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A CONSENSUS MODEL FOR GROUP DECISION MAKING WITH HESITANT FUZZY INFORMATION

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ABSTRACT. This article presents a more improved consensus-based method for dealing with multi-person decision making (MPDM) that uses hesitant fuzzy preference relations (HFPR's) that aren't in the usual format. We proposed a Lukasiewicz transitivity (T_L-transitivity)-based technique for establishing normalised hesitant fuzzy preference relations (NHFPR's) at the most essential level, after that, a model based on consensus is constructed. After that, a transitive closure formula is created in order to build T_L -consistent hesitant fuzzy preference relations (HFPR's) and symmetrical matrices. Afterwards, a consistency analysis is performed to determine the degree of consistency of the data given by the decision makers (DMs), as a result, the consistency weights must be assigned to them. After combining consistency weights and preset(predefined) priority weights, the final priority weights vector of DMs is obtained (if there are any). The consensus process determines either data analysis and selection of a suitable alternative should be done directly or externally. The enhancement process aims to improve the DM's consensus measure, despite the implementation of an indicator for locating sluggish points, in the circumstance that an unfavorable agreement is achieved. Finally, a comparison case demonstrates the relevance and effectiveness of the proposed system. The conclusions indicate that the suggested strategy can provide insight into the MPDM system.

1. Introduction

Decision-making is an essential component of human existence. Several of them urge decision-makers to make "rational" or "great" judgments [1] based on a set of criteria for assessing particular possibilities. When selecting a preferences possibility in practical domains with a set of criteria that varies and may be classified, there is frequently a critera conflict [2]. Multi-criteria decision support approaches are often used for decision makers to evaluate choice options. One way decision support is implemented is as a means to evaluate the use of environmental, economic, and ecological factors.

Multi-person decision making, also referred to as MPDM, is an important method for achieving optimal choice results in modern society. However, given varying levels of expertise and interpretation, DM's contributions to the evaluation will vary depending on their abilities. This therefore presents a challenge when trying to come together and agree on a conclusion. This is a key challenge in the decision-making process as it begins with assessing, given that decisions are typically best suited to many different levels of consideration. One of the difficulties involved with consensus decision-making is achieving unanimous and acceptable outcomes. Since a

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GDM is a consensus-driven process, various strategies for reaching an effective consensus have been posited and investigated, resulting in an excessive percentage of data. Zhang et al. [3] have developed a cost-effective model for maximum support degree consensus processing, which can also be applied for reluctant information and linguistic judgments where the degree of certainty is low. Li et al. [4] proposed an effective interactive technique for achieving consensus at a low and unpredictable cost, fixing communication problems between human participants. A probabilistic hesitant fuzzy preference relation is a type of family of decision problems in which each member has its own associated rate of greater or less preference. Li and Wang [5] presented an automatic iterative approach to obtain a consensus level. Tian and Wang et al. and Zhang et al. [6, 7] developed consensus-reaching models of human behaviour, as did Herrera-Viedma et al. [8], who studied model analysis in fuzzy environments.

They used a consensus model with heterogeneous large-scale GDM with satisfaction and individual concerns to examine how individuals interact with one another. The hesitant fuzzy preference relation (HFPR) concept created by Xia and Xu [9] is currently being employed in MPDM as an efficient and simple approach for exchanging alternative facts across groups of DM's. The consensus-building mechanism, which is based on preference relations, is regularly utilized as a apparatus in MPDM strategies. The fuzzy preference relation (HFPR) concept made by Xia and Xu is as of now being used as an proficient and basic instrument for a gather of DMs to trade elective actualities. Many academics have explored the HFPR suggested in [9] in the context of GDM [12, 13, 14], but significant disagreements persist, for example, while offering the decision degree to which an alternative x_1 is superior to another alternative x_2 for a group of three DMs. If none of the DMs are able to agree on their appraisal, the preferred degree of x_1 to x_2 can be a set made up of their combined judgements expressed as the hesitant fuzzy element (HFE). The HFPR has the immediate benefit of providing the DMs with a set of values displaying the outcomes of the evaluation. However, the HFEs supply a limited amount of components, which might make reaching an agreement difficult. Most consensus models, for example, are focused on calculating the distance between two HFPRs, and determining an effective distance between them is quite challenging [15].

Many scholars have created various ways for obtaining the priority weight vector, as well as consensus reaching models . Zhu et al. [16] were the first to propose a and b normalisation procedures. The normalization-based strategy demanded that any two HFEs have the same amount of elements. This strategy was highly regarded by a number of decision-making academics. Zhang and Wu [17] developed goal programming models for the incomplete hesitant multiplicative preference relation (HFPR) and estimated the priority weight vectors using a and b normalisation approaches. Meng et al. [18] developed a unique consistency technique for hesitant multilicative preference relations, which yielded the hesitant fuzzy priority weight vector. Zhang et al. [19] established various preference relations based on q-rung orthopair fuzzy sets and studied a method for extracting priority weights from such relations. Since many researchers have been compelled to use these two procedures indefinitely, a plethora of other normalisation methods have been developed. For example, Xu et al. [20] created a consensus model for tackling water allocation management problems using an additive consistency-based normalising technique.

Because of their limited competence and experience, DMs may struggle to construct entire preference connections during paired judgments on alternatives. Approaches that aid in the management of HFPRs with insufficient information are required.

Zhang et al. [21] suggested a technique for guessing missing HFPR components. Khalid and Beg [22] suggested an algorithm for the DMs that makes use of a hesitant upper bound condition. In paired evaluations of DM's preferences, the consistency of fuzzy preference relations (HFPR) is crucial in the decision-making process. The distance metric between normalised HFPR and consistent HFPR is crucial in determining the consistency degree of HFPR. Zhu et al. [23] developed a regression approach and methodology for transforming HFPR into a fuzzy preference relation with the maximum level of consistency.

In this paper, we define additive reciprocity which is an effect way to calculate the DM's in the MPDM problems. Additive reciprocity is more sufficient ans easy method to determine the unknown and missing values. This work proposes a consensus-based solution for dealing with the MPDM problem utilising consistent HFPRs. The authors present an enhanced approach for suggesting agreement in group decision making based on T_L-consistency in the context of HFPRs. They also suggest a technique for accelerating the execution of a higher consensus level on an easy path. When an FPR has missing preferences, the proposed technique estimates more appropriate and consistent values. The attribute of consistency is linked to the transitivity property, for which numerous relevant forms or conditions have been proposed. In the first phase, we use the T_L-transitivity characteristic to assess the missing preferences of IFPRs. Then, we propose modified consistency matrices of experts that must meet T_L-consistency. The experts are assigned a level of relevance based on accuracy weights combined with confidence weights.

A new approach for normalising the HFPRs is developed using additive reciprocity and then extended to the MPDM issue utilising consistency and consensus metrics. Section 4 includes a comparison example to evaluate the efficacy of the suggested strategy. This text is arranged as follows: in Section 2, some fundamental definitions are presented to assist the reader in understanding the work; in Section 3, a novel process is proposed and its efficiency is evaluated. Section 5 compares the findings achieved using our suggested approach to those found in the literature. The final part has some conclusions.

2. Preliminaries

A fuzzy set is a notion established by L. A. Zadeh [24] in 1965 to describe how an item is more or less tied to a certain category to which we desire to adapt. Some basic information is provided in this part to assist everyone understand the topic better.

