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# INVERSE SCATTERING TRANSFORM FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH SELF-CONSISTENT SOURCE

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**Annotation.** In this work is shown that the "finite density" solution of the nonlinear Schrodinger equation with self-consistent source, can be found by the inverse scattering problem for the Dirac's type operator.

**Keywords:** nonlinear Schrodinger equation, Zakharov-Shabat system, nonzero boundary condition, inverse scattering theory, self-consistent source.

**Introduction.** The nonlinear Schrödinger equation (NSE)

$$iu_t - 2\chi|u|^2 u + u_{xx} = 0, \; \chi = const$$

with various boundary conditions models a wide class of nonlinear phenomena in physics. In the work [1], V. Zakharov and A. Shabat showed that NLS equation can be applied in the study of optical self-focusing and splitting of optical beams. This equation belongs to the class of equations that can be solvable using the inverse scattering method for a Dirac-type operator. This was shown in the works of V.E. Zakharov and A.B. Shabat [1], L.A. Takhtadjan and L.D. Fadeev [2], M. Ablowitz, D. Kaup, A. Newell and H. Segur [3].

In [4], V.K. Melnikov obtained evolutions of scattering data with respect to t for a self-adjoint Dirac operator with a potential that is an NSE solution with a self-consistent source of integral type. However, we note that in the above works NSE was considered in the class of "rapidly decreasing" functions, i.e. conditions that vanish in a certain way as the coordinate tends to infinity.

In connection with the application to specific physical problems, it became necessary to consider NLS not only in the class of rapidly decreasing functions, but also in classes of functions of a special form. First, in the work of V.E.Zakharov and A.B. Shabat [5], NSE was integrated in the class of "finite density" functions, i.e., functions for which $u(x,t)\to e^{i\alpha+2it}$, $u_x(x,t)\to 0$ as $x\to\infty$. The n-soliton solution of the NSE in the case of a finite density was found in [6].

**Formulation of the problem.** We consider the integration of the following system of equations

$$iu_t-2u|u|^2+u_{xx}=-2i\sum_{n=1}^{N}(\phi_{1,n}^*\psi_{2,n}^*+\phi_{2,n}\psi_{1,n}), \quad (1)$$

$$\frac{\partial\varphi_{1,n}}{\partial x}-u^*\varphi_{2,n}+i\xi_n\varphi_{1,n}=\frac{\partial\varphi_{2,n}}{\partial x}-u\varphi_{1,n}-i\xi_n\varphi_{2,n}=0,\ \ n=1,2,...,N, \quad (2)$$

$$\frac{\partial\psi_{1,n}}{\partial x}-u\psi_{2,n}-i\xi_n\psi_{1,n}=\frac{\partial\psi_{2,n}}{\partial x}-u^*\psi_{1,n}+i\xi_n\psi_{2,n}=0,\ \ n=1,2,...,N, \quad (3)$$

with initial value

$$u(x,0)=u_0(x), \quad (4)$$

where the bar means complex conjugation and $\xi_j$, $j=1,2,\ldots,N$ are the eigenvalues and function $u_0(x)$ satisfy the following properties:

1. $\int_{-\infty}^{0}(1-x)\left|u(x,t)-\rho e^{i\alpha}\right|dx+\int_{0}^{\infty}(1+x)\left|u(x,t)-\rho e^{i\beta}\right|dx<\infty$,

2. The equation

$$L(0)y \equiv i\begin{pmatrix} \frac{d}{dx} & -\bar{u}_0(x) \\ u_0(x) & -\frac{d}{dx} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \xi \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \; x \in R$$

can have $N$ number of eigenvalues. Here, the function $\bar{u}_0(x)$ is a complex conjugation of $u_0(x)$.

We also assume that the eigenfunctions $\Phi_n = (\varphi_{1,n}, \varphi_{2,n})^T$ $(\Psi_n = (\psi_{1,n}, \psi_{2,n})^T)$ corresponding to $\xi_n(t)$ this eigenvalues satisfy the following normalizing conditions

$$\det\{\Psi_k^T(s,t), \Phi_k(s,t)\} = \omega_k(t), \; k = 1,2,...,N, \tag{5}$$

Here $\omega_k(t)$, $j = 1,2,...,N$ are given and the continuous functions of $t$.

