A NOTE ON THE EXPONENTIAL DIOPHANTINE EQUATION $(44m^2+1)^x+(5m^2-1)^y=(7m)^z$

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Abstract

Let m be a positive integer. We show that the exponential Diophantine equation $(44m^2+1)^x+(5m^2-1)^y=(7m)^z$ has only the positive integer solution $(x,y,z)=(1,1,2)$ under some conditions. The proof is based on elementary methods, Baker's method and linear forms in p-adic logarithms.

1. Introduction

Let A, B, C be fixed coprime positive integers with $\min(A,B,C)>1$. The ternary exponential Diophantine equation

$$A^x+B^y=C^z \tag{1}$$

in positive integers x, y, z has been actively studied by many authors. It is known that the number of solutions (x,y,z) of Equation (1) is finite.

In the last decade, many of the recent works on Equation (1) concerned the case where

$$A=am^2+1,\ B=bm^2-1,\ C=cm,$$

and a, b, c are fixed positive integers such that $a+b=c^2$, $2\nmid c$. Clearly, in this case Equation (1) always has a solution $(x,y,z)=(1,1,2)$.

In 2020 Terai and Shinsho [12] proposed the following conjecture.

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Conjecture 1. Let m be a positive integer greater than one. Let $a, b, c > 1$ be positive integers satisfying $a + b = c^2$. Then the equation

$$(am^2 + 1)^x + (bm^2 - 1)^y = (cm)^z$$

has only the positive integer solution $(x, y, z) = (1, 1, 2)$.

Although, in general, the above conjecture is widely open, it has been confirmed by several authors under some conditions on m, a, b, c:

- (N. Terai [8]) $(a, b, c) = (4, 5, 3)$, $m \leq 20$ or $m \not\equiv 3 \pmod 6$.
- (J.-L. Su and X.-X. Li [7]) $(a, b, c) = (4, 5, 3)$, $m > 90$, $m \equiv 0 \pmod 3$.
- (C. Bertok [2]) $(a, b, c) = (4, 5, 3)$, $20 < m \leq 90$.
- (M. Alan [1]) $(a, b, c) = (18, 7, 5)$, $m \not\equiv 23, 47, 63$ or $87 \pmod{120}$.
- (N. Terai [10]) $(a, b, c) = (4, 21, 5)$, m satisfies some conditions.
- (N. Terai [9]) $(a, b, c) = (10, 15, 5)$, for all m.
- (N. Terai and T. Hibino [11]) $(a, b, c) = (12, 13, 5)$, $m \not\equiv 17, 33 \pmod{40}$.
- (N. Terai and Y. Shinsho [13]) $(a, b, c) = (4, 45, 7)$, $m \equiv -1 \pmod 3$, $m \equiv 2 \pmod 5$ or $m \equiv \pm 1, \pm 2 \pmod 7$.
- (S. Fei and J. Luo [4]) $(a, b, c) = (28, 21, 7)$, for all m.

In this paper we consider the exponential Diophantine equation

$$(44m^2 + 1)^x + (5m^2 - 1)^y = (7m)^z. \tag{2}$$

We prove the following result.

Theorem 1. *Let m be a positive integer. When m is odd, we suppose that*

$$m \equiv 2 \pmod 5 \text{ or } m \equiv 0, \pm 1, \pm 3 \pmod 7. \tag{3}$$

Then Equation (2) has only the positive integer solution $(x, y, z) = (1, 1, 2)$.

2. Preliminaries

In this section, we give some lemmas that will be useful for the proof of the main result.

Lemma 1 (Terai-Shinsho [12]). *Let r be an odd integer with $r \geq 3$. Then the equation*

$$4^x + (r^2 - 4)^y = r^z$$

has only the positive integer solution $(x, y, z) = (1, 1, 2)$.

