On the zeros of Dirichlet Eta Function

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Abstract

We consider the analytic continuation of Riemann's Zeta Function derived from Dirichlet Eta Function $\eta(s)$ which is evaluated at $s = \frac{1}{2} + \sigma + i\omega$, where σ, ω are real and compute inverse Fourier transform of $\Gamma(\frac{s}{2})\eta(s)$ and derive a related function $E_p(t)$. We study the properties of $E_p(t)$ and a promising new method is presented which could be used to show that the Fourier Transform of $E_p(t)$ given by $E_{p\omega}(\omega)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line.

Keywords: Riemann, Hypothesis, Zeta, Eta, exponential functions

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1. Introduction

It is well known that Riemann's Zeta function given by $\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}$ converges in the half-plane where the real part of s is greater than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole s-plane apart from a simple pole at $s = 1$ and that $\zeta(s)$ satisfies a symmetric functional equation given by $\xi(s) = \xi(1-s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \int_0^\infty e^{-u}u^{s-1}du$ is the Gamma function. [4] [5] We can see that if Riemann's Xi function has a zero in the critical strip, then Riemann's Zeta function also has a zero at the same location. Riemann made his conjecture in his 1859 paper, that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with real part of $s = \frac{1}{2}$, which is called the Riemann Hypothesis.[1]

Hardy and Littlewood later proved that infinitely many of the zeros of $\zeta(s)$ are on the critical line with real part of $s = \frac{1}{2}$.[2] It is well known that $\zeta(s)$ does not have non-trivial zeros when real part of $s = \frac{1}{2} + \sigma + i\omega$, given by $\frac{1}{2} + \sigma \geq 1$ and $\frac{1}{2} + \sigma \leq 0$. In this paper, **critical strip** $0 < Re[s] < 1$ corresponds to $0 \leq |\sigma| < \frac{1}{2}$.

In this paper, a **new method** is discussed and a specific solution is presented to prove Riemann's Hypothesis. If the specific solution presented in this paper is incorrect, it is **hoped** that the new method discussed in this paper will lead to a correct solution by other researchers.

In Section 2 to Section 4, we prove Riemann's hypothesis by taking the analytic continuation of Riemann's Zeta Function derived from Dirichlet Eta function $\eta(s)$ and compute inverse Fourier transform of $\Gamma(\frac{s}{2})\eta(s)$ and show that $\Gamma(\frac{s}{2})\eta(s)$ does not have zeros for finite and real ω when $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip **excluding** the critical line.

In Section 7, it is shown that the new method is **not** applicable to Hurwitz zeta function and related functions and **does not** contradict the existence of their non-trivial zeros away from the critical line with real part of $s = \frac{1}{2}$.

We present an **outline** of the new method below.

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1.1. *Step 1: Dirichlet Eta function*

We use the analytic continuation of Riemann's zeta function given by $\zeta(s) = \dfrac{\eta(s)}{1-2^{1-s}}$ where $\zeta(s) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$ diverges for $Re[s] \leq 1$ and $\eta(s) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s}$ is Dirichlet Eta function which converges for $Re[s] > 0$. (link and Titchmarsh pp16-17)

We see that if $\eta(s)$ has a zero in the critical strip, then $\zeta(s)$ also has a zero at the same location. We evaluate $A(s) = \Gamma(\frac{s}{2})\eta(s)$ at $s = \frac{1}{2} + \sigma + i\omega$ in Eq. 10 for $0 < \sigma < \frac{1}{2}$ and compute its inverse Fourier Transform $a(t)$ in Eq. 14.

In Section 1.6 and Section 1.7, it is shown that, if $\eta(\frac{1}{2} + \sigma + i\omega)$ has a zero at $\omega = \omega_0$ in the critical strip, then the Fourier transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ given by $E_{p\omega}(\omega)$ **also** has a zero at $\omega = \omega_0$, where $E_0(t) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} (e^{-\pi \frac{n^2}{4} e^{-2t}} - e^{-\pi n^2 e^{-2t}}) e^{-\frac{t}{2}}$ derived using $a(t)$.

Statement 1: We assume that the analytic continuation of Riemann's zeta function derived from Dirichlet Eta function given by $E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ and then prove that this leads to a **contradiction** for $0 < |\sigma| < \frac{1}{2}$.

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1.2. *Step 2: On the zeros of a related function* $G(\omega, t_2, t_0)$

Let us consider $0 < \sigma < \frac{1}{2}$ at first. Let us consider a new function $g(t, t_2, t_0) = f(t, t_2, t_0)e^{-\sigma t}u(-t) + f(t, t_2, t_0)e^{\sigma t}u(t)$, where $f(t, t_2, t_0) = e^{-2\sigma t_0} f_1(t, t_2, t_0) + e^{2\sigma t_0} f_2(t, t_2, t_0)$ and $f_1(t, t_2, t_0) = e^{\sigma t_0} E_p^{'}(t + t_0, t_2)$ and $f_2(t, t_2, t_0) = e^{-\sigma t_0} E_p^{'}(t - t_0, t_2)$ and $E_p^{'}(t, t_2) = e^{-\sigma t_2} E_p(t - t_2) - e^{\sigma t_2} E_p(t + t_2)$ and t_0, t_2 are real and $g(t, t_2, t_0)$ is a real function of variable t and $u(t)$ is Heaviside unit step function. We can see that $g(t, t_2, t_0)h(t) = f(t, t_2, t_0)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

In Section 2.1, we will show that the Fourier transform of the **even function** $g_{even}(t, t_2, t_0) = \frac{1}{2}[g(t, t_2, t_0) + g(-t, t_2, t_0)]$ given by $G_R(\omega, t_2, t_0)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$, for every value of t_0, for each nonzero value of t_2, where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign, to satisfy Statement 1, where $\omega_z(t_2, t_0)$ is real and finite.

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1.3. *Step 3: On the zeros of the function* $G_R(\omega, t_2, t_0)$

In Section 2.3, we compute the Fourier transform of the function $g(t, t_2, t_0)$ and compute its real part given by $G_R(\omega, t_2, t_0)$ and we can write as follows.

$$G_R(\omega,t_2,t_0) = e^{-2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega\tau)d\tau$$
$$+e^{2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\cos(\omega\tau)d\tau \qquad (1)$$

We require $G_R(\omega,t_2,t_0) = 0$ for $\omega = \omega_z(t_2,t_0)$ for every value of t_0, for **each non-zero value** of t_2, to satisfy **Statement 1**. In general $\omega_z(t_2,t_0) \neq \omega_0$. Hence we can see that $P(t_2,t_0) = G_R(\omega_z(t_2,t_0),t_2,t_0) = 0$.

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1.4. *Step 4: Zero Crossing function* $\omega_z(t_2,t_0)$ *is an even function of variable* t_0

In Section 2.4, we show the result in Eq. 2 and that $\omega_z(t_2,t_0) = \omega_z(t_2,-t_0)$. It is shown that $P(t_2,t_0) = G_R(\omega_z(t_2,t_0),t_2,t_0) = P_{odd}(t_2,t_0) + P_{odd}(t_2,-t_0) = 0$ and that $P_{odd}(t_2,t_0)$ is an **odd** function of t_0, for each non-zero value of t_2 as follows.

$$P_{odd}(t_2,t_0) = [\cos(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{t_0} E_0^{'}(\tau,t_2)e^{-2\sigma\tau}\cos(\omega_z(t_2,t_0)\tau)d\tau$$
$$+\sin(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{t_0} E_0^{'}(\tau,t_2)e^{-2\sigma\tau}\sin(\omega_z(t_2,t_0)\tau)d\tau]$$
$$+e^{2\sigma t_0}[\cos(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{t_0} E_{0n}^{'}(\tau,t_2)\cos(\omega_z(t_2,t_0)\tau)d\tau + \sin(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{t_0} E_{0n}^{'}(\tau,t_2)\sin(\omega_z(t_2,t_0)\tau)d\tau] \qquad (2)$$

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1.5. *Step 5: Final Step*

In Section 6, it is shown that $\omega_z(t_2,t_0)$ is a **continuous** function of variable t_0 and t_2, for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$. In Section 5, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

In Section 3, we set $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$, such that $\omega_z(t_{2c},t_{0c})t_{0c} = \frac{\pi}{2}$ and substitute in the equation for $P_{odd}(t_2,t_0)$ in Eq. 2 and show that this leads to the result in Eq. 3. We use $E_0'(t,t_2) = E_0(t-t_2) - E_0(t+t_2)$ and $E_{0n}'(t,t_2) = E_0'(-t,t_2)$.

$$\int_{0}^{t_{0c}}(E_0(\tau-t_{2c}) - E_0(\tau+t_{2c}))(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau))\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau = 0 \qquad (3)$$

We show that **each** of the terms in the integrand in Eq. 3 are **greater than zero**, in the interval $0 < \tau < t_{0c}$ and the integrand is zero at $\tau = 0$ and $\tau = t_{0c}$, where $t_{0c} > 0$.

Hence the result in Eq. 3 leads to a **contradiction** for $0 < \sigma < \frac{1}{2}$.

We show this result for $0 < \sigma < \frac{1}{2}$ and then use the property $\xi(\frac{1}{2}+\sigma+i\omega) = \xi(\frac{1}{2}-\sigma-i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$. Hence we produce a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

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1.6. ***Analytic continuation of Riemann Zeta function derived from Dirichlet Eta function***

We consider Riemann's Xi function $\xi(s)$, where $s = \frac{1}{2} + \sigma + i\omega$. Using the functional equation of Riemann's zeta function given by $\zeta(s) = \zeta(1-s)\Gamma(1-s)\sin(\frac{s\pi}{2})\pi^{(s-1)}2^s$, we get $\xi(s) = \xi(1-s)$. Titchmarsh pp16-17) Using $\zeta(s) = \frac{\eta(s)}{1-2^{1-s}}$, we write as follows.

$$\xi(s) = \zeta(s)\Gamma(\frac{s}{2})\pi^{\frac{-s}{2}}\frac{s(s-1)}{2} = \xi(1-s)$$
$$\xi(s) = \frac{\eta(s)}{1-2^{1-s}}\Gamma(\frac{s}{2})\pi^{\frac{-s}{2}}\frac{s(s-1)}{2} \tag{4}$$

We define a related analytic continuation $E(s)$ as follows. Given $\xi(s) = \xi(1-s)$, we see that $E(s) = E(1-s)$ is analytic in the region $0 < Re[s] < 1$ and has simple poles at $s = 0$ and $s = 1$.

$$E(s) = \frac{\xi(s)(1-2^{1-s})(2^s-1)}{s(s-1)}$$
$$E(1-s) = \frac{\xi(1-s)(1-2^s)(2^{1-s}-1)}{(1-s)(-s)} = \frac{\xi(s)(2^s-1)(1-2^{1-s})}{(s-1)(s)} = E(s) \tag{5}$$

We substitute $\xi(s)$ from Eq. 4 and $\zeta(s) = \dfrac{\eta(s)}{1-2^{1-s}}$ in Eq. 5 and cancel the common terms $s(s-1)$ and $(1-2^{1-s})$ as follows.

$$E(s) = \frac{\eta(s)}{1-2^{1-s}}\Gamma(\frac{s}{2})\pi^{\frac{-s}{2}}\frac{s(s-1)}{2}\frac{(1-2^{1-s})(2^s-1)}{s(s-1)}$$
$$E(s) = \frac{\eta(s)}{1-2^{1-s}}\Gamma(\frac{s}{2})\pi^{\frac{-s}{2}}\frac{1}{2}(1-2^{1-s})(2^s-1)$$
$$E(s) = \eta(s)\Gamma(\frac{s}{2})\frac{\pi^{\frac{-s}{2}}}{2}(2^s-1) \tag{6}$$

We evaluate $E(s)$ at $s = \frac{1}{2} + \sigma + i\omega$ and use $K^{i\omega} = e^{i\omega\log(K)}$ as follows.

$$E(\frac{1}{2}+\sigma+i\omega) = E_{p\omega}(\omega) = \eta(\frac{1}{2}+\sigma+i\omega)\Gamma(\frac{\frac{1}{2}+\sigma+i\omega}{2})\frac{\pi^{\frac{-(\frac{1}{2}+\sigma)}{2}}}{2}e^{\frac{-i\omega}{2}\log(\pi)}(2^{\frac{1}{2}+\sigma}e^{i\omega\log(2)}-1)$$

(7)

We define $A_\omega(\omega) = \eta(\frac{1}{2} + \sigma + i\omega)\Gamma(\frac{\frac{1}{2}+\sigma+i\omega}{2})$, and we can rearrange the terms as follows.

$$E_{p\omega}(\omega) = A_\omega(\omega)\frac{\pi^{\frac{-(\frac{1}{2}+\sigma)}{2}}}{2}e^{\frac{-i\omega}{2}\log(\pi)}(2^{\frac{1}{2}+\sigma}e^{i\omega\log(2)} - 1)$$

(8)

We define $a(t)$ as the Inverse Fourier Transform of $A_\omega(\omega)$. We compute the Inverse Fourier Transform of $E_{p\omega}(\omega)$ given by $E_p(t)$ as follows, using time shifting property.

$$E_p(t) = \frac{\pi^{\frac{-(\frac{1}{2}+\sigma)}{2}}}{2}[2^{\frac{1}{2}+\sigma}a(t - \frac{\log(\pi)}{2} + \log(2)) - a(t - \frac{\log(\pi)}{2})]$$

(9)

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1.7. *Derivation of $a(t)$ and $E_p(t)$*

We start with the gamma function $\Gamma(\frac{s}{2}) = \int_0^\infty y^{\frac{s}{2}-1}e^{-y}dy$. We evaluate $A(s) = \Gamma(\frac{s}{2})\eta(s)$ at $s = \frac{1}{2} + \sigma + i\omega$ below. We substitute $y = \pi n^2 x$ and $dy = \pi n^2 dx$ in Eq. 10 and get $y^{\frac{s}{2}-1}dy = (\pi n^2)^{\frac{s}{2}-1}x^{\frac{s}{2}-1}\pi n^2 dx = \pi^{\frac{s}{2}}n^s(\pi n^2)^{-1}x^{\frac{s}{2}-1}\pi n^2 dx = \pi^{\frac{s}{2}}n^s x^{\frac{s}{2}-1}dx$.

$$A(s) = \Gamma(\frac{s}{2})\eta(s) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^s}\int_0^\infty y^{\frac{s}{2}-1}e^{-y}dy = \pi^{\frac{s}{2}}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^s}n^s\int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx \qquad (10)$$

For $Re[s] > 0$, the gamma function is analytic in the complex plane (link) and $\eta(s)$ converges and hence $|A(s)| = |\Gamma(\frac{s}{2})\eta(s)|$ converges.

We can interchange the order of integration and summation in Eq. 10 using Fubini's theorem given that the integrands in Eq. 10 before the interchange and the integrands in Eq. 13 after the interchange are absolutely integrable with exponential asymptotic fall-off rate (Appendix A.8) and hence the integral in Eq. 13 converges and equal the corresponding expressions in Eq. 10 and we write as follows. (link)

$$A(s) = \pi^{\frac{s}{2}}\int_0^\infty \sum_{n=1}^{\infty}(-1)^{n-1}e^{-\pi n^2 x}x^{\frac{s}{2}-1}dx \qquad (11)$$

Now we substitute $x = e^{-2t}$ and $dx = -2e^{-2t}dt = -2xdt$ and write Eq. 11 as follows.

$$A(s) = 2\pi^{\frac{s}{2}}\int_{-\infty}^\infty \sum_{n=1}^{\infty}(-1)^{n-1}e^{-\pi n^2 e^{-2t}}e^{-st}dt \qquad (12)$$

We substitute $s = \frac{1}{2} + \sigma + i\omega$ in Eq. 12 as follows.

$$A(\frac{1}{2} + \sigma + i\omega) = A_\omega(\omega) = 2\pi^{\frac{\frac{1}{2}+\sigma}{2}}e^{\frac{i\omega}{2}\log\pi}\int_{-\infty}^\infty \sum_{n=1}^{\infty}(-1)^{n-1}e^{-\pi n^2 e^{-2t}}e^{-\frac{t}{2}}e^{-\sigma t}e^{-i\omega t}dt \qquad (13)$$

The integrand in Eq. 13 is absolutely integrable given asymptotic exponential fall-off rate. (Appendix A.8) We see that the inverse Fourier transform of $A_\omega(\omega)$ is given by $a(t)$ as follows, using the time shifting property.

$$a(t) = a_0(t + \frac{\log \pi}{2}), \quad a_0(t) = 2\pi^{\frac{1}{4}+\frac{\sigma}{2}} \sum_{n=1}^{\infty} (-1)^{n-1} e^{-\pi n^2 e^{-2t}} e^{-\frac{t}{2}} e^{-\sigma t} \tag{14}$$

We know that $\Gamma(\frac{s}{2})$ does not have zeros for any value of s (link) and the gamma function is analytic in the complex plane for $Re[s] > 0$ (link). If $\eta(s)$ has a zero at $\omega = \omega_0$ in the critical strip, then $A(\frac{1}{2} + \sigma + i\omega)$ in Eq. 10 has a zero at $\omega = \omega_0$ and the Fourier transform of $a(t)$ given by $A_\omega(\omega)$ in Eq. 13 has a zero at $\omega = \omega_0$ (**Result E.0**)

Now we substitute $a(t)$ in Eq. 14 in Eq. 9 copied below and cancel the common terms $\frac{\log(\pi)}{2}$ and $2\pi^{\frac{1}{4}+\frac{\sigma}{2}}$ as follows. We use $2^{\frac{1}{2}+\sigma}2^{-(\frac{1}{2}+\sigma)} = 1$ in the first term in $E_p(t)$ below.

