cdots b_{m+1-j,m+1-j}, \quad 1\le j\le m,\\
&c_m \rightsquigarrow c_1(0101)^2 d_3\cdots d_m,\\
&d_m \rightsquigarrow (0101)^2 d_3\cdots d_m,\\
&0101 \rightsquigarrow 0101,
\end{aligned}$$

where $a_m = 0^m$, $b_{m,j} = a_m j$ with $j = 1, 2, \dots, m$, $c_m = a_m m(m-1)$, and $d_m = a_m m(m-1)(m-2)$. Define $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_m(x)$, $D_m(x)$) to be the generating function for the number of nodes at level $n \ge 1$ for the subtree of $\mathcal{T}(B; a_m)$ (respectively, $\mathcal{T}(B; b_{m,j})$, $\mathcal{T}(B; c_m)$, $\mathcal{T}(B; d_m)$), where its root stays at level 0. Define $B_m(x) = \sum_{j=1}^{m} B_{m,j}(x)$. Thus,

$$\begin{aligned}
A_m(x) &= x + xA_{m+1} + xB_m(x),\\
B_{m,j}(x) &= x + x\sum_{i=1}^{j} C_i(x) + xA_{m+2-j}(x) + xB_{m+1-j}(x),\\
C_m(x) &= x + \frac{2x^2}{1-x} + xC_1(x) + x\sum_{i=3}^{m} D_i(x),\\
D_m(x) &= x + \frac{2x^2}{1-x} + x\sum_{i=3}^{m-1} D_i(x),
\end{aligned}$$

where $1 \le j \le m$. Define $F(v) = \sum_{m\ge1} F_m(x)v^{m-1}$ where $F \in \{A, B, C\}$ and $D(v) = \sum_{m\ge3} D_m(x)v^{m-3}$. Then, by the fact that $C_1(x) = \frac{x}{(1-x)^2}$, the above recurrence can be written as

$$\begin{aligned}
A(v) &= \frac{x}{1-v} + \frac{x}{v}(A(v) - A(0)) + xB(v),\\
B(v) &= \frac{x}{(1-v)^2} + \frac{x}{(1-v)^2}C(v) + \frac{x}{v(1-v)}(A(v) - A(0)) + \frac{x}{1-v}B(v),\\
C(v) &= \frac{x}{(1-x)^2} + \frac{xv}{1-v}\left(1 + \frac{2x}{1-x} + \frac{x}{(1-x)^2}\right) + \frac{xv^2}{1-v}D(v),\\
D(v) &= \frac{x}{1-v} + \frac{2x^2}{(1-x)(1-v)} + \frac{xv^2}{1-v}D(v).
\end{aligned}$$

It is not hard to get explicit formulas for $D(v)$ and $C(v)$. Then by solving the second equation for $B(v)$ and substituting it into the first equation, we obtain

$$\begin{aligned}(v^2-v+x)A(v)+(v-1)xA(0)&\\ +\frac{xv(-2x^2+2x-1+(x^4+x^3+2x^2-5x+3)v)}{(vx+v-1)(1-x)^2(1-v)^2}&\\ +\frac{x(1-x)^2v^3(-(2x+3)+(1+x)v)}{(vx+v-1)(1-x)^2(1-v)^2}&=0.\end{aligned}$$

Hence, by taking $v=\frac{1-\sqrt{1-4x}}{2}$ and solving for $A(0)$, we complete the proof. □

Theorem 12 (Class 208). *We have* $\{100,110,120,201\}\stackrel{\mathbf{I}}{\sim}\{100,110,120,210\}$. *Moreover, the generating function* $F_{\{100,110,120,201\}}(x)$ *is given by*

$$\frac{\frac{x}{1-x}+x\sum_{j\geq0}\frac{x^{2j+1}(1-w_j-x)(2(1+x)w_j^2-w_j-x)\prod_{i=0}^{j-1}(x-w_i)}{(1-w_j(1+x))\prod_{i=0}^{j}(w_i^2+(x-1)w_i+x)\prod_{i=0}^{j}(1-2w_i)\prod_{i=0}^{j}(1-w_i)}}{1-2x\sum_{j\geq0}\frac{x^{2j+1}(1-w_j-x)\prod_{i=0}^{j-1}(x-w_i)}{\prod_{i=0}^{j}(w_i^2+(x-1)w_i+x)\prod_{i=0}^{j}(1-2w_i)\prod_{i=0}^{j-1}(1-w_i)}},$$

where $w_i=\frac{x^{i+1}}{1-x(1-x^i)/(1-x)}$ *for all* $i\geq0$.

Proof. By Table 1, we see that the generating tree $\mathcal{T}(\{100,110,120,201\})$ is the same as the generating tree $\mathcal{T}(\{100,110,120,210\})$. Thus, $\{100,110,120,201\}\stackrel{\mathbf{I}}{\sim}\{100,110,120,210\}$. Now fix $B=\{100,110,120,201\}$ (for example). Define $A_m(x)$ (respectively, $B_{m,j}(x)$) to be the generating function for the number of nodes at level $n\geq1$ for the subtree of $\mathcal{T}'(B;a_m)$ (respectively, $\mathcal{T}'(B;b_{m,j})$), where its root stays at level 0.

Define $A(v)=\sum_{m\geq1}A_m(x)v^{m-1}$ and $B(v,u)=\sum_{m\geq1}\sum_{j=1}^{m}B_m(x)v^{m-1}u^{m-j}$. Then, as before, the rules of the generating tree $\mathcal{T}(B)$ can be written as

$$A(v)=\frac{x}{1-v}+\frac{x}{v}(A(v)-A(0))+xB(v,1), \tag{11}$$

$$\begin{aligned}B(v,u)&=\frac{x}{(1-v)(1-uv)}+\frac{2x}{uv(1-v)}(A(uv)-A(0))\\&\quad+\frac{x}{u(1-v)}(B(v,u)-B(v,0))+\frac{x}{1-v}B(uv,1).\end{aligned} \tag{12}$$

By finding $A(v)$ from (11) and substituting it into (12) after replacing v by v/u, we obtain

$$\begin{aligned}B(v/u,u)&=\frac{x}{(1-v)^2}+\frac{2x}{(1-v)(v-x)}\left(\frac{x}{1-v}-A(0)+xB(v,1)\right)\\&\quad+\frac{x}{u-v}(B(v/u,u)-B(v/u,0))+\frac{xu}{u-v}B(v,1).\end{aligned} \tag{13}$$

By taking $u=v+x$, we have

$$B(v/(v+x),0)=\frac{x}{(1-v)^2}+\frac{2x}{(1-v)(v-x)}\left(\frac{x}{1-v}-A(0)+xB(v,1)\right)+(v+x)B(v,1)$$

which, by replacing v with $vx/(1-v)$, implies

$$B(v,0) = \frac{(2(1+x)v^2 - 2(2+x)v + 2)A(0) + vx - x}{(1-v)(1-2v)(1-v(1+x))} - \frac{xB(vx/(1-v),1)}{(1-v)(1-2v)}.$$

Thus, by setting $u = 1$ into (12) and using expression of $B(v, 0)$, we obtain

$$B(v,1) = -\frac{(v-x)x^2}{(v^2+vx-v+x)(1-2v)(1-v)}B(vx/(1-v),1) + \frac{2x(1-v-x)}{(v^2+vx-v+x)(1-2v)}A(0) + \frac{x(1-v-x)(2v^2x+2v^2-v-x)}{(1-v)(1-2v)(1-v(1+x))(v^2+vx-v+x)}.$$

By iterating this equation (here we assume that $|x| < 1$), we have

$$B(v,1) = 2A(0)\sum_{j\geq 0}\frac{x^{2j+1}(1-v_j-x)\prod_{i=0}^{j-1}(x-v_i)}{\prod_{i=0}^{j}(v_i^2+(x-1)v_i+x)\prod_{i=0}^{j}(1-2v_i)\prod_{i=0}^{j-1}(1-v_i)} + \sum_{j\geq 0}\frac{x^{2j+1}(1-v_j-x)(2(1+x)v_j^2-v_j-x)\prod_{i=0}^{j-1}(x-v_i)}{(1-v_j(1+x))\prod_{i=0}^{j}(v_i^2+(x-1)v_i+x)\prod_{i=0}^{j}(1-2v_i)\prod_{i=0}^{j}(1-v_i)}$$

where $v_i = \frac{vx^i}{1-v(1-x^i)/(1-x)}$ for all $i \geq 0$.

By (11) with $v = x$, we have that $A(0) = \frac{x}{1-x} + xB(x, 1)$, which completes the proof. $\square$

4. Left to Right Maximum and Combinatorial Arguments

In this section, we use bijections based on decompositions of restricted inversion sequences according to left to right maximum to finish the proof for the rest of Theorem 1. In order to do that, we need the following definitions and notation.

A *left to right maximum (LRmax)* in an inversion sequence $e = e_0e_1 \dots e_n$ is an entry e_i such that $e_i \geq e_j$ for all $j < i$. If $e_i = m$ we say m is the *value* of the LRmax. Thus for $e = 01102211$, the LRmax are e_0, e_1, e_2, e_4, e_5 with values $0, 1, 1, 2, 2$ respectively. So any inversion sequence $e \in \mathbf{I}_n$ can be decomposed uniquely as $m_1\pi_1 \cdots m_k\pi_k$ where $m_1, \dots, m_k$ are all of the LRmax entries in e; thus $0 \leq m_1 \leq \cdots \leq m_k \leq n$ and $m_i > \pi_i$ (entrywise). We call this the *WLRmax* decomposition of e (W for weakly).