Let α be an algebraic number of degree $d \geq 1$ with the minimal polynomial

$$a_0X^d + +a_1X^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d}(X - \alpha^{(i)}),$$

where the a_i's are relatively prime integers with $a_0 > 0$ and the $\alpha^{(i)}$s are the conjugates of α. Then the *logarithmic height* of α is defined as

$$h(\alpha) = \frac{1}{d}\big(\log |a_0| + \sum_{i=1}^{d} \log\left(\max\{|\alpha^{(i)}|, 1\}\right)\big).$$

Let α_1, α_2 be two real algebraic numbers with $|\alpha_1|$, $|\alpha_2| \geq 1$ and b_1, b_2 be positive integers. We consider the linear form

$$\Lambda_1 = b_2 \log \alpha_2 - b_1 \log \alpha_1.$$

Let A_1 and A_2 be real numbers greater than 1 with

$$\log A_i \geq \max\left\{h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D}\right\} \ (i = 1, 2),$$

where $D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$. Set

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$

With the above notation, we cite a result due to Laurent [6, Corollary 2] with $m = 10$ and $C_2 = 25.2$. Recall that two nonzero complex numbers α, β are *multiplicatively independent* if the only solution of the equation $\alpha^x \beta^y = 1$ in integers x, y is $x = y = 0$.

Lemma 2 (Laurent [6]). *Let Λ_1 be given as above with $\alpha_1 > 1$ and $\alpha_2 > 1$. Suppose that α_1 and α_2 are multiplicatively independent. Then*

$$\log |\Lambda_1| \geq -25.2D^4 \left(\max\left\{ \log b' + 0.38, \frac{10}{D}\right\}\right)^2 \log A_1 \log A_2.$$

We will also need a result on linear forms in p-adic logarithms due to Bugeaud. Here we just use a special case $y_1 = y_2 = 1$ in the notation from [3, p. 375]. Let p be an odd prime and a_1 and a_2 be non-zero integers prime to p. Let g denote the smallest positive integer such that

$$v_p(a_1^g - 1) \geq 1, \ v_p(a_2^g - 1) \geq 1,$$

where we denote the p-adic valuation by $v_p(\cdot)$. Assume that there exists a real number E such that

$$\frac{1}{p-1} < E \leq v_p(a_1^g - 1).$$

We consider the integer

$$\Lambda_2 = a_1^{b_1} - a_2^{b_2}.$$

Lemma 3 (Bugeaud [3]). *Let $A_1 > 1$, $A_2 > 1$ be real numbers such that*

$$\log A_i \geq \max\{\log |a_i|, E \log p\} \ (i = 1, 2)$$

and put

$$b' = \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1}.$$

If a_1 and a_2 are multiplicatively independent then we have the upper estimate

$$v_p(\Lambda_2) \leq \frac{36.1g}{E^3(\log p)^4}(\max\{\log b' + \log (E \log p) + 0.4, 6E \log p, 5\})^2 \log A_1 \log A_2.$$

The next lemma shows that for a possible solution (x, y, z) of the exponential Diophantine equation $(am^2+1)^x + (bm^2-1)^y = (cm)^z$ there is an upper and lower bound for z depending on $\max\{x, y\}$ and m.

Lemma 4 (Alan [1]). *Let a, b, c and $m > 1$ be positive integers such that $a + b = c^2$ and (x, y, z) be a positive integer solution of the exponential Diophantine equation $(am^2 + 1)^x + (bm^2 - 1)^y = (cm)^z$. If $M = \max\{x, y\} > 1$, then*

$$\left(2 - \frac{\log\left(\frac{c^2}{\min\left(a, b - \frac{1}{m^2}\right)}\right)}{\log (cm)}\right) M < z < 2M.$$

Finally for the case $m \equiv 0 \pmod 7$ we will require a result from [5] that gives an upper bound for m.

Lemma 5 (Fu-Yang [5]). *Let a, b, c, m be positive integers such that $a+b = c^2$, $2 \mid a$, $2 \nmid c$, and $m > 1$. If $c \mid m$ and $m > 36c^3 \log c$, then $(am^2+1)^x + (bm^2-1)^y = (cm)^z$ has only the solution $(x, y, z) = (1, 1, 2)$.*

3. Proof

In this section, we give a proof of Theorem 1. The proof follows in a series of lemmas. First we consider the case $m = 1$.

Lemma 6. *The equation*

$$45^x + 4^y = 7^z$$

has only the positive integer solution $(x, y, z) = (1, 1, 2)$.

Proof. This follows from Lemma 1 with $r = 7$. $\square$

By Lemma 6, we may assume that $m \geq 2$.