Definition 1. *Fuzzy Set* [24]*:The fuzzy set is defined as follows: If X is a discourse universe and x is a specific element of X, then a fuzzy set A defined on X can be expressed as a collection of ordered pairs $A = \{(x, \mu_A(x)), x \in X\}$. As part of the membership function $A : X \rightarrow [0, 1]$, the degree to which x relates to A is denoted as $\mu_A(x)$.*

Definition 2. *Hesitant Fuzzy Set* [25]*:Let X represent a universal set. An HFS on X is defined as a function that returns a subset of $[0, 1]$ when applied to X. The membership degree of $x \in X$ is represented by this value, which is referred to as the hesitant fuzzy element (HFE).*

Definition 3. *Fuzzy Preference Relation* [26]*: If the set of alternatives* $A = \{A_1, A_2, ..., A_n\}$ *exists, then the fuzzy preference relation* U *on* $A \times A$ *with the requirement* $u_{ij} \geq 0$, $u_{ij} + u_{ji} = 1$ *exists, is called* ***additive reciprocity*** [27], $i, j = 1, 2, ..., n$*, the degree to which the option* A_i *is prior to the alternative* A_j *is denoted by* u_{ij}*.* A_i *and* A_j*, represented by* $A_i \prec A_j$ *are indifference at* $u_{ij} = 0.5$ *;* $0 \leq u_{ij} < 0.5$ *denotes that* A_j *is preferable over* A_i *which is indicated by* $A_j \succ A_i$*, the lower the value of* u_{ij}*, the better, the stronger the preference for* A_j *over* A_i*, the better;* $0.5 < u_{ij} \leq 1$ *denotes that* A_i *is preferable to* A_j*, which is symbolised by* $A_i \succ A_j$*, the greater* u_{ij}*'s value, the higher the preference for* A_i *over* A_j*, the better. It should be emphasised that in a fuzzy preference relation, the value* u_{ij} *is a number between* 0 *and* 1*.*

Definition 4. *Hesitant fuzzy preference relation* [28]*:let* $A = \{A_1, A_2, ..., A_n\}$ *is a fixed set, then a matrix* $H = (h_{ij})_{n \times n} \sqsubset A \times A$ *presents a hesitant fuzzy preference relation* H *on* A*, where* $h_{ij} = \{h_{ij}^{\alpha} \mid \alpha = 1, 2, ...l_{h_{ij}}\}$ *is an HFE reflecting all the possible degrees to which* A_i *is preferred over* A_j*, in* h_{ij} h_{ij}^{α} *is the* α*th element, the number of values in* h_{ij} *is represented by* $l_{h_{ij}}$*. Furthermore,* h_{ij} *must meet the following requirements:*

$$\begin{Bmatrix} h_{ij}^{\alpha} + h_{ji}^{\alpha} = 1, \ i, j = 1, 2, ..., n \\ h_{ii} = \{0.5\}, \ i = 1, 2, ..., n \\ l_{h_{ij}} = l_{h_{ji}}, \ i, j = 1, 2, ..., n \end{Bmatrix}$$

Definition 5. *Incomplete Fuzzy Preference Relation* [11]*:If* $U = (u_{ij})_{n \times n}$ *is a preference relation, then* U *is referred to as a incomplete fuzzy preference relation, if the DM is unable to provide some of its elements, it is represented by the unknown number* x*, and the DM will be able to offer the remaining.*

Definition 6. *Additive Transitivity:if* $\forall \ i, k, j \ \in X$ *holds, then* R *is known as additive transitive:*

$$R(x, z) = R(x, y) + R(y, z) - 0.5$$

Definition 7. *Lukasiewicz Transitivity:* U *is said to be* T_L*-transitive if it holds* $\forall \ i, k \neq j \in \{1, 2, ..., n\} : \ u_{ik} \geq \max(u_{ij} + u_{jk} - 1, 0)$.

Definition 8. *Consistent Fuzzy Prefrence Relation:A Fuzzy Prefrence Relation* U *is said to be* T_L*-transivity(consistent), if for* $\forall \ i, k \neq j \in \{1, 2, ..., n\} : \ u_{ik} \geq \max(u_{ij} + u_{jk} - 1, 0)$ *(*T_L*-transivity) is satisfied.*

3. Methodology

Throughout this section, we offered a superior technique for dealing with MPDM issues using HFPRs, which included the following phases: normalisation, consistency measures, consensus measures, consensus enhancing process, allocating priority weights to DMs, and selection process.

3.1. **Normalization.** Within this part, it is proposed that a new approach for normalising HFPRs be developed, because for any two hesitant fuzzy preference values $(HFPVs)$, h_{ij} and h_{lm}, $|h_{ij}| \neq \ |h_{lm}|$ for $i, j, l, m \in \{1, 2, ..., n\}$ in the majority of cases, Sets of pairwise comparisons with their cardinalities at the ijth and lmth places are represented by $|h_{ij}|$ & $|h_{lm}|$. We use additive reciprocity to discover the unknown preferences of IFPRs in the first stage. Regarding the normalising of the given HFPRs, we propose a new method for calculating the components that will be added to HFPVs. Other intermediate values cannot be added to the estimated element since it is just the maximum or lowest entry of HFPV. The suggested technique is based on additive reciprocity, in which we generate the incomplete fuzzy preference relation(s) IFPRs using unknown preference values for the components to be added.

3.2. **Determine Missing Values.** An $IFPR$ based on T_L-consistency can only be completed if each option among the known preference values is investigated at least once. As a result, the system must request that the expert create a sufficient number of preferences, with each possibility being examined at least once in order for the $IFPR$ to become a complete FPR. The sequence in which the missing preference values are measured has an effect on the final outcome. For the purpose of constructing the unknown or missing preference values in an $IFPR\, R = (r_{ij})_{n\times n}$, the below mentioned sets are used to denote the alternative's pairs for known and unknown or missing preference values

$$K_e = \{(i,j)|r_{ij} \ is\ known\ value\}, \tag{3.1}$$

$$U_e = \{(i,j)|r_{ij} \ is\ unknown\ value\}, \tag{3.2}$$

where $r_{ij} \in [0,1]$ are the preference values of a_i over a_j, $r_{ij} + r_{ji} = 1 \Longrightarrow r_{ii} = 0.5\ \forall\ i \in \{1,2,....,n\}$. As a result, based on T_L-transitivity $r_{ik} \geq max\ (r_{ik} + r_{kj} - 1, 0)$, to estimate the unknown preference value r_{ij} of alternative a_i over alternative a_j, the following set can be defined.

$$E_{ij} = \{k \neq i,j|(i,k) \in Ke, (k,j) \in K_e \ and\ (i,j) \in U_e\} \tag{3.3}$$

for $i,j,k \in \{1,2,3,...,n\}$. Using the Equation (3.3), the final value of r_{ij} is calculated as follows:

$$r_{ij} = \begin{cases} ave_{k\in E_{ij}}\ (\max(r_{ik} + r_{kj} - 1, 0)), & if\ |E_{ij}| \neq 0 \\ 0.5, & otherwise \end{cases} \tag{3.4}$$

and

$$r_{ji} = 1 - r_{ij} \qquad (\textbf{additive reciprocity}) \tag{3.5}$$

Finally, the entire FPR $R^* = (r^*_{ij})_{n\times n}$ is produced. Now, two additional sets of known and unknown elements, $\acute{K_e}$ and $\acute{U_e}$, are defined as follows:

$$\acute{K_e} = K_e\ \cup \{(i,j)\},\ \text{and}\ \acute{U_e} = U_e\ - \{(i,j)\} \tag{3.6}$$

As a result, a normalised hesitant fuzzy preference relation (NHFPR) $H^* = (h^*_{ij})_{n\times n}$ with

$$h^{*\beta}_{ij} + h^{*\beta}_{ji} = 1,\ h^*_{ij} = \{h^{*\beta}_{ij}\ |\beta = 1,2,...,|h^*_{ij}|\}\ \ for\ |h^*_{ij}| = |h^*_{ji}|$$

is constructed. Because of the growth in complexity, there are several decision-making procedures that occur in multi-person situations in the real world and because of the unpredictability of the socioeconomic environment, it is more difficult for a single decision maker can assess all of the interconnected components of a decision-making challenge.