$$\begin{aligned}
E_p(t) &= \frac{\pi^{\frac{-(\frac{1}{2}+\sigma)}{2}}}{2}[2^{\frac{1}{2}+\sigma} a(t - \frac{\log(\pi)}{2} + \log(2)) - a(t - \frac{\log(\pi)}{2})] \\
E_p(t) &= \frac{\pi^{-(\frac{1}{4}+\frac{\sigma}{2})}}{2}[2^{\frac{1}{2}+\sigma} a_0(t - \frac{\log(\pi)}{2} + \frac{\log(\pi)}{2} + \log(2)) - a_0(t - \frac{\log(\pi)}{2} + \frac{\log(\pi)}{2})] \\
E_p(t) &= \frac{\pi^{-(\frac{1}{4}+\frac{\sigma}{2})}}{2}[2^{\frac{1}{2}+\sigma} a_0(t + \log(2)) - a_0(t)], \quad a_0(t + \log(2)) = 2 * 2^{-(\frac{1}{2}+\sigma)} \pi^{\frac{1}{4}+\frac{\sigma}{2}} \sum_{n=1}^{\infty} (-1)^{n-1} e^{-\pi \frac{n^2}{4} e^{-2t}} e^{-\frac{t}{2}} e^{-\sigma t} \\
E_p(t) &= \sum_{n=1}^{\infty} (-1)^{n-1} e^{-\pi \frac{n^2}{4} e^{-2t}} e^{-\frac{t}{2}} e^{-\sigma t} - \sum_{n=1}^{\infty} (-1)^{n-1} e^{-\pi n^2 e^{-2t}} e^{-\frac{t}{2}} e^{-\sigma t} \\
E_p(t) &= E_0(t) e^{-\sigma t}, \quad E_0(t) = \sum_{n=1}^{\infty} (-1)^{n-1} (e^{-\pi \frac{n^2}{4} e^{-2t}} - e^{-\pi n^2 e^{-2t}}) e^{-\frac{t}{2}}
\end{aligned} \tag{15}$$

We see that $E_0(t)$ is the inverse Fourier transform of $E(\frac{1}{2} + i\omega)$ (set $\sigma = 0$ in Eq. 7 and Eq. 9) and it obeys $E_0(t) = E_0(-t)$ given that $E(s) = E(1-s)$ using Eq. 5(We use the result in Appendix A.7). (**Result E.1**)

Using Eq. 8, we have derived the analytic continuation of Riemann's zeta function derived from Dirichlet Eta function given by $E_{p\omega}(\omega) = \eta(\frac{1}{2} + \sigma + i\omega) B(\omega)$ where
$B(\omega) = \Gamma(\frac{\frac{1}{2}+\sigma+i\omega}{2}) \frac{\pi^{\frac{-(\frac{1}{2}+\sigma)}{2}}}{2} e^{\frac{-i\omega}{2}\log(\pi)} (2^{\frac{1}{2}+\sigma} e^{i\omega \log(2)} - 1)$.

We see that, if $\eta(\frac{1}{2} + \sigma + i\omega)$ has a zero at $\omega = \omega_0$ in the critical strip, then the Fourier transform of the function $E_p(t) = E_0(t) e^{-\sigma t}$ given by $E_{p\omega}(\omega)$ **also** has a zero at $\omega = \omega_0$, where $E_0(t) = \sum_{n=1}^{\infty} (-1)^{n-1} (e^{-\pi \frac{n^2}{4} e^{-2t}} - e^{-\pi n^2 e^{-2t}}) e^{-\frac{t}{2}}$.

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2. An Approach towards Riemann's Hypothesis

Theorem 1: The analytic continuation of Riemann's zeta function derived from Dirichlet Eta function given by $E_{p\omega}(\omega) = \eta(\frac{1}{2} + \sigma + i\omega)B(\omega)$ does not have zeros for any real value of $-\infty < \omega < \infty$, for $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line, where $B(\omega) = \Gamma(\frac{\frac{1}{2}+\sigma+i\omega}{2})\frac{\pi^{\frac{-(\frac{1}{2}+\sigma)}{2}}}{2}e^{\frac{-i\omega}{2}\log(\pi)}(2^{\frac{1}{2}+\sigma}e^{i\omega\log(2)} - 1)$ given that $E_0(t) = E_0(-t)$ is an even function of variable t, where $E_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$, $E_p(t) = E_0(t)e^{-\sigma t}$ and $E_0(t) = \sum_{n=1}^{\infty}(-1)^{n-1}(e^{-\pi\frac{n^2}{4}e^{-2t}} - e^{-\pi n^2 e^{-2t}})e^{-\frac{t}{2}}$.

Proof: We assume that Riemann Hypothesis is false and prove its truth using proof by contradiction.

Statement 1: Let us assume that the analytic continuation of Riemann's zeta function derived from Dirichlet Eta function given by $E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ where ω_0 is real and finite and $0 < |\sigma| < \frac{1}{2}$, corresponding to the critical strip excluding the critical line. We will prove that this assumption leads to a **contradiction**.

We will prove it for $0 < \sigma < \frac{1}{2}$ first and then use the property $\xi(\frac{1}{2} + \sigma + i\omega) = \xi(\frac{1}{2} - \sigma - i\omega)$ to show the result for $-\frac{1}{2} < \sigma < 0$ and hence show the result for $0 < |\sigma| < \frac{1}{2}$.

We know that $\omega_0 \neq 0$, because $\zeta(s)$ has no zeros on the real axis between 0 and 1, when $s = \frac{1}{2} + \sigma + i\omega$ is real, $\omega = 0$ and $0 \leq |\sigma| < \frac{1}{2}$. [3] (Titchmarsh pp30-31). This is shown in detail in first two paragraphs in Appendix A.1.

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2.1. ***New function*** $g(t, t_2, t_0)$

Let us consider the function $E_p^{'}(t, t_2) = e^{-\sigma t_2}E_p(t - t_2) - e^{\sigma t_2}E_p(t + t_2) = (E_0(t - t_2) - E_0(t + t_2))e^{-\sigma t} = E_0^{'}(t, t_2)e^{-\sigma t}$, where t_2 is non-zero and real, and $E_0^{'}(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$ (**Definition 1**). Its Fourier transform is given by $E_{p\omega}^{'}(\omega, t_2) = E_{p\omega}(\omega)(e^{-\sigma t_2}e^{-i\omega t_2} - e^{\sigma t_2}e^{i\omega t_2})$ which has a zero at the **same** $\omega = \omega_0$, using Statement 1 and linearity and time shift properties of the Fourier transform (link). (**Result 2.1.1**).

Let us consider the function $f(t, t_2, t_0) = e^{-2\sigma t_0}f_1(t, t_2, t_0) + e^{2\sigma t_0}f_2(t, t_2, t_0)$ where $f_1(t, t_2, t_0) = e^{\sigma t_0}E_p^{'}(t + t_0, t_2)$ and $f_2(t, t_2, t_0) = f_1(t, t_2, -t_0) = e^{-\sigma t_0}E_p^{'}(t - t_0, t_2)$ where t_0 is real and we can see that the Fourier Transform of this function $F(\omega, t_2, t_0) = E_{p\omega}^{'}(\omega, t_2)(e^{-2\sigma t_0}e^{\sigma t_0}e^{i\omega t_0} + e^{2\sigma t_0}e^{-\sigma t_0}e^{-i\omega t_0}) = E_{p\omega}^{'}(\omega, t_2)(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0})$ also has a zero at the **same** $\omega = \omega_0$, using Result 2.1.1.
(**Result 2.1.2**)

Let us consider a new function $g(t, t_2, t_0) = g_-(t, t_2, t_0)u(-t) + g_+(t, t_2, t_0)u(t)$ where $g(t, t_2, t_0)$ is a real function of variable t and $u(t)$ is Heaviside unit step function and $g_-(t, t_2, t_0) = f(t, t_2, t_0)e^{-\sigma t}$

and $g_+(t,t_2,t_0) = f(t,t_2,t_0)e^{\sigma t}$. We can see that $g(t,t_2,t_0)h(t) = f(t,t_2,t_0)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

We can write the above equations as follows.

$$\begin{aligned}
E_p^{'}(t,t_2) = e^{-\sigma t_2}E_p(t-t_2) - e^{\sigma t_2}E_p(t+t_2) = (E_0(t-t_2) - E_0(t+t_2))e^{-\sigma t} = E_0^{'}(t,t_2)e^{-\sigma t} \\
f_1(t,t_2,t_0) = e^{\sigma t_0}E_p^{'}(t+t_0,t_2) \\
f_2(t,t_2,t_0) = f_1(t,t_2,-t_0) = e^{-\sigma t_0}E_p^{'}(t-t_0,t_2) \\
f(t,t_2,t_0) = e^{-2\sigma t_0}f_1(t,t_2,t_0) + e^{2\sigma t_0}f_2(t,t_2,t_0) = e^{-\sigma t_0}E_p^{'}(t+t_0,t_2) + e^{\sigma t_0}E_p^{'}(t-t_0,t_2) \\
g(t,t_2,t_0) = [f(t,t_2,t_0)e^{-\sigma t}]u(-t) + [f(t,t_2,t_0)e^{\sigma t}]u(t) \\
g(t,t_2,t_0)h(t) = f(t,t_2,t_0), \quad h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]
\end{aligned} \tag{16}$$

We can show that $E_p(t), E_p^{'}(t,t_2), h(t)$ are absolutely integrable functions and go to zero as $t \to \pm\infty$. Hence their respective Fourier transforms given by $E_{p\omega}(\omega), E_{p\omega}^{'}(\omega,t_2), H(\omega)$ are finite for real ω and go to zero as $|\omega| \to \infty$, as per Riemann Lebesgue Lemma (link). We can show that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ are absolutely **integrable** functions. These results are shown in Appendix A.1.

In Section 2.3 and Section 2.4, it is shown that $g(t,t_2,t_0)$ is a Fourier transformable function and its Fourier transform given by $G(\omega,t_2,t_0) = e^{-2\sigma t_0}G_1(\omega,t_2,t_0) + e^{2\sigma t_0}G_1(\omega,t_2,-t_0)$ converges. (Eq. 24 and Eq. 27)

If we take the Fourier transform of the equation $g(t,t_2,t_0)h(t) = f(t,t_2,t_0)$ where $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$, using Result 2.1.2, we get $\frac{1}{2\pi}[G(\omega,t_2,t_0) * H(\omega)] = F(\omega,t_2,t_0) = E_{p\omega}^{'}(\omega,t_2)(e^{-\sigma t_0}e^{i\omega t_0} + e^{\sigma t_0}e^{-i\omega t_0}) = F_R(\omega,t_2,t_0) + iF_I(\omega,t_2,t_0)$ as per **convolution theorem** (link), where $*$ denotes convolution operation given by $F(\omega,t_2,t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega',t_2,t_0)H(\omega-\omega')d\omega'$. (**Result 2.1.a**)

We see that $H(\omega) = H_R(\omega) = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}] = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and is the Fourier transform of the function $h(t)$ (link). $G(\omega,t_2,t_0) = G_R(\omega,t_2,t_0) + iG_I(\omega,t_2,t_0)$ is the Fourier transform of the function $g(t,t_2,t_0)$. We can write $g(t,t_2,t_0) = g_{even}(t,t_2,t_0) + g_{odd}(t,t_2,t_0)$ where $g_{even}(t,t_2,t_0)$ is an even function and $g_{odd}(t,t_2,t_0)$ is an odd function of variable t.

If Statement 1 is true, then we require the Fourier transform of the function $f(t,t_2,t_0)$ given by $F(\omega,t_2,t_0)$ to have a zero at $\omega = \omega_0$ for **every value** of t_0, for each non-zero value of t_2, using Result 2.1.2. This implies that the **real** part of the Fourier transform of the **even function** $g_{even}(t,t_2,t_0) = \frac{1}{2}[g(t,t_2,t_0) + g(-t,t_2,t_0)]$ given by $G_R(\omega,t_2,t_0)$(Details in Appendix B.2) must have **at least one zero** at $\omega = \omega_z(t_2,t_0) \neq 0$ where $\omega_z(t_2,t_0)$ is real, where $G_R(\omega,t_2,t_0)$ crosses the zero line to the opposite sign, explained below. We note that $\omega_z(t_2,t_0)$ can be different from ω_0 in general.

Because $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ is real and does not have zeros for any finite value of ω, **if** $G_R(\omega,t_2,t_0)$ does not have at least one zero for some $\omega = \omega_z(t_2,t_0) \neq 0$, where $G_R(\omega,t_2,t_0)$ crosses the zero line to the opposite sign, **then** the **real part** of $F(\omega,t_2,t_0)$ given by $F_R(\omega,t_2,t_0) = \frac{1}{2\pi}[G_R(\omega,t_2,t_0) * H(\omega)]$, obtained by the convolution of $H(\omega)$ and $G_R(\omega,t_2,t_0)$, **cannot** possibly have zeros for any non-zero finite value of ω, which contradicts Result 2.1.2 and **Statement 1**. This is shown in detail in Lemma 1.

The proof for Lemma 1 below is shown for **a fixed value** of $t_0 = t_{0f}$ and $t_2 = t_{2f}$, in the interval $|t_0| < \infty$ and $0 < |t_2| < \infty$ (**Interval A**), where $G_R(\omega, t_2, t_0)$ is a function of ω **only**. The proof continues to hold for our choice of **each and every combination** of **fixed values** of t_0 and t_2 in interval A, where $G_R(\omega, t_2, t_0)$ is a function of ω **only**.

Lemma 1: Let $t_0, t_2 \in \Re$ be fixed values and $t_2 \neq 0$ and $E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ using Statement 1. Then the Fourier transform of the **even function** $g_{even}(t, t_2, t_0)$ given by $G_R(\omega, t_2, t_0)$ must have **at least one zero** at $\omega = \omega_z(t_2, t_0) \neq 0$, where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign and $\omega_z(t_2, t_0)$ is real.

Proof: If $E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$ to satisfy Statement 1, then $F(\omega, t_2, t_0)$ has a zero at $\omega = \omega_0$, using Result 2.1.2 and its real part given by $F_R(\omega, t_2, t_0)$ has a zero at $\omega = \omega_0$, where $\omega_0 \neq 0$(**Result 2.1.3**).

We do not have a closed form solution for $G_R(\omega, t_2, t_0)$ and do not know the exact location of its zeros at $\omega = \omega_z(t_2, t_0)$. For a specific choice of t_2, t_0, **only one** of the 2 cases is possible:
Case A: $G_R(\omega, t_2, t_0)$ does not have a zero crossing for any choice of $\omega \neq 0$ or
Case B: $G_R(\omega, t_2, t_0)$ has at least one zero crossing for a specific $\omega \neq 0$.
If Statement 1 is true, **then** Case B is the **only** possibility and Case A is **ruled out**, as shown below.

We want to show the **Result 2.1.5** that $G_R(\omega, t_2, t_0)$ **must have at least one** zero crossing at **some value** of $\omega = \omega_z(t_2, t_0) \neq 0$ (**Case B**), to satisfy **Statement 1**, for this choice of fixed t_0, t_2.

To show Result 2.1.5, we **assume the opposite Case A**, that $G_R(\omega, t_2, t_0)$ **does not** have at least one zero for **any** value of $\omega \neq 0$, where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign (zero crossing) and will show that $F_R(\omega, t_2, t_0)$ does not have at least one zero at finite $\omega \neq 0$ for this case, which **contradicts** Result 2.1.3 and Statement 1 and hence we **rule out** Case A and arrive at Case B (Case B is the same as Result 2.1.5).

This **does not** mean that, proof of Lemma 1 will work **only if** $G_R(\omega, t_2, t_0)$ does not have a zero crossing for any value of $\omega \neq 0$, for any choice of t_2, t_0. The device **Proof by Contradiction** is used here to **rule out** Case A and arrive at Case B. (Details of other cases in Section 2.1.1)

It is noted that, for **Case B**, we **do not** use Eq. 17 to Eq. 20 and related arguments, because Case B is the desired Result 2.1.5. (**Note 1**)

The arguments above and following proof continue to hold for our choice of **each and every combination** of **fixed values** of t_0 and t_2 in interval A, where $G_R(\omega, t_2, t_0)$ is a function of ω **only**.

Given that $H(\omega)$ is real, using Result 2.1.a, we write the convolution theorem only for the real parts as follows.

$$F_R(\omega, t_2, t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_R(\omega', t_2, t_0) H(\omega - \omega') d\omega' \tag{17}$$

We can show that the above integral converges for real ω, given that the integrand is absolutely integrable because $G(\omega, t_2, t_0)$ and $H(\omega)$ have fall-off rate of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ because the first derivatives of $g(t, t_2, t_0)$ and $h(t)$ are discontinuous at $t = 0$.(Details in Appendix A.2 and Appendix A.5)

We substitute $H(\omega) = \frac{2\sigma}{(\sigma^2+\omega^2)}$ in Eq. 17 and we get

$$F_R(\omega, t_2, t_0) = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} G_R(\omega', t_2, t_0) \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' \tag{18}$$

We can split the integral in Eq. 18 using $\int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_{0}^{\infty}$, as follows.

$$F_R(\omega, t_2, t_0) = \frac{\sigma}{\pi} [\int_{-\infty}^{0} G_R(\omega', t_2, t_0) \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega' + \int_{0}^{\infty} G_R(\omega', t_2, t_0) \frac{1}{(\sigma^2 + (\omega - \omega')^2)} d\omega'] \tag{19}$$

We see that $G_R(-\omega, t_2, t_0) = G_R(\omega, t_2, t_0)$ because $g(t, t_2, t_0)$ is a real function of variable t. (Details in Appendix B.1) We can substitute $\omega' = -\omega''$ in the first integral in Eq. 19 and substituting $\omega'' = \omega'$ in the result, we can write as follows.

$$F_R(\omega, t_2, t_0) = \frac{\sigma}{\pi} \int_{0}^{\infty} G_R(\omega', t_2, t_0) [\frac{1}{(\sigma^2 + (\omega - \omega')^2)} + \frac{1}{(\sigma^2 + (\omega + \omega')^2)}] d\omega' \tag{20}$$

We note that t_0 and t_2 are **fixed** in Eq. 20 and $G_R(\omega, t_2, t_0)$ is a function of ω **only** and the integrand in Eq. 20 is integrated over the variable ω **only**.