Lemma 7. *If (x, y, z) is a positive integer solution of Equation* (2)*, then y is odd.*

Proof. It follows from Equation (2) that $z \geq 2$. Taking Equation (2) modulo m^2 implies that $1 + (-1)^y \equiv 0 \pmod{m^2}$ and hence y is odd. $\square$

Lemma 8. *If m is even then Equation* (2) *has only the positive integer solution $(x, y, z) = (1, 1, 2)$.*

Proof. If $z \leq 2$, then $(x, y, z) = (1, 1, 2)$ is the only positive integer solution of Equation (2). Thus we may assume that $z \geq 3$. Taking Equation (2) modulo m^3 implies that

$$1 + 44m^2x - 1 + 5m^2y \equiv 0 \pmod{m^3}.$$

So

$$44x + 5y \equiv 0 \pmod{m},$$

which is impossible, since y is odd and m is even. Hence for $z \geq 3$, Equation (2) has no positive integer solution when m is even. $\square$

3.1. The Case m Is Odd and $m \equiv 2 \pmod 5$ or $m \equiv \pm 1, \pm 3 \pmod 7$

By Lemma 8, we may suppose that m is odd and $m \geq 3$. Let (x, y, z) be a solution of Equation (2).

Lemma 9. *If m is odd and $m \equiv 2 \pmod 5$ or $m \equiv \pm 1, \pm 3 \pmod 7$, then $y = 1$ and x is odd.*

Proof. We follow closely an argument in [13]. We first show that x is odd and z is even by considering the following case analysis.

a) $m \equiv 2 \pmod 5$. Taking Equation (2) modulo 5 implies that

$$2^x + (-1)^y \equiv (-1)^z \pmod{5}.$$

Since y is odd, we have $2^x \equiv 1 + (-1)^z \pmod 5$. This shows that z is even. Then $1 = \left(\frac{2}{5}\right)^{x-1} = (-1)^{x-1}$, where $\left(\frac{*}{*}\right)$ denotes the Jacobi symbol. Hence x is odd.

b) $m \equiv \pm 1, \pm 3 \pmod 7$. Since $m^2 \equiv 1 \pmod 7$ or $m^2 \equiv 2 \pmod 7$, taking Equation (2) modulo 7 implies that

$$3^x + 4^y \equiv 0 \pmod{7} \text{ or } 5^x + 2^y \equiv 0 \pmod{7},$$

that is, $\left(\frac{3}{7}\right)^x = \left(\frac{-4^y}{7}\right)$ or $\left(\frac{5}{7}\right)^x = \left(\frac{-2^y}{7}\right)$. This shows that x is odd. In these cases, we have that $\left(\frac{5m^2-1}{44m^2+1}\right) = 1$ and $\left(\frac{7m}{44m^2+1}\right) = -1$. Indeed,

$$\left(\frac{5m^2-1}{44m^2+1}\right) = \left(\frac{49m^2}{44m^2+1}\right) = 1$$

and

$$\begin{aligned}\left(\frac{7m}{44m^2+1}\right) &= \left(\frac{7}{44m^2+1}\right)\left(\frac{m}{44m^2+1}\right) = \left(\frac{44m^2+1}{7}\right)\left(\frac{44m^2+1}{m}\right)\\ &= \left(\frac{2m^2+1}{7}\right) = -1,\end{aligned}$$

since $m^2 \equiv 1, 2 \pmod 7$. Hence z is even from Equation (2).

Suppose that $y \geq 2$. Taking Equation (2) modulo 8 implies that

$$5^x \equiv (7m)^z \equiv 1 \pmod 8,$$

so x is even, but this contradicts the fact that x is odd as seen from above. Hence, $y = 1$. □

From Lemma 9, it follows that $y = 1$ and x is odd. If $x = 1$, then clearly $z = 2$. From now on, we may suppose that $x \geq 3$. Thus our theorem is reduced to solving Pillai's equation

$$c^z - a^x = b \tag{4}$$

with $x \geq 3$, where $a = 44m^2 + 1$, $b = 5m^2 - 1$ and $c = 7m$.

Next we obtain a lower bound for x.