3.3. **Consistency Analysis.** Some consistency measurements, such as the amount of consistency between two alternatives, the degree of consistency among alternatives, and HFPR's level of consistency, are specified in this subsection. The term consistency index(CI) refers to the degree of consistency with a value between 0 and 1. Assume H_q is the HFPR for the decision maker $D_q (1 \leq q \leq l)$, then after receiving the NHFPR H^{*q}, The following transitive closure formula may be used to obtain T_L-consistent HFPR $\tilde{H}^{*q}$:

$$\tilde{h}_{ij}^{*q\beta} = \max_{k \neq i,j} (h_{ij}^{*q\beta}, \max(h_{ik}^{*q\beta}, h_{kj}^{*q\beta} - 1, 0)) \ , \ \tilde{h}_{ij}^{*q\beta} + \tilde{h}_{ji}^{*q\beta} = 1 \tag{3.7}$$

where $h_{ij}^{*q} = \{h_{ij}^{*\beta} \ | \beta = 1, 2, ..., |h_{ij}^{*}|\}$. After analysing the distance between them in the following manner, we can calculate the HFPR H^{*q} consistency level based on its similarity to the corresponding T_L-consistency $\tilde{H}^{*q}$.

1. T_L consistency index($T_L CI$) for a pair of alternatives ranked as follows:

$$T_L CI(h_{ij}^{*q}) = 1 - \frac{1}{|h_{ij}^{*}|} \sum_{\beta=1}^{|h_{ij}^{*}|} d(\ h_{ij}^{*q\beta}, \tilde{h}_{ij}^{*q\beta} \ ) \tag{3.8}$$

where $d(\ h_{ij}^{*q\beta}, \tilde{h}_{ij}^{*q\beta} \ )$ denotes the distance obtained by $d(\ h_{ij}^{*q\beta}, \tilde{h}_{ij}^{*q\beta} \ ) = | \ h_{ij}^{*q\beta} - \tilde{h}_{ij}^{*q\beta} \ |$. The greater the level of $T_L CI(h_{ij}^{*q})$, the more consistent h_{ij}^{*q} is in comparison to the other HFPVs when it comes to the options a_i and a_j.

2. $T_L CI$ for alternatives a_i, $1 \leq i \leq n$,as follows:

$$T_L CI(a_i) = \frac{1}{2(n-1)} \sum_{j=1, j \neq i}^{n} (T_L CI \ (h_{ij}^{*q}) + T_L CI(h_{ji}^{*q}) \ ) \tag{3.9}$$

with $T_L CI(a_i) \in [0, 1]$. If $T_L CI(a_i) = 1$, then the alternative a_i preference values are totally consistent; on the other hand, the lower $T_L CI(a_i)$, the more inconsistency there is in these preference values.

3. Finally, $T_L CI$ against NHFPR the average operator is used to assess H^{*q}:

$$T_L CI(H^{*q}) = \frac{1}{n} \sum_{i=1}^{n} T_L CI(a_i) \tag{3.10}$$

with $T_L CI(H^{*q}) \in [0, 1]$. If $T_L CI(H^{*q}) = 1$, NHFPR H^{*q} is totally consistent, if $T_L CI(H^{*q})$ is less, H^{*q} is more inconsistent. Equation (3.10) assigns DM D_q to the consistency index, however the global consistency index(CI) may be calculated using the average operator and is provided as:

$$CI = \frac{1}{l} \sum_{q=1}^{l} T_L CI(H^{*q}) \tag{3.11}$$

with $CI \in [0, 1]$. The $T_L CI$ is calculated in three steps, each comprising Equations (3.8)-(3.10), Experts who produced the HFPR with better consistency indices should be given more weights. As a result, the following relationship may be used to assign consistency weights to experts:

$$Cw(Dq) = \frac{T_L CI(H^{*q})}{\sum_{q=1}^{l} T_L CI(H^{*q})} \tag{3.12}$$

with $Cw(Dq) \in [0,1]$ & $\sum_{q=1}^{l} Cw(Dq) = 1$.

3.4. **Consensus Analysis.** Some levels for estimating global consensus among decision makers are defined in this subsection. It is critical to determine the amount of consensus among decision makers after examining NHFPRs $H^{*q}, q = 1, 2, ..., l$. After aggregating the similarity matrices $S^{qr} = (s_{ij}^{qr})_{n\times n}$ for each pair of decision makers $(D_q, D_r)(q = 1, 2, ..., l-1; r = q+1, ..., l)$, $S = (s_{ij})_{n\times n}$ is a collective similarity matrix that may be built as follows:

$$S = (s_{ij})_{n\times n} = (\frac{2}{l(l-1)} \sum_{q=1}^{l-1} \sum_{r=q+1}^{l} (1 - \frac{1}{|h_{ij}^*|} \sum_{\beta=1}^{|h_{ij}^*|} d(\ h_{ij}^{*q\beta}, h_{ij}^{*r\beta}\ )))_{n\times n} \tag{3.13}$$

where $1 - \frac{1}{|h_{ij}^*|} \sum_{\beta=1}^{|h_{ij}^*|} d(\ h_{ij}^{*q\beta}, h_{ij}^{*r\beta}\ ) = s_{ij}^{qr}$ & $d(\ h_{ij}^{*q\beta}, h_{ij}^{*r\beta}\ ) = |\ h_{ij}^{*q\beta} - h_{ij}^{*r\beta}\ |$, $\{\beta = 1, 2, ..., |h_{ij}^*|\}$. The following levels are used to determine the degree of global consensus among decision-makers:

1. The degree of consensus on a pair of alternatives (a_i, a_j)

$$cd_{ij} = s_{ij} \tag{3.14}$$

2. The degree of consensus on alternatives a_i, as stated by CD_i:

$$CD_i = \frac{1}{2(n-1)} \sum_{j=1, j\neq i}^{n} (s_{ij} + s_{ji}) \tag{3.15}$$

3. The degree of consensus on the relation represented by CR is specified at the third level to determine the global consensus degree among all DMs:

$$CR = \frac{1}{n} \sum_{i=1}^{n} CD_i \tag{3.16}$$

If all specialists reach a level of global consensus, a comparison with the threshold consensus degree η is required, the nature of the problem is typically predetermined. If $CR \leq \eta$ is achieve, it means that there has been a sufficient level of consensus obtained, and the process of decision-making may begin. Aside from that, the consensus degree is unstable, and experts are being pressed to alter their preferences.