In Appendix A.2, it is shown that $G(\omega', t_2, t_0)$ is finite for real ω' and goes to zero as $|\omega'| \to \infty$. We can see that for $\omega' \to \infty$, the integrand in Eq. 20 goes to zero. For finite $\omega \geq 0$, and $0 \leq \omega' < \infty$, we can see that the term $\frac{1}{(\sigma^2+(\omega-\omega')^2)} + \frac{1}{(\sigma^2+(\omega+\omega')^2)} > 0$, for $0 < \sigma < \frac{1}{2}$. We see that $G_R(\omega', t_2, t_0)$ is **not** an all zero function of variable ω' (Details in Section 2.2). (**Result 2.1.4**)

- **Case 1:** $G_R(\omega', t_2, t_0) \geq 0$ **for all finite** $\omega' \geq 0$

We see that $F_R(\omega, t_2, t_0) > 0$ for all finite $\omega \geq 0$, using Result 2.1.4. We see that $F_R(-\omega, t_2, t_0) = F_R(\omega, t_2, t_0)$ because $f(t, t_2, t_0)$ is a real function (Appendix B.1) and link). Hence $F_R(\omega, t_2, t_0) > 0$ for all finite $\omega \leq 0$.

This **contradicts** Statement 1 and Result 2.1.3 which requires $F_R(\omega, t_2, t_0)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega', t_2, t_0)$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) > 0$ where it crosses the zero line and becomes negative, where $\omega_z(t_2, t_0)$ is finite.

- **Case 2:** $G_R(\omega', t_2, t_0) \leq 0$ **for all finite** $\omega' \geq 0$

We see that $F_R(\omega, t_2, t_0) < 0$ for all finite $\omega \geq 0$, using Result 2.1.4. We see that $F_R(-\omega, t_2, t_0) = F_R(\omega, t_2, t_0)$ because $f(t, t_2, t_0)$ is a real function (Appendix B.1) and link). Hence $F_R(\omega, t_2, t_0) < 0$ for all finite $\omega \leq 0$.

This **contradicts** Statement 1 and Result 2.1.3 which requires $F_R(\omega, t_2, t_0)$ to have at least one zero at finite $\omega \neq 0$. Therefore $G_R(\omega', t_2, t_0)$ must have **at least one zero** at $\omega' = \omega_z(t_2, t_0) > 0$, where it crosses the zero line and becomes positive, where $\omega_z(t_2, t_0)$ is finite.

We have shown that, $G_R(\omega, t_2, t_0)$ must have **at least one zero** at finite $\omega = \omega_z(t_2, t_0) \neq 0$ where it crosses the zero line to the opposite sign, to satisfy **Statement 1**, for specific choices of fixed t_0, t_2. We call this **Result 2.1.5**.

The arguments above and the proof continue to hold for our choice of **each and every combination** of **fixed values** of t_0 and t_2 in interval A, where $G_R(\omega, t_2, t_0)$ is a function of ω **only**.

In the rest of the sections, we consider only the **first** zero crossing to the right of origin, where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign. Hence $0 < \omega_z(t_2, t_0) < \infty$, for all $|t_0| < \infty$, for each non-zero value of t_2, to satisfy **Statement 1**.

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*2.1.1. **Discussion of Lemma 1***

Result 2.1.5: $G_R(\omega, t_2, t_0)$ must have **at least one zero** at finite $\omega = \omega_z(t_2, t_0) \neq 0$ where it crosses the zero line to the opposite sign, to satisfy **Statement 1**.

For each fixed value of t_0, t_2, only 2 cases are possible for $G_R(\omega, t_2, t_0)$. **Case A:** $G_R(\omega, t_2, t_0)$ does not have a zero crossing for any choice of $\omega \neq 0$. **Case B:** $G_R(\omega, t_2, t_0)$ has at least one zero crossing for a specific $\omega \neq 0$. Proof of Lemma 1 assumes Case A and uses **Proof by Contradiction** to rule out Case A and arrive at Case B, for each choice of fixed t_0, t_2. This does not mean that Proof of Lemma 1 does not work for Case B. For Case B, we **do not** use Proof of Lemma 1 and jump to the end of the proof because we already have the desired Result 2.1.5 which is the same as Case B.

The logic used is this proof is as follows: **If** Statement 1 is true(RH is false), **then** Result 2.1.5 is true (Case B), for **each and every** combination of **fixed** values of t_0, t_2 in interval A ($|t_0| < \infty$ and $0 < |t_2| < \infty$)and hence Case A is **ruled out** and only Case B is possible for $G_R(\omega, t_2, t_0)$. Then we proceed with Result 2.1.5 to Section 2.3, 2.4 and Section 3, to produce a **contradiction** of Statement 1 in Eq. 51 and thus prove the truth of RH.

Alternate Method: We present an alternate method of analyzing all possible cases of $G_R(\omega, t_2, t_0)$ below. We can arrive at Result 2.1.5, for **each and every** combination of **fixed** values of t_0, t_2 in interval A, using Proof of Lemma 1 for Case C and Case D or using Case E, as explained below.

It is noted that $F_R(\omega, t_2, t_0)$ and $G_R(\omega, t_2, t_0)$ may have more zeros than $F(\omega, t_2, t_0)$ and $G(\omega, t_2, t_0)$ respectively. That **does not** affect the proof of Lemma 1, as explained below.

We do not have a closed form solution for $G_R(\omega, t_2, t_0)$ and do not know the exact location of its zeros at $\omega = \omega_z(t_2, t_0)$, for each fixed choice of t_2, t_0. We consider 3 possible cases of $G_R(\omega, t_2, t_0)$ below.

- **Case C:** We consider the case that $G_R(\omega, t_2, t_0)$ **does not** have a zero crossing, for any value of $\omega \neq 0$, for **each and every** choice of t_2, t_0 in Interval A and we use Proof of Lemma 1 for each and every choice of t_2, t_0, to show that it leads to a **contradiction** of Statement 1, and hence prove Result 2.1.5, for each and every choice of t_2, t_0.

Hence Case C is **ruled out**, **if** Statement 1 is true. (**Result 2.1.1.c**)

- **Case D:** We consider the case $G_R(\omega, t_2, t_0)$ has a zero crossing at $\omega = \omega_z(t_2, t_0)$, for **specific** choices of $t_2 = t_2^{'}, t_0 = t_0^{'}$.(**Not** for all possible choices of t_2, t_0)

For Case D, this means that $G_R(\omega, t_2, t_0)$ has **at least one zero crossing** at $\omega = \omega_z(t_2, t_0)$, for **specific** choices of $t_2 = t_2^{'}, t_0 = t_0^{'}$, which is the desired **Result 2.1.5** and hence we **do not** go through the arguments in this proof and we can jump to end of Proof of Lemma 1 (using Note 1). In this case, we **have not** assumed Statement 1 and yet arrived at Result 2.1.5, for **specific** choices of $t_2 = t_2^{'}, t_0 = t_0^{'}$.

For Case D, there is **at least one** choice of $t_2 = t_{2f}, t_0 = t_{0f}$ for which $G_R(\omega, t_2, t_0)$ **does not** have a zero crossing, for any value of $\omega \neq 0$. For this choice of $t_2 = t_{2f}, t_0 = t_{0f}$, we use Proof of Lemma 1 to show that it leads to a **contradiction** of Statement 1, and hence prove Result 2.1.5.

Hence Case D is **ruled out**, **if** Statement 1 is true. (**Result 2.1.1.d**)

- **Case E:** We consider the case $G_R(\omega, t_2, t_0)$ has at least one zero crossing at $\omega = \omega_z(t_2, t_0)$, for **each and every** choices of t_2, t_0 in Interval A. We call this **Statement 3**.

For Case E, this means that $G_R(\omega, t_2, t_0)$ has **at least one zero crossing** at $\omega = \omega_z(t_2, t_0)$, for **each and every** choices of t_2, t_0 which is the desired **Result 2.1.5** and hence we **do not** go through the arguments in this proof and we can jump to end of Proof of Lemma 1 (using Note 1). In this case, we **have not** assumed Statement 1 and yet arrived at Result 2.1.5, for **each and every** choices of t_2, t_0.

For Case E, we see that we arrive at Result 2.1.5 by **assuming** Statement 3 only. Then we proceed with Result 2.1.5 to Section 2.3, 2.4 and Section 3, to produce a **contradiction** of Statement 3 in Eq. 51. Hence Statement 3 is false and Case E is **ruled out**.

There are **only 3** possible cases for $G_R(\omega, t_2, t_0)$ given by Case C,D and E. We have ruled out Case E in above para. **If** Statement 1 is true, Case C and Case D have been **ruled out** using Result 2.1.1.d and Result 2.1.1.e. Given that Case C,D and E are the **only 3** possible cases for $G_R(\omega, t_2, t_0)$, this means **Statement 1 is false**.

Thus we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$ and hence prove the truth of Riemann's Hypothesis.

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2.2. $G_R(\omega', t_2, t_0)$ *is not an all zero function of variable* ω'

If $G_R(\omega', t_2, t_0)$ is an all zero function of variable ω', for each given value of $t_0, t_2 \in \Re$ and $t_2 \neq 0$ (**Statement 2**), then $F_R(\omega, t_2, t_0)$ in Eq. 17 is an all zero function of ω, for real ω. Hence $2f_{even}(t, t_2, t_0) = f(t, t_2, t_0) + f(-t, t_2, t_0)$ is an **all-zero** function of t, given that the Fourier transform of $f_{even}(t, t_2, t_0)$ is given by $F_R(\omega, t_2, t_0)$, using symmetry properties of Fourier transform(Appendix B.2) and link). Hence $f(t, t_2, t_0)$ is an **odd function** of variable t.(**Result 2.2**).

From Eq. 16 we see that $E_p^{'}(t, t_2) = e^{-\sigma t_2}E_p(t - t_2) - e^{\sigma t_2}E_p(t + t_2) = [E_0(t - t_2) - E_0(t + t_2)]e^{-\sigma t}$. Hence $f_1(t, t_2, t_0) = e^{\sigma t_0}E_p^{'}(t + t_0, t_2) = e^{\sigma t_0}[E_0(t + t_0 - t_2) - E_0(t + t_0 + t_2)]e^{-\sigma t}e^{-\sigma t_0}$ and we use

$e^{\sigma t_0}e^{-\sigma t_0} = 1$ and hence $f_1(t,t_2,t_0) = [E_0(t+t_0-t_2) - E_0(t+t_0+t_2)]e^{-\sigma t}$ and $f_2(t,t_2,t_0) = f_1(t,t_2,-t_0) = [E_0(t-t_0-t_2) - E_0(t-t_0+t_2)]e^{-\sigma t}$. Hence we can write $f(t,t_2,t_0) = e^{-2\sigma t_0}f_1(t,t_2,t_0) + e^{2\sigma t_0}f_2(t,t_2,t_0)$ in Eq. 16, as follows.

$$f(t,t_2,t_0) = e^{-2\sigma t_0}[E_0(t+t_0-t_2) - E_0(t+t_0+t_2)]e^{-\sigma t} + e^{2\sigma t_0}[E_0(t-t_0-t_2) - E_0(t-t_0+t_2)]e^{-\sigma t} \quad (21)$$

Case 1: For $t_0 \neq 0$ and $t_2 \neq 0$, it is shown that Result 2.2 is false. We will compute $f(t,t_2,t_0)$ in Eq. 21 at $t = 0$ and show that it does not equal zero.

We see that $f(0,t_2,t_0) = e^{-2\sigma t_0}[E_0(t_0-t_2) - E_0(t_0+t_2)] + e^{2\sigma t_0}[E_0(-t_0-t_2) - E_0(-t_0+t_2)]$ $= -2\sinh(2\sigma t_0)[E_0(t_0-t_2) - E_0(t_0+t_2)]$. We use the fact that $E_0(t_0) = E_0(-t_0)$ (Details in Appendix A.7) and hence $E_0(t_0-t_2) = E_0(-t_0+t_2)$ and $E_0(t_0+t_2) = E_0(-t_0-t_2)$.

If Result 2.2 is true, then we require $f(0,t_2,t_0) = -2\sinh(2\sigma t_0)[E_0(t_0-t_2) - E_0(t_0+t_2)] = 0$. For our choice of $0 < \sigma < \frac{1}{2}$ and $t_0 \neq 0$, this implies that $E_0(t_0-t_2) = E_0(t_0+t_2)$. Given that $t_0 \neq 0$ and $t_2 \neq 0$, we set $t_2 = Kt_0$ for real $K \neq 0$ and we get $E_0((1-K)t_0) = E_0((1+K)t_0)$. This is **not** possible for $t_0 \neq 0$ because $E_0(t_0)$ is **strictly decreasing** for $t_0 > 0$ (Details in Section 5) and $1-K \neq 1+K$ or $1-K \neq -(1+K)$ for $K \neq 0$. Hence Result 2.2 is false and Statement 2 is false and $G_R(\omega',t_2,t_0)$ is **not** an all zero function of variable ω'.

Case 2: For $t_0 = 0$ and $t_2 \neq 0$, we have $f(t,t_2,t_0) = 2[E_0(t-t_2) - E_0(t+t_2)]e^{-\sigma t} = 2D(t)e^{-\sigma t}$ in Eq. 21, where $D(t) = E_0(t-t_2) - E_0(t+t_2)$. We see that $D(t) + D(-t) = E_0(t-t_2) - E_0(t+t_2) + E_0(-t-t_2) - E_0(-t+t_2)$. Given that $E_0(t) = E_0(-t)$, we have $D(t) + D(-t) = E_0(t-t_2) - E_0(t+t_2) + E_0(t+t_2) - E_0(t-t_2) = 0$ and hence $D(t) = E_0(t-t_2) - E_0(t+t_2)$ is an **odd** function of variable t (**Result 2.2.1**).

If Result 2.2 is true, then we require $f(t,t_2,t_0) = 2D(t)e^{-\sigma t}$ to be an **odd** function of variable t. Using Result 2.2.1, we require $D(t)$ to be an **odd** function of variable t. This is possible only for $\sigma = 0$. This is **not** possible for our choice of $0 < \sigma < \frac{1}{2}$. Hence Result 2.2 is false and Statement 2 is false and $G_R(\omega',t_2,t_0)$ is **not** an all zero function of variable ω'.

Case 3: For $t_2 = 0$ and $|t_0| < \infty$, we have $E_p^{'}(t,t_2) = e^{-\sigma t_2}E_p(t-t_2) - e^{\sigma t_2}E_p(t+t_2) = 0$ and $f(t,t_2,t_0) = g(t,t_2,t_0) = 0$ for all t in Eq. 16 and Lemma 1 is not applicable for this case.

.

2.3. *On the zeros of a related function* $G(\omega,t_2,t_0)$

In this section, we compute the Fourier transform of the function $g_{even}(t,t_2,t_0) = \frac{1}{2}[g(t,t_2,t_0) + g(-t,t_2,t_0)]$ given by $G_R(\omega,t_2,t_0)$(using Appendix B.2). We require $G_R(\omega,t_2,t_0) = 0$ for $\omega = \omega_z(t_2,t_0)$ for **every value** of t_0, for each non-zero value of t_2, to satisfy **Statement 1**, using Lemma 1 in Section 2.1.

We **define** $g_1(t,t_2,t_0) = f_1(t,t_2,t_0)e^{-\sigma t}u(-t) + f_1(t,t_2,t_0)e^{\sigma t}u(t) = e^{\sigma t_0}E_p^{'}(t+t_0,t_2)e^{-\sigma t}u(-t) + e^{\sigma t_0}E_p^{'}(t+t_0,t_2)e^{\sigma t}u(t)$, using Eq. 16 (**Definition 3**). First we compute the Fourier transform of the function $g_1(t,t_2,t_0)$ given by $G_1(\omega,t_2,t_0) = G_{1R}(\omega,t_2,t_0) + iG_{1I}(\omega,t_2,t_0)$ as follows.

$$G_1(\omega,t_2,t_0)=\int_{-\infty}^{\infty}g_1(t,t_2,t_0)e^{-i\omega t}dt=\int_{-\infty}^{0}g_1(t,t_2,t_0)e^{-i\omega t}dt+\int_{0}^{\infty}g_1(t,t_2,t_0)e^{-i\omega t}dt$$
$$G_1(\omega,t_2,t_0)=\int_{-\infty}^{0}e^{\sigma t_0}E_p^{'}(t+t_0,t_2)e^{-\sigma t}e^{-i\omega t}dt+\int_{0}^{\infty}e^{\sigma t_0}E_p^{'}(t+t_0,t_2)e^{\sigma t}e^{-i\omega t}dt \tag{22}$$

We use $E_p^{'}(t,t_2)=E_0^{'}(t,t_2)e^{-\sigma t}$ from Eq. 16, where $E_0^{'}(t,t_2)=E_0(t-t_2)-E_0(t+t_2)$, using Definition 1 in Section 2.1 and we get $E_p^{'}(t+t_0,t_2)=E_0^{'}(t+t_0,t_2)e^{-\sigma t}e^{-\sigma t_0}$ and write Eq. 22 as follows. Then we substitute $t=-t$ in the second integral in first line of Eq. 23.

$$G_1(\omega,t_2,t_0)=\int_{-\infty}^{0}E_0^{'}(t+t_0,t_2)e^{-2\sigma t}e^{-i\omega t}dt+\int_{0}^{\infty}E_0^{'}(t+t_0,t_2)e^{-i\omega t}dt$$
$$G_1(\omega,t_2,t_0)=\int_{-\infty}^{0}E_0^{'}(t+t_0,t_2)e^{-2\sigma t}e^{-i\omega t}dt+\int_{-\infty}^{0}E_0^{'}(-t+t_0,t_2)e^{i\omega t}dt \tag{23}$$

We define $E_{0n}^{'}(t,t_2)=E_0^{'}(-t,t_2)$ (**Definition 2**) and get $E_0^{'}(-t+t_0,t_2)=E_{0n}^{'}(t-t_0,t_2)$ and write Eq. 23 as follows. The integral in Eq. 24 converges, given that $E_0(t)e^{-2\sigma t}$ is an absolutely **integrable** function and its t_0,t_2 shifted versions are absolutely **integrable**, using $E_0^{'}(t,t_2)=E_0(t-t_2)-E_0(t+t_2)$ in Definition 1 in Section 2.1 and Definition 2. (Details in Appendix A.1)

$$G_1(\omega,t_2,t_0)=\int_{-\infty}^{0}E_0^{'}(t+t_0,t_2)e^{-2\sigma t}e^{-i\omega t}dt+\int_{-\infty}^{0}E_{0n}^{'}(t-t_0,t_2)e^{i\omega t}dt=G_{1R}(\omega,t_2,t_0)+iG_{1I}(\omega,t_2,t_0) \tag{24}$$

The above equations can be expanded as follows using the identity $e^{i\omega t}=\cos(\omega t)+i\sin(\omega t)$. Comparing the **real parts** of $G_1(\omega,t_2,t_0)$, we have

$$G_{1R}(\omega,t_2,t_0)=\int_{-\infty}^{0}E_0^{'}(t+t_0,t_2)e^{-2\sigma t}\cos{(\omega t)}dt+\int_{-\infty}^{0}E_{0n}^{'}(t-t_0,t_2)\cos{(\omega t)}dt \tag{25}$$

.