Say $\mathbf{m_1} < \mathbf{m_2} < \cdots < \mathbf{m_k}$ are the first occurrences of the (distinct) LRmax values of an inversion sequence e. Then e can be decomposed as $\mathbf{m_1}\pi_\mathbf{1}\mathbf{m_2}\pi_\mathbf{2} \dots \mathbf{m_k}\pi_\mathbf{k}$

for some $k \geq 1$ with $\mathbf{m_i} \geq \pi_\mathbf{i}$ (entrywise) for $1 \leq i \leq k$. This is the *LRmax decomposition* of e. For example, 0 0 0③3 1 3④2⑨3 is an LRmax decomposition with the first occurrence of each left to right maximum value circled (except for the initial 0).

An inversion sequence e also has a unique decomposition as $e = 0^{r_1} m_2^{r_2} \pi_2 \dots m_k^{r_k} \pi_k$ with $r_i \geq 1$ for all i where $m_2, \dots, m_k$ are the first occurrences of the nonzero LR-max values and the first letter of π_i (if present) is less than m_i for $i = 2, \dots, k$. We call this the *strict LRmax decomposition* of e. For example, $00033429939 = 0^3 3^2 \pi_2 4^1 \pi_3 9^2 \pi_4$ with $\pi_2 = \epsilon, \pi_3 = 2, \pi_4 = 39$.

4.1. Class 30: $\{000, 010, 101, 120\} \overset{\mathbf{I}}{\sim} \{000, 010, 110, 120\}$

By the LRmax decomposition of any inversion sequence in either $\mathbf{I}_n(\{000, 010, 101, 120\})$ or $\mathbf{I}_n(\{000, 010, 110, 120\})$, we have the following lemma.

Lemma 1. *Suppose $e \in \mathbf{I}_n$ and $\mathbf{m_1}\pi_\mathbf{1}\mathbf{m_2}\pi_\mathbf{2} \dots \mathbf{m_k}\pi_\mathbf{k}$ is the LRmax decomposition of e. Then*

(1) $e \in \mathbf{I}_n(\{000, 010, 101, 120\})$ if and only if the following hold

- *(i) If a letter m_i appears in π_i, then it appears as the leftmost letter of π_i, for all $i = 1, 2, \dots, k$;*
- *(ii) $\mathbf{m_1}\pi_\mathbf{1} < \mathbf{m_2}\pi_\mathbf{2} < \dots < \mathbf{m_k}\pi_\mathbf{k}$;*
- *(iii) Each π_i avoids $\{000, 010, 101, 120\}$.*

(2) $e \in \mathbf{I}_n(\{000, 010, 110, 120\})$ if and only if the following hold

- *(i) If a letter m_i appears in π_i, then it appears as the rightmost letter of π_i, for all $i = 1, 2, \dots, k$;*
- *(ii) $\mathbf{m_1}\pi_\mathbf{1} < \mathbf{m_2}\pi_\mathbf{2} < \dots < \mathbf{m_k}\pi_\mathbf{k}$;*
- *(iii) Each π_i avoids $\{000, 010, 110, 120\}$.*

Now, we are ready to define a recursive bijection

$$f : \mathbf{I}_n(\{000, 010, 101, 120\}) \mapsto \mathbf{I}_n(\{000, 010, 110, 120\}).$$

We define $f(a) = a$, for any letter $0 \leq a \leq n$. For any inversion sequence $e \in \mathbf{I}_n(\{000, 010, 101, 120\})$ with LRmax decomposition $\mathbf{m_1}\pi_\mathbf{1} \cdots \mathbf{m_k}\pi_\mathbf{k}$, we define $f(e) = \mathbf{m_1}\beta_\mathbf{1} \cdots \mathbf{m_k}\beta_\mathbf{k}$, where

- if $\pi_i = m_i\pi_i'$, then β_i is defined to be $f(\pi_i')m_i$,
- otherwise, we define β_i as $f(\pi_i)$.

For example, if $e = 0022116655487$, then

$$f(e) = 0022f(11)66f(554)8f(7) = 0022116654587.$$

By Lemma 1, we have that $e \in \mathbf{I}_n(\{000, 010, 101, 120\})$ if and only if $f(e) \in \mathbf{I}_n(\{000, 010, 110, 120\})$.

4.2. Class 36: $\{000, 010, 100, 201\} \overset{\mathbf{I}}{\sim} \{000, 010, 100, 210\}$

By the WLRmax decomposition of an inversion sequence in either $\mathbf{I}_n(\{000, 010, 100, 201\})$ or $\mathbf{I}_n(\{000, 010, 100, 210\})$, we have the following lemma.

Lemma 2. *Suppose $e \in \mathbf{I}_n$ and $m_1\pi_1m_2\pi_2 \ldots m_k\pi_k$ is the WLRmax decomposition of e. Then*
(1) $e \in \mathbf{I}_n(\{000, 010, 100, 201\})$ if any only if

(i) Each letter m_i appears at most twice as a LRmax;

(ii) All the letters in $\pi_1\pi_2 \cdots \pi_k$ are distinct;

(iii) All the letters that are smaller than m_i in $\pi_i \cdots \pi_k$ form a decreasing sequence.

(2) $e \in \mathbf{I}_n(\{000, 010, 100, 210\})$ if and only if

(i) Each letter m_i appears at most twice as a LRmax;

(ii) All the letters in $\pi_1\pi_2 \cdots \pi_k$ are distinct;

(iii) All the letters smaller than m_i in $\pi_i \cdots \pi_k$ form an increasing sequence.

Now, we are ready to define a bijection

$$f : \mathbf{I}_n(\{000, 010, 100, 201\}) \mapsto \mathbf{I}_n(\{000, 010, 100, 210\}).$$

Let $e \in \mathbf{I}_n(\{000, 010, 100, 201\})$ with WLRmax decomposition $m_1\pi_1 \cdots m_k\pi_k$. We reorder the letters of e such that $m_1, \ldots, m_k$ stay in their positions, and reorder the letters of $\pi_1 \cdots \pi_k$ such that $\pi_i \cdots \pi_k$ with $i = 1, 2, \ldots, k$ forms an increasing sequence. The result is $f(e)$. For example, if $e = 00212548893$ then $f(e) = 00212538894$. Clearly, e avoids $\{000, 010, 100, 201\}$ if and only if $f(e)$ avoids $\{000, 010, 100, 210\}$.

4.3. Class 37: $\{000, 010, 101, 201\} \overset{\mathbf{I}}{\sim} \{000, 010, 101, 210\}$

Using the strict LRmax decomposition, we have the following characterization of inversion sequences avoiding $\{000, 010, 101, 201\}$ and $\{000, 010, 101, 210\}$, respectively.

Lemma 3. *Suppose $e \in \mathbf{I}_n$ and $0^{r_1} m_2^{r_2} \pi_2 \dots m_k^{r_k} \pi_k$ is the strict LRmax decomposition of e. Then $e \in \mathbf{I}_n(\{000, 010, 101, 201\})$ (resp., $e \in \mathbf{I}_n(\{000, 010, 101, 210\})$) if and only if*
(i) each letter in e appears at most twice;
(ii) if a letter x appears twice in e, then the two occurrences of x are adjacent;
(iii) the entries in $\pi_i \dots \pi_k$ that are less than m_i are weakly decreasing (resp., weakly increasing) for $i = 1, \dots, k$.

Here is the bijection. In the strict LRmax decomposition $0^{r_1} m_2^{r_2} \pi_2 \dots m_k^{r_k} \pi_k$ of $e \in \mathbf{I}_n(\{000, 010, 101, 201\})$, write $\pi_2 \dots \pi_k$ as a list $a_1^{s_1} a_2^{s_2} \dots a_\ell^{s_\ell}$ with superscripts $s_i = 2$ to indicate repeated entries (otherwise $s_i = 1$ and is omitted). For example, for

$$e = 0\,0\,\textcircled{2}\,2\,\textcircled{4}\,\underline{3}\,\textcircled{7}\,\underline{6\,6}\,\textcircled{10}\,\underline{9\,9\,8\,8\,5\,1} \in \mathbf{I}_{16}(\{000, 010, 101, 201\})$$

with the first occurrence of each nonzero LRmax circled and the π's underlined, we have $m_1 = 0$, $r_1 = 2$, $m_2 = 2$, $r_2 = 2$, $m_3 = 4$, $m_4 = 7$, $m_5 = 10$ and $\pi_2 = \epsilon$, $\pi_3 = 3$, $\pi_4 = 6\,6$, $\pi_5 = 9\,9\,8\,8\,5\,1$ and $a_1^{s_1} a_2^{s_2} \dots a_\ell^{s_\ell} = 3\,6^2\,9^2\,8^2\,5\,1$. In the expression $a_1^{s_1} a_2^{s_2} \dots a_\ell^{s_\ell}$, arrange $a_1, \dots, a_\ell$ in increasing order while keeping the superscripts frozen in place. The example yields $1\,3^2\,5^2\,6^2\,8\,9$. Replace each "2" superscript with a duplicate entry to get $1\,3\,3\,5\,5\,6\,6\,8\,9$ and split this list into a new set of π's, say $(\pi_i')_{i=2}^k$ with π_i' of the same length as π_i for each i. Then the desired inversion sequence in $\mathbf{I}_n(\{000, 010, 101, 210\})$ has strict LRmax decomposition $0^{r_1} m_2^{r_2} \pi_2' \dots m_k^{r_k} \pi_k'$. In the example, we get

$$0\,0\,\textcircled{2}\,2\,\textcircled{4}\,\underline{1}\,\textcircled{7}\,\underline{3\,3}\,\textcircled{10}\,\underline{5\,5\,6\,6\,8\,9} \in \mathbf{I}_{16}(\{000, 010, 101, 210\}) \tag{14}$$

with the first nonzero LRmax entries still circled and each π_i' underlined for clarity.