The main goal of this work is to study the integration of the nonlinear Schrodinger equation via inverse scattering problem in the class of $u(x,t)$ function, which is sufficiently smooth and tends to its limits rapidly enough when $x \to \pm\infty$ and satisfies the condition

$$\int_{-\infty}^{0}(1-x)\left|u(x,t)-\rho e^{i\alpha-2i\rho^2 t}\right|dx + \int_{0}^{\infty}(1+x)\left|u(x,t)-\rho e^{i\beta-2i\rho^2 t}\right|dx +$$

$$+\int_{-\infty}^{\infty}\sum_{k=1}^{2}\left|\frac{\partial^k u(x,t)}{\partial x^k}\right|dx < \infty, \;\; \rho > 0. \tag{6}$$

Let the function $u(x,t)$ be a solution of equation (1), from the class of functions (5). Consider an operator with a potential $u(x,t)$ that is a solution to the problem under consideration and find the evolution from $t$ the scattering data.

**Necessary information from scattering theory.** Consider the system of linear equations on the real line $R$

$$(L - \xi I) f = 0, \tag{7}$$

where $f = f(x,\xi)$ is vector-column function and

$$L(t)=i\begin{pmatrix} \frac{\partial}{\partial x} & -\bar{u}(x,t) \\ u(x,t) & -\frac{\partial}{\partial x} \end{pmatrix},\ t\geq 0.$$

There we present some necessary facts for our further exposition from the theory of the direct and inverse scattering problem for the system of equations (7).

We define the Jost solutions of the system (7) with the following asymptotic values

$$\begin{gathered} \varphi\sim\begin{pmatrix} 1 \\ \frac{i(\xi-p)}{\rho}e^{i\alpha-2i\rho^2t} \end{pmatrix}e^{-ipx},\ \text{as}\quad x\to-\infty, \\ \bar{\varphi}\sim\begin{pmatrix} -\frac{i(\xi-p)}{\rho}e^{-i\alpha+2i\rho^2t} \\ 1 \end{pmatrix}e^{ipx},\ \text{as}\quad x\to-\infty, \\ \psi\sim\begin{pmatrix} -\frac{i(\xi-p)}{\rho}e^{-i\beta+2i\rho^2t} \\ 1 \end{pmatrix}e^{ipx},\ \text{as}\quad x\to-\infty, \\ \bar{\psi}\sim\begin{pmatrix} 1 \\ \frac{i(\xi-p)}{\rho}e^{i\beta-2i\rho^2t} \end{pmatrix}e^{-ipx},\ \text{as}\quad x\to\infty, \end{gathered} \tag{8}$$

where

$$p(\xi)=\sqrt{\xi^2-\rho^2}, \tag{9}$$

here and below we will use the standard Pauli matrices

$$\sigma_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For real $\xi$ $(\xi^2\geq\rho^2)$, path of square root is fixed by the condition $sign\,p(\xi)=sign\,\xi$. The Riemann surface $\Gamma$ of a function $p(\xi)$ consists of two

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instances $\Gamma_+$ and $\Gamma_-$ a complex plane $C$ with cuts along the real axis from $-\infty$ to $-\rho$ and from $\rho$ to $\infty$ with properly identified cut edges (see [5]). The function $p(\xi)$ is introduced on $\Gamma$ the formula (8), where $\pm \operatorname{Im} p \geq 0$ on the sheets $\Gamma_\pm$. In what follows, for convenience, we will often omit the dependence of the function $p(\xi)$ on $\xi$. Thus, in formulas where and is involved, it is always assumed that $p(\xi)$ is a function of $\xi$.

It can be shown that

$$\frac{d}{dx}\det(\varphi,\bar{\varphi})=0 \text{ and } \frac{d}{dx}\det(\psi,\bar{\psi})=0. \tag{10}$$

From (8) and (10) it follows that

$$\det(\varphi,\bar{\varphi})=\frac{2p(\xi-p)}{\rho^2},\ \det(\psi,\bar{\psi})=\frac{2p(\xi-p)}{\rho^2}. \tag{11}$$

For real $p$ and $\xi$ pairs of vector functions $\{\varphi,\bar{\varphi}\}$ and $\{\psi,\bar{\psi}\}$ form a fundamental system of solutions to (7), so, there is a functions $a(t,\xi)$, $b(t,\xi)$ that for solutions $\{\varphi,\bar{\varphi}\}$ and $\{\psi,\bar{\psi}\}$

$$\varphi(x,t,\xi)=a(t,\xi)\bar{\psi}(x,t,\xi)+b(t,\xi)\psi(x,t,\xi),\ \text{as } \xi \text{ из } R^1\setminus[-\rho,\rho]. \tag{12}$$