2.4. Zero crossing function $\omega_z(t_2,t_0)$ is an even function of variable t_0, for a given t_2

Now we consider Eq. 16 and the function $f(t,t_2,t_0)=e^{-2\sigma t_0}f_1(t,t_2,t_0)+e^{2\sigma t_0}f_2(t,t_2,t_0)=e^{-\sigma t_0}E_p^{'}(t+t_0,t_2)+e^{\sigma t_0}E_p^{'}(t-t_0,t_2)$ where $f_1(t,t_2,t_0)=e^{\sigma t_0}E_p^{\prime}(t+t_0,t_2)$ and $f_2(t,t_2,t_0)=f_1(t,t_2,-t_0)=e^{-\sigma t_0}E_p^{'}(t-t_0,t_2)$ and $g(t,t_2,t_0)h(t)=f(t,t_2,t_0)$ where $g(t,t_2,t_0)=f(t,t_2,t_0)e^{-\sigma t}u(-t)+f(t,t_2,t_0)e^{\sigma t}u(t)$ and $h(t)=[e^{\sigma t}u(-t)+e^{-\sigma t}u(t)]$. We can write the above equations and $g_1(t,t_2,t_0)$ from Definition 3 in Section 2.3, as follows. We define $g_2(t,t_2,t_0)$ below and write $g(t,t_2,t_0)$ as follows.

$$
\begin{aligned}
g_1(t,t_2,t_0) = f_1(t,t_2,t_0)e^{-\sigma t}u(-t) + f_1(t,t_2,t_0)e^{\sigma t}u(t), \quad g_1(t,t_2,t_0)h(t) = f_1(t,t_2,t_0)\\
g_2(t,t_2,t_0) = f_2(t,t_2,t_0)e^{-\sigma t}u(-t) + f_2(t,t_2,t_0)e^{\sigma t}u(t), \quad g_2(t,t_2,t_0)h(t) = f_2(t,t_2,t_0)\\
g(t,t_2,t_0) = e^{-2\sigma t_0}g_1(t,t_2,t_0) + e^{2\sigma t_0}g_2(t,t_2,t_0)
\end{aligned}
\tag{26}
$$

We use the fact that, for $t < 0$, $e^{-2\sigma t_0}g_1(t,t_2,t_0) + e^{2\sigma t_0}g_2(t,t_2,t_0) = e^{-2\sigma t_0}f_1(t,t_2,t_0)e^{-\sigma t} + e^{2\sigma t_0}f_2(t,t_2,t_0)e^{-\sigma t} = e^{-2\sigma t_0}e^{\sigma t_0}E'_p(t+t_0,t_2)e^{-\sigma t} + e^{2\sigma t_0}e^{-\sigma t_0}E'_p(t-t_0,t_2)e^{-\sigma t} = [e^{-\sigma t_0}E'_p(t+t_0,t_2) + e^{\sigma t_0}E'_p(t-t_0,t_2)]e^{-\sigma t} = f(t,t_2,t_0)e^{-\sigma t} = g(t,t_2,t_0)$ using Eq. 26 and the paragraph before Eq. 26.

For $t = 0, u(t) = u(-t) = \frac{1}{2}$ and hence $g_1(0,t_2,t_0) = f_1(0,t_2,t_0)$, $g_2(t,t_2,t_0) = f_2(0,t_2,t_0)$ and hence $e^{-2\sigma t_0}g_1(0,t_2,t_0) + e^{2\sigma t_0}g_2(0,t_2,t_0) = e^{-2\sigma t_0}f_1(0,t_2,t_0) + e^{2\sigma t_0}f_2(0,t_2,t_0) = f(0,t_2,t_0)$ and $g(0,t_2,t_0) = \frac{f(0,t_2,t_0)}{h(0)} = f(0,t_2,t_0)$ and hence $g(t,t_2,t_0) = e^{-2\sigma t_0}g_1(t,t_2,t_0) + e^{2\sigma t_0}g_2(t,t_2,t_0)$ at $t = 0$.

Similarly, for $t > 0$, $e^{-2\sigma t_0}g_1(t,t_2,t_0) + e^{2\sigma t_0}g_2(t,t_2,t_0) = e^{-2\sigma t_0}f_1(t,t_2,t_0)e^{\sigma t} + e^{2\sigma t_0}f_2(t,t_2,t_0)e^{\sigma t} = e^{-2\sigma t_0}e^{\sigma t_0}E'_p(t+t_0,t_2)e^{\sigma t} + e^{2\sigma t_0}e^{-\sigma t_0}E'_p(t-t_0,t_2)e^{\sigma t} = [e^{-\sigma t_0}E'_p(t+t_0,t_2) + e^{\sigma t_0}E'_p(t-t_0,t_2)]e^{\sigma t} = f(t,t_2,t_0)e^{\sigma t} = g(t,t_2,t_0)$. Hence $g(t,t_2,t_0) = e^{-2\sigma t_0}g_1(t,t_2,t_0) + e^{2\sigma t_0}g_2(t,t_2,t_0)$ for all t in Eq. 26 .

We compute the Fourier transform of the function $g(t,t_2,t_0)$ in Eq. 26 and compute its real part $G_R(\omega,t_2,t_0)$ using the procedure in Section 2.3, similar to Eq. 25 and we can write as follows in Eq. 27. We use $G_{2R}(\omega,t_2,t_0) = G_{1R}(\omega,t_2,-t_0)$ given that $f_2(t,t_2,t_0) = f_1(t,t_2,-t_0)$ and $g_2(t,t_2,t_0) = g_1(t,t_2,-t_0)$ and $G_2(\omega,t_2,t_0) = G_1(\omega,t_2,-t_0)$. We substitute $t = \tau$ in the equation for $G_{1R}(\omega,t_2,t_0)$ below, copied from Eq. 25.

$$
\begin{aligned}
G_R(\omega,t_2,t_0) = e^{-2\sigma t_0}G_{1R}(\omega,t_2,t_0) + e^{2\sigma t_0}G_{2R}(\omega,t_2,t_0) = e^{-2\sigma t_0}G_{1R}(\omega,t_2,t_0) + e^{2\sigma t_0}G_{1R}(\omega,t_2,-t_0)\\
G_{1R}(\omega,t_2,t_0) = \int_{-\infty}^{0}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega\tau)d\tau\\
G_R(\omega,t_2,t_0) = e^{-2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega\tau)d\tau\\
+e^{2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\cos(\omega\tau)d\tau
\end{aligned}
\tag{27}
$$

We require $G_R(\omega,t_2,t_0) = 0$ for $\omega = \omega_z(t_2,t_0)$ for every value of t_0, for each non-zero value of t_2, to satisfy **Statement 1**, using Lemma 1 in Section 2.1. In general $\omega_z(t_2,t_0) \neq \omega_0$. Hence we can see that $P(t_2,t_0) = G_R(\omega_z(t_2,t_0),t_2,t_0) = 0$ and we can rearrange the terms in Eq. 27 as follows. We take the first and fourth terms in $G_R(\omega,t_2,t_0)$ in Eq. 27 and include them in the first line in Eq. 28. We take the second and third terms in Eq. 27 and include them in the second line in Eq. 28.

$$
\begin{aligned}
P(t_2,t_0) = G_R(\omega_z(t_2,t_0),t_2,t_0) = \int_{-\infty}^{0}[e^{-2\sigma t_0}E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + e^{2\sigma t_0}E_{0n}^{'}(\tau+t_0,t_2)]\cos(\omega_z(t_2,t_0)\tau)d\tau\\
+\int_{-\infty}^{0}[e^{2\sigma t_0}E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + e^{-2\sigma t_0}E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega_z(t_2,t_0)\tau)d\tau = 0
\end{aligned}
$$

(28)

We use the fact that $f(t,t_2,t_0) = e^{-\sigma t_0}E_p^{'}(t+t_0,t_2) + e^{\sigma t_0}E_p^{'}(t-t_0,t_2) = f(t,t_2,-t_0)$ in Eq. 16, is **unchanged** by the substitution $t_0 = -t_0$. **If** $f(t,t_2,t_0) = f(t,t_2,-t_0)$ is unchanged by the substitution $t_0 = -t_0$, **then** $g(t,t_2,t_0) = g(t,t_2,-t_0)$ is unchanged by the substitution $t_0 = -t_0$, using the fact that $g(t,t_2,t_0)h(t) = f(t,t_2,t_0)$ and $h(t) = [e^{\sigma t}u(-t) + e^{-\sigma t}u(t)]$.

Hence the Fourier transform of $g(t,t_2,t_0)$ given by $G(\omega,t_2,t_0) = G(\omega,t_2,-t_0)$ and its real part given by $G_R(\omega,t_2,t_0) = G_R(\omega,t_2,-t_0)$ is **unchanged** by the substitution $t_0 = -t_0$ and the zero crossing in $G_R(\omega,t_2,-t_0)$ given by $\omega_z(t_2,-t_0)$ is the **same** as the zero crossing in $G_R(\omega,t_2,t_0)$ given by $\omega_z(t_2,t_0)$ and we get $\omega_z(t_2,t_0) = \omega_z(t_2,-t_0)$ and hence $\omega_z(t_2,t_0)$ is an **even** function of variable t_0, for each non-zero value of t_2.

We can write Eq. 28 as follows, where $P_{odd}(t_2,t_0)$ is an **odd** function of variable t_0, for each non-zero value of t_2. We use $\omega_z(t_2,t_0) = \omega_z(t_2,-t_0)$.

$$P(t_2,t_0) = P_{odd}(t_2,t_0) + P_{odd}(t_2,-t_0) = 0$$

$$P_{odd}(t_2,t_0) = \int_{-\infty}^{0} [e^{-2\sigma t_0}E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + e^{2\sigma t_0}E_{0n}^{'}(\tau+t_0,t_2)]\cos(\omega_z(t_2,t_0)\tau)d\tau \tag{29}$$

.

3. Final Step

We expand $P_{odd}(t_2,t_0)$ in Eq. 29 as follows, using the substitution $\tau + t_0 = \tau'$. We get $\tau = \tau' - t_0$ and $d\tau = d\tau'$ and substitute back $\tau' = \tau$ in the second line below. We use $e^{-2\sigma t_0}e^{2\sigma t_0} = 1$ and use the identity $\cos(\omega_z(t_2,t_0)(\tau - t_0)) = \cos(\omega_z(t_2,t_0)t_0)\cos(\omega_z(t_2,t_0)\tau) + \sin(\omega_z(t_2,t_0)t_0)\sin(\omega_z(t_2,t_0)\tau)$ below.

$$\begin{aligned}
P_{odd}(t_2,t_0) &= \int_{-\infty}^{t_0} [e^{-2\sigma t_0}E_0^{'}(\tau^{'},t_2)e^{-2\sigma\tau^{'}}e^{2\sigma t_0} + e^{2\sigma t_0}E_{0n}^{'}(\tau^{'},t_2)]\cos(\omega_z(t_2,t_0)(\tau^{'} - t_0)d\tau^{'} \\
P_{odd}(t_2,t_0) &= [\cos(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{t_0} E_0^{'}(\tau,t_2)e^{-2\sigma\tau}\cos(\omega_z(t_2,t_0)\tau)d\tau \\
&+ \sin(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{t_0} E_0^{'}(\tau,t_2)e^{-2\sigma\tau}\sin(\omega_z(t_2,t_0)\tau)d\tau] \\
&+e^{2\sigma t_0}[\cos(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{t_0} E_{0n}^{'}(\tau,t_2)\cos(\omega_z(t_2,t_0)\tau)d\tau + \sin(\omega_z(t_2,t_0)t_0)\int_{-\infty}^{t_0} E_{0n}^{'}(\tau,t_2)\sin(\omega_z(t_2,t_0)\tau)d\tau]
\end{aligned} \tag{30}$$

In Section 2.1, it is shown that $0 < \omega_z(t_2,t_0) < \infty$, for all $|t_0| < \infty$, for each non-zero value of t_2. In this section, we consider $t_0 > 0$ and $t_2 > 0$ only.

In Section 6, it is shown that $\omega_z(t_2,t_0)$ is a **continuous** function of variable t_0 and t_2, for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$.

In Section 5, it is shown that $E_0(t)$ is **strictly decreasing** for $t > 0$.

Given that $\omega_z(t_2, t_0)$ is a continuous function of both t_0 and t_2, we can find a suitable value of $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$. Given that $\omega_z(t_2, t_0)$ is a continuous function of t_0 and t_2 and given that t_0 is a continuous function, we see that the **product** of two continuous functions $\omega_z(t_2, t_0)t_0$ is a **continuous** function and is positive for $t_0 > 0$ because $0 < \omega_z(t_2, t_0) < \infty$.

We see that $\omega_z(t_2, t_0)t_0$ is a **continuous** function of variable t_0 and t_2, and that $\omega_z(t_2, t_0)t_0 = \frac{\pi}{2}$ can be reached for specific values of t_0 and $t_2 = 2t_0$, as finite t_0 increases without bounds.(Details in Section 4). As t_0, t_2 increase from zero to a larger and larger finite value without bounds, the continuous function $\omega_z(t_2, t_0)t_0$ starts from zero and will pass through $\frac{\pi}{2}$, for specific values of t_0 and $t_2 = 2t_0$.