To reverse the map, list the nonempty π's as boxed entries with m_i marked above the sequence of boxes for π_i, so that each box is associated with an m_i. The preceding example (14) gives

$$\overset{4}{\boxed{1}} \qquad \overset{7}{\boxed{3}\boxed{3}} \qquad \overset{10}{\boxed{5}\boxed{5}\boxed{6}\boxed{6}\boxed{8}\boxed{9}}.$$

Now form a set S of the distinct letters in $\pi_2 \dots \pi_k$, here $S = \{1, 3, 5, 6, 8, 9\}$, and erase the contents of the boxes, within an "x" is inserted in each box that contained the second occurrence of a repeated letter:

$$\overset{4}{\boxed{}} \qquad \overset{7}{\boxed{}\boxed{\text{x}}} \qquad \overset{10}{\boxed{}\boxed{\text{x}}\boxed{}\boxed{\text{x}}\boxed{}\boxed{}}.$$

Next, fill in the blank boxes left to right in turn with the letters of S using the largest available letter that is less than the m_i associated with the box. The example yields

$$\overset{4}{\boxed{3}} \qquad \overset{7}{\boxed{6}\boxed{\text{x}}} \qquad \overset{10}{\boxed{9}\boxed{\text{x}}\boxed{8}\boxed{\text{x}}\boxed{5}\boxed{1}}.$$

Lastly, replace each "x" with the letter immediately to its left. This yields the π_i sequence of the original inversion sequence.

4.4. Class 168: $\{010, 100, 101, 201\} \overset{\mathbf{I}}{\sim} \{010, 100, 101, 210\}$

This case is very similar to Case 37, see Subsection 4.3. Using the strict LRmax decomposition, we have the following characterization of inversion sequences avoiding $\{010, 100, 101, 201\}$ and $\{010, 100, 101, 210\}$, respectively.

Lemma 4. *Suppose* $e \in \mathbf{I}_n$ *and* $0^{r_1} m_2^{r_2} \pi_2 \dots m_k^{r_k} \pi_k$ *is the strict LRmax decomposition of* e. *Then* $e \in \mathbf{I}_n(\{010, 100, 101, 201\})$ (*respectively,* $e \in \mathbf{I}_n(\{010, 100, 101, 210\})$) *if and only if*
(i) *the letters in* $\pi_2 \dots \pi_k$ *are all distinct;*
(ii) *no* m_i *appears as a letter in* $\pi_2 \dots \pi_k$*;*
(iii) *the entries in the concatenation* $\pi_i \dots \pi_k$ *that are less than* m_i *are decreasing* (*respectively, increasing*) *for* $i = 1, \dots, k$.

Here is the bijection. In the strict LRmax decomposition $0^{r_1} m_2^{r_2} \pi_2 \dots m_k^{r_k} \pi_k$ of $e \in \mathbf{I}_n(\{010, 100, 101, 201\})$, rearrange (if necessary) the letters of $\pi_2 \dots \pi_k$ so that they are increasing. For example, with the π_i's underlined and the first occurrence of each nonzero LRmax circled,

$$\begin{aligned} &0\,0\,0\,\textcircled{3}\,3\,3\,3\,\underline{2}\,\textcircled{7}\,7\,\underline{6}\,\textcircled{8}\,\underline{5\,4\,1} \in \mathbf{I}_{15}(\{010, 100, 101, 201\}) \\ &\quad\mapsto 0\,0\,0\,\textcircled{3}\,3\,3\,3\,\underline{1}\,\textcircled{7}\,7\,\underline{2}\,\textcircled{8}\,\underline{4\,5\,6} \in \mathbf{I}_{15}(\{010, 100, 101, 210\})\,. \end{aligned}$$

The inverse mapping is obtained much as in Case 37 (but without the complication of repeated entries).

4.5. Class 156: $\{010, 100, 101, 120\} \overset{\mathbf{I}}{\sim} \{010, 100, 110, 120\}$

Lemma 5. *Suppose* $e \in \mathbf{I}_n$ *avoids* 100 *and* b *starts a* 110 *or* 101 *pattern in* e. *Then* b *is a left to right maximum in* e.

Proof. If b starts a 110 in e but is not a left to right maximum, then there is a $c > b$ and an $a < b$ such that $cbba$ is a subsequence of e. But then cbb is a forbidden 100. Similarly for 101. □

Lemma 6. *Suppose* $e \in \mathbf{I}_n(\{010, 100, 120\})$ *and* $\mathbf{m_1}\pi_\mathbf{1}\mathbf{m_2}\pi_\mathbf{2} \dots \mathbf{m_k}\pi_\mathbf{k}$ *is the LRmax decomposition of* e. *Then*
(i) $\mathbf{m_1}\pi_\mathbf{1} < \mathbf{m_2}\pi_\mathbf{2} < \cdots < \mathbf{m_k}\pi_\mathbf{k}$ (*entrywise*).
(ii) *If* e *avoids* 101*, then all occurrences of* $\mathbf{m_i}$ *in* e *occur as a single run at the start of* $\mathbf{m_i}\pi_\mathbf{i}$, $\mathbf{1} \le \mathbf{i} \le \mathbf{k}$.
(iii) *If* e *avoids* 110*, then all but the first occurrence of* $\mathbf{m_i}$ *occur as a single* (*possibly vacuous*) *run at the end of* π_i, $1 \le i \le k$.

Proof. (i) Certainly, $\mathbf{m_i}\pi_\mathbf{i} < \mathbf{m_{i+1}}$. If $a \leq \mathbf{m_i}$ for some entry a in π_{i+1}, then $\mathbf{m_i m_{i+1} a}$ is a 120 if $a < \mathbf{m_i}$ and a 010 if $a = \mathbf{m_i}$. Parts (ii) and (iii) are clear. □

In view of Lemmas 5 and 6, the following is a bijection from $\mathbf{I}_n(\{010, 100, 101, 120\})$ to $\mathbf{I}_n(\{010, 100, 110, 120\})$: in the LRmax decomposition $0\pi_1\mathbf{m_2}\pi_\mathbf{2}\ldots\mathbf{m_k}\pi_\mathbf{k}$ of $e \in \mathbf{I}_n(\{010, 100, 101, 120\})$, for each $i = 2, \ldots, k$, move all occurrences of $\mathbf{m_i}$ in π_i from the start of π_i to the end. For example,

$$\begin{array}{lll} 0\,0\,0\;\;3\;\;3\,3\,2\,1\;\;6\;\;4\,5\;\;8\;\;8\,7\;\;9\;\;9\;\;10 & = & \\ 0\,0\,0\,\textcircled{3}\,3\,3\,2\,1\,\textcircled{6}\,4\,5\,\textcircled{8}\,8\,7\,\textcircled{9}\,9\,\textcircled{10} & \to & 0\,0\,0\,\textcircled{3}\,2\,1\,3\,3\,\textcircled{6}\,4\,5\,\textcircled{8}\,7\,8\,\textcircled{9}\,9\,\textcircled{10}. \end{array}$$

To reverse the map, move all occurrences of $\mathbf{m_i}$ in π_i from the end of π_i to the start.

4.6. Class 133: $\{000, 101, 120, 201\} \overset{\mathbf{I}}{\sim} \{000, 101, 120, 210\}$

We have the following characterizations whose straightforward proofs are left to the reader.

Lemma 7. *Suppose $e \in \mathbf{I}_n$ and $\mathbf{m_1}\pi_\mathbf{1}\mathbf{m_2}\pi_\mathbf{2}\ldots\mathbf{m_k}\pi_\mathbf{k}$ is the LRmax decomposition of e. Then $e \in \mathbf{I}_n(\{000, 101, 120, 201\})$ if and only if*
(i) $\mathbf{m_i} \leq \pi_\mathbf{i+1}$ for $1 \leq i \leq k-1$;
(ii) each π_i is weakly decreasing;
(iii) no entry occurs three or more times;
(iv) if $e_i = m$ is both a LRmax and a descent top, then e_i is the last occurrence of m in e.

Lemma 8. *Suppose $e \in \mathbf{I}_n$ and $\mathbf{m_1}\pi_\mathbf{1}\mathbf{m_2}\pi_\mathbf{2}\ldots\mathbf{m_k}\pi_\mathbf{k}$ is the LRmax decomposition of e. Then $e \in \mathbf{I}_n(\{000, 101, 120, 210\})$ if and only if*
(i) $\mathbf{m_i} \leq \pi_\mathbf{i+1}$ for $1 \leq i \leq k-1$;
(ii) for each i, if π_i has a descent, then it has only one, it occurs right at the start and the descent top is the second occurrence of $\mathbf{m_i}$;
(iii) no entry occurs three or more times;
(iv) if $e_i = m$ is both a LRmax and a descent top, then e_i is the last occurrence of m in e.