Lemma 10. *If (x, y, z) is a positive integer solution of Equation (4), then*

$$x \geq \frac{1}{44}(m^2 - 5).$$

Proof. Since $x \geq 3$, from Equation (4) we get

$$(7m)^z = (44m^2+1)^x + 5m^2 - 1 \geq (44m^2+1)^3 + 5m^2 - 1 > (7m)^3.$$

Hence $z \geq 4$. Taking Equation (4) modulo m^4 implies that

$$1 + 44m^2x + 5m^2 - 1 \equiv 0 \pmod{m^4},$$

so $44x + 5 \equiv 0 \pmod{m^2}$ and the lemma follows. □

Lemma 11. *If (x, y, z) is a positive integer solution of Equation (4), then*

$$x < 2521 \log(7m).$$

Proof. The proof follows the argument as in [1], [11], and [13]. Without loss of generality we may assume that $z > 2$. Consider the linear form of two logarithms

$$\Lambda = z \log c - x \log a.$$

Using the inequality $\log(1+t) < t$ for $t > 0$, we have

$$0 < \Lambda = \log\left(\frac{c^z}{a^x}\right) = \log\left(1 + \frac{b}{a^x}\right) < \frac{b}{a^x}. \tag{5}$$

Hence, we get

$$\log \Lambda < \log b - x \log a. \tag{6}$$

On the other hand, it follows from Lemma 2 that

$$\log \Lambda \geq -25.2(\max\{\log b' + 0.38, 10\})^2 \log a \log c, \tag{7}$$

where

$$b' = \frac{x}{\log c} + \frac{z}{\log a}.$$

Observe that

$$a^{x+1} - c^z = a(c^z - b) - c^z = (a-1)c^z - ab > 44m^2 \cdot 49m^2 - (44m^2+1)(5m^2-1) > 0,$$

since $z > 2$. Thus $b' < \frac{2x+1}{\log c}$. Put $M = \frac{x}{\log c}$. Now combining (6) and (7) we obtain

$$x \log a < \log b + 25.2\left(\max\left\{\log\left(2M + \frac{1}{\log c}\right) + 0.38, 10\right\}\right)^2 \log a \log c.$$

Since $\log b < \log a \log c$ and $\log c = \log 7m > 2$ for $m \geq 3$, we can rewrite the above inequality as

$$M < 1 + 25.2(\max\{\log(2M+0.5) + 0.38, 10\})^2.$$

If $\log(2M+0.5) + 0.38 > 10$, then $M \geq 7532$. But the inequality

$$M < 1 + 25.2(\log(2M+0.5) + 0.38)^2$$

implies $M \leq 1867$. Therefore, $\max\{\log(2M+0.5)+0.38, 10\} = 10$ yields $M < 2521$ and hence $x < 2521 \log(7m)$. □

We are now ready to prove Theorem 1. It follows from Lemmas 10, 11 that

$$\frac{1}{44}(m^2 - 5) < 2521 \log(7m).$$

Hence we obtain $m \le 990$. From (5) we have the inequality

$$\left|\frac{\log a}{\log c} - \frac{z}{x}\right| < \frac{b}{xa^x \log c}.$$

Since $\frac{a^x \log c}{b} > \frac{44m^2 x}{5m^2} > 2x$, we get that

$$\left|\frac{\log a}{\log c} - \frac{z}{x}\right| < \frac{1}{2x^2},$$

which implies that $\frac{z}{x}$ is a convergent in the simple continued fraction expansion to $\frac{\log a}{\log c}$. Let $\frac{z}{x} = \frac{p_k}{q_k}$, where $\frac{p_k}{q_k}$ is the k-th convergent of the simple continued fraction expansion to $\frac{\log a}{\log c}$. Note that $q_k \le x$, since $(p_k, q_k) = 1$. It follows then that

$$\frac{1}{q_k^2(a_{k+1}+2)} < \left|\frac{\log a}{\log c} - \frac{p_k}{q_k}\right| < \frac{b}{xa^x \log c} < \frac{b}{q_k a^{q_k} \log c},$$

where a_{k+1} is the $(k+1)$-st partial quotient to $\frac{\log a}{\log c}$. Thus q_k and a_{k+1} satisfy the inequality

$$a_{k+1} + 2 > \frac{a^{q_k} \log c}{bq_k}. \tag{8}$$

Finally, using a computer program we checked that there do not exist any convergent $\frac{p_k}{q_k}$ of $\frac{\log a}{\log c}$ satisfying (8) when $q_k < 2521 \log 7m$ in the range $3 \le m \le 990$.