We set $t_0 = t_{0c} > 0$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ in Eq. 30 as follows. We use the fact that $\cos(\omega_z(t_{2c}, t_{0c})t_{0c}) = 0$, $\sin(\omega_z(t_{2c}, t_{0c})t_{0c}) = 1$ and $\omega_z(t_{2c}, -t_{0c}) = \omega_z(t_{2c}, t_{0c})$ shown in Section 2.4.

$$P_{odd}(t_{2c}, t_{0c}) = \int_{-\infty}^{t_{0c}} E_0^{'}(\tau, t_{2c})e^{-2\sigma\tau}\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau + e^{2\sigma t_{0c}}\int_{-\infty}^{t_{0c}} E_{0n}^{'}(\tau, t_{2c})\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau \tag{31}$$

We compute $P_{odd}(t_2, -t_0)$ in Eq. 30 as follows. We use $\omega_z(t_2, -t_0) = \omega_z(t_2, t_0)$ (Details in Section 2.4).

$$\begin{aligned} P_{odd}(t_2, -t_0) = [\cos(\omega_z(t_2, t_0)t_0)\int_{-\infty}^{-t_0} E_0^{'}(\tau, t_2)e^{-2\sigma\tau}\cos(\omega_z(t_2, t_0)\tau)d\tau \\ - \sin(\omega_z(t_2, t_0)t_0)\int_{-\infty}^{-t_0} E_0^{'}(\tau, t_2)e^{-2\sigma\tau}\sin(\omega_z(t_2, t_0)\tau)d\tau] \\ +e^{-2\sigma t_0}[\cos(\omega_z(t_2, t_0)t_0)\int_{-\infty}^{-t_0} E_{0n}^{'}(\tau, t_2)\cos(\omega_z(t_2, t_0)\tau)d\tau - \sin(\omega_z(t_2, t_0)t_0)\int_{-\infty}^{-t_0} E_{0n}^{'}(\tau, t_2)\sin(\omega_z(t_2, t_0)\tau)d\tau] \end{aligned} \tag{32}$$

We set $t_0 = t_{0c} > 0$ and $t_2 = t_{2c} = 2t_{0c}$ such that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ in Eq. 32 as follows. We use $\cos(\omega_z(t_{2c}, t_{0c})t_{0c}) = 0$, $\sin(\omega_z(t_{2c}, t_{0c})t_{0c}) = 1$.

$$P_{odd}(t_{2c}, -t_{0c}) = -\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau, t_{2c})e^{-2\sigma\tau}\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau - e^{-2\sigma t_{0c}}\int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau, t_{2c})\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau \tag{33}$$

We compute $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0$ in Eq. 29, at $t_0 = t_{0c}$ and $t_2 = t_{2c}$ using Eq. 31 and Eq. 33.

$$\begin{aligned} \int_{-\infty}^{t_{0c}} E_0^{'}(\tau, t_{2c})e^{-2\sigma\tau}\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau + e^{2\sigma t_{0c}}\int_{-\infty}^{t_{0c}} E_{0n}^{'}(\tau, t_{2c})\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau \\ - \int_{-\infty}^{-t_{0c}} E_0^{'}(\tau, t_{2c})e^{-2\sigma\tau}\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau - e^{-2\sigma t_{0c}}\int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau, t_{2c})\sin(\omega_z(t_{2c}, t_{0c})\tau)d\tau = 0 \end{aligned}$$

$$(34)$$

We split the first two integrals in the left hand side of Eq. 34 using $\int_{-\infty}^{t_{0c}} = \int_{-\infty}^{-t_{0c}} + \int_{-t_{0c}}^{t_{0c}}$ as follows.

$$\begin{aligned}
&[\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau, t_{2c}) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{-t_{0c}}^{t_{0c}} E_0^{'}(\tau, t_{2c}) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau] \\
&+ e^{2\sigma t_{0c}}[\int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + \int_{-t_{0c}}^{t_{0c}} E_{0n}^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau] \\
&- \int_{-\infty}^{-t_{0c}} E_0^{'}(\tau, t_{2c}) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - e^{-2\sigma t_{0c}} \int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = 0
\end{aligned} \tag{35}$$

We cancel the common integral $\int_{-\infty}^{-t_{0c}} E_0^{'}(\tau, t_{2c}) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ in Eq. 35 and rearrange the terms as follows, using $2\sinh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}$.

$$\begin{aligned}
&\int_{-t_{0c}}^{t_{0c}} E_0^{'}(\tau, t_{2c}) e^{-2\sigma\tau} \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau + e^{2\sigma t_{0c}} \int_{-t_{0c}}^{t_{0c}} E_{0n}^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
&= -2\sinh(2\sigma t_{0c}) \int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau
\end{aligned} \tag{36}$$

We can combine the integrals in the left hand side of Eq. 36 as follows.

$$\begin{aligned}
&\int_{-t_{0c}}^{t_{0c}} [E_0^{'}(\tau, t_{2c}) e^{-2\sigma\tau} + E_{0n}^{'}(\tau, t_{2c}) e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
&= -2\sinh(2\sigma t_{0c}) \int_{-\infty}^{-t_{0c}} E_{0n}^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau
\end{aligned} \tag{37}$$

We denote the right hand side of Eq. 37 as RHS. We can split the integral in the left hand side of Eq. 37 using $\int_{-t_{0c}}^{t_{0c}} = \int_{-t_{0c}}^{0} + \int_{0}^{t_{0c}}$ as follows.

$$\begin{aligned}
&\int_{-t_{0c}}^{0} [E_0^{'}(\tau, t_{2c}) e^{-2\sigma\tau} + E_{0n}^{'}(\tau, t_{2c}) e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
&+ \int_{0}^{t_{0c}} [E_0^{'}(\tau, t_{2c}) e^{-2\sigma\tau} + E_{0n}^{'}(\tau, t_{2c}) e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS
\end{aligned} \tag{38}$$

We substitute $\tau = -\tau$ in the first integral in Eq. 38 as follows. We use $E_0^{'}(-\tau, t_{2c}) = E_{0n}^{'}(\tau, t_{2c})$ and $E_{0n}^{'}(-\tau, t_{2c}) = E_0^{'}(\tau, t_{2c})$ using Definition 2 in Section 2.3.

$$\begin{aligned}
&\int_{t_{0c}}^{0} [E_{0n}^{'}(\tau, t_{2c}) e^{2\sigma\tau} + E_0^{'}(\tau, t_{2c}) e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\
&+ \int_{0}^{t_{0c}} [E_0^{'}(\tau, t_{2c}) e^{-2\sigma\tau} + E_{0n}^{'}(\tau, t_{2c}) e^{2\sigma t_{0c}}] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS
\end{aligned}$$

$$(39)$$

Given that $\int_{t_{0c}}^{0} = -\int_{0}^{t_{0c}}$, we can simplify Eq. 39 as follows.

$$\int_0^{t_{0c}} [E_0^{'}(\tau, t_{2c})(e^{-2\sigma\tau} - e^{2\sigma t_{0c}}) + E_{0n}^{'}(\tau, t_{2c})(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau = RHS \tag{40}$$

We substitute $\tau = -\tau$ in the right hand side of Eq. 37 as follows. We use $E_{0n}^{'}(-\tau, t_{2c}) = E_0^{'}(\tau, t_{2c})$ using Definition 2 in Section 2.3.

$$RHS = 2\sinh(2\sigma t_{0c}) \int_{t_{0c}}^{\infty} E_0^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \tag{41}$$

We split the integral on the right hand side in Eq. 41 using $\int_{t_{0c}}^{\infty} = \int_0^{\infty} - \int_0^{t_{0c}}$, as follows.

$$RHS = 2\sinh(2\sigma t_{0c})[\int_0^{\infty} E_0^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau - \int_0^{t_{0c}} E_0^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \tag{42}$$

We consolidate the integrals of the form $\int_0^{t_{0c}} E_0^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau$ in Eq. 40 and Eq. 42 as follows. We use $2\sinh(2\sigma t_{0c}) = e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}$.

$$\begin{aligned}\int_0^{t_{0c}} [E_0^{'}(\tau, t_{2c})(e^{-2\sigma\tau} - e^{2\sigma t_{0c}} + e^{2\sigma t_{0c}} - e^{-2\sigma t_{0c}}) + E_{0n}^{'}(\tau, t_{2c})(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2\sinh(2\sigma t_{0c}) \int_0^{\infty} E_0^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau\end{aligned} \tag{43}$$

We cancel the common term $e^{2\sigma t_{0c}}$ in the first integral in Eq. 43 as follows.

$$\begin{aligned}\int_0^{t_{0c}} [E_0^{'}(\tau, t_{2c})(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}}) + E_{0n}^{'}(\tau, t_{2c})(-e^{2\sigma\tau} + e^{2\sigma t_{0c}})] \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau \\ = 2\sinh(2\sigma t_{0c}) \int_0^{\infty} E_0^{'}(\tau, t_{2c}) \sin(\omega_z(t_{2c}, t_{0c})\tau) d\tau\end{aligned} \tag{44}$$

We substitute $E_0^{'}(\tau, t_{2c}) = E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ (using Definition 1 in Section 2.1) and $E_{0n}^{'}(\tau, t_{2c}) = E_0^{'}(-\tau, t_{2c}) = E_0(-\tau - t_{2c}) - E_0(-\tau + t_{2c})$ (using Definition 2 in Section 2.3). We see that $E_0(-\tau - t_{2c}) = E_0(\tau + t_{2c})$ and $E_0(-\tau + t_{2c}) = E_0(\tau - t_{2c})$ given that $E_0(\tau) = E_0(-\tau)$(Details in Appendix A.7). Hence we see that $E_{0n}^{'}(\tau, t_{2c}) = E_0(\tau + t_{2c}) - E_0(\tau - t_{2c}) = -E_0^{'}(\tau, t_{2c})$ (**Result 3.1**) and write Eq. 44 as follows.

$$\begin{aligned}\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(e^{-2\sigma\tau} - e^{-2\sigma t_{0c}} + e^{2\sigma\tau} - e^{2\sigma t_{0c}}) \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau \\ = 2\sinh{(2\sigma t_{0c})} \int_0^{\infty} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau\end{aligned} \tag{45}$$

We substitute $2\cosh{(2\sigma\tau)} = e^{2\sigma\tau} + e^{-2\sigma\tau}$ and $2\cosh{(2\sigma t_{0c})} = e^{2\sigma t_{0c}} + e^{-2\sigma t_{0c}}$ and cancel the common factor of 2 in Eq. 45 as follows.

$$\begin{aligned}\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh{(2\sigma\tau)} - \cosh{(2\sigma t_{0c})}) \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau \\ = \sinh{(2\sigma t_{0c})} \int_0^{\infty} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})) \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau\end{aligned} \tag{46}$$

Next Step:

We denote the right hand side of Eq. 46 as $RHS^{'}$. We substitute $\tau - t_{2c} = \tau'$ and $\tau + t_{2c} = \tau''$ in the right hand side of Eq. 46 and then substitute $\tau' = \tau$ and $\tau'' = \tau$ in the second line below. We use the identity $\sin{(\omega_z(t_{2c}, t_{0c})(\tau + t_{2c}))} = \sin{(\omega_z(t_{2c}, t_{0c})\tau)} \cos{(\omega_z(t_{2c}, t_{0c})t_{2c})} + \cos{(\omega_z(t_{2c}, t_{0c})\tau)} \sin{(\omega_z(t_{2c}, t_{0c})t_{2c})}$ below.

$$\begin{aligned}RHS^{'} = \sinh{(2\sigma t_{0c})}[\int_{-t_{2c}}^{\infty} E_0(\tau') \sin{(\omega_z(t_{2c}, t_{0c})(\tau' + t_{2c}))} d\tau' - \int_{t_{2c}}^{\infty} E_0(\tau'') \sin{(\omega_z(t_{2c}, t_{0c})(\tau'' - t_{2c}))} d\tau'']\\ RHS^{'} = \sinh{(2\sigma t_{0c})}[\cos{(\omega_z(t_{2c}, t_{0c}))t_{2c})} \int_{-t_{2c}}^{\infty} E_0(\tau) \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau \\ + \sin{(\omega_z(t_{2c}, t_{0c})t_{2c})} \int_{-t_{2c}}^{\infty} E_0(\tau) \cos{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau \\ - \cos{(\omega_z(t_{2c}, t_{0c}))t_{2c})} \int_{t_{2c}}^{\infty} E_0(\tau) \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau + \sin{(\omega_z(t_{2c}, t_{0c})t_{2c})} \int_{t_{2c}}^{\infty} E_0(\tau) \cos{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau]\end{aligned} \tag{47}$$

In Eq. 47, given that $\omega_z(t_{2c}, t_{0c})t_{0c} = \frac{\pi}{2}$ and $t_{2c} = 2t_{0c}$ and hence $\omega_z(t_{2c}, t_{0c})t_{2c} = 2\frac{\pi}{2} = \pi$ and $\sin{(\omega_z(t_{2c}, t_{0c})t_{2c})} = 0$ and $\cos{(\omega_z(t_{2c}, t_{0c})t_{2c})} = -1$. Hence we cancel common terms and write Eq. 47 and Eq. 46 as follows.

$$\begin{aligned}\int_0^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh{(2\sigma\tau)} - \cosh{(2\sigma t_{0c})}) \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau \\ = -\sinh{(2\sigma t_{0c})}[\int_{-t_{2c}}^{\infty} E_0(\tau) \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau - \int_{t_{2c}}^{\infty} E_0(\tau) \sin{(\omega_z(t_{2c}, t_{0c})\tau)} d\tau]\end{aligned} \tag{48}$$

We use $\int_{-t_{2c}}^{\infty} E_0(\tau)\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau = \int_{-t_{2c}}^{t_{2c}} E_0(\tau)\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau + \int_{t_{2c}}^{\infty} E_0(\tau)\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau$ and cancel the common term $\int_{t_{2c}}^{\infty} E_0(\tau)\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau$ in the right hand side $RHS^{''}$ of Eq. 48 as follows.

$$RHS^{''} = -\sinh(2\sigma t_{0c})\int_{-t_{2c}}^{t_{2c}} E_0(\tau)\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau \tag{49}$$

Given that $E_0(\tau)$ is an **even** function of variable τ (Details in Appendix A.7) and $E_0(\tau)\sin(\omega_z(t_{2c},t_{0c})\tau)$ is an **odd** function of variable τ, we get $\int_{-t_{2c}}^{t_{2c}} E_0(\tau)\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau = 0$ explained below.

We see that $I = \int_{-t_{2c}}^{t_{2c}} E_0(\tau)\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau = \int_{-t_{2c}}^{0} E_0(\tau)\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau$
$+ \int_{0}^{t_{2c}} E_0(\tau)\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau$. We substitute $\tau = -\tau$ in the first integral and get
$I = \int_{t_{2c}}^{0} E_0(\tau)\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau + \int_{0}^{t_{2c}} E_0(\tau)\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau$
$= -\int_{0}^{t_{2c}} E_0(\tau)\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau + \int_{0}^{t_{2c}} E_0(\tau)\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau = 0$. Hence $RHS^{''} = 0$ in Eq. 49. We write Eq. 48 as follows.

$$\int_{0}^{t_{0c}} (E_0(\tau - t_{2c}) - E_0(\tau + t_{2c}))(\cosh(2\sigma\tau) - \cosh(2\sigma t_{0c}))\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau = 0 \tag{50}$$

We can multiply Eq. 50 by a factor of -1 as follows.

$$\int_{0}^{t_{0c}} [E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})](\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau))\sin(\omega_z(t_{2c},t_{0c})\tau)d\tau = 0 \tag{51}$$

In Eq. 51, given that $\omega_z(t_{2c},t_{0c})t_{0c} = \frac{\pi}{2}$, as τ varies over the interval $(0, t_{0c})$, $\omega_z(t_{2c},t_{0c})\tau = \frac{\pi\tau}{2t_{0c}}$ varies from $(0, \frac{\pi}{2})$ and the sinusoidal function is > 0, in the interval $0 < \tau < t_{0c}$, for $t_{0c} > 0$.

In Eq. 51, we see that the integral on the left hand side is > 0 for $t_{0c} > 0$, because each of the terms in the integrand are > 0, in the interval $0 < \tau < t_{0c}$ as follows. Given that $E_0(t)$ is a **strictly decreasing** function for $t > 0$(Details in Section 5), we see that $E_0(\tau - t_{2c}) - E_0(\tau + t_{2c})$ is > 0 (Details in Section 3.1) in the interval $0 < \tau < t_{0c}$. The term $(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau))$ is > 0 in the interval $0 < \tau < t_{0c}$.

The integrand is zero at $\tau = 0$ due to the term $\sin(\omega_z(t_{2c},t_{0c})\tau)$ and the integrand is zero at $\tau = t_{0c}$ due to the term $\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau)$ and hence the integral **cannot** equal zero, as required by the right hand side of Eq. 51. Hence this leads to a **contradiction**, for $0 < \sigma < \frac{1}{2}$.

For $\sigma = 0$, both sides of Eq. 51 is zero, given the term $(\cosh(2\sigma t_{0c}) - \cosh(2\sigma\tau)) = 0$ and **does not** lead to a contradiction.

We have shown this result for $0 < \sigma < \frac{1}{2}$. Given that $E_p(t) = E_0(t)e^{-\sigma t}$ is real, its Fourier transform $E_{p\omega}(\omega) = E_{pR\omega}(\omega) + iE_{pI\omega}(\omega)$ has symmetry properties and hence $E_{pR\omega}(-\omega) = E_{pR\omega}(\omega)$ and $E_{pI\omega}(-\omega) = -E_{pI\omega}(\omega)$ (Symmetry property of Fourier Transform) also have a zero at $\omega = \omega_0$ and hence $E_{p\omega}(-\omega) = E_{pR\omega}(\omega) - iE_{pI\omega}(\omega)$ **also** has a zero at $\omega = \omega_0$ to satisfy Statement A.

If $E_{p\omega}(\omega)$ and $\eta(\frac{1}{2} + \sigma + i\omega)$ has a zero at $\omega = \omega_0$ to satisfy Statement 1, then $E_{p\omega}(-\omega)$ and $\eta(\frac{1}{2} + \sigma - i\omega)$ also has a zero at $\omega = \omega_0$(using last paragraph) and $\eta(\frac{1}{2} - \sigma + i\omega)$ also has a zero at $\omega = \omega_0$ using the functional equation for Dirichlet Eta function derived in Appendix A.9 which relates $\eta(s)$ and $\eta(1-s)$. Hence the results in above sections hold for $-\frac{1}{2} < \sigma < 0$ and for $0 < |\sigma| < \frac{1}{2}$.

Hence we have produced a **contradiction** of **Statement 1** that the Fourier Transform of the function $E_p(t) = E_0(t)e^{-\sigma t}$ has a zero at $\omega = \omega_0$ for $0 < |\sigma| < \frac{1}{2}$.

Hence the assumption in **Statement 1** that the analytic continuation of Riemann's zeta function derived from Dirichlet eta function given by $E_{p\omega}(\omega)$ has a zero at $\omega = \omega_0$, where ω_0 is real and finite, leads to a **contradiction** for the region $0 < |\sigma| < \frac{1}{2}$ which corresponds to the critical strip excluding the critical line. Hence $\zeta(s)$ does not have non-trivial zeros in the critical strip excluding the critical line and we have proved Riemann's Hypothesis.

.

3.1. ***Result*** $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$

It is shown in Section 5 that $E_0(t)$ is **strictly decreasing** for $t > 0$. In this section, it is shown that $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$, for $0 < t < t_{0c}$ and $t_{2c} = 2t_{0c}$ in Eq. 51 .

Given that $E_0(t)$ is a **strictly decreasing** function for $t > 0$ and $E_0(t)$ is an **even** function of variable t (Details in Appendix A.7), and $t_{2c} = 2t_{0c}$, we see that, in the interval $0 < t < t_{0c}$, $E_0(t + t_{2c}) = E_0(t + 2t_{0c})$ ranges from $E_0(3t_{0c}) < E_0(t + t_{2c}) < E_0(2t_{0c})$(**Result 3.1.1**) and $E_0(t - t_{2c}) = E_0(t - 2t_{0c})$ which ranges from $E_0(-2t_{0c}) < E_0(t - t_{2c}) < E_0(-t_{0c})$ respectively. Given that $E_0(t) = E_0(-t)$, we see that $E_0(2t_{0c}) < E_0(t - t_{2c}) < E_0(t_{0c})$ in the interval $0 < t < t_{0c}$ (**Result 3.1.2**).