Here is the bijection, with an obvious inverse. Given $e \in \mathbf{I}_n(\{000, 101, 120, 201\})$ with LRmax decomposition $\mathbf{m_1}\pi_\mathbf{1}\mathbf{m_2}\pi_\mathbf{2}\ldots\mathbf{m_k}\pi_\mathbf{k}$, for each $i \geq 2$, reverse π_i unless the first entry of π is the second occurrence of $\mathbf{m_i}$, in which case leave this entry intact and reverse the rest of π_i. For example,

$$\begin{array}{lll} 0\,0\;\;1\;\;1\;\;2\;\;5\;\;4\,4\,3\;\;9\;\;9\,8\,8\,7\,6\,6 & = & \\ 0\,0\,\textcircled{1}\,1\,\textcircled{2}\,\textcircled{5}\,4\,4\,3\,\textcircled{9}\,9\,8\,8\,7\,6\,6 & \to & 0\,0\,\textcircled{1}\,1\,\textcircled{2}\,\textcircled{5}\,3\,4\,4\,\textcircled{9}\,9\,6\,6\,7\,8\,8. \end{array}$$

4.7. Class 164: $\{010, 100, 120, 201\} \overset{\mathbf{I}}{\sim} \{010, 110, 120, 201\}$

We have the following characterizations.

Lemma 9. *Suppose* $e \in \mathbf{I}_n$ *and* $\mathbf{m_1}\pi_1\mathbf{m_2}\pi_2 \dots \mathbf{m_k}\pi_k$ *is the LRmax decomposition of* e*. Then* $e \in \mathbf{I}_n(\{010, 100, 120, 201\})$ *if and only if*
(i) $\mathbf{m_i} < \pi_{\mathbf{i+1}}$ *for* $1 \le i \le k-1$*, and*
(ii) *each* $\mathbf{m_i}\pi_\mathbf{i}$ *starts with one or more occurrences of* $\mathbf{m_i}$ *and thereafter is decreasing, except that it may end with zero or more occurrences of* m_i*.*

Proof. If condition (i) is not met, then either $a = m_i$ occurs in π_{i+1} and $m_i m_{i+1} a$ is a 010, or $a < m_i$ occurs in π_{i+1} and $m_i m_{i+1} a$ is a 120. If $e \in \mathbf{I}_n(\{010, 100, 120, 201\})$ and $m_i = a$ say, then the entries other than a in π_i are decreasing for otherwise, m_i starts a 100 or a 201, and all occurrences of a in π_i are at the start or the end for else a is the "2" of a 120. Thus condition (ii) holds. We leave the reader to show the converse: that if the two conditions hold, then $e \in \mathbf{I}_n(\{010, 100, 120, 201\})$. □

The next lemma has an analogous proof.

Lemma 10. *Suppose* $e \in \mathbf{I}_n$ *and* $\mathbf{m_1}\pi_1\mathbf{m_2}\pi_2 \dots \mathbf{m_k}\pi_k$ *is the LRmax decomposition of* e*. Then* $e \in \mathbf{I}_n(\{010, 110, 120, 201\})$ *if and only if*
(i) $\mathbf{m_i} < \pi_{\mathbf{i+1}}$ *for* $1 \le i \le k-1$*, and*
(ii) *each* π_i *has the form* $m_i^r w\, m_i^t$ *with* $r, t \ge 0$*, where* $m_i > w$ *and* w *is decreasing except that the last letter of* w *may be repeated indefinitely.*

Here is the bijection, with an obvious inverse. Suppose $e \in \mathbf{I}_n(\{010, 100, 120, 201\})$ with LRmax decomposition $\mathbf{m_1}\pi_1\mathbf{m_2}\pi_2 \dots \mathbf{m_k}\pi_k$. By Lemma 9, for each $i \ge 2$, π_i has the form $m_i^r u_1 \dots u_s m_i^t$ with $m_i > u_1 > \dots > u_s$ for some $r, s, t \ge 0$. Replace π_i with $u_1 \dots u_{s-1} u_s^{r+1} m_i^t$ if $s \ge 1$ and leave π unchanged if $s = 0$. For example,

$$0\,0\,0\,\textcircled{3}\,3\,3\,2\,1\,3\,3\,\textcircled{7}\,7\,6\,5\,4\,7\,\textcircled{8}\,\textcircled{10}\,10\,9\,10 \\ \to 0\,0\,0\,\textcircled{3}\,2\,1\,1\,1\,3\,3\,\textcircled{7}\,6\,5\,4\,4\,7\,\textcircled{8}\,\textcircled{10}\;9\,9\,10\,.$$

4.8. Class 166: $\{010, 101, 120, 201\} \overset{\mathbf{I}}{\sim} \{010, 101, 120, 210\}$

We have the following characterizations whose straightforward proofs are left to the reader.

Lemma 11. *Suppose* $e \in \mathbf{I}_n$ *and* $\mathbf{m_1}\pi_1\mathbf{m_2}\pi_2 \dots \mathbf{m_k}\pi_k$ *is the LRmax decomposition of* e*. Then* $e \in \mathbf{I}_n(\{010, 101, 120, 201\})$ *if and only if*
(i) *for* $1 \le i \le k-1$*, we have* $\mathbf{m_i} < \pi_{\mathbf{i+1}}$*, and*
(ii) *each* $\mathbf{m_i}\pi_\mathbf{i}$ *is weakly decreasing.*

Lemma 12. *Suppose* $e \in \mathbf{I}_n$ *and* $\mathbf{m_1}\pi_1\mathbf{m_2}\pi_2 \dots \mathbf{m_k}\pi_k$ *is the LRmax decomposition of* e*. Then* $e \in \mathbf{I}_n(\{010, 101, 120, 201\})$ *if and only if*

(i) for $1 \leq i \leq k-1$*, we have* $\mathbf{m_i} < \pi_{\mathbf{i+1}}$*, and*
(ii) for each $i \geq 2$*, all occurences of* $b := \mathbf{m_i}$ *in* π_i *occur at the start and the rest of* π_i *is weakly increasing.*

Here is the bijection, with an obvious inverse. For each π_i, set $b = \mathbf{m_i}$, leave all occurences of b at the start of π_i intact, and reverse the remainder of π_i to change it from weakly decreasing to weakly increasing. For example, with the first occurrence of each noninitial left-to-right maximum circled,

$$0\,0\,0\,ⓒ3\,3\,2\,2\,2\,1\,ⓒ6\,5\,4\,4 \to 0\,0\,0\,ⓒ3\,3\,1\,2\,2\,2\,ⓒ6\,4\,4\,5\,.$$

4.9. Class 206: $\{101, 100, 120, 201\} \overset{\mathbf{I}}{\sim} \{101, 110, 120, 201\}$

By finding the generating trees for all pairs in Class 206 (see Table 1), we can state the following result.

Theorem 13 **(Class 206).** *We have*
(1) *The rules of the generating tree* $\mathcal{T}(\{100, 101, 120, 201\})$ *are*

$$\begin{aligned} a_m &\rightsquigarrow a_{m+1} b_{m,1} \cdots b_{m,m}, \\ b_{m,j} &\rightsquigarrow a_{m+1-j} c_{m+2-j,2} \cdots c_{m,j} b_{m+1,j} b_{m+1-j,1} \cdots b_{m+1-j,m+1-j}, \quad 1 \leq j \leq m, \\ c_{m,j} &\rightsquigarrow a^2_{m+2-j} c_{m+3-j,2} \cdots c_{m,j-1} b_{m+1-j,1} \cdots b_{m+1-j,m+1-j}, \quad 2 \leq j \leq m, \end{aligned}$$

where $a_m = 0^m$*,* $b_{m,j} = a_m j$*, and* $c_{m,j} = a_m j(j-1)$*.*
(2) *The rules of the generating tree* $\mathcal{T}(\{100, 101, 120, 210\})$ *are*

$$\begin{aligned} a_m &\rightsquigarrow a_{m+1} b_{m,1} \cdots b_{m,m}, \\ b_{m,j} &\rightsquigarrow a_{m+1-j} c_{m+2-j,2} \cdots c_{m,j} b_{m+1,j} b_{m+1-j,1} \cdots b_{m+1-j,m+1-j}, \quad 1 \leq j \leq m, \\ c_{m,j} &\rightsquigarrow a^2_{m+2-j} c_{m+3-j,2} \cdots c_{m,j-1} b_{m+1-j,1} \cdots b_{m+1-j,m+1-j}, \quad 2 \leq j \leq m, \end{aligned}$$

where $a_m = 0^m$*,* $b_{m,j} = a_m j$*, and* $c_{m,j} = a_m j0$*.*
(3) *The rules of the generating tree* $\mathcal{T}(\{101, 110, 120, 201\})$ *are*

$$\begin{aligned} a_m &\rightsquigarrow a_{m+1} b_{m,1} \cdots b_{m,m}, \\ b_{m,j} &\rightsquigarrow a_{m+2-j} c_{m+1-j,1} \cdots c_{m,j} b_{m+1-j,1} \cdots b_{m+1-j,m+1-j}, \quad 1 \leq j \leq m, \\ c_{m,j} &\rightsquigarrow c_{m+2-j,1} \cdots c_{m-1,j-1} c_{m+2-j,1} a_{m+2-j} b_{m+1-j,1} \cdots b_{m+1-j,m+1-j}, \\ &\qquad 1 \leq j \leq m, \end{aligned}$$

where $a_m = 0^m$*,* $b_{m,j} = a_m j$*, and* $c_{m,j} = a_m j(j-1)$*.*

We have the following characterizations whose straightforward proofs are left to the reader.