In view of the above proof, we have proved the following:

Proposition 1. *Let m be a positive integer with $m \ge 3$. Put $a = 44m^2 + 1$, $b = 5m^2 - 1$ and $c = 7m$. Then Pillai's equation*

$$c^z - a^x = b$$

has no positive integer solutions x, z with $x \ge 3$.

3.2. The Case m is Odd and $7 \mid m$

From Lemma 5, we may assume that $m \le 24028$ when $m \equiv 0 \pmod 7$.

Lemma 12. *Let (x, y, z) be a positive integer solution of Equation (2). Suppose that $m \equiv 0 \pmod 7$. Then the only positive integer solution of Equation (2) is $(x, y, z) = (1, 1, 2)$.*

Proof. Obviously $(1, 1, 2)$ is the only solution of Equation (2) for $M = \max\{x, y\} = 1$. Suppose that $M > 1$. From Lemma 4 for $m \ge 7$ we have that

$$1.35M < \left(2 - \frac{\log\left(\frac{49}{4}\right)}{\log 49}\right)M < z < 2M.$$

Hence $z \geq 3$. Taking Equation (2) modulo m, we see that y is odd. Here we apply Lemma 3. For this we set $p = 7$, $a_1 = 44m^2 + 1$, $a_2 = 1 - 5m^2$, $b_1 = x$, $b_2 = y$, and

$$\Lambda \coloneqq (44m^2 + 1)^x - (1 - 5m^2)^y.$$

Then we may take $g = 1$, $E = 2$, $A_1 = 44m^2 + 1$, $A_2 = 5m^2 - 1$. We get

$$2z \leq \frac{36.1(\max\{\log b' + \log(2\log 7) + 0.4, 12\log 7, 5\})^2 \log(44m^2+1)\log(5m^2-1)}{8(\log 7)^4},$$

where

$$b' = \frac{x}{\log(5m^2-1)} + \frac{y}{\log(44m^2+1)}.$$

Suppose that $z \geq 4$. We will show that this leads to a contradiction. Taking Equation (2) modulo m^4, we find

$$44x + 5y \equiv 0 \pmod{m^2}.$$

Then $M \geq \frac{m^2}{49}$. Since $z > \left(2 - \frac{\log(\frac{49}{4})}{\log(7m)}\right)M$ and $b' < \frac{M}{\log m}$ we have that

$$\begin{aligned} 2\left(2 - \frac{\log(\frac{49}{4})}{\log(7m)}\right)M \leq \frac{36.1}{8(\log 7)^4}&\left(\max\{\log\left(\frac{M}{\log m}\right) + \log(2\log 7) + 0.4, 12\log 7\}\right)^2 \\ &\times \log(44m^2+1)\log(5m^2-1). \qquad (9) \end{aligned}$$

Let

$$h = \max\{\log\left(\frac{M}{\log m}\right) + \log(2\log 7) + 0.4, 12\log 7\}.$$

Suppose that $\log\left(\frac{M}{\log m}\right) + \log(2\log 7) + 0.4 \geq 12\log 7$. Then the inequality $\log M \geq 12\log 7 - \log(2\log 7) - 0.4$ implies that $M \geq 2383998120$. On the other hand, from Inequality (9) we have that

$$2M \leq 0.32(\log M + 1.76)^2 \log(44 \cdot 24028^2 + 1)\log(5 \cdot 24028^2 - 1),$$

which implies that $M < 10061$, a contradiction. Hence $h = 12\log 7$ and therefore from Inequality (9) we get

$$\frac{2m^2}{49}\left(2 - \frac{\log(\frac{49}{4})}{\log(7m)}\right) \leq 172\log(44m^2+1)\log(5m^2-1).$$

This yields that $m \leq 795$. Hence

$$M \leq \frac{172\log(44m^2+1)\log(5m^2-1)}{2\left(2 - \frac{\log(\frac{49}{4})}{\log(7m)}\right)}$$

and therefore all x, y, z are bounded. Using a program in Maple we found that there is no (m, x, y, z) under consideration satisfying Equation (2). We conclude that $z \leq 3$. In this case, one can easily show that $(x, y, z) = (1, 1, 2)$. Thus there is no positive integer solution of Equation (2) other than $(x, y, z) = (1, 1, 2)$ when $7 \mid m$. $\square$

4. Example

In this section we verify that when $1 \leq m \leq 18$, Equation (2) has only the positive integer solution $(x, y, z) = (1, 1, 2)$.