3.5. **Enhancement Mechanism.** The enhancement mechanism acts as a moderator in the consensus-building process, providing decision makers with detailed information to help them improve their results. In case of inadequate consensus level, we must determine the places where preference values will be updated, so that decision-makers can attain an appropriate level of consensus In this regard, the following is a definition of an identifier:

$$I^q = \{(i,j)|\ cd_{ij}\ < CR\ \ \&\ h_{ij}^{q\beta}\ is\ \ a\ known\ value\} \tag{3.17}$$

Once the positions have been established using an identifier, the enhancement mechanism recommends that the relevant DM D_q augment the element $h_{ij}^{q\beta}$ of HFPV h_{ij}^q, if it is less than the mean value $h_{ij,ave}^{*q\beta}$ of the participant's opinions, or to decline if the value is more than the mean, and to remain the same if the value is equal to the mean. To automatically improve the consensus, the DMs would not have to give their updated components in the automated procedure. In such a case, the new element $h^{q\beta}_{ij,new}$ may be evaluated using the following equation, new for $cd_{ij} < CR$:

$$h^{q\beta}_{ij,new} = \lambda h^{q\beta}_{ij} + (1-\lambda) h^{*q\beta}_{ij,ave} \tag{3.18}$$

where $\lambda \in [0,1]$ is known as the optimization parameter. The new evaluated values will undoubtedly be closer to the mean values than the previous ones,and so the degree of consensus will increase.

3.6. **Rating of Decision Makers.** To analyse decision makers' final priority evaluations, emerging consistency weights and predefined priority weights are employed, which are as follows:

$$w(D_q) = \frac{w_q \times Cw(Dq)}{\sum_{q=1}^{l} w_q \times Cw(Dq)} \tag{3.19}$$

where $w_q, 1 \leq q \leq l$, denotes the decision maker's predefined priority weights, while $\sum_{q=1}^{l} w(D_q) = 1$. If the DMs do not use predefined priority weights, then, as the final priority rating, their consistency weights will be used.

3.7. **Aggregated NHFPR.** On a regular basis, the preference level associated with each DM may be weighted differently. After reviewing decision maker's priority ratings, their views are to be consolidated into a global one. Using the weighted average operator, we create the collective consistent NHFPR H^{*c} as follows:

$$H^{*c} = (h^{*c}_{ij})_{n\times n} = (\sum_{q=1}^{l} w(D_q) \times \tilde{h}^{*q}_{ij})_{n\times n},\ for 1 \leq i \leq n, \quad 1 \leq j \leq n. \tag{3.20}$$

3.8. **Ranking of Alternatives.** When the DMs attain an appropriate degree of consensus, the process of ranking the alternatives begins and the best one is chosen. The ranking value $v(a_i)$ for alternative $a_i, (i = 1, 2, ...n)$, is defined in this context, as follows:

$$v(a_i) = \frac{2}{n(n-1)} \sum_{j=1, j\neq i}^{n} \left(\frac{1}{|h^{*c}_{ij}|} \sum_{\beta=1}^{|h^{*c}_{ij}|} h^{*c\beta}_{ij}\right) \tag{3.21}$$

with $\sum_{i=1}^{n} v(a_i) = 1$.

4. Comparative Example

A bank wants to put a particular amount of money into the best option. To reduce the risks of making judgments in this highly competitive and fast-paced industry, the company's leader enlists the assistance of a group of experts in the decision-making process in the hopes of reaching a consensus. The following is a panel with three options:

- a_1 is the automobile industry,
- a_2 is the food industry,
- a_3 is the computer industry

Three specialists have been consulted,$D_q, q = 1, 2, 3$ from three consulting departments is tasked with assessing the three options. $a_i, i = 1, 2, 3$.Following pairwise comparisons, the DMs $D_q, q = 1, 2, 3$ offer the following HFPRs $H^q, q = 1, 2, 3$, respectively with threshold consensus level $\eta = 0.8$.

$$
\begin{aligned}
H^1 &= \begin{bmatrix} \{0.5\} & \{0.3, 0.5\} & \{0.5, 0.6, 0.7\} & \{0.4\} \\ \{0.7, 0.5\} & \{0.5\} & \{0.4, 0.6\} & \{0.6, 0.7\} \\ \{0.5, 0.4, 0.3\} & \{0.6, 0.4\} & \{0.5\} & \{0.6, 0.8\} \\ \{0.6\} & \{0.4, 0.3\} & \{0.4, 0.2\} & \{0.5\} \end{bmatrix} \\
H^2 &= \begin{bmatrix} \{0.5\} & \{0.7, 0.6\} & \{0.8, 0.6\} & \{0.4\} \\ \{0.3, 0.4\} & \{0.5\} & \{0.8, 0.7, 0.5\} & \{0.4, 0.3\} \\ \{0.2, 0.4\} & \{0.2, 0.3, 0.5\} & \{0.5\} & \{0.7, 0.6\} \\ \{0.6\} & \{0.6, 0.7\} & \{0.3, 0.4\} & \{0.5\} \end{bmatrix} \\
H^3 &= \begin{bmatrix} \{0.5\} & \{0.7, 0.6, 0.4\} & \{0.6\} & \{0.4, 0.3\} \\ \{0.3, 0.4, 0.6\} & \{0.5\} & \{0.5\} & \{0.6\} \\ \{0.4\} & \{0.5\} & \{0.5\} & \{0.9, 0.5, 0.4\} \\ \{0.6, 0.7\} & \{0.4\} & \{0.1, 0.5, 0.6\} & \{0.5\} \end{bmatrix}
\end{aligned}
$$

Normalization:

Equation (3.1)–(3.6) were utilised to generate NHFPRs in order to normalise the provided data

$$
H^{*1} = \begin{bmatrix} \{0.5\} & \{0.3, 0.5, 0.9\} & \{0.5, 0.6, 0.7\} & \{0.4, 0.3, 0.4\} \\ \{0.7, 0.5, 0.1\} & \{0.5\} & \{0.4, 0.6, 0.6\} & \{0.6, 0.7, 0.3\} \\ \{0.5, 0.4, 0.3\} & \{0.6, 0.4, 0.4\} & \{0.5\} & \{0.6, 0.8, 0\} \\ \{0.6, 0.7, 0.6\} & \{0.4, 0.3, 0.7\} & \{0.4, 0.2, 1\} & \{0.5\} \end{bmatrix}
$$

$$
\begin{aligned}
H^{*2} &= \begin{bmatrix} \{0.5\} & \{0.7, 0.6, 0.1\} & \{0.8, 0.6, 0.6\} & \{0.4, 0.1, 0.4\} \\ \{0.3, 0.4, 0.9\} & \{0.5\} & \{0.8, 0.7, 0.5\} & \{0.4, 0.3, 0.2\} \\ \{0.2, 0.4, 0.4\} & \{0.2, 0.3, 0.5\} & \{0.5\} & \{0.7, 0.6, 0.6\} \\ \{0.6, 0.9, 0.6\} & \{0.6, 0.7, 0.8\} & \{0.3, 0.4, 0.4\} & \{0.5\} \end{bmatrix} \\
H^{*3} &= \begin{bmatrix} \{0.5\} & \{0.7, 0.6, 0.4\} & \{0.6, 0, 0.6\} & \{0.4, 0.3, 0\} \\ \{0.3, 0.4, 0.6\} & \{0.5\} & \{0.5, 0, 0.2\} & \{0.6, 0, 0.6\} \\ \{0.4, 1, 0.4\} & \{0.5, 1, 0.8\} & \{0.5\} & \{0.9, 0.5, 0.4\} \\ \{0.6, 0.7, 1\} & \{0.4, 1, 0.4\} & \{0.1, 0.5, 0.6\} & \{0.5\} \end{bmatrix}
\end{aligned}
$$