Lemma 13. *Suppose* $e \in \mathbf{I}_n$ *and* $\mathbf{m_1}\pi_\mathbf{1}\mathbf{m_2}\pi_\mathbf{2}\dots\mathbf{m_k}\pi_\mathbf{k}$ *is the LRmax decomposition of* e*. Then* $e \in \mathbf{I}_n(\{101, 100, 120, 201\})$ *if and only if*
(i) *for* $1 \le i \le k-1$*, we have* $\mathbf{m_i} \le \pi_\mathbf{i+1}$ *and, further,* $\mathbf{m_i} < \pi_\mathbf{i+1}$ *unless every entry of* π_i *is equal to* m_i*, and*
(ii) *each* $\mathbf{m_i}\pi_\mathbf{i}$ *starts with one or more occurrences of* $\mathbf{m_i}$ *and thereafter is strictly decreasing.*

Lemma 14. *Suppose* $e \in \mathbf{I}_n$ *and* $\mathbf{m_1}\pi_\mathbf{1}\mathbf{m_2}\pi_\mathbf{2}\dots\mathbf{m_k}\pi_\mathbf{k}$ *is the LRmax decomposition of* e*. Then* $e \in \mathbf{I}_n(\{101, 110, 120, 201\})$ *if and only if*
(i) *for* $1 \le i \le k-1$*, we have* $\mathbf{m_i} \le \pi_\mathbf{i+1}$ *and, further,* $\mathbf{m_i} < \pi_\mathbf{i+1}$ *unless every entry of* π_i *is equal to* m_i*, and*
(ii) *each* $\mathbf{m_i}\pi_\mathbf{i}$ *consists of a sequence of one or more letters that are strictly decreasing except that the last letter may be repeated indefinitely.*

Here is the bijection, with an obvious inverse. For each π_i, if all its letters are $\mathbf{m_i}$ leave π_i intact, otherwise, let $a < \mathbf{m_i}$ denote its last letter, set $b = \mathbf{m_i}$, and transfer all (if any) occurrences of b at the start of π_i to the end, changing them from b to a. For example, with the first occurrence of each noninitial left-to-right maximum circled

$$0\,0\,\textcircled{2}\,2\,2\,1\,0\,\textcircled{5}\,4\,3\,\textcircled{8}\,8\,8\,\textcircled{9}\,9\,8 \rightarrow 0\,0\,\textcircled{2}\,1\,0\,0\,0\,\textcircled{5}\,4\,3\,\textcircled{8}\,8\,8\,\textcircled{9}\,8\,8\,.$$

4.10. Class 207: $\{100, 101, 110, 210\} \overset{\mathbf{I}}{\sim} \{100, 101, 110, 201\}$

We have the following characterization of inversion sequences avoiding $\{100, 101, 110\}$.

Lemma 15. *Suppose* $e \in \mathbf{I}_n$ *and* $0^{r_1} m_2^{r_2}\pi_2 \dots m_k^{r_k}\pi_k$ *is the strict LRmax decomposition of* e*. Then* $e \in \mathbf{I}_n(\{100, 101, 110\})$ *if and only if*
(i) *for* $2 \le i \le k$*, either* m_i *occurs only once in* e *or each occurrence of* m_i *is a weak right-to-left min in* e (*that is, less than equal to all following letters*)*, and*
(ii) *the concatenation* $\pi_2 \dots \pi_k$ *consists of distinct letters.*

Lemma 16. *Suppose* $e \in \mathbf{I}_n(\{100, 101, 110\})$ *and* $0^{r_1} m_2^{r_2}\pi_2 \dots m_k^{r_k}\pi_k$ *is the strict LRmax decomposition of* e*. Then*
(i) $e \in \mathbf{I}_n(\{100, 101, 110, 210\})$ *if and only if the concatenation* $\pi_2 \dots \pi_k$ *is* (*strictly*) *increasing, and*
(ii) $e \in \mathbf{I}_n(\{100, 101, 110, 201\})$ *if and only if, for* $i = 2, \dots, k$*, the entries in the concatenation* $\pi_2 \dots \pi_k$ *that are less than* m_i *form a decreasing list.*

Here is the bijection. In the strict LRmax decomposition $0^{r_1} m_2^{r_2}\pi_2 \dots m_k^{r_k}\pi_k$ of $e \in \mathbf{I}_n(\{100, 101, 110, 210\})$, consider the π's as a set of filled boxes with one entry in each box. Remove the entries from all boxes and refill the boxes left to right, using the largest available entry that is less than m_i when the box is in π_i.

For example, with the first occurrence of each left-to-right max circled and the π's underlined,

$$0^5\,\textcircled{6}\,\underline{0\,1}\,\textcircled{7}\,\textcircled{8}\,\underline{2\,3}\,\textcircled{12}\,\underline{4\ 5\ 10}\ \textcircled{15}\,\underline{11\,13\,14}\,\textcircled{16}\,16\,\textcircled{17}$$
$$\mapsto 0^5\,\textcircled{6}\,\underline{5\,4}\,\textcircled{7}\,\textcircled{8}\,\underline{3\,2}\,\textcircled{12}\,\underline{11\,10\,1}\,\textcircled{15}\,\underline{14\ 13\ 0}\ \textcircled{16}\,16\,\textcircled{17}.$$

The inverse is obvious: identify the π's and then rearrange their entries in increasing order.

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5. Appendix

Table 2: Sequences $\{|\mathbf{I}_n(B)|\}_{n=0}^{9}$, where B any set of four patterns in P.

Beginning of Table 2		
Class	B	$\{\|\mathbf{I}_n(B)\|\}_{n=0}^{9}$
1	{000,001,010,012}, {000,001,011,012}	1,2,1,0,0,0,0,0,0
2	{000,001,010,011}, {001,010,011,012}	1,2,1,1,1,1,1,1,1
T3	{000,001,012,110}	1,2,2,0,0,0,0,0,0
4	{000,001,012,021}, {000,001,012,100}, {000,001,012,101}, {000,001,012,102}, {000,001,012,120}, {000,001,012,201}, {000,001,012,210}, {000,010,011,012}	1,2,2,1,0,0,0,0,0
5	{000,001,011,120}, {001,011,012,100}	1,2,2,1,1,1,1,1,1
6	{000,001,010,021}, {000,001,010,100}, {000,001,010,101}, {000,001,010,102}, {000,001,010,110}, {000,001,010,120}, {000,001,010,201}, {000,001,010,210}, {000,001,011,021}, {000,001,011,100}, {000,001,011,101}, {000,001,011,102}, {000,001,011,110}, {000,001,011,201}, {000,001,011,210}, {001,010,011,021}, {001,010,011,100}, {001,010,011,101}, {001,010,011,102}, {001,010,011,110}, {001,010,011,120}, {001,010,011,201}, {001,010,011,210}, {001,010,012,021}, {001,010,012,100}, {001,010,012,101}, {001,010,012,102}, {001,010,012,110}, {001,010,012,120}, {001,010,012,201}, {001,010,012,210}, {001,011,012,021}, {001,011,012,101}, {001,011,012,102}, {001,011,012,110}, {001,011,012,120}, {001,011,012,201}, {001,011,012,210}	1,2,2,2,2,2,2,2,2
T7	{000,011,012,021}	1,2,3,0,0,0,0,0,0
8	{000,011,012,100}, {000,011,012,101}, {000,011,012,102}, {000,011,012,110}, {000,011,012,120}, {000,011,012,201}, {000,011,012,210}	1,2,3,1,0,0,0,0,0
T9	{000,010,012,021}	1,2,3,2,0,0,0,0,0
10	{001,011,100,120}, {001,012,100,110}	1,2,3,2,2,2,2,2,2
11	{000,010,012,100}, {000,010,012,110}	1,2,3,3,1,0,0,0,0

Continuation of Table 2		
Class	B	$\{\lvert\mathbf{I}_n(B)\rvert\}_{n=0}^{9}$
12	{000,010,012,101}, {000,010,012,102}, {000,010,012,120}, {000,010,012,201}, {000,010,012,210}	1,2,3,3,2,1,0,0,0
T13	{000,001,021,120}	1,2,3,3,2,2,2,2,2
14	{000,001,021,110}, {000,001,110,120}, {001,011,021,100}, {001,011,021,120}, {001,011,100,101}, {001,011,100,102}, {001,011,100,110}, {001,011,100,201}, {001,011,100,210}, {001,011,101,120}, {001,011,102,120}, {001,011,110,120}, {001,011,120,201}, {001,011,120,210}, {001,012,021,100}, {001,012,021,110}, {001,012,100,101}, {001,012,100,102}, {001,012,100,120}, {001,012,100,201}, {001,012,100,210}, {001,012,101,110}, {001,012,102,110}, {001,012,110,120}, {001,012,110,201}, {001,012,110,210}	1,2,3,3,3,3,3,3,3
15	{000,001,021,100}, {000,001,021,101}, {000,001,021,102}, {000,001,021,201}, {000,001,021,210}, {000,001,100,120}, {000,001,101,120}, {000,001,102,120}, {000,001,120,201}, {000,001,120,210}	1,2,3,4,4,4,4,4,4