Example 1. Let m be a positive integer with $1 \leq m \leq 18$. Then the Diophantine equation

$$(44m^2 + 1)^x + (5m^2 - 1)^y = (7m)^z$$

has only the positive integer solution $(x, y, z) = (1, 1, 2)$.

Proof. It follows from Theorem 1 that the above equation has only the positive integer solution $(x, y, z) = (1, 1, 2)$ in all cases $1 \leq m \leq 18$ except for the following two cases $m = 5, 9$:

$$(a)\ (3 \cdot 367)^x + (4 \cdot 31)^y = 35^z$$
$$(b)\ (5 \cdot 23 \cdot 31)^x + (4 \cdot 101)^y = 63^z$$

(a) `Case 1`: $y = 1$. Then it follows from Proposition 1 that the equation

$$35^z - (3 \cdot 367)^x = 124$$

has only the positive integer solution $z = 2, x = 1$.

`Case 2`: $y \geq 2$. Then taking (a) modulo 8 implies that $5^x \equiv 3^z \pmod 8$. By $(\frac{124}{367}) = 1$ and $(\frac{35}{367}) = -1$, we see that z is even and hence x is even, say $z = 2Z$, $x = 2X$. Then

$$(35^Z + (3 \cdot 367)^X)(35^Z - (3 \cdot 367)^X) = (4 \cdot 31)^y.$$

Thus we have the following two cases:

$$\begin{cases} 35^Z \pm (3 \cdot 367)^X = 2^{2y-1} \\ 35^Z \mp (3 \cdot 367)^X = 2 \cdot 31^y \end{cases}$$

or

$$\begin{cases} 35^Z \pm (3 \cdot 367)^X = 2^{2y-1} \cdot 31^y \\ 35^Z \mp (3 \cdot 367)^X = 2. \end{cases}$$

We deal with the first system. Adding these two equations yields

$$35^Z = 2^{2y-2} + 31^y.$$

Taking the above equation modulo 5 implies that $(-1)^{y-1} + 1 \equiv 0 \pmod 5$ and y is even, which is impossible.

Now we consider the second system. Adding these two equations yields

$$35^Z = 2^{2y-2}31^y + 1$$

Again taking the above equation modulo 5 we see that y is even, which is impossible.
(b) `Case 1`: $y = 1$. Then it follows from Proposition 1 that the equation

$$63^z - (5 \cdot 23 \cdot 31)^x = 404$$

has only the positive integer solution $z = 2, x = 1$.

`Case 2`: $y \geq 2$. Then taking (b) modulo 8 implies that $5^x \equiv (-1)^z \pmod 8$. By $(\frac{404}{23}) = 1$ and $(\frac{63}{23}) = -1$, we see that z is even and hence x is even, say $z = 2Z$, $x = 2X$. Then

$$(63^Z + (5 \cdot 23 \cdot 31)^X)(63^Z - (5 \cdot 23 \cdot 31)^X) = (4 \cdot 101)^y.$$

Thus we have the following two cases:

$$\begin{cases} 63^Z \pm (5 \cdot 23 \cdot 31)^X = 2^{2y-1} \\ 35^Z \mp (5 \cdot 23 \cdot 31)^X = 2 \cdot 101^y \end{cases}$$

or

$$\begin{cases} 63^Z \pm (5 \cdot 23 \cdot 31)^X = 2^{2y-1} \cdot 101^y \\ 63^Z \mp (5 \cdot 23 \cdot 31)^X = 2. \end{cases}$$

We deal with the first system. Adding these two equations yields

$$63^Z = 2^{2y-2} + 101^y.$$

Taking the above equation modulo 3 implies that y is odd. On the other hand, taking the same equation modulo 7, yields $4^{y-1} \equiv 4^y \pmod 7$, which is impossible.

Now we consider the second system. Adding these two equations yields

$$63^Z = 2^{2y-2}101^y + 1.$$

Again taking the above equation modulo 3 we see that y is odd. On the other hand, taking the same equation modulo 31, yields $1 \equiv 2^{5y-2} + 1 \pmod{31}$, which is impossible. $\square$

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