Using Result 3.1.1 and Result 3.1.2, we see that $E_0(t-t_{2c}) > E_0(t+t_{2c})$, in the interval $0 < t < t_{0c}$.

Hence $E_0(t - t_{2c}) - E_0(t + t_{2c}) > 0$ for $0 < t < t_{0c}$ in Eq. 51 , for $t_{0c} > 0$ and $t_{2c} = 2t_{0c}$.

.

4. $\boldsymbol{\omega_z(t_2, t_0)t_0 = \frac{\pi}{2}}$ can be reached for specific $\boldsymbol{t_0, t_2}$

It is noted that we **do not** use $\lim_{t_0 \to \infty}$ in this section. Instead we consider real $t_0 > 0$ which increases to a larger and larger finite value without bounds. We use $0 < \sigma < \frac{1}{2}$ below.

We copy $P_{odd}(t_2, t_0)$ from the first line in Eq. 30 using $\tau' = \tau$ and copy the first line in Eq 29 derived assuming Statement 1, concisely as follows.

$$
\begin{aligned}
P_{odd}(t_2, t_0) = \int_{-\infty}^{t_0} E_0^{'}(\tau, t_2) e^{-2\sigma\tau} \cos\left(\omega_z(t_2, t_0)(\tau - t_0)\right) d\tau + e^{2\sigma t_0} \int_{-\infty}^{t_0} E_{0n}^{'}(\tau, t_2) \cos\left(\omega_z(t_2, t_0)(\tau - t_0)\right) d\tau \\
P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) = 0
\end{aligned}
\tag{52}
$$

We note that $E_0^{'}(\tau, t_2) = E_0(\tau - t_2) - E_0(\tau + t_2)$ and $E_{0n}^{'}(\tau, t_2) = E_0^{'}(-\tau, t_2) = -E_0^{'}(\tau, t_2) = E_0(\tau + t_2) - E_0(\tau - t_2)$ (using Result 3.1 in Section 3). We choose $t_2 = 2t_0$ and we choose t_1 such that $E_0(t)$ approximates zero for $|t| > t_1$, given that $E_0(t)$ has an asymptotic **exponential** fall-off rate of $o[e^{-1.5|t|}]$ (Details in Appendix A.4). We choose $t_0 >> t_1$ and hence $E_0(\tau - t_2) = E_0(\tau - 2t_0)$ approximates zero in the interval $(-\infty, t_0]$ for τ, given that $0 < E_0(\tau - 2t_0) \leq E_0(-t_0)$ because $E_0(-t_0) = E_0(t_0) \approx 0$ and $E_0(\tau - 2t_0)$ is **strictly decreasing** for $\tau > 2t_0$ and strictly increasing for

$\tau < 2t_0$. (Details in Section 5)

Hence in the interval $(-\infty, t_0]$, we see that $E_0^{'}(\tau, t_2) \approx -E_0(\tau + t_2)$ and $E_{0n}^{'}(\tau, t_2) \approx E_0(\tau + t_2)$, for sufficiently large t_0. We can write Eq. 52 as follows. We use $\omega_z(t_2, -t_0) = \omega_z(t_2, t_0)$ (Details in Section 2.4). We **note that** $t_2 = 2t_0$ in the rest of this section and we continue to use the notation $\omega_z(t_2, t_0)$ where $t_2 = 2t_0$.

$$\begin{aligned} P_{odd}(t_2, t_0) &\approx -\int_{-\infty}^{t_0} E_0(\tau + 2t_0) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)(\tau - t_0)) d\tau \\ &\quad + e^{2\sigma t_0} \int_{-\infty}^{t_0} E_0(\tau + 2t_0) \cos(\omega_z(t_2, t_0)(\tau - t_0)) d\tau \\ P_{odd}(t_2, -t_0) &= \int_{-\infty}^{-t_0} E_0^{'}(\tau, t_2) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)(\tau + t_0)) d\tau \\ &\quad + e^{-2\sigma t_0} \int_{-\infty}^{-t_0} E_{0n}^{'}(\tau, t_2) \cos(\omega_z(t_2, t_0)(\tau + t_0)) d\tau \end{aligned} \tag{53}$$

We see that the term $P_{odd}(t_2, -t_0)$ in Eq. 53 approaches a value very close to zero, as real t_0 increases to a larger and larger finite value without bounds, due to the terms $e^{-2\sigma t_0}$ and the integrals $\int_{-\infty}^{-t_0}$, given $0 < \sigma < \frac{1}{2}$ and $t_0 > 0$ and given that the integrands are absolutely integrable and finite because the terms $E_0^{'}(\tau, t_2) e^{-2\sigma\tau}$ and $E_{0n}^{'}(\tau, t_2) = -E_0^{'}(\tau, t_2)$ have exponential asymptotic fall-off rate as $|\tau| \to \infty$(Details in Section Appendix C.1.1) Hence we can ignore $P_{odd}(t_2, -t_0)$ for sufficiently large t_0 and write Eq. 52, using Eq. 53 and $t_2 = 2t_0$.

$$\begin{aligned} Q(t_0) = P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0) &\approx -\int_{-\infty}^{t_0} E_0(\tau + 2t_0) e^{-2\sigma\tau} \cos(\omega_z(t_2, t_0)(\tau - t_0)) d\tau \\ &\quad + e^{2\sigma t_0} \int_{-\infty}^{t_0} E_0(\tau + 2t_0) \cos(\omega_z(t_2, t_0)(\tau - t_0)) d\tau \approx 0 \end{aligned} \tag{54}$$

We substitute $\tau + 2t_0 = t$, $\tau = t - 2t_0$ and $d\tau = dt$ in Eq. 54 and write as follows.

$$\begin{aligned} Q(t_0) &\approx -e^{4\sigma t_0} \int_{-\infty}^{3t_0} E_0(t) e^{-2\sigma t} \cos(\omega_z(t_2, t_0)(t - 3t_0)) dt \\ &\quad + e^{2\sigma t_0} \int_{-\infty}^{3t_0} E_0(t) \cos(\omega_z(t_2, t_0)(t - 3t_0)) dt \approx 0 \end{aligned} \tag{55}$$

We multiply Eq. 55 by $e^{-3\sigma t_0}$ and ignore the last integral for sufficiently large t_0, given that $e^{2\sigma t_0} e^{-3\sigma t_0} = e^{-\sigma t_0}$ and $|\int_{-\infty}^{3t_0} E_0(t) \cos(\omega_z(t_2, t_0)(t - 3t_0)) dt| \leq \int_{-\infty}^{3t_0} |E_0(t)| dt \leq \int_{-\infty}^{\infty} |E_0(t)| dt$ is finite (link and Appendix A.1) and expand as follows.

$$S(t_0) = Q(t_0) e^{-3\sigma t_0} \approx -e^{\sigma t_0} \int_{-\infty}^{3t_0} E_0(t) e^{-2\sigma t} \cos(\omega_z(t_2, t_0)(t - 3t_0)) dt = -e^{\sigma t_0} R(t_0) \approx 0$$

$$R(t_0) = \cos(\omega_z(t_2, t_0) 3t_0) \int_{-\infty}^{3t_0} E_0(t) e^{-2\sigma t} \cos(\omega_z(t_2, t_0) t) dt + \sin(\omega_z(t_2, t_0) 3t_0) \int_{-\infty}^{3t_0} E_0(t) e^{-2\sigma t} \sin(\omega_z(t_2, t_0) t) dt$$

(56)

In Section 2.1, it is shown that $0 < \omega_z(t_2, t_0) < \infty$, for all $|t_0| < \infty$, for each non-zero value of t_2. For $t_0 > 0$, we see that $\omega_z(t_2, t_0)t_0 > 0$. In Section 6, it is shown that $\omega_z(t_2, t_0)$ is a **continuous** function of variable t_0 and t_2, for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$. Hence $\omega_z(t_2, t_0)t_0$ is a positive continuous function.

We **require** $\omega_z(t_2, t_0)t_0 = \frac{\pi}{2}$ in Section 3 for a specific $t_0 = t_{0c}$ and $t_2 = t_{2c} = 2t_{0c}$. To show that $\omega_z(t_2, t_0)t_0 = \frac{\pi}{2}$ can be reached, we **assume the opposite** case that $\omega_z(t_2, t_0)t_0 < \frac{\pi}{2}$ for all $0 < t_0 < \infty$ and $t_2 = 2t_0$ (**Statement C**) and show that this leads to a **contradiction**.

Let $\omega_z(t_2, t_0)t_0 = KF(t_2, t_0)$, where $0 < K < \frac{\pi}{2}$ and $0 < F(t_2, t_0) \le 1$ is a positive continuous function for $0 < t_0 < \infty$ and $t_2 = 2t_0$, such that $\omega_z(t_2, t_0)t_0 < \frac{\pi}{2}$. Hence $\omega_z(t_2, t_0) = \frac{KF(t_2,t_0)}{t_0}$.

We choose t_3 such that $E_0(t)e^{-2\sigma t}$ is vanishingly small and approximates zero for $|t| > t_3$ (**Result 4.a**), given that $E_0(t)e^{-2\sigma t}$ has an asymptotic **exponential** fall-off rate of $o[e^{-0.5|t|}]$ (Details in Appendix A.4).We choose $t_0 >> t_3$ and note that t_3 is **independent** of t_0. As t_0 increase without bounds, in the interval $|t| \le t_3$, we see that the term $\cos(\omega_z(t_2, t_0)t) \approx 1$ and $\sin(\omega_z(t_2, t_0)t) \approx \omega_z(t_2, t_0)t \approx 0$ (**Result 4.b**), given that $\omega_z(t_2, t_0)t = \frac{KF(t_2,t_0)t}{t_0} \le \frac{KF(t_2,t_0)t_3}{t_0} << 1$, because $t_0 >> t_3$ and $F(t_2, t_0) \le 1$. Hence we write Eq. 56 as follows, using Result 4.a and Result 4.b.

$$R(t_0) \approx \cos(\omega_z(t_2, t_0)3t_0) \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t} \cos(\omega_z(t_2, t_0)t)dt \approx \cos(3KF(t_2, t_0)) \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}dt \tag{57}$$

For sufficiently large t_0, the integral $R(t_0) \approx \cos(3KF(t_2, t_0)) \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}dt$ remains finite, because $\cos(\omega_z(t_2, t_0)3t_0)$ oscillates in the interval $[-1, 1]$ and $\int_{-\infty}^{\infty} E_0(t)e^{-2\sigma t}dt > 0$ (Details in Appendix A.1) and **does not** approach zero exponentially, as real t_0 increases to a larger and larger finite value without bounds. This is explained in detail in Section 4.1.

The term $e^{\sigma t_0}$ in $S(t_0) = -e^{\sigma t_0}R(t_0)$ in Eq. 56 increases to a larger and larger finite value **exponentially** as t_0 increases, and hence the term $S(t_0)$ approaches a larger and larger finite value exponentially, given that $R(t_0)$ **does not** approach zero exponentially and hence $S(t_0)$ and $Q(t_0)$ in Eq. 55 and $P_{odd}(t_2, t_0) + P_{odd}(t_2, -t_0)$ in Eq. 52 **cannot** equal zero, to satisfy Statement 1, in this case.

Hence **Statement C** is **false** and hence $\omega_z(t_2, t_0)t_0 = \frac{\pi}{2}$ can be reached for specific values of t_0 and $t_2 = 2t_0$, as finite t_0 increases without bounds, given that $\omega_z(t_2, t_0)t_0$ is a **continuous** function of variable t_0 and t_2, for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$.

.

4.1. $A(t_0) = \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t} \cos(\omega_z(t_2, t_0)t)dt$ ***does not have exponential fall off rate***

We compute the **minimum** value of the integral $A(t_0) = \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t} \cos(\omega_z(t_2, t_0)t)dt$ in Eq. 56 , for sufficiently large t_3 and $t_0 >> t_3$ and $0 < \sigma < \frac{1}{2}$. We note that $t_2 = 2t_0$ and note that t_3 is **independent** of t_0 below. We split $A(t_0)$ as follows.

$$A(t_0) = B(t_3,t_0) + C(t_3,t_0) + D(t_3,t_0)$$
$$B(t_3,t_0) = \int_{-\infty}^{-t_3} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt, \quad C(t_3,t_0) = \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt$$
$$D(t_3,t_0) = \int_{t_3}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt \tag{58}$$

We see that $E_0(t)e^{-2\sigma t} > 0$ for $|t| < \infty$ and $E_0(t)e^{-2\sigma t}$ is an absolutely integrable function (Details in Appendix A.1) and hence $C_0(t_3) = \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}dt > 0$ (**Result 4.1.1**).

Given that $\omega_z(t_2,t_0) = \frac{KF(t_2,t_0)}{t_0}$ where $0 < K < \frac{\pi}{2}$ and $0 < F(t_2,t_0) \leq 1$ in previous subsection and $t_0 >> t_3$, we see that $\omega_z(t_2,t_0)t = \frac{KF(t_2,t_0)t}{t_0} \leq \frac{KF(t_2,t_0)t_3}{t_0} << 1$ in the interval $|t| \leq t_3$ and hence $\cos(\omega_z(t_2,t_0)t) \approx 1$ and $\cos(\omega_z(t_2,t_0)t) > \frac{1}{2}$ in the interval $|t| \leq t_3$. Hence we can write $C(t_3,t_0) = \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt > \frac{C_0(t_3)}{2} > 0$, using Result 4.1.1. (**Result 4.1.2**).

We see that $|B(t_3,t_0)| = |\int_{-\infty}^{-t_3} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt| \leq \int_{-\infty}^{-t_3} |E_0(t)e^{-2\sigma t}|dt \approx 0$ (link) and $|D(t_3,t_0)| = |\int_{t_3}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt| \leq \int_{t_3}^{3t_0} |E_0(t)e^{-2\sigma t}|dt \approx 0$, for sufficiently large t_3 and $t_0 >> t_3$, given that $E_0(t)e^{-2\sigma t}$ has an asymptotic **exponential** fall-off rate of $o[e^{-0.5|t|}]$ (Details in Appendix A.4) and $E_0(t)e^{-2\sigma t} > 0$ for $|t| < \infty$ (Details in Appendix A.1).

As we increase t_3 to t_3' and t_0 to $t_0' >> t_3'$, we see that $C(t_3',t_0') > C(t_3,t_0) > 0$, using Result 4.1.1 and Result 4.1.2, given that $E_0(t)e^{-2\sigma t} > 0$ for $|t| < \infty$ (**Result 4.1.3**).

As we increase t_3 to t_3' and t_0 to $t_0' >> t_3'$, we see that $|B(t_3',t_0')| < |B(t_3,t_0)|$ and $|D(t_3',t_0')| < |D(t_3,t_0)|$ approach zero (**Result 4.1.4**), given that $E_0(t)e^{-2\sigma t}$ has an asymptotic **exponential** fall-off rate of $o[e^{-0.5|t|}]$ (Details in Appendix A.4) and $E_0(t)e^{-2\sigma t} > 0$ for $|t| < \infty$ (Details in Appendix A.1).

Hence we see that $A(t_0) = \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt > \frac{C_0(t_3)}{2} - |B(t_3,t_0)| - |D(t_3,t_0)| \approx \frac{C_0(t_3)}{2} > 0$ using Result 4.1.2, Result 4.1.3 and Result 4.1.4.

For example, we choose $t_3 = 10$ such that $E_0(t)e^{-2\sigma t}$ is vanishingly small and approximates zero for $|t| > t_3$. Given that $E_0(t) > 0$ for $|t| < \infty$ (Details in Appendix A.6) and the term $e^{-2\sigma t}$ has a minimum value of $e^{-|t|}$ for $0 < \sigma < \frac{1}{2}$, we see that the integral $C_0(t_3) = \int_{-t_3}^{t_3} E_0(t)e^{-2\sigma t}dt > 2\int_0^{t_3} E_0(t)e^{-|t|}dt > C_{00} = 0.42$ where C_{00} is computed by considering the first 5 terms $n = 1,2,3,4,5$ in $E_0(t) = \sum_{n=1}^{\infty}[4\pi^2n^4e^{4t} - 6\pi n^2e^{2t}]e^{-\pi n^2e^{2t}}e^{\frac{t}{2}}$. Hence $C_0(t_3) > 0.42$. (Matlab simulation)

Hence we see that $A(t_0) = \int_{-\infty}^{3t_0} E_0(t)e^{-2\sigma t}\cos(\omega_z(t_2,t_0)t)dt > \frac{C_0(t_3)}{2} - |B(t_3,t_0)| - |D(t_3,t_0)| \approx 0.21$. As t_0 increases without bounds, we see that $A(t_0)$ **does not** have exponential fall off rate.

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5. Strictly decreasing $E_0(t)$ for $t > 0$

We show that $E_0(t)$ is strictly decreasing for $t > 0$, by showing that $\frac{dE_0(-t)}{dt} < 0$ for $0 < t < \infty$.