Continuation of Table 2		
Class	B	$\{\lvert\mathbf{I}_n(B)\rvert\}_{n=0}^{9}$
16	{000,001,100,110}, {000,001,101,110}, {000,001,102,110}, {000,001,110,201}, {000,001,110,210}, {000,010,011,021}, {001,010,021,100}, {001,010,021,101}, {001,010,021,102}, {001,010,021,110}, {001,010,021,120}, {001,010,021,201}, {001,010,021,210}, {001,010,100,101}, {001,010,100,102}, {001,010,100,110}, {001,010,100,120}, {001,010,100,201}, {001,010,100,210}, {001,010,101,102}, {001,010,101,110}, {001,010,101,120}, {001,010,101,201}, {001,010,101,210}, {001,010,102,110}, {001,010,102,120}, {001,010,102,201}, {001,010,102,210}, {001,010,110,120}, {001,010,110,201}, {001,010,110,210}, {001,010,120,201}, {001,010,120,210}, {001,010,201,210}, {001,011,021,101}, {001,011,021,102}, {001,011,021,110}, {001,011,021,201}, {001,011,021,210}, {001,011,101,102}, {001,011,101,110}, {001,011,101,201}, {001,011,101,210}, {001,011,102,110}, {001,011,102,201}, {001,011,102,210}, {001,011,110,201}, {001,011,110,210}, {001,011,201,210}, {001,012,021,101}, {001,012,021,102}, {001,012,021,120}, {001,012,021,201}, {001,012,021,210}, {001,012,101,102}, {001,012,101,120}, {001,012,101,201}, {001,012,101,210}, {001,012,102,120}, {001,012,102,201}, {001,012,102,210}, {001,012,120,201}, {001,012,120,210}, {001,012,201,210}, {010,011,012,021}	1,2,3,4,5,6,7,8,9
17	{000,001,100,210}, {000,001,101,210}, {000,001,102,210}, {000,001,201,210}, {000,010,011,102}	1,2,3,5,7,9,11,13,15
T18	{010,011,012,210}	1,2,3,5,8,12,17,23,30

Continuation of Table 2		
Class	B	$\{\lvert\mathbf{I}_n(B)\rvert\}_{n=0}^{9}$
19	{000,001,100,101}, {000,001,100,102}, {000,001,100,201}, {000,001,101,102}, {000,001,101,201}, {000,001,102,201}, {000,010,011,120}, {010,011,012,100}, {010,011,012,101}, {010,011,012,102}, {010,011,012,110}, {010,011,012,120}, {010,011,012,201}	1,2,3,5,8,13,21,34,55
20	{000,010,011,100}, {000,010,011,101}, {000,010,011,110}, {000,010,011,201}, {000,010,011,210}	1,2,3,5,9,17,33,65,129
T21	{000,010,102,120}	1,2,4,10,26,66,172,457,1225
T22	{000,010,102,110}	1,2,4,10,26,67,177,475,1287
T23	{000,010,100,102}	1,2,4,10,26,68,187,523,1486
T24	{000,010,101,102}	1,2,4,10,26,70,195,557,1619
T25	{000,010,102,210}	1,2,4,10,27,73,202,568,1612
T26	{000,010,102,201}	1,2,4,10,27,73,203,577,1667
T27	{000,010,100,120}	1,2,4,10,27,78,241,779,2617
T28	{000,010,100,110}	1,2,4,10,27,79,245,801,2743
T29	{000,010,101,110}	1,2,4,10,27,79,245,802,2757
30	{000,010,101,120}, {000,010,110,120}	1,2,4,10,27,79,247,816,2822
T31	{000,010,100,101}	1,2,4,10,27,81,263,920,3441
T32	{000,010,120,201}	1,2,4,10,28,85,278,964,3493
T33	{000,010,120,210}	1,2,4,10,28,85,278,965,3505
T34	{000,010,110,201}	1,2,4,10,28,86,283,987,3609
T35	{000,010,110,210}	1,2,4,10,28,86,283,988,3625
36	{000,010,100,201}, {000,010,100,210}	1,2,4,10,28,87,295,1071, 4121
37	{000,010,101,201}, {000,010,101,210}	1,2,4,10,28,88,302,1116, 4386
T38	{000,010,201,210}	1,2,4,10,29,95,341,1308, 5263
39	{000,012,021,101}, {000,012,021,110}	1,2,4,3,0,0,0,0,0
T40	{000,012,101,110}	1,2,4,3,1,0,0,0,0
41	{000,012,021,100}, {000,012,021,102}, {000,012,021,120}, {000,012,021,201}, {000,012,021,210}, {000,012,100,110}	1,2,4,4,0,0,0,0,0
42	{000,012,100,101}, {000,012,102,110}, {000,012,110,120}, {000,012,110,201}, {000,012,110,210}	1,2,4,4,1,0,0,0,0
43	{000,012,101,102}, {000,012,101,120}, {000,012,101,201}, {000,012,101,210}	1,2,4,4,2,1,0,0,0

Continuation of Table 2		
Class	B	$\{\lvert\mathbf{I}_n(B)\rvert\}_{n=0}^{9}$
44	{000,012,100,102}, {000,012,100,120}, {000,012,100,201}, {000,012,100,210}	1,2,4,5,1,0,0,0,0
45	{000,012,102,120}, {000,012,102,201}, {000,012,102,210}, {000,012,120,201}, {000,012,120,210}, {000,012,201,210}	1,2,4,5,2,1,0,0,0
46	{000,011,021,102}, {000,011,102,120}, {001,021,100,110}, {001,021,100,120}, {001,021,110,120}, {001,100,110,120}	1,2,4,5,6,7,8,9,10
47	{000,011,021,120}, {000,011,100,102}, {000,011,101,102}, {000,011,102,110}, {000,011,102,201}, {000,011,102,210}, {001,021,100,101}, {001,021,100,102}, {001,021,100,201}, {001,021,100,210}, {001,021,101,110}, {001,021,101,120}, {001,021,102,110}, {001,021,102,120}, {001,021,110,201}, {001,021,110,210}, {001,021,120,201}, {001,021,120,210}, {001,100,101,110}, {001,100,101,120}, {001,100,102,110}, {001,100,102,120}, {001,100,110,201}, {001,100,110,210}, {001,100,120,201}, {001,100,120,210}, {001,101,110,120}, {001,102,110,120}, {001,110,120,201}, {001,110,120,210}, {011,012,021,100}	1,2,4,6,8,10,12,14,16

Continuation of Table 2		
Class	B	$\{\lvert\mathbf{I}_n(B)\rvert\}_{n=0}^{9}$
48	{000,011,021,100}, {000,011,021,101}, {000,011,021,110}, {000,011,021,201}, {000,011,021,210}, {001,021,101,102}, {001,021,101,201}, {001,021,101,210}, {001,021,102,201}, {001,021,102,210}, {001,021,201,210}, {001,100,101,210}, {001,100,102,210}, {001,100,201,210}, {001,101,102,110}, {001,101,102,120}, {001,101,110,201}, {001,101,110,210}, {001,101,120,201}, {001,101,120,210}, {001,102,110,201}, {001,102,110,210}, {001,102,120,201}, {001,102,120,210}, {001,110,201,210}, {001,120,201,210}, {010,012,021,100}, {010,012,021,101}, {010,012,021,102}, {010,012,021,110}, {010,012,021,120}, {010,012,021,201}, {010,012,021,210}, {011,012,021,101}, {011,012,021,102}, {011,012,021,110}, {011,012,021,120}, {011,012,021,201}, {011,012,021,210}	1,2,4,7,11,16,22,29,37
T49	{011,012,100,210}	1,2,4,7,12,19,28,39,52
50	{000,011,100,120}, {000,011,101,120}, {000,011,110,120}, {000,011,120,201}, {000,011,120,210}, {001,100,101,102}, {001,100,101,201}, {001,100,102,201}, {011,012,100,201}	1,2,4,7,12,20,33,54,88
51	{011,012,100,101}, {011,012,100,102}, {011,012,100,110}, {011,012,100,120}	1,2,4,7,13,23,41,72,126
T52	{011,012,201,210}	1,2,4,8,14,22,32,44,58
T53	{010,012,100,110}	1,2,4,8,14,23,36,55,83
54	{001,101,102,210}, {001,101,201,210}, {001,102,201,210}, {010,012,100,210}, {010,012,110,210}, {011,012,101,210}, {011,012,102,210}, {011,012,110,210}, {011,012,120,210}	1,2,4,8,15,26,42,64,93
55	{010,012,100,101}, {010,012,100,102}, {010,012,100,120}, {010,012,100,201}, {010,012,101,110}, {010,012,102,110}, {010,012,110,120}, {010,012,110,201}, {011,012,101,201}, {011,012,102,201}, {011,012,110,201}, {011,012,120,201}	1,2,4,8,15,27,47,80,134