Consistency Analysis:

The HFPR's consistency levels as stated by decision makers were measured using expressions (3.7)–(3.12).

$$
\tilde{h}_{ij}^{*q\beta} = \max_{k \neq i,j}(h_{ij}^{*q\beta}, \max(h_{ik}^{*q\beta}, h_{kj}^{*q\beta} - 1, 0)) \ , \ \tilde{h}_{ij}^{*q\beta} + \tilde{h}_{ji}^{*q\beta} = 1
$$

$$\tilde{H}^{*1} = \begin{bmatrix} \{0.5\} & \{0.3,0.5,0.9\} & \{0.5,0.6,0.7\} & \{0.4,0.4,0.4\} \\ \{0.7,0.5,0.1\} & \{0.5\} & \{0.4,0.6,0.6\} & \{0.6,0.7,0.3\} \\ \{0.5,0.4,0.3\} & \{0.6,0.4,0.4\} & \{0.5\} & \{0.6,0.8,0\} \\ \{0.6,0.6,0.6\} & \{0.4,0.3,0.7\} & \{0.4,0.2,1\} & \{0.5\} \end{bmatrix}$$

$$\tilde{H}^{*2} = \begin{bmatrix} \{0.5\} & \{0.7,0.6,0.2\} & \{0.8,0.6,0.6\} & \{0.5,0.2,0.4\} \\ \{0.3,0.4,0.8\} & \{0.5\} & \{0.8,0.7,0.5\} & \{0.5,0.3,0.3\} \\ \{0.2,0.4,0.4\} & \{0.2,0.3,0.5\} & \{0.5\} & \{0.7,0.6,0.2\} \\ \{0.5,0.8,0.6\} & \{0.5,0.7,0.7\} & \{0.3,0.4,0.8\} & \{0.5\} \end{bmatrix}$$

$$\tilde{H}^{*3} = \begin{bmatrix} \{0.5\} & \{0.7,0.6,0.4\} & \{0.6,0,0.6\} & \{0.5,0.3,0\} \\ \{0.3,0.4,0.6\} & \{0.5\} & \{0.5,0,0.2\} & \{0.6,0,0.6\} \\ \{0.4,1,0.4\} & \{0.5,1,0.8\} & \{0.5\} & \{0.9,0.5,0.4\} \\ \{0.5,0.7,1\} & \{0.4,1,0.4\} & \{0.1,0.5,0.6\} & \{0.5\} \end{bmatrix}$$

(i). In NHFPRs H^{*q}, $q = 1,2,3$, the consistency measures of pairs of alternatives are:

$$T_LCI(h_{ij}^{*1}) = \begin{bmatrix} 1 & 1 & 1 & 0.9667 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0.9667 & 1 & 1 & 1 \end{bmatrix}, \ T_LCI(h_{ij}^{*2}) = \begin{bmatrix} 1 & 0.9667 & 1 & 0.9333 \\ 0.9667 & 1 & 1 & 0.9333 \\ 1 & 1 & 1 & 0.8667 \\ 0.9333 & 0.9333 & 0.8667 & 1 \end{bmatrix}$$

$$T_LCI(h_{ij}^{*3}) = \begin{bmatrix} 1 & 1 & 1 & 0.9667 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0.9667 & 1 & 1 & 1 \end{bmatrix}$$

(ii). T_LCI alternatives $a_1, a_2, a_3, \&\ a_4$ are:

$$T_LCI(a_1) = (0.9889, 0.9667, 0.9889), \ T_LCI(a_2) = (1, 0.9667, 1)$$

$$T_LCI(a_3) = (1, 0.9556, 1)\ , \ T_LCI(a_3) = (0.9889, 0.9111, 0.9889)$$

(iii). T_LCI against NHFPRs H^{*q} are:

$$\begin{aligned} T_LCI(H^{*1}) &= 0.9945, \ T_LCI(H^{*2}) = 0.95, \\ T_LCI(H^{*3}) &= 0.9945 \end{aligned}$$

The global consistency index (CI) may be calculated using (3.11) as follows:

$$CI = 0.9797$$

Now, using (3.12),estimate the consistency weights of the DMs D_1, D_2 *and* D_3 as follows:

$$\begin{aligned} Cw(D_1) &= 0.3384, Cw(D_2) = 0.3232 \\ Cw(D_3) &= 0.3384 \end{aligned}$$

Now $\sum_{q=1}^{l} Cw(D_q) = 1.$

Consensus measures:

$$s_{ij}^{12} = \begin{bmatrix} 1 & 0.5667 & 0.8667 & 0.9333 \\ 0.5667 & 1 & 0.8 & 0.7667 \\ 0.8667 & 0.8 & 1 & 0.7 \\ 0.9333 & 0.7667 & 0.7 & 1 \end{bmatrix}, \; s_{ij}^{13} = \begin{bmatrix} 1 & 0.6667 & 0.7333 & 0.8667 \\ 0.6667 & 1 & 0.6333 & 0.6667 \\ 0.7333 & 0.6333 & 1 & 0.6667 \\ 0.8667 & 0.6667 & 0.6667 & 1 \end{bmatrix}$$

$$s_{ij}^{23} = \begin{bmatrix} 1 & 0.9 & 0.7333 & 0.8 \\ 0.9 & 1 & 0.5667 & 0.7 \\ 0.7333 & 0.5667 & 1 & 0.8333 \\ 0.8 & 0.7 & 0.8333 & 1 \end{bmatrix}$$

Similarity matrix

$$S = \begin{bmatrix} 1 & 0.7111 & 0.7778 & 0.8667 \\ 0.7111 & 1 & 0.6667 & 0.7111 \\ 0.7778 & 0.6667 & 1 & 0.7333 \\ 0.8667 & 0.7111 & 0.7333 & 1 \end{bmatrix}$$

(i). Consensus degree on a pair of alternatives (a_i, a_j) using the similarity matrix S :

$$cd_{ij} = s_{ij}, \; i, j = 1, 2, 3, 4$$

(ii). Consensus degree on alternatives (a_i)

$$\begin{aligned} CD_1 &= 0.7852, \quad CD_2 = 0.6963, \\ CD_3 &= 0.7259, \quad CD_4 = 0.7704, \end{aligned}$$

(iii). Consensus degree on the relation

$$CR = 0.7445$$

Final weights of DMs:

The consistency weights $Cw(D_q), q = 1, 2, 3$ will be utilised as the DMs final weights by using (3.19),because no pre-determined weights are involved.As a result,

$$\begin{aligned} w(D_1) &= 0.3384, \quad w(D_2) = 0.3232 \\ w(D_3) &= 0.3384, \end{aligned}$$