We set $y = \pi e^{2t}$ in $E_0(-t)$ in the second line in Eq. A.5 and then take the first derivative of $E_0(y)$ as follows. We see that $\frac{dy}{dt} = 2\pi e^{2t} = 2y$ and $\frac{dE_0(-t)}{dt} = \frac{dE_0(-t)}{dy}\frac{dy}{dt} = \frac{dE_0(y)}{dy}2y$ and hence we will show that $\frac{dE_0(y)}{dy} < 0$ for $\pi < y < \infty$.

$$\begin{aligned}
E_0(-t) &= \sum_{n=odd}^{\infty}(e^{-\pi\frac{n^2}{4}e^{2t}} - e^{-\pi n^2 e^{2t}} - e^{-\pi\frac{(n+1)^2}{4}e^{2t}} + e^{-\pi(n+1)^2e^{2t}})e^{\frac{t}{2}} \\
E_0(y) &= (\pi)^{-\frac{1}{4}}\sum_{n=odd}^{\infty} e^{-\frac{n^2}{4}y}y^{\frac{1}{4}} - e^{-n^2y}y^{\frac{1}{4}} - e^{-\frac{(n+1)^2}{4}y}y^{\frac{1}{4}} + e^{-(n+1)^2y}y^{\frac{1}{4}} \\
\frac{dE_0(y)}{dy} &= (\pi)^{-\frac{1}{4}}\sum_{n=odd}^{\infty} e^{-\frac{n^2}{4}y}y^{\frac{1}{4}}(\frac{1}{4y} - \frac{n^2}{4}) - e^{-n^2y}y^{\frac{1}{4}}(\frac{1}{4y} - n^2) \\
&-e^{-\frac{(n+1)^2}{4}y}y^{\frac{1}{4}}(\frac{1}{4y} - \frac{(n+1)^2}{4}) + e^{-(n+1)^2y}y^{\frac{1}{4}}(\frac{1}{4y} - (n+1)^2)
\end{aligned} \tag{59}$$

We take the common term $e^{-\frac{n^2}{4}y}y^{\frac{1}{4}}$ out and use $(n+1)^2 = n^2 + 2n + 1$ and rearrange the terms in Eq. 59 as follows.

$$\begin{aligned}
\frac{dE_0(y)}{dy} &= (\pi)^{-\frac{1}{4}}\sum_{n=odd}^{\infty} e^{-\frac{n^2}{4}y}y^{\frac{1}{4}}[(\frac{1}{4y} - \frac{n^2}{4}) - e^{-\frac{3n^2}{4}y}(\frac{1}{4y} - n^2) \\
-e^{-\frac{(2n+1)}{4}y}&(\frac{1}{4y} - \frac{(n+1)^2}{4}) + e^{-\frac{3n^2}{4}y}e^{-(2n+1)y}(\frac{1}{4y} - (n+1)^2)]
\end{aligned} \tag{60}$$

We compute the **maximum** value of $\frac{dE_0(y)}{dy}$ in Eq. 60, by computing the maximum value of positive terms and minimum value of absolute value of negative terms. We ignore the negative terms inside the brackets $-e^{-\frac{3n^2}{4}y}\frac{1}{4y}$, $-e^{-\frac{(2n+1)}{4}y}\frac{1}{4y}$ and $-(n+1)^2e^{-\frac{3n^2}{4}y}e^{-(2n+1)y}$ because we want the maximum value of $\frac{dE_0(y)}{dy}$ in the interval $\pi < y < \infty$.

$$\begin{aligned}
\frac{dE_0(y)}{dy} &< (\pi)^{-\frac{1}{4}}\sum_{n=odd}^{\infty} e^{-\frac{n^2}{4}y}y^{\frac{1}{4}}[(\frac{1}{4y} - \frac{n^2}{4}) + e^{-\frac{3n^2}{4}y}n^2 \\
&+e^{-\frac{(2n+1)}{4}y}\frac{(n+1)^2}{4} + e^{-\frac{3n^2}{4}y}e^{-(2n+1)y}\frac{1}{4y}]
\end{aligned} \tag{61}$$

We see that $y = \pi e^{2t}$ is in the range $y = [\pi, \infty)$ for $0 \leq t < \infty$, and in the range $y = [\pi, y_a)$ for $0 \leq t < t_a = 0.1$, where $y_a = \pi e^{2t_a} = 3.8371$.

- It is shown in Section 5.0.1 that $\frac{dE_0(y)}{dy} < 0$ for $y_a \leq y < \infty$ for $y_a = 3.8371$.
- It is shown in Section 5.0.2 that $\frac{d^2E_0(y)}{dy^2} < 0$ for $\pi \leq y < y_a$ and hence $\frac{dE_0(y)}{dy} < 0$ for $\pi < y < y_a$.

- Hence $\frac{dE_0(y)}{dy} < 0$ for $\pi < y < \infty$. Given $y = \pi e^{2t}$ and $\frac{dy}{dt} = 2\pi e^{2t} = 2y$ and $\frac{dE_0(-t)}{dt} = \frac{dE_0(-t)}{dy}\frac{dy}{dt} = \frac{dE_0(y)}{dy}2y$, we see that $\frac{dE_0(-t)}{dt} < 0$ for $t > 0$. Hence $E_0(t) = E_0(-t)$ is strictly decreasing for $t > 0$.

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5.0.1. $\frac{dE_0(y)}{dy} < 0$ ***for*** $y_a \leq y < \infty$ ***for*** $y_a = 3.8371$

We see that the **maximum** value of the **first term** inside brackets $(\frac{1}{4y} - \frac{n^2}{4})$ in Eq. 61 occurs at $n = 1$ and $y = y_a = 3.8371$ given by $D_1 = \frac{1}{4y_a} - \frac{1}{4} = \frac{1}{4*3.8371} - \frac{1}{4} = -0.1848$.

We consider the **second term** inside brackets in Eq. 61 given by $I(y,n) = n^2 e^{-\frac{3n^2}{4}y}$. It is a **strictly decreasing** function in the region $y_a \leq y < \infty$, with **maximum** value at $y = y_a$, for each n.

We set $y = y_a = 3.8371$ and compute $\frac{dI(y_a,n)}{dn} = e^{-\frac{3n^2}{4}y_a}[2n + n^2(-\frac{6ny_a}{4})]$ which has an inflection point at $2n + n^2(-\frac{6ny_a}{4}) = 0$. Given that $I(y_a,n) > 0$ for all finite n and goes to zero as $n \to \infty$ due to the term $e^{-\frac{3n^2}{4}y_a}$, this inflection point is a **maximum** point. We cancel common term n and get $2 + n^2(-\frac{6y_a}{4}) = 0$ which has roots at $n^2 = \frac{4}{3y_a}$ given by $n = \pm 0.5895$. Hence we choose $n = 0.5895$ as a positive solution and $I(y_a,n)$ is **strictly decreasing** for $n > 0.5895$ and the nearest positive integer is $n = 1$, where $I(y_a,n)$ has a **maximum** value for all positive integer n. (**Result E.5.1**)

Hence the **maximum** value of $I(y,n)$ in the interval $y_a \leq y < \infty$, is at $y = y_a$ and $n = 1$ given by $I(y_a,1) = e^{-\frac{3}{4}y_a} = 0.0563 = D_2$.

We consider the **third term** inside brackets in Eq. 61 given by $J(y,n) = \frac{(n+1)^2}{4}e^{-\frac{(2n+1)}{4}y}$ which is **strictly decreasing** function in the interval $y_a \leq y < \infty$, with **maximum** value at $y = y_a$, for each n.

We set $y = y_a = 3.8371$ and compute $\frac{dJ(y_a,n)}{dn} = e^{-\frac{(2n+1)}{4}y_a}[\frac{2(n+1)}{4} + \frac{(n+1)^2}{4}(-\frac{(2y_a)}{4})]$ which has an inflection point at $\frac{2(n+1)}{4} + \frac{(n+1)^2}{4}(-\frac{(2y_a)}{4}) = 0$. Given that $J(y_a,n) > 0$ for all finite n and goes to zero as $n \to \infty$ due to the term $e^{-\frac{(2n+1)}{4}y_a}$, this inflection point is a **maximum** point. We cancel common term $\frac{2(n+1)}{4}$ and get $1 - (n+1)\frac{y_a}{4} = 0$ which has roots at $n + 1 = \frac{4}{y_a} = 1.0424$ given by $n = 0.0424$. Hence $J(y_a,n)$ is **strictly decreasing** for $n > 0.0424$ and the nearest positive integer is $n = 1$ where $J(y_a,n)$ has a **maximum** value for all positive integer n.(**Result E.5.2**)

Hence the **maximum** value of $J(y,n)$ in the interval $y_a \leq y < \infty$, is at $y = y_a$ and $n = 1$ given by $J(y_a,1) = e^{-\frac{3}{4}y_a} = 0.0563 = D_3$.

The fourth term in Eq. 61 given by $e^{-\frac{3n^2}{4}y}e^{-(2n+1)y}\frac{1}{4y}$ has a maximum at $n = 1$ and $y = y_a$ given by $e^{-\frac{3}{4}y_a}e^{-3y_a}\frac{1}{4y_a} = 3.6706 * 10^{-8} < 10^{-7} = D_4$.

Hence the maximum value of the terms in square bracket in Eq. 61 for $y_a \leq y < \infty$ and for $n = 1$, is given by $D_1 + D_2 + D_3 + D_4 = -0.1848 + 0.0563 + 0.0563 + 10^{-7} \approx -0.0722 < 0$. This summation is negative for $n > 1$, given Result E.5.1 and Result E.5.2 and $D_2 + D_3 + D_4$ is a smaller positive value and D_1 is more negative than the case for $n = 1$. Hence $\frac{dE_0(y)}{dy} < 0$ for $y_a \leq y < \infty$, given summation of negative terms for each odd n and given that $e^{-\frac{n^2}{4}y}y^{\frac{1}{4}} > 0$ for all finite n and y.

.

5.0.2. $\frac{d^2E_0(y)}{dy^2} < 0$ *for* $\pi \leq y < y_a$ *and hence* $\frac{dE_0(y)}{dy} < 0$ *for* $\pi < y < y_a$

We compute the second derivative $\frac{d^2E_0(-t)}{dt^2}$ from Eq. 59 as follows.

We set $y = \pi e^{2t}$ in Eq. 59 as follows.

$$
\begin{aligned}
E_0(y) &= (\pi)^{-\frac{1}{4}} \sum_{n=odd}^{\infty} e^{-\frac{n^2}{4}y} y^{\frac{1}{4}} - e^{-n^2 y} y^{\frac{1}{4}} - e^{-\frac{(n+1)^2}{4}y} y^{\frac{1}{4}} + e^{-(n+1)^2 y} y^{\frac{1}{4}} \\
\frac{dE_0(y)}{dy} &= (\pi)^{-\frac{1}{4}} \sum_{n=odd}^{\infty} e^{-\frac{n^2}{4}y} y^{\frac{1}{4}} (\frac{1}{4y} - \frac{n^2}{4}) - e^{-n^2 y} y^{\frac{1}{4}} (\frac{1}{4y} - n^2) \\
&\quad - e^{-\frac{(n+1)^2}{4}y} y^{\frac{1}{4}} (\frac{1}{4y} - \frac{(n+1)^2}{4}) + e^{-(n+1)^2 y} y^{\frac{1}{4}} (\frac{1}{4y} - (n+1)^2)
\end{aligned}
\tag{62}
$$

We compute the second derivative $\frac{d^2E_0(y)}{dy^2}$ as follows.

$$
\begin{aligned}
\frac{d^2E_0(y)}{dy^2} &= (\pi)^{-\frac{1}{4}} \sum_{n=odd}^{\infty} e^{-\frac{n^2}{4}y} y^{\frac{1}{4}} (-\frac{1}{4y^2} + (\frac{1}{4y} - \frac{n^2}{4})^2) - e^{-n^2 y} y^{\frac{1}{4}} (-\frac{1}{4y^2} + (\frac{1}{4y} - n^2)^2) \\
&\quad - e^{-\frac{(n+1)^2}{4}y} y^{\frac{1}{4}} (-\frac{1}{4y^2} + (\frac{1}{4y} - \frac{(n+1)^2}{4})^2) + e^{-(n+1)^2 y} y^{\frac{1}{4}} (-\frac{1}{4y^2} + (\frac{1}{4y} - (n+1)^2)^2)
\end{aligned}
\tag{63}
$$

We simplify it as follows.

$$
\begin{aligned}
\frac{d^2E_0(y)}{dy^2} &= (\pi)^{-\frac{1}{4}} \sum_{n=odd}^{\infty} e^{-\frac{n^2}{4}y} y^{\frac{1}{4}} (-\frac{1}{4y^2} + \frac{1}{16y^2} - \frac{n^2}{8y} + \frac{n^4}{16}) \\
&\quad - e^{-n^2 y} y^{\frac{1}{4}} (-\frac{1}{4y^2} + \frac{1}{16y^2} - \frac{n^2}{2y} + n^4) \\
&\quad - e^{-\frac{(n+1)^2}{4}y} y^{\frac{1}{4}} (-\frac{1}{4y^2} + \frac{1}{16y^2} - \frac{(n+1)^2}{8y} + \frac{(n+1)^4}{16}) \\
&\quad + e^{-(n+1)^2 y} y^{\frac{1}{4}} (-\frac{1}{4y^2} + \frac{1}{16y^2} - \frac{(n+1)^2}{2y} + (n+1)^4)
\end{aligned}
\tag{64}
$$

We compute the **maximum** value of $\frac{d^2E_0(y)}{dy^2}$ with $y = \pi e^{2t}$ in the range $y = [\pi, y_a)$ for $0 \leq t < t_a = 0.1$, where $y_a = 3.8371$, by computing the maximum value of positive terms and minimum value of absolute value of negative terms. Let the maximum value of y be $y_{max} = y_a = \pi e^{2t_a}$ and minimum value of y be $y_{min} = \pi$ in the interval $y = [\pi, y_a)$.

The first term in curved brackets in Eq. 64 at $n = 1$ is given by $-\frac{1}{4y^2}+\frac{1}{16y^2}-\frac{n^2}{8y}+\frac{n^4}{16} = -\frac{3}{16y^2}-\frac{1}{8y}+\frac{1}{16}$ and the **maximum value** of the whole first term in the interval $y = [y_{min}, y_{max})$ is given by $e^{-\frac{1}{4}y_{min}}(y_{max})^{\frac{1}{4}}\frac{1}{16} - e^{-\frac{1}{4}y_{max}}(y_{min})^{\frac{1}{4}}(\frac{3}{16y_{max}^2} + \frac{1}{8y_{max}})$ and similarly we compute the other 3 terms at $n = 1, 3, 5, 7, 9$. The **maximum** value of $\frac{d^2E_0(y)}{dy^2}$ in Eq. 64 at $n = 1, 3, 5, 7, 9$ in the interval $y = [y_{min}, y_{max})$ is given by -0.0097 which is **negative**. (**Result E.5.5**) Matlab simulation)

We note that $-\frac{1}{4y^2} + \frac{1}{16y^2} = -\frac{3}{16y^2}$ and ignore the negative terms in Eq. 64 because we are computing the maximum value of $\frac{d^2E_0(y)}{dy^2}$ for $n \geq 11$ given by $[\frac{d^2E_0(y)}{dy^2}]_2$.

$$
\begin{aligned}
[\frac{d^2E_0(y)}{dy^2}]_2 < (\pi)^{-\frac{1}{4}} \sum_{n=11,13,...}^{\infty} & e^{-\frac{n^2}{4}y}y^{\frac{1}{4}}\frac{n^4}{16} + e^{-n^2y}y^{\frac{1}{4}}(\frac{3}{16y^2} + \frac{n^2}{2y}) \\
& + e^{-\frac{(n+1)^2}{4}y}y^{\frac{1}{4}}(\frac{3}{16y^2} + \frac{(n+1)^2}{8y}) + e^{-(n+1)^2y}y^{\frac{1}{4}}(n+1)^4
\end{aligned}
\tag{65}
$$

We compute the maximum value of $[\frac{d^2E_0(y)}{dy^2}]_2$ in Eq. 65 for $n \geq 11$ by setting first term as $e^{-\frac{n^2}{4}y_{min}}(y_{max})^{\frac{1}{4}}$ and $n+1 < 1.1n$, $n^2 < 10e^{0.1n^2} = 10[1 + 0.1n^2 + \frac{0.01}{2}n^4 + ...]$ and
$n^4 < 200e^{0.1n^2} = 200[1 + 0.1n^2 + \frac{0.01}{2}n^4 + ...]$ as follows.

$$
\begin{aligned}
[\frac{d^2E_0(y)}{dy^2}]_2 < (\pi)^{-\frac{1}{4}}y_{max}^{\frac{1}{4}} \sum_{n=11,13,...}^{\infty} & e^{-\frac{n^2}{4}y_{min}}200e^{0.1n^2}\frac{1}{16} + e^{-n^2y_{min}}(\frac{3}{16y_{min}^2} + 10e^{0.1n^2}\frac{1}{2y_{min}}) \\
& + e^{-\frac{(n+1)^2}{4}y_{min}}(\frac{3}{16y_{min}^2} + 10e^{0.1n^2}\frac{(1.1)^2}{8y_{min}}) + e^{-(n+1)^2y_{min}}200e^{0.1n^2}(1.1)^4
\end{aligned}
\tag{66}
$$

We use $n + 1 > n$ for the exponent term and simplify above equation as follows.

$$
\begin{aligned}
[\frac{d^2E_0(y)}{dy^2}]_2 < (\pi)^{-\frac{1}{4}}y_{max}^{\frac{1}{4}} \sum_{n=11,13,...}^{\infty} & e^{-n^2(\frac{1}{4}y_{min}-0.1)}\frac{200}{16} + e^{-n^2y_{min}}\frac{3}{16y_{min}^2} + e^{-n^2(y_{min}-0.1)}\frac{10}{2y_{min}} \\
& + e^{-n^2\frac{1}{4}y_{min}}\frac{3}{16y_{min}^2} + e^{-n^2(\frac{1}{4}y_{min}-0.1)}\frac{(1.1)^2 * 10}{8y_{min}} + e^{-n^2(y_{min}-0.1)}200(1.1)^4
\end{aligned}
\tag{67}
$$