The coefficients $a(\xi,t)$ and $b(\xi,t)$ are called transition coefficients. From relations (10) and (11) we obtain

$$|a(\xi,t)|^2-|b(\xi,t)|^2=1, \tag{13}$$

where the functions $a(\xi)$ and $b(\xi)$ are independent of $x$ and

$$a(\xi,t)=\frac{\rho^2}{2p(\xi-p)}\det(\varphi(x,\xi,t),\psi(x,\xi,t)), \tag{14}$$

$$b(\xi,t)=\frac{\rho^2}{2p(\xi-p)}\det(\varphi(x,\xi,t),\bar{\psi}(x,\xi,t)).$$

The function $a(\xi,t)$ admit an analytic continuation in $\xi$ into the plane $\Gamma_+$. The function $a(\xi,t)$ has the asymptotics

$$a(\xi,t) = 1 + O\left(\frac{1}{|\xi|}\right), \text{ as } \operatorname{Im}\xi > 0 \tag{15}$$

and

$$a(\xi,t) = e^{i(\alpha-\beta)} + O\left(\frac{1}{|\xi|}\right), \text{ as } \operatorname{Im}\xi < 0. \tag{16}$$

Besides, in the plane $\Gamma_+$ the function $a(\xi,t)$ has a finite number of zeros at the points $\xi_k$ $(k = 1, 2, ..., N)$, and these points are the eigenvalues of the operator $L$.

It follows from representation (14) that if $a(\xi_n,t) = 0$, then the columns $\varphi(x,\xi,t)$ and $\psi(x,\xi,t)$ are linearly dependent at $\xi = \xi_n$, i.e.,

$$\varphi(x,\xi_n,t) = c_n(t)\psi(x,\xi_n,t), \quad n = 1,2,...,N. \tag{17}$$

Note that the vector-functions

$$h_n(x,t) = \frac{\left.\frac{d}{d\xi}(\varphi - c_n\psi)\right|_{\xi=\xi_n}}{\dot{a}(\xi_n,t)}, \quad n = 1,2,...,N.$$

The following integral representations hold for the Jost solutions

$$\psi(x,\xi,t) = \begin{pmatrix} -\frac{i(\xi-p)}{\rho} e^{-i\beta+2i\rho^2 t} \\ 1 \end{pmatrix} e^{ipx} + \int_{-\infty}^{x} K(x,y,t) \begin{pmatrix} -\frac{i(\xi-p)}{\rho} e^{-i\beta+2i\rho^2 t} \\ 1 \end{pmatrix} dy, \tag{18}$$

where

$$K^{\pm}(x,y,t) = \begin{pmatrix} K_{11}^{\pm}(x,y,t) & K_{12}^{\pm}(x,y,t) \\ K_{21}^{\pm}(x,y,t) & K_{22}^{\pm}(x,y,t) \end{pmatrix}.$$

In representation (18) the kernel $K(x,y,t)$ does not dependent on $\xi$ and the related to the potential $u(x,t)$ as the following:

$$2K_{21}(x,x,t)=\rho e^{i\beta-2i\rho^2 t}-u(x,t), \tag{19}$$

It is well known that the components of the kernel $K^{+}(x,y,t)$ for $y>x$ are solutions of the system of Gelfand-Levitan-Marchenko integral equations:

$$K^{+}(x,y)+F(x+y)+\int_x^{\infty}K^{+}(x,s)F(s+y)ds=0,\ y\ge x \tag{20}$$

where

$$F(x)=\begin{pmatrix}F_1(x) & F_2^*(x)\\ F_2(x) & F_1(x)\end{pmatrix},\quad e(x,z)=e^{\frac{i}{2}\left(z-\frac{\rho^2}{z}\right)x}$$

$$\begin{pmatrix}F_1(x)\\ F_2(x)\end{pmatrix}=\frac{1}{4\pi}\int_{-\infty}^{\infty}r(z,t)e(x,z)\begin{pmatrix}\frac{-i\rho e^{i\beta-2i\rho^2 t}}{z}\\ 1\end{pmatrix}dz-\frac{1}{2}\sum_{n=1}^{N}\frac{c_n(t)}{\dot a(z_n,t)z_n}\cdot\begin{pmatrix}\rho e^{-i\beta+2i\rho^2 t}\\ iz_n\end{pmatrix}e(x,z_n).$$

**Definition.** The set of the quantities $\{a(\xi,t),\, b(\xi,t),\xi_n(t),c_n(t),n=1,2,...,N\}$ is called the scattering data for equation (7).