Collective NHFPR construction :
After applying (3.20), we get the collective NHFPR H^{*c}

$$H^{*c} = \begin{bmatrix} \{0.5\} & \{0.55, 0.56, 0.51\} & \{0.63, 0.39, 0.63\} & \{0.47, 0.3, 0.27\} \\ \{0.45, 0.44, 0.49\} & \{0.5\} & \{0.57, 0.43, 0.43\} & \{0.56, 0.33, 0.38\} \\ \{0.37, 0.61, 0.37\} & \{0.43, 0.57, 0.57\} & \{0.5\} & \{0.73, 0.63, 0.15\} \\ \{0.53, 0.7, 0.73\} & \{0.44, 0.67, 0.62\} & \{0.27, 0.37, 0.85\} & \{0.5\} \end{bmatrix}$$

Final Ranking of alternatives:

The equation (3.21) is used to get the final ranking order of the alternatives after analysing the ranking values.

$$v(a_1) = 0.2394, \quad v(a_2) = 0.2267, \quad v(a_3) = 0.2461 \quad \& \quad v(a_4) = 0.2878$$

$\sum_{i=1}^{n} v(a_i) = 1$.As a result,the preferred order of alternatives is

$$a_4 \succ a_3 \succ a_1 \succ a_2$$

Enhancement mechanism:
As

$$0.7445 = CR < \eta(given\ \eta = 0.8)$$

As a result,DMs must alter their preferences with the help of (3.17).The following are the mean values of the expert's preferences:

$$H_{ave} = \begin{bmatrix} \{0.5\} & \{0.57, 0.57, 0.47\} & \{0.63, 0.4, 0.63\} & \{0.4, 0.23, 0.27\} \\ \{0.43, 0.43, 0.53\} & \{0.5\} & \{0.57, 0.43, 0.43\} & \{0.53, 0.33, 0.37\} \\ \{0.37, 0.6, 0.37\} & \{0.43, 0.57, 0.57\} & \{0.5\} & \{0.73, 0.63, 0.33\} \\ \{0.6, 0.77, 0.73\} & \{0.47, 0.67, 0.63\} & \{0.27, 0.37, 0.67\} & \{0.5\} \end{bmatrix}$$

The identifier (3.17) now offers the following set of options for improving relevant preference values.

$$I^q = \{(1,2), (2,1), (2,3), (3,2), (2,4), (4,2), (3,4), (4,3)\}.$$

Assume that the DMs using (3.18) were pleased with the suggestions and improved their preference relations as a result:

$$H_{new}^{1} = \begin{bmatrix} \{0.5\} & \{0.435, 0.535\} & \{0.5, 0.6, 0.7\} & \{0.4\} \\ \{0.565, 0.465\} & \{0.5\} & \{0.485, 0.515\} & \{0.565, 0.515\} \\ \{0.5, 0.4, 0.3\} & \{0.515, 0.485\} & \{0.5\} & \{0.665, 0.715\} \\ \{0.6\} & \{0.435, 0.485\} & \{0.335, 0.285\} & \{0.5\} \end{bmatrix}$$

$$H_{new}^{2} = \begin{bmatrix} \{0.5\} & \{0.635, 0.585\} & \{0.8, 0.6\} & \{0.4\} \\ \{0.365, 0.415\} & \{0.5\} & \{0.685, 0.565, 0.465\} & \{0.465, 0.315\} \\ \{0.2, 0.4\} & \{0.315, 0.435, 0.535\} & \{0.5\} & \{0.715, 0.615\} \\ \{0.6\} & \{0.535, 0.685\} & \{0.285, 0.385\} & \{0.5\} \end{bmatrix}$$

$$H_{new}^{3} = \begin{bmatrix} \{0.5\} & \{0.635, 0.585, 0.435\} & \{0.6\} & \{0.4, 0.3\} \\ \{0.365, 0.415, 0.565\} & \{0.5\} & \{0.535\} & \{0.565\} \\ \{0.4\} & \{0.465\} & \{0.5\} & \{0.815, 0.565, 0.365\} \\ \{0.6, 0.7\} & \{0.435\} & \{0.185, 0.435, 0.635\} & \{0.5\} \end{bmatrix}$$

Normalization:

Equation (3.1)–(3.6) were utilised to generate NHFPRs in order to normalise the provided data

$$H_{new}^{*1} = \begin{bmatrix} \{0.5\} & \{0.435, 0.54, 0.19\} & \{0.5, 0.6, 0.7\} & \{0.4, 0.18, 0.4\} \\ \{0.56, 0.46, 0.81\} & \{0.5\} & \{0.48, 0.51, 0.51\} & \{0.56, 0.515, 0.21\} \\ \{0.5, 0.4, 0.3\} & \{0.52, 0.49, 0.49\} & \{0.5\} & \{0.665, 0.715, 0\} \\ \{0.6, 0.82, 0.6\} & \{0.44, 0.485, 0.79\} & \{0.335, 0.285, 1\} & \{0.5\} \end{bmatrix}$$

$$H_{new}^{*2} = \begin{bmatrix} \{0.5\} & \{0.6, 0.59, 0.1\} & \{0.8, 0.6, 0.6\} & \{0.4, 0.11, 0.4\} \\ \{0.4, 0.41, 0.9\} & \{0.5\} & \{0.68, 0.565, 0.465\} & \{0.46, 0.31, 0.17\} \\ \{0.2, 0.4, 0.4\} & \{0.32, 0.435, 0.535\} & \{0.5\} & \{0.73, 0.62, 0.62\} \\ \{0.6, 0.89, 0.6\} & \{0.54, 0.69, 0.83\} & \{0.28, 0.38, 0.38\} & \{0.5\} \end{bmatrix}$$

$$H_{new}^{*3} = \begin{bmatrix} \{0.5\} & \{0.6, 0.585, 0.44\} & \{0.6, 0, 0.6\} & \{0.4, 0.3, 0\} \\ \{0.4, 0.415, 0.56\} & \{0.5\} & \{0.535, 0, 0.18\} & \{0.565, 0, 0.565\} \\ \{0.4, 1, 0.4\} & \{0.465, 1, 0.82\} & \{0.5\} & \{0.8, 0.56, 0.365\} \\ \{0.6, 0.7, 1\} & \{0.435, 1, 0.435\} & \{0.2, 0.44, 0.635\} & \{0.5\} \end{bmatrix}$$

Consistency Analysis:

The HFPR's consistency levels as stated by decision makers were measured using expressions (3.7)–(3.12).

$$\tilde{h}_{ij}^{*q\beta} = \max_{k \neq i,j}(h_{ij}^{*q\beta}, \max(h_{ik}^{*q\beta}, h_{kj}^{*q\beta} - 1, 0)) \ , \ \tilde{h}_{ij}^{*q\beta} + \tilde{h}_{ji}^{*q\beta} = 1$$

$$\tilde{H}^{*1}_{new} = \begin{bmatrix} \{0.5\} & \{0.5, 0.54, 0.2\} & \{0.5, 0.6, 0.7\} & \{0.4, 0.31, 0.4\} \\ \{0.5, 0.46, 0.8\} & \{0.5\} & \{0.485, 0.52, 0.52\} & \{0.57, 0.51, 0.21\} \\ \{0.5, 0.4, 0.3\} & \{0.515, 0.48, 0.48\} & \{0.5\} & \{0.665, 0.715, 0\} \\ \{0.6, 0.69, 0.6\} & \{0.43, 0.49, 0.79\} & \{0.335, 0.285, 1\} & \{0.5\} \end{bmatrix}$$