Continuation of Table 2		
Class	B	$\{\lvert\mathbf{I}_n(B)\rvert\}_{n=0}^{9}$
56	{010,012,101,210}, {010,012,102,210}, {010,012,120,210}, {010,012,201,210}	1,2,4,8,16,31,57,99,163
57	{000,011,100,101}, {000,011,100,110}, {000,011,100,201}, {000,011,100,210}, {000,011,101,110}, {000,011,101,201}, {000,011,101,210}, {000,011,110,201}, {000,011,110,210}, {000,011,201,210}, {001,101,102,201}, {010,011,021,100}, {010,011,021,101}, {010,011,021,102}, {010,011,021,110}, {010,011,021,120}, {010,011,021,201}, {010,011,021,210}, {010,012,101,102}, {010,012,101,120}, {010,012,101,201}, {010,012,102,120}, {010,012,102,201}, {010,012,120,201}, {011,012,101,102}, {011,012,101,110}, {011,012,101,120}, {011,012,102,110}, {011,012,102,120}, {011,012,110,120}	1,2,4,8,16,32,64,128,256
T58	{010,011,102,120}	1,2,4,9,20,45,100,221,484
T59	{010,011,102,210}	1,2,4,9,21,50,119,281,656
T60	{010,011,102,201}	1,2,4,9,21,50,120,289,697
61	{010,011,100,102}, {010,011,101,102}, {010,011,102,110}	1,2,4,9,21,51,126,316,799
62	{000,010,021,100}, {000,010,021,101}, {000,010,021,102}, {000,010,021,110}, {000,010,021,120}, {000,010,021,201}, {000,010,021,210}	1,2,4,9,21,51,127,323,835
63	{010,011,120,201}, {010,011,120,210}	1,2,4,9,22,57,153,421,1179
64	{010,011,100,120}, {010,011,101,120}, {010,011,110,120}	1,2,4,9,22,58,161,467,1402
T65	{010,011,201,210}	1,2,4,9,23,64,186,551,1645
66	{010,011,100,201}, {010,011,100,210}, {010,011,101,201}, {010,011,101,210}, {010,011,110,201}, {010,011,110,210}	1,2,4,9,23,65,198,639,2160
67	{010,011,100,101}, {010,011,100,110}, {010,011,101,110}	1,2,4,9,23,66,210,733,2781
68	{012,021,100,101}, {012,021,100,110}, {012,021,101,110}	1,2,5,10,17,26,37,50,65
T69	{012,100,101,110}	1,2,5,10,19,33,56,93,154
T70	{012,100,110,210}	1,2,5,11,21,35,53,75,101

Continuation of Table 2		
Class	B	$\{\lvert \mathbf{I}_n(B)\rvert\}_{n=0}^{9}$
71	{012,021,100,102}, {012,021,100,120}, {012,021,100,201}, {012,021,100,210}, {012,021,101,102}, {012,021,101,120}, {012,021,101,201}, {012,021,101,210}, {012,021,102,110}, {012,021,110,120}, {012,021,110,201}, {012,021,110,210}	1,2,5,11,21,36,57,85,121
T72	{012,100,110,201}	1,2,5,11,21,36,58,90,137
T73	{000,021,102,120}	1,2,5,11,21,51,127,323,835
74	{012,100,102,110},{012,100,110,120}	1,2,5,11,22,39,66,108,175
T75	{012,100,101,210}	1,2,5,11,22,40,67,105,156
T76	{012,101,110,210}	1,2,5,11,22,41,72,120,191
77	{012,100,101,201},{012,101,110,201}	1,2,5,11,22,41,73,126,213
T78	{011,021,102,120}	1,2,5,11,22,42,79,149,284
T79	{000,021,102,110}	1,2,5,11,22,52,128,324,836
80	{012,100,101,102},{012,100,101,120}	1,2,5,11,23,45,85,156,281
81	{011,021,100,102}, {012,101,102,110}, {012,101,110,120}	1,2,5,11,23,47,95,191,383
T82	{011,100,102,120}	1,2,5,11,25,55,121,263,569
T83	{000,021,101,102}	1,2,5,11,25,60,148,374,962
84	{012,100,201,210}, {012,110,201,210}	1,2,5,12,25,46,77,120,177
T85	{000,021,100,102}, {000,021,102,201}, {000,021,102,210}	1,2,5,12,25,60,148,374,962
86	{012,100,102,210}, {012,100,120,210}, {012,101,201,210}, {012,102,110,210}, {012,110,120,210}	1,2,5,12,26,51,92,155,247
87	{012,100,102,201}, {012,100,120,201}, {012,102,110,201}, {012,110,120,201}	1,2,5,12,26,51,93,161,269
T88	{012,100,102,120}	1,2,5,12,27,56,110,207,378
89	{012,101,102,210}, {012,101,120,210}	1,2,5,12,27,57,113,211,373
90	{011,021,100,120}, {011,021,101,102}, {011,021,102,110}, {011,021,102,201}, {011,021,102,210}, {011,102,120,201}, {011,102,120,210}, {012,021,102,120}, {012,021,102,201}, {012,021,102,210}, {012,021,120,201}, {012,021,120,210}, {012,021,201,210}, {012,101,102,201}, {012,101,120,201}, {012,102,110,120}	1,2,5,12,27,58,121,248,503
91	{011,101,102,120}, {011,102,110,120}	1,2,5,12,28,64,144,320,704
T92	{012,101,102,120}	1,2,5,12,28,65,151,351,816
T93	{011,100,102,210}	1,2,5,12,29,69,162,375,857
T94	{011,100,102,201}	1,2,5,12,29,70,169,408,985

Continuation of Table 2		
Class	B	$\{\lvert\mathbf{I}_n(B)\rvert\}_{n=0}^{9}$
T95	{000,021,101,110}	1,2,5,12,29,71,177,449,1157
96	{000,021,101,120},{000,021,110,120}	1,2,5,12,29,72,182,468,1220
97	{011,100,101,102}, {011,100,102,110}	1,2,5,12,30,75,190,483,1235
T98	{000,102,110,120}	1,2,5,12,30,77,200,528,1408
T99	{000,101,102,120}	1,2,5,12,30,77,201,532,1424
T100	{000,101,102,110}	1,2,5,12,31,81,216,583,1590
T101	{000,100,102,120}	1,2,5,13,31,80,207,542,1439
102	{012,102,201,210}, {012,120,201,210}	1,2,5,13,32,73,156,318,629
T103	{011,102,201,210}	1,2,5,13,32,75,170,377,824
104	{000,021,100,120}, {000,021,120,201}, {000,021,120,210}	1,2,5,13,32,81,207,537,1409
T105	{000,102,120,201}	1,2,5,13,32,84,215,566,1494
T106	{000,102,120,210}	1,2,5,13,32,84,217,575,1528
107	{012,102,120,201}, {012,102,120,210}	1,2,5,13,33,80,185,411,885
108	{011,021,100,101}, {011,021,100,110}, {011,021,100,201}, {011,021,100,210}, {011,021,101,120}, {011,021,110,120}, {011,021,120,201}, {011,021,120,210}, {011,101,102,210}, {011,102,110,210}	1,2,5,13,33,81,193,449,1025
109	{011,101,102,201}, {011,102,110,201}	1,2,5,13,33,82,201,489,1185
110	{000,021,100,110}, {000,021,110,201}, {000,021,110,210}	1,2,5,13,33,84,215,556,1453
111	{000,102,110,201}	1,2,5,13,33,87,228,609,1636
T112	{011,101,102,110}	1,2,5,13,34,89,233,610,1597
T113	{000,102,110,210}	1,2,5,13,34,90,240,645,1745
114	{000,021,100,101}, {000,021,101,201}, {000,021,101,210}	1,2,5,13,35,96,267,750,2123
115	{011,100,120,201},{011,100,120,210}	1,2,5,13,35,96,268,758,2167
T116	{000,101,102,210}	1,2,5,13,36,101,288,827, 2389
117	{011,100,101,120}, {011,100,110,120}	1,2,5,13,36,103,306,935, 2933
T118	{000,100,101,102}	1,2,5,13,36,104,308,934, 2881
T119	{000,101,102,201}	1,2,5,13,37,107,321,979, 3042
T120	{000,101,110,120}	1,2,5,13,38,117,378,1275, 4451
T121	{000,100,102,210}	1,2,5,14,38,107,304,868, 2494