**Evolution of scattering data.** If the potential $u(x,t)$ in the system of equations (7) depends on $t$, then its solution $f$ must also depend on $t$. Let this time dependence have the form

$$\frac{\partial}{\partial t}f=A(x,t,\xi)f, \tag{21}$$

where

$$A=\begin{pmatrix}-2i\xi^2+i|u|^2 & 2u\xi+iu_x\\ -2u^*\xi+iu_x^* & 2i\xi^2-i|u|^2\end{pmatrix}.$$

The compatibility condition for linear systems (7) and (21) is

$$\frac{\partial L}{\partial t}+[L,A]=G, \quad (22)$$

where $G=\begin{pmatrix} 0 & \bar{g} \\ g & 0 \end{pmatrix}$.

Let $\varphi(x,\xi,t)$ be a Jost solution of the equation $L(t)\varphi=\xi\varphi$. By differentiating this relation with respect to $t$, we obtain the equation

$$\frac{\partial L}{\partial t}\varphi+L\frac{\partial \varphi}{\partial t}=\xi\frac{\partial \varphi}{\partial t}. \quad (23)$$

By substituting $\frac{\partial L}{\partial t}$ (22) into (23), we obtain the equation

$$(L-\xi)(\frac{\partial \varphi}{\partial t}-A\varphi)=-iG\varphi, \quad (24)$$

whose solution we seek in the form

$$\frac{\partial \varphi}{\partial t}-A\varphi=\alpha(x,t)\psi+\beta(x,t)\varphi. \quad (25)$$

For the functions $\alpha(x,t)$ and $\beta(x,t)$, we obtain the equation

$$\sigma_3\frac{\partial \alpha}{\partial x}\psi+\sigma_3\frac{\partial \beta}{\partial x}\varphi=-G\varphi. \quad (26)$$

By multiplying Eq. (26) by $\sigma_1\varphi$ and $\sigma_1\psi$, we obtain

$$\frac{\partial \alpha}{\partial x}=\frac{i\rho^2}{2p(\xi-p)}\frac{\sigma_1\varphi G\varphi}{a}, \quad \frac{\partial \beta}{\partial x}=\frac{i\rho^2}{2p(\xi-p)}\frac{\sigma_1\psi G\psi}{a}. \quad (27)$$

Relation (8) implies that $\frac{\partial \varphi}{\partial t}-A\varphi\to(2i\xi p+i\rho^2)\varphi$ as $x\to-\infty$ therefore, from (21) we have $a(x,t)\to 0,\ \beta(x,t)\to 2i\xi p+i\rho^2$ as $x\to-\infty$. By solving (27), we obtain

$$\alpha(x,t)=\frac{1}{a}\int_{-\infty}^{x}\sigma_1\varphi G\varphi ds, \quad \beta(x,t)=-\frac{1}{a}\int_{-\infty}^{x}\sigma_1\psi G\psi ds-2i\xi p-i\rho^2.$$

Therefore, relation (13) can be represented in the form

$$\frac{\partial\varphi}{\partial t}-A\varphi=\frac{1}{a}\int_{-\infty}^{x}\sigma_1\varphi G\varphi ds\cdot\psi-\left(\frac{1}{a}\int_{-\infty}^{x}\sigma_1\psi G\psi ds+2i\xi p+i\rho^2\right)\cdot\varphi \qquad (28)$$

By using (25) and by passing to the limit as $x\to\infty$ in (28), we obtain

$$\dot{a}=-\frac{i\rho^2}{2p(\xi-p)}\int_{-\infty}^{\infty}\sigma_1\psi G\varphi ds,$$

$$\dot{b}=-(2i\xi p+i\rho^2)b+\frac{i\rho^2}{2p(\xi-p)a}\int_{-\infty}^{\infty}\sigma_1\varphi G\varphi ds-\frac{i\rho^2 b}{2p(\xi-p)a}\int_{-\infty}^{\infty}\sigma_1\psi G\varphi ds.$$

As in the continuous spectrum, one can show that

$$\dot{c}_n=-(4i\xi_n p_n+2i\rho^2)c_n-\frac{i\rho^2}{2p_n(\xi_n-p_n)}\int_{-\infty}^{\infty}\sigma_1 h_n R\varphi_n ds,$$

$$\frac{d\xi_n}{dt}=\frac{\int_{-\infty}^{\infty}\sigma_1\varphi G\varphi dx}{2\int_{-\infty}^{\infty}\varphi_{n1}\varphi_{n2}dx},\ n=1,2,3,\dots,N.$$