$$\tilde{H}^{*2}_{new} = \begin{bmatrix} \{0.5\} & \{0.64, 0.6, 0.2\} & \{0.8, 0.6, 0.6\} & \{0.4, 0.215, 0.4\} \\ \{0.36, 0.4, 0.8\} & \{0.5\} & \{0.69, 0.54, 0.465\} & \{0.44, 0.315, 0.24\} \\ \{0.2, 0.4, 0.4\} & \{0.31, 0.43, 0.535\} & \{0.5\} & \{0.72, 0.62, 0.62\} \\ \{0.6, 0.78, 0.6\} & \{0.53, 0.685, 0.76\} & \{0.28, 0.38, 0.38\} & \{0.5\} \end{bmatrix}$$

$$\tilde{H}^{*3}_{new} = \begin{bmatrix} \{0.5\} & \{0.64, 0.59, 0.44\} & \{0.6, 0, 0.6\} & \{0.414, 0.3, 0\} \\ \{0.36, 0.41, 0.56\} & \{0.5\} & \{0.535, 0, 0.2\} & \{0.565, 0, 0.565\} \\ \{0.4, 1, 0.4\} & \{0.465, 1, 0.8\} & \{0.5\} & \{0.8, 0.57, 0.385\} \\ \{0.585, 0.7, 1\} & \{0.435, 1, 0.435\} & \{0.2, 0.43, 0.615\} & \{0.5\} \end{bmatrix}$$

(i). In NHFPRs H^{*q}, q = 1, 2, 3,the consistency measures of pairs of alternatives are:

$$T_LCI(h^{*1}_{ij})_{new} = \begin{bmatrix} 1 & 1 & 1 & 0.955 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0.955 & 1 & 1 & 1 \end{bmatrix}, \quad T_LCI(h^{*2}_{ij})_{new} = \begin{bmatrix} 1 & 0.97 & 1 & 0.965 \\ 0.97 & 1 & 1 & 0.9767 \\ 1 & 1 & 1 & 1 \\ 0.965 & 0.9767 & 1 & 1 \end{bmatrix},$$

$$T_LCI(h^{*3}_{ij})_{new} = \begin{bmatrix} 1 & 1 & 1 & 0.995 \\ 1 & 1 & 0.9933 & 1 \\ 1 & 0.9933 & 1 & 0.9933 \\ 0.995 & 1 & 0.9933 & 1 \end{bmatrix}$$

(ii). T_LCI alternatives a_1, a_2, a_3 and a_4 are:

$$T_LCI(a_1)_{new} = (0.985, 0.9783, 0.9983), \; T_LCI(a_2)_{new} = (1, 0.9822, 0.9978)$$
$$T_LCI(a_3)_{new} = (1, 1, 0.9953) \;, T_LCI(a_4)_{new} = (0.985, 0.9806, 0.9961)$$

(iii). T_LCI against NHFPRs H^{*q} are:

$$T_LCI(H^{*1})_{new} = 0.9925, \qquad T_LCI(H^{*2})_{new} = 0.9852,$$
$$T_LCI(H^{*3})_{new} = 0.9969$$

The global consistency index (CI) may be calculated using (3.11) as follows:

$$CI_{new} = 0.9915$$

Now, using (3.12),estimate the consistency weights of the DMs D_1, D_2 *and* D_3 as follows:

$$Cw(D_1)_{new} = 0.3337, Cw(D_2)_{new} = 0.3312$$
$$Cw(D_3)_{new} = 0.3351$$

Now $\sum_{q=1}^{l} Cw(D_q)_{new} = 1.$

Consensus measures:

$$s^{12}_{ij_{new}} = \begin{bmatrix} 1 & 0.9 & 0.8667 & 0.9767 \\ 0.9 & 1 & 0.9 & 0.8867 \\ 0.8667 & 0.9 & 1 & 0.745 \\ 0.9767 & 0.8867 & 0.745 & 1 \end{bmatrix}, \; s^{13}_{ij_{new}} = \begin{bmatrix} 1 & 0.835 & 0.7333 & 0.8267 \\ 0.835 & 1 & 0.7 & 0.71 \\ 0.7333 & 0.7 & 1 & 0.7783 \\ 0.8267 & 0.71 & 0.7783 & 1 \end{bmatrix},$$

$$s^{23}_{ij_{new}} = \begin{bmatrix} 1 & 0.9017 & 0.7333 & 0.8033 \\ 0.9017 & 1 & 0.6667 & 0.73 \\ 0.7333 & 0.6667 & 1 & 0.8667 \\ 0.8033 & 0.73 & 0.8667 & 1 \end{bmatrix}$$

Similarity matrix

$$S_{new} = \begin{bmatrix} 1 & 0.8789 & 0.7778 & 0.8689 \\ 0.8789 & 1 & 0.7556 & 0.7756 \\ 0.7778 & 0.7556 & 1 & 0.7967 \\ 0.8689 & 0.7756 & 0.7967 & 1 \end{bmatrix}$$

(i). Consensus degree on a pair of alternatives (a_i, a_j) based on the similarity matrix S :

$$cd_{ij} = s_{ij}, \; i, j = 1, 2, 3, 4$$

(ii). Consensus degree on alternatives (a_i)

$$\begin{aligned} CD_{1\ new} &= 0.8419, \quad CD_{2\ new} = 0.8034, \\ CD_{3\ new} &= 0.7767, \quad CD_{4\ new} = 0.8137, \end{aligned}$$

(iii). Consensus degree on the relation

$$CR_{new} = 0.8089$$

Final weights of DMs:

The consistency weights $Cw(D_q), q = 1, 2, 3$ will be utilised as the DMs final weights by using (3.19),because no pre-determined weights are involved. As a result,

$$\begin{aligned} w(D_1)_{new} &= 0.3337, \quad w(D_2)_{new} = 0.3312, \\ w(D_3)_{new} &= 0.3351, \end{aligned}$$

Collective NHFPR construction :

After applying (3.20),we get the collective NHFPR H^{*c} .

$$H^{*c}_{new} = \begin{bmatrix} \{0.5\} & \{0.57, 0.56, 0.28\} & \{0.631, 0.4, 0.64\} & \{0.4, 0.28, 0.26\} \\ \{0.43, 0.44, 0.72\} & \{0.5\} & \{0.57, 0.36, 0.39\} & \{0.52, 0.27, 0.34\} \\ \{0.369, 0.6, 0.36\} & \{0.43, 0.64, 0.61\} & \{0.5\} & \{0.73, 0.62, 0.33\} \\ \{0.6, 0.72, 0.74\} & \{0.48, 0.73, 0.66\} & \{0.27, 0.38, 0.67\} & \{0.5\} \end{bmatrix}$$

Final Ranking of alternatives:

The equation (3.21) is used to get the final ranking order of the alternatives after analysing the ranking values.

$v(a_1)_{new} = 0.2234, \quad v(a_2)_{new} = 0.2244, \quad v(a_3)_{new} = 0.2605$ and $v(a_4)_{new} = 0.2917$

$\sum_{i=1}^{n} v(a_i) = 1$.As a result,the preferred order of alternatives is

$$a_4 \succ a_3 \succ a_2 \succ a_1$$

5. **Comparison**

By comparing our results to those of Xu et al. [29] based on consistency measure, consensus measure, and final ranking, we clearly verify the suggested technique. According to Xu et al. [29], the initial consistency levels, consensus level, and ultimate ranking of alternatives are as follows: $CR = 0.7554$, $a_4 \succ a_3 \succ a_2 \succ a_1$ respectively.