Continuation of Table 2		
Class	B	$\{\lvert\mathbf{I}_n(B)\rvert\}_{n=0}^{9}$
T122	{000,102,201,210}	1,2,5,14,38,108,301,854, 2425
T123	{000,100,102,201}	1,2,5,14,38,110,323,972, 2969
124	{000,021,100,201}, {000,021,100,210}, {000,021,201,210}	1,2,5,14,39,111,317,911, 2627
T125	{011,120,201,210}	1,2,5,14,40,115,331,950, 2713
T126	{011,100,201,210}	1,2,5,14,41,122,365,1094, 3281
127	{011,101,120,201}, {011,101,120,210}, {011,110,120,201}, {011,110,120,210}	1,2,5,14,41,123,375,1156, 3590
128	{010,021,100,101}, {010,021,100,102}, {010,021,100,110}, {010,021,100,120}, {010,021,100,201}, {010,021,100,210}, {010,021,101,102}, {010,021,101,110}, {010,021,101,120}, {010,021,101,201}, {010,021,101,210}, {010,021,102,110}, {010,021,102,120}, {010,021,102,201}, {010,021,102,210}, {010,021,110,120}, {010,021,110,201}, {010,021,110,210}, {010,021,120,201}, {010,021,120,210}, {010,021,201,210}, {011,021,101,110}, {011,021,101,201}, {011,021,101,210}, {011,021,110,201}, {011,021,110,210}, {011,021,201,210}	1,2,5,14,42,132,429,1430, 4862
T129	{011,101,110,120}	1,2,5,14,42,132,431,1452, 5026
130	{011,100,101,201}, {011,100,101,210}, {011,100,110,201}, {011,100,110,210}	1,2,5,14,42,133,441,1521, 5425
T131	{000,100,101,110}	1,2,5,14,42,134,451,1590, 5834
T132	{000,100,101,120}	1,2,5,14,42,136,462,1632, 5960
133	{000,101,120,201}, {000,101,120,210}	1,2,5,14,43,142,495,1800, 6781
T134	{000,101,110,201}	1,2,5,14,43,142,496,1811, 6854
T135	{000,101,110,210}	1,2,5,14,43,142,497,1827, 7008

Continuation of Table 2		
Class	B	$\{\lvert\mathbf{I}_n(B)\rvert\}_{n=0}^{9}$
T136	{011,100,101,110}	1,2,5,14,43,144,523,2048, 8597
T137	{000,100,110,120}	1,2,5,14,44,149,533,2002, 7810
T138	{000,110,120,201}	1,2,5,14,45,154,562,2144, 8480
T139	{000,110,120,210}	1,2,5,14,45,154,564,2161, 8616
T140	{010,100,102,120}	1,2,5,15,49,167,581,2049, 7301
T141	{010,102,110,120}	1,2,5,15,49,167,582,2058, 7357
T142	{010,101,102,120}	1,2,5,15,49,167,583,2068, 7423
T143	{010,100,102,110}	1,2,5,15,49,167,583,2071, 7455
T144	{010,101,102,110}	1,2,5,15,49,168,593,2135, 7797
T145	{010,100,101,102}	1,2,5,15,49,170,614,2285, 8700
T146	{000,100,120,201}	1,2,5,15,49,174,650,2533, 10195
T147	{000,100,120,210}	1,2,5,15,49,174,652,2549, 10311
T148	{010,102,120,201}	1,2,5,15,50,174,614,2178, 7758
T149	{010,102,120,210}	1,2,5,15,50,174,616,2201, 7919
T150	{010,102,110,201}	1,2,5,15,50,174,616,2202, 7933
T151	{010,102,110,210}	1,2,5,15,50,174,617,2211, 7983
152	{010,100,102,210}, {011,101,201,210}, {011,110,201,210}	1,2,5,15,50,176,638,2354, 8789
T153	{010,100,102,201}	1,2,5,15,50,176,639,2371, 8953
T154	{010,101,102,210}	1,2,5,15,50,177,649,2431, 9230
T155	{010,101,102,201}	1,2,5,15,50,177,651,2460, 9489

Continuation of Table 2		
Class	B	$\{\lvert\mathbf{I}_n(B)\rvert\}_{n=0}^{9}$
156	{010,100,101,120}, {010,100,110,120}	1,2,5,15,50,180,685,2723, 11207
T157	{010,101,110,120}	1,2,5,15,50,180,686,2736, 11325
T158	{010,100,101,110}	1,2,5,15,50,180,689,2781, 11773
T159	{000,120,201,210}	1,2,5,15,50,181,693,2767, 11408
T160	{000,100,110,201}	1,2,5,15,50,181,697,2818, 11845
T161	{000,100,101,210}	1,2,5,15,50,181,697,2822, 11905
T162	{010,102,201,210}	1,2,5,15,51,184,679,2529, 9474
T163	{000,110,201,210}	1,2,5,15,51,189,744,3059, 12993
164	{010,100,120,201}, {010,110,120,201}	1,2,5,15,51,189,745,3077, 13180
T165	{000,101,201,210}	1,2,5,15,51,189,746,3084, 13204
166	{010,100,110,201}, {010,100,110,210}, {010,100,120,210}, {010,101,110,201}, {010,101,120,201}, {010,101,120,210}, {010,110,120,210}, {011,101,110,201}, {011,101,110,210}	1,2,5,15,51,189,746,3091, 13311
T167	{010,101,110,210}	1,2,5,15,51,189,747,3109, 13511
168	{000,100,101,201}, {000,100,110,210}	1,2,5,15,51,191,772,3320, 15032
169	{010,100,101,201}, {010,100,101,210}	1,2,5,15,51,191,773,3334, 15161
T170	{010,110,201,210}	1,2,5,15,52,199,813,3477, 15387
T171	{010,120,201,210}	1,2,5,15,52,199,815,3510, 15711
172	{010,100,201,210}, {010,101,201,210}	1,2,5,15,52,201,841,3726, 17213
T173	{000,100,201,210}	1,2,5,16,58,230,965,4216, 18970
174	{021,100,102,120}, {021,101,102,120}, {021,102,110,120}	1,2,6,18,52,152,464,1486, 4946

Continuation of Table 2		
Class	B	$\{\lvert\mathbf{I}_n(B)\rvert\}_{n=0}^{9}$
T175	{021,100,102,110}	1,2,6,18,52,153,470,1508, 5010
T176	{021,101,102,110}	1,2,6,18,53,158,486,1550, 5109
T177	{021,100,101,102}	1,2,6,18,55,173,560,1858, 6291
178	{021,102,110,201}, {021,102,110,210}	1,2,6,19,57,168,506,1585, 5165
179	{021,102,120,201}, {021,102,120,210}	1,2,6,19,58,174,528,1649, 5328
180	{021,100,102,201}, {021,100,102,210}	1,2,6,19,59,183,580,1893, 6347
181	{021,101,102,201}, {021,101,102,210}	1,2,6,19,60,191,619,2048, 6909
T182	{021,100,101,110}	1,2,6,19,61,198,651,2171, 7345
183	{021,100,101,120}, {021,100,110,120}, {021,101,110,120}	1,2,6,19,61,199,661,2234, 7668
T184	{100,102,110,120}	1,2,6,19,63,212,726,2521, 8863
T185	{100,101,102,120}	1,2,6,19,63,213,733,2558, 9034
T186	{101,102,110,120}	1,2,6,19,63,215,749,2650, 9490
T187	{100,101,102,110}	1,2,6,19,64,222,788,2842, 10378
T188	{021,102,201,210}	1,2,6,20,66,213,683,2211, 7291
189	{100,102,120,201}, {102,110,120,201}	1,2,6,20,68,231,788,2711, 9423
190	{021,100,110,201}, {021,100,110,210}	1,2,6,20,68,232,794,2732, 9468
191	{021,100,120,201}, {021,100,120,210}, {021,101,120,201}, {021,101,120,210}, {021,110,120,201}, {021,110,120,210}, {100,102,120,210}, {101,102,120,201}, {101,102,120,210}, {102,110,120,210}	1,2,6,20,68,233,805,2807, 9879
T192	{100,102,110,201}	1,2,6,20,68,234,816,2882, 10294
T193	{100,102,110,210}	1,2,6,20,69,240,842,2979, 10625

Continuation of Table 2		
Class	B	$\{\lvert \mathbf{I}_n(B)\rvert\}_{n=0}^{9}$
T194	{101,102,110,210}	1,2,6,20,69,242,858,3068, 11050
T195	{101,102,110,201}	1,2,6,20,69,242,859,3080, 11140
196	{021,100,101,201}, {021,100,101,210}, {021,101,110,201}, {021,101,110,210}	1,2,6,20,70,252,924,3432, 12870
T197	{100,101,102,210}	1,2,6,20,71,260,970,3662, 13938
T198	{100,101,102,201}	1,2,6,20,72,272,1064,4272, 17504
T199	{100,101,110,120}	1,2,6,20,73,282,1140,4770, 20526
T200	{102,110,201,210}	1,2,6,21,74,258,897,3131, 11007
T201	{102,120,201,210}	1,2,6,21,75,265,927,3230, 11268
T202	{100,102,201,210}	1,2,6,21,76,277,1016,3756, 13998
203	{021,120,201,210}, {101,102,201,210}	1,2,6,21,77,287,1079,4082, 15522
204	{021,100,201,210}, {021,110,201,210}	1,2,6,21,78,297,1144,4433, 17238
T205	{021,101,201,210}	1,2,6,21,80,322,1347,5798, 25512
206	{100,101,120,201}, {100,101,120,210}, {101,110,120,201}	1,2,6,21,80,323,1363,5950, 26671
207	{100,101,110,201}, {100,101,110,210}, {101,110,120,210}	1,2,6,21,80,324,1375,6052, 27425
208	{100,110,120,201}, {100,110,120,210}	1,2,6,21,82,343,1509,6893, 32419
T209	{101,120,201,210}	1,2,6,22,88,372,1644,7518, 35266
210	{100,101,201,210}, {100,110,201,210}, {101,110,201,210}	1,2,6,22,90,394,1806,8558, 41586
211	{100,120,201,210}, {110,120,201,210}	1,2,6,22,90,396,1833,8801, 43441
212	{000,100,102,110}	1,2,5,13,33,87,231,621, 1686
End of Table 2		