**Theorem 1.** If the function $u(x,t)$ is a solution of the equation (1) in the class of functions (3), then the scattering data of the system (7) with the function $u(x,t)$ depend on $t$ as follows:

$$\dot{a}=-\frac{i\rho^2}{2p(\xi-p)}\int_{-\infty}^{\infty}\sigma_1\varphi G\varphi ds,$$

$$\dot{b}=(2i\xi p+i\rho^2)b+\frac{i\rho^2}{2p(\xi-p)a}\int_{-\infty}^{\infty}\sigma_1\varphi G\varphi ds-\frac{i\rho^2}{2p(\xi-p)a}\int_{-\infty}^{\infty}\sigma_1\psi G\psi ds,$$

$$\dot{c}_n=-(4i\xi_n p_n+2i\rho^2)c_n-\frac{i\rho^2}{2p_n(\xi_n-p_n)}\int_{-\infty}^{\infty}\sigma_1 h_n R\varphi_n ds,$$

$$\frac{d\xi_n}{dt}=\frac{\int_{-\infty}^{\infty}\sigma_1\varphi_n G\varphi_n dx}{2\int_{-\infty}^{\infty}\varphi_{n1}\varphi_{n2}dx},\ n=1,2,3,\ldots,N.$$

The obtained relations determine completely the evolution of the scattering data for the system (7), which allow as to find the solution of the problem (1)-(3) by using the inverse scattering problem method.

**Corollary.** If we get $g(x,t)=-2i\sum_{n=1}^{N}(\phi_{1,n}^{*}\psi_{2,n}^{*}+\phi_{2,n}\psi_{1,n})$ then

$$\int_{-\infty}^{\infty}\sigma_1\varphi G\varphi dx=0,\quad \int_{-\infty}^{\infty}\sigma_1\psi G\varphi dx=0,$$

$$\int_{-\infty}^{\infty}\sigma_1 h_n G\varphi_n dx=i\beta_n\omega_n(t),\quad \int_{-\infty}^{\infty}\sigma_1\varphi_n G\varphi_n dx=\omega_n(t).$$

In this case

$$\frac{d\xi_n}{dt}=0,\ \dot{c}_n=-(4i\xi_n p_n+2i\rho^2-i\beta_n(\omega_n(t)+\omega_n^{*}(t))c_n.$$

**Example.** Let

$$u_0=\sqrt{2}e^{-4it}\frac{e^{-x}+ie^{x}}{e^{-x}+e^{x}}.$$

Where $\rho,\alpha,\beta,p_1$ and $c$ are positive numbers. In this case, the scattering data system of equations (7) with potential $u_0$ has

$$a(t,\xi)=\frac{\xi-p-1-i}{\xi+p-1+i},\ b(t,\xi)=0,\ c_1=\frac{i(1+i)}{\sqrt{2}}e^{-4it},\ \xi=1,p=i$$

Using results theorem 1, we can find

$$\frac{d\xi_1}{dt}=0,\ c_1(t)=e^{-4t+4it}\cdot\exp\int_0^t i\beta_n(\omega_n(\tau)-\omega_n^{*}(\tau))d\tau.$$

Solving the inverse problem we get

$$u(x,t)=\sqrt{2}e^{-4it}\frac{e^{-x}+ie^{-x+g(t)}}{e^{-x}+e^{-x+g(t)}},$$

$$\varphi_1=\frac{\alpha_1}{e^{-x}+e^{x}},\quad \varphi_2=\alpha_1\frac{-(1-i)}{\sqrt{2}}e^{-4it}\cdot\frac{1}{e^{-x}+e^{x}},$$

$$\psi_1=\alpha_2\left(-\frac{i(1-i)}{\sqrt{2}}e^{4it}\cdot\frac{1}{e^{-x}+e^{x}}\right)-\frac{1}{e^{-x}+e^{x}}\left(-2x+a_{\xi}^{'}(e^{-2x}-ie^{2x})+i\right)\cdot\frac{2\omega_1}{2i(1-i)},$$

$$\psi_2=\alpha_2\frac{1}{e^{-x}+e^{x}}-\frac{\sqrt{2}\omega_1}{2}e^{-4it}\cdot\frac{1}{e^{-x}+e^{x}}\left(i-2x+a_{\xi}^{'}(ie^{-2x}-e^{2x})\right),$$

where $g(t)=i\int_0^t\beta_n(\omega_n(\tau)-\omega_n^*(\tau))d\tau$, $\beta_1=\alpha_1\alpha_2$.

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