Additive reciprocity is incorporated into our suggested strategy to evaluate the HFPV's unidentified components throughout the normalisation process and create the consistent HFPRs in accordance.

TABLE: The results were compared of the Xu et al. results and the suggested method

$Methods$	$T_LCI(H^{*1})$	$T_LCI(H^{*2})-$	$T_LCI(H^{*3})-$	CR	$Ranking\ Order$
$Xu\ et\ al.$ [29]	0.9445	0.8722	0.9501	0.7554	$a_4 \succ a_3 \succ a_2 \succ a_1$
$Before\ Enhancement$	0.9945	0.95	0.9945	0.7745	$a_4 \succ a_3 \succ a_1 \succ a_2$
$After\ Enhancement$	0.9925	0.9852	0.9969	0.8089	$a_4 \succ a_3 \succ a_2 \succ a_1$

6. Conclusions

A consensus-based technique for dealing with the MPDM problem using consistent HFPRs is proposed in this paper. In this regard, the notion of HFPRs was derived from the work of Xu [20], and we propose an effective additive reciprocity-based strategy for normalising HFPRs.

The above example demonstrates a step-by-step procedure for normalising the HFPR. Following a consistency study, DMs were assigned consistency weights. To carry greater weight in the aggregation process, DMs with a high level of consistency should be given larger weights. Furthermore, an improvement Mechanism is provided to aid in the implementation of a higher degree of agreement on a straightforward path. When the DMs have reached an adequate level of consensus, the technique moves on to the selection step, which includes methods for aggregating and ranking to determine the best solution. A comparative case is created to highlight the feasibility and efficacy of the proposed method. The findings provide us a better understanding of the MPDM process.

7. Future Work

The main future work of this problem is that we can extend this problem by utilizing dual hesitant fuzzy set and by applying aggregation operator we can make this problem more compact. This problem can also be extended by using triangular fuzzy number.

References

[1] Wątróbski, J., Jankowski, J., Ziemba, P., Karczmarczyk, A., & Zioło, M. (2019). Generalised framework for multi-criteria method selection. *Omega*, 86, 107-124.

[2] Stojčić, M., Zavadskas, E. K., Pamučar, D., Stević, Ž., & Mardani, A. (2019). Application of MCDM methods in sustainability engineering: A literature review 2008–2018. *Symmetry*, 11(3), 350.

[3] Zhang, C., Zhang, H., & Wang, J. (2018). Personalized restaurant recommendation method combining group correlations and customer preferences. *Information Sciences*, 454, 128-143.

[4] Li, Y., Zhang, H., & Dong, Y. (2017). The interactive consensus reaching process with the minimum and uncertain cost in group decision making.

[5] Li, J., Wang, J. Q., & Hu, J. H. (2019). Consensus building for hesitant fuzzy preference relations with multiplicative consistency. *Computers & Industrial Engineering*, 128, 387-400.

[6] Tian, Z. P., Wang, J. Q., Zhang, H. Y., & Wang, T. L. (2018). Signed distance-based consensus in multi-criteria group decision-making with multi-granular hesitant unbalanced linguistic information. *Computers & Industrial Engineering*, 124, 125-138.

[7] Zhang, H., Dong, Y., & Herrera-Viedma, E. (2017). Consensus building for the heterogeneous large-scale GDM with the individual concerns and satisfactions. *IEEE Transactions on Fuzzy Systems*, 26(2), 884-898.

[8] Herrera-Viedma, E., Cabrerizo, F. J., Kacprzyk, J., & Pedrycz, W. (2014). A review of soft consensus models in a fuzzy environment. *Information Fusion*, 17, 4-13. *Applied Soft Computing*, 60, 202-212.

[9] Xia, M., & Xu, Z. (2013). Managing hesitant information in GDM problems under fuzzy and multiplicative preference relations. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 21(06), 865-897.

[10] Zhang, C., Liao, H., & Luo, L. (2019). Additive consistency-based priority-generating method of q-rung orthopair fuzzy preference relation. *International Journal of Intelligent Systems*, 34(9), 2151-2176.

[11] Xu, Z. S. (2004). Goal programming models for obtaining the priority vector of incomplete fuzzy preference relation. *International journal of approximate reasoning*, 36(3), 261-270.

[12] He, Y., & Xu, Z. (2017). A consensus reaching model for hesitant information with different preference structures. *Knowledge-Based Systems*, 135, 99-112.

[13] Wu, Z., Jin, B., & Xu, J. (2018). Local feedback strategy for consensus building with probability-hesitant fuzzy preference relations. *Applied Soft Computing*, 67, 691-705.

[14] Xu, Y., Chen, L., Rodríguez, R. M., Herrera, F., & Wang, H. (2016). Deriving the priority weights from incomplete hesitant fuzzy preference relations in group decision making. *Knowledge-Based Systems*, 99, 71-78.

[15] Meng, F., & An, Q. (2017). A new approach for group decision making method with hesitant fuzzy preference relations. *Knowledge-Based Systems*, 127, 1-15.

[16] Zhu, B., Xu, Z., & Xu, J. (2013). Deriving a ranking from hesitant fuzzy preference relations under group decision making. *IEEE Transactions on Cybernetics*, 44(8), 1328-1337.

[17] Zhang, Z., & Wu, C. (2014). A decision support model for group decision making with hesitant multiplicative preference relations. *Information Sciences*, 282, 136-166.

[18] Meng, F., Tang, J., An, Q., & Chen, X. (2019). A new procedure for hesitant multiplicative preference relations. *International Journal of Intelligent Systems*, 34(5), 819-857.

[19] Li, H., Yin, S., & Yang, Y. (2019). Some preference relations based on q-rung orthopair fuzzy sets. *International Journal of Intelligent Systems*, 34(11), 2920-2936.
[20] Xu, Y., Cabrerizo, F. J., & Herrera-Viedma, E. (2017). A consensus model for hesitant fuzzy preference relations and its application in water allocation management. *Applied Soft Computing*, 58, 265-284.
[21] Zhang, Z., Wang, C., & Tian, X. (2015). Multi-criteria group decision making with incomplete hesitant fuzzy preference relations. *Applied Soft Computing*, 36, 1-23.
[22] Khalid, A., & Beg, I. (2017). Incomplete hesitant fuzzy preference relations in group decision making. *International Journal of Fuzzy Systems*, 19(3), 637-645.
[23] Zhu, B. (2013). Studies on consistency measure of hesitant fuzzy preference relations. *Procedia Computer Science*, 17, 457-464.
[24] Zadeh, L. A., Klir, G. J., & Yuan, B. (1996). *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers* (Vol. 6). World Scientific.
[25] Torra, V. (2010). Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6), 529-539.
[26] Tanino, T. (1988). Fuzzy preference relations in group decision making. *Non-conventional preference relations in decision making*, 54-71.
[27] Orlovsky, S. (1978). Decision-making with a fuzzy preference relation. *Fuzzy sets and systems*, 1(3), 155-167.
[28] Xia, M., Xu, Z., & Chen, N. (2013). Some hesitant fuzzy aggregation operators with their application in group decision making. *Group Decision and Negotiation*, 22(2), 259-279.
[29] Xu, Y., Cabrerizo, F. J., & Herrera-Viedma, E. (2017). A consensus model for hesitant fuzzy preference relations and its application in water allocation management. Applied Soft Computing, 58, 265-284.

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