We use the complementary error function given by $erfc(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-u^2}du$ link) and the fact that $\sum_{n=11,13,...}^{\infty} e^{-n^2K} < \int_{11}^{\infty} e^{-t^2K}dt = \frac{1}{\sqrt{K}}\int_{11\sqrt{K}}^{\infty} e^{-u^2}du = \frac{\sqrt{\pi}}{2\sqrt{K}}erfc(11\sqrt{K})$ using the substitution $t\sqrt{K} = u$ and $dt\sqrt{K} = du$ and write Eq. 67 as follows.

$$
\begin{aligned}
[\frac{d^2E_0(y)}{dy^2}]_2 < (\pi)^{-\frac{1}{4}} y_{max}^{\frac{1}{4}} [\frac{200}{16} \frac{\sqrt{\pi}}{2\sqrt{(\frac{1}{4}y_{min} - 0.1)}} erfc(11\sqrt{(\frac{1}{4}y_{min} - 0.1)}) + \frac{3}{16y_{min}^2} \frac{\sqrt{\pi}}{2\sqrt{y_{min}}} erfc(11\sqrt{y_{min}}) \\
+ \frac{10}{2y_{min}} \frac{\sqrt{\pi}}{2\sqrt{(y_{min} - 0.1)}} erfc(11\sqrt{(y_{min} - 0.1)}) + \frac{3}{16y_{min}^2} \frac{\sqrt{\pi}}{2\sqrt{\frac{1}{4}y_{min}}} erfc(11\sqrt{\frac{1}{4}y_{min}} \\
+ \frac{(1.1)^2 * 10}{8y_{min}} \frac{\sqrt{\pi}}{2\sqrt{(\frac{1}{4}y_{min} - 0.1)}} erfc(11\sqrt{(\frac{1}{4}y_{min} - 0.1)}) \\
+ 200(1.1)^4 \frac{\sqrt{\pi}}{2\sqrt{(y_{min} - 0.1)}} erfc(11\sqrt{(y_{min} - 0.1)})]
\end{aligned}
\tag{68}
$$

We compute Eq. 68 numerically and get $[\frac{d^2E_0(y)}{dy^2}]_2 < 8.65 * 10^{-37}$. The **maximum** value of $[\frac{d^2E_0(y)}{dy^2}]_2$ in Eq. 67 at $n = 11, 13, ..$ in the interval $y = [y_{min}, y_{max})$ is given by $8.65 * 10^{-37}$ which is **positive**. (**Result E.5.6**) Matlab simulation)

Using Result E.5.5 and E.5.6, we get the **maximum** value of $\frac{d^2E_0(y)}{dy^2}$ in Eq. 64 at $n = 1, 3, 5, ..$ in the interval $y = [y_{min}, y_{max})$ is given by $-0.0097 + 8.65 * 10^{-37} \approx -0.0097$ which is **negative**. (**Result E.5.7**)

Hence we have shown that $\frac{d^2E_0(y)}{dy^2} < 0$, for $\pi \leq y < y_a$ and hence $\frac{dE_0(y)}{dy} < 0$ for $\pi < y < y_a$ given that $\frac{dE_0(y)}{dy} = 0$ at $y = \pi$.

It is shown in Section 5.0.1 that $\frac{dE_0(y)}{dy} < 0$ for $y_a \leq y < \infty$ for all finite n.

Hence $\frac{dE_0(y)}{dy} < 0$ for $\pi < y < \infty$. We see that $y = \pi e^{2t}$ and $\frac{dy}{dt} = 2\pi e^{2t} = 2y$ and $\frac{dE_0(-t)}{dt} = \frac{dE_0(-t)}{dy}\frac{dy}{dt} = \frac{dE_0(y)}{dy} 2y$ and hence $\frac{dE_0(-t)}{dt} < 0$ for $t > 0$. Hence $E_0(t) = E_0(-t)$ is strictly decreasing for $t > 0$.

.

6. $\omega_z(t_2, t_0)$ is a continuous function of t_0 and t_2

It is shown in **Lemma 1** in Section 2.1 that $G_R(\omega, t_2, t_0) = 0$ at $\omega = \omega_z(t_2, t_0)$ where it crosses the zero line to the opposite sign, if Statement 1 is true (Result 2.1.5) and that $\omega_z(t_2, t_0)$ is **finite and non-zero** for all $|t_0| < \infty$ and for each non-zero value of t_2 and that $\omega_z(t_2, t_0)$ is an even function of variable t_0, for a given value of t_2(Details in Section 2.4). For a given t_2 and t_0, $\omega_z(t_2, t_0)$ can have more than one value, corresponding to multiple zero crossings in $G_R(\omega, t_2, t_0)$, but we consider only the first zero crossing to the right of origin in the section below, where $G_R(\omega, t_2, t_0)$ crosses the zero line to the opposite sign, as detailed in **Lemma 1** in Section 2.1.

We consider the Fourier transform of the even part of $g(t, t_2, t_0)$ given by $G_R(\omega, t_2, t_0)$ in the section below and show that, under this Fourier transformation, as we change t_0 and t_2, the zero crossing in $G_R(\omega, t_2, t_0)$ given by $\omega_z(t_2, t_0)$ is a continuous function of t_0 and t_2, for all $0 < t_0 < \infty$ and $0 < t_2 < \infty$.

.

6.1. *Discussion of First Method*

Consider the **segment** S in $G_R(\omega, t_2, t_0)$ in the neighborhood around the first zero crossing where $\frac{dG_R(\omega,t_2,t_0)}{d\omega} \leq 0$. (Segment S is the portion between the majenta lines in example plot)

- In the **segment** S, $G_R(\omega, t_2, t_0)$ in Eq. 27 copied in Eq. 69 is a **continuous** function of ω, for **each** value of t_0 and t_2 as shown in Appendix C.1 and $\frac{dG_R(\omega,t_2,t_0)}{d\omega} \leq 0$ in the neighborhood around the **first zero crossing**.

- If we **fix** the X-coordinate ω and t_2, $G_R(\omega, t_2, t_0)$ is a **continuous** function of t_0, as shown in Appendix C.3. Hence, for **each** fixed value of ω and t_2, as we change t_0 by an infinitesimal δt_0, $G_R(\omega, t_2, t_0 - \delta t_0)$ and $G_R(\omega, t_2, t_0 + \delta t_0)$, move towards $G_R(\omega, t_2, t_0)$ in a **continuous** manner, as $\delta t_0 \to 0$. (**Result 6.1**)

$$
\begin{aligned}
G_R(\omega, t_2, t_0) = e^{-2\sigma t_0} \int_{-\infty}^{0} [E_0'(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}'(\tau - t_0, t_2)] \cos{(\omega\tau)} d\tau \\
+ e^{2\sigma t_0} \int_{-\infty}^{0} [E_0'(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}'(\tau + t_0, t_2)] \cos{(\omega\tau)} d\tau
\end{aligned}
\tag{69}
$$

- Every point in the segment S (plot), moves continuously, as we change t_0 by an infinitesimal δt_0, for each fixed value of ω and t_2.

Using Result 6.1 and Result 2.1.5 in Section 2.1, we can see that this also applies to the first **zero crossing** in $G_R(\omega, t_2, t_0)$ in the segment S, given by $\omega_z(t_2, t_0)$ where $G_R(\omega, t_2, t_0) = 0$ in Eq. 69. The **zero crossing** moves **continuously**, as we change t_0 by an infinitesimal δt_0. This is explained in detail in the section below.

.

6.1.1. *Zero Crossings in* $G_R(\omega, t_2, t_0)$ *move continuously as a function of* t_0 *for a given* t_2*.*

This is shown by an **example** plot. **Red** plot corresponds to $G_R(\omega, t_2, t_0)$ with zero crossing at point P_0, **green** plot corresponds to $G_R(\omega, t_2, t_0 + \delta t_0)$ with zero crossing at point P_{11} and **Blue** plot corresponds to $G_R(\omega, t_2, t_0 - \delta t_0)$ with zero crossing at point P_{21}.

We **define** the **point** P_{12} in $G_R(\omega, t_2, t_0 + \delta t_0)$ as the point which has the **fixed X-coordinate** $\omega = \omega_z(t_2, t_0)$. We **define** the **point** P_{22} in $G_R(\omega, t_2, t_0 - \delta t_0)$ as the point which has the **fixed X-coordinate** $\omega = \omega_z(t_2, t_0)$.

We **define** the **point** P_{11} in $G_R(\omega, t_2, t_0 + \delta t_0)$ as the **zero crossing point** which has the **fixed Y-coordinate** which equals zero. We **define** the **point** P_{21} in $G_R(\omega, t_2, t_0 - \delta t_0)$ as the **zero crossing point** which has the **fixed Y-coordinate** which equals zero.

Given Result 6.1 and Result 2.1.5 in Section 2.1, as we change t_0 by an infinitesimal δt_0, $G_R(\omega, t_2, t_0 + \delta t_0)$ moves towards $G_R(\omega, t_2, t_0)$ in Eq. 69 in a **continuous** manner, for **each fixed** value of ω and t_2, including the zero crossing point. The **point** P_{12} in $G_R(\omega, t_2, t_0 + \delta t_0)$ which corresponds to the **fixed X-coordinate** $\omega = \omega_z(t_2, t_0)$, moves towards corresponding point P_0 in $G_R(\omega, t_2, t_0)$, for the **same** $\omega = \omega_z(t_2, t_0)$ in a **continuous** manner, as $\delta t_0 \to 0$. Given that P_0 is a **zero crossing point** in $G_R(\omega, t_2, t_0)$, this is equivalent to the **zero crossing point** P_{11} in $G_R(\omega, t_2, t_0 + \delta t_0)$ moving towards corresponding **zero crossing** point P_0 in $G_R(\omega, t_2, t_0)$ in a **continuous** manner, as $\delta t_0 \to 0$.

Similarly, as we change t_0 by an infinitesimal δt_0, $G_R(\omega, t_2, t_0 - \delta t_0)$ moves towards $G_R(\omega, t_2, t_0)$ in Eq. 69 in a **continuous** manner as follows. The **point** P_{22} in $G_R(\omega, t_2, t_0 - \delta t_0)$ which corresponds to the **fixed X-coordinate** $\omega = \omega_z(t_2, t_0)$, moves towards corresponding point P_0 in $G_R(\omega, t_2, t_0)$, for the **same** $\omega = \omega_z(t_2, t_0)$ in a **continuous** manner, as $\delta t_0 \to 0$. Given that P_0 is a **zero crossing point** in $G_R(\omega, t_2, t_0)$, this is equivalent to the **Zero crossing point** P_{21} in $G_R(\omega, t_2, t_0 - \delta t_0)$ moving towards corresponding **zero crossing** point P_0 in $G_R(\omega, t_2, t_0)$ in a **continuous** manner, as $\delta t_0 \to 0$.

$$\begin{aligned} G_R(\omega_z(t_2, t_0), t_2, t_0) = e^{-2\sigma t_0} \int_{-\infty}^{0} [E_0^{'}(\tau + t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)] \cos(\omega_z(t_2, t_0)\tau) d\tau \\ + e^{2\sigma t_0} \int_{-\infty}^{0} [E_0^{'}(\tau - t_0, t_2) e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)] \cos(\omega_z(t_2, t_0)\tau) d\tau = 0 \end{aligned} \tag{70}$$

As $\delta t_0 \to 0$, zero crossing point P_{11} in $G_R(\omega, t_2, t_0 + \delta t_0)$ given by $\omega_z(t_2, t_0 + \delta t_0)$ moves towards corresponding **zero crossing** point P_0 in $G_R(\omega, t_2, t_0)$ given by $\omega_z(t_2, t_0)$, in a **continuous** manner, in Eq. 70.

Similarly, the zero crossing point P_{21} in $G_R(\omega, t_2, t_0 - \delta t_0)$ given by $\omega_z(t_2, t_0 - \delta t_0)$ moves towards corresponding **zero crossing** point P_0 in $G_R(\omega, t_2, t_0)$ given by $\omega_z(t_2, t_0)$, in a **continuous** manner, in Eq. 70, as $\delta t_0 \to 0$. (example plot).

Hence we deduce that $\omega_z(t_2, t_0)$ is **continuous** in the interval $[t_0 - \delta t_0, t_0 + \delta t_0]$ in the **segment** S, around the first zero crossing at $\omega = \omega_z(t_2, t_0)$ (example plot).

Using arguments in the above paras, we see that $\omega_z(t_2, t_0)$ is a **continuous** function of t_0, for all $0 < t_0 < \infty$, for **each** fixed value of t_2.

It is shown in Section Appendix C.4 that $G_{R,2r}(\omega, t_2, t_0)$ is a **continuous** function of t_2. We can use arguments similar to the above paras and show that $\omega_z(t_2, t_0)$ is a **continuous** function of t_2, for all $0 < t_2 < \infty$, for **each** fixed value of t_0.

Hence $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 and t_2 for $0 < t_0 < \infty$, and $0 < t_2 < \infty$. This is shown in detail in the next section using Implicit Function Theorem.

.

6.2. *Second Method using Implicit Function Theorem*

In this section, it is shown that $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 and t_2 for $0 < t_0 < \infty$, and $0 < t_2 < \infty$, in the steps below using **Implicit Function Theorem**.

- It is shown in Appendix C.1 that $G_R(\omega, t_2, t_0)$ and $G_{R,2r}(\omega, t_2, t_0)$ are partially differentiable at least twice with respect to ω, for some value of $r \in W$ (element of set of whole numbers including zero.)

- It is shown in Appendix C.3 that $G_{R,2r}(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to t_0. It is shown in Appendix C.4 that $G_{R,2r}(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to t_2.

- In Appendix C.7, it is shown that, **if** $G_R(\omega, t_2, t_0) = 0$ at $\omega = \pm\omega_z(t_2, t_0)$, for each fixed choice of positive $t_0, t_2 \in \Re$ and $(2r+1)$ is the highest order of the zero at $\omega = \pm\omega_z(t_2, t_0)$ for some value of $r \in W$ (element of set of whole numbers including zero), **then** $G_{R,2r}(\omega, t_2, t_0) = \frac{\partial^{2r} G_R(\omega, t_2, t_0)}{\partial \omega^{2r}} = 0$ at $\omega = \pm\omega_z(t_2, t_0)$ and $\frac{\partial G_{R,2r}(\omega, t_2, t_0)}{\partial \omega} = \frac{\partial^{2r+1} G_R(\omega, t_2, t_0)}{\partial \omega^{2r+1}} \neq 0$ at $\omega = \pm\omega_z(t_2, t_0)$.

- It is shown in Appendix C.5 that the zero crossing in $G_{R,2r}(\omega, t_2, t_0)$ given by $\omega_z(t_2, t_0)$, is a **continuous** function of t_0, for a given t_2, for $0 < t_0 < \infty$, using **Implicit Function Theorem** in $\Re^2$.

- It is shown in Appendix C.6 that $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 and t_2, for $0 < t_0 < \infty$ and $0 < t_2 < \infty$, using **Implicit Function Theorem** in $\Re^3$.

.

7. Hurwitz Zeta Function and related functions

We can show that the new method is **not** applicable to Hurwitz zeta function and related zeta functions and **does not** contradict the existence of their non-trivial zeros away from the critical line given by $Re[s] = \frac{1}{2}$. The new method requires the **symmetry** relation $\xi(s) = \xi(1-s)$ and hence $\xi(\frac{1}{2}+i\omega) = \xi(\frac{1}{2}-i\omega)$ when evaluated at the critical line $s = \frac{1}{2}+i\omega$. This means $\xi(\frac{1}{2}+i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ and $E_0(t) = E_0(-t)$ (Details in Appendix A.7) where $E_0(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$ and this condition is satisfied for Riemann's Zeta function.

It is **not** known that Hurwitz Zeta Function given by $\zeta(s, a) = \sum\limits_{m=0}^{\infty} \frac{1}{(m+a)^s}$ satisfies a symmetry relation similar to $\xi(s) = \xi(1-s)$ where $\xi(s)$ is an entire function, for $a \neq 1$ and hence the condition $E_0(t) = E_0(-t)$ is **not** known to be satisfied [6]. Hence the new method is **not** applicable to Hurwitz zeta function and **does not** contradict the existence of their non-trivial zeros away from the critical line.

Dirichlet L-functions satisfy a symmetry relation $\xi(s, \chi) = \epsilon(\chi)\xi(1-s, \bar{\chi})$ [7] which does **not** translate to $E_0(t) = E_0(-t)$ required by the new method and hence this proof is **not** applicable to them. This proof does not need or use Euler product.

We know that $\zeta(s) = \sum\limits_{m=1}^{\infty} \frac{1}{m^s}$ diverges for $Re[s] \leq 1$. Hence we derive a convergent and entire function $\xi(s)$ using the well known theorem $F(x) = 1 + 2\sum\limits_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}(1 + 2\sum\limits_{n=1}^{\infty} e^{-\pi \frac{n^2}{x}})$, where $x > 0$ is real [4](link) and then derive $E_0(t) = \sum_{n=1}^{\infty}[4\pi^2 n^4 e^{4t} - 6\pi n^2 e^{2t}]e^{-\pi n^2 e^{2t}} e^{\frac{t}{2}}$. In the case of **Hurwitz zeta** function and **other zeta functions** with non-trivial zeros away from the critical line, it is **not** known if a corresponding relation similar to $F(x)$ exists, which enables derivation of a convergent and entire function $\xi(s)$ and results in $E_0(t)$ as a Fourier transformable, real, even and analytic function. Hence the new method presented in this paper is **not** applicable to Hurwitz zeta function and related zeta functions.

The proof of Riemann Hypothesis presented in this paper is **only** for the specific case of Riemann's Zeta function and **only** for the **critical strip** $0 \leq |\sigma| < \frac{1}{2}$. This proof requires both $E_p(t)$ and $E_{p\omega}(\omega)$ to be Fourier transformable where $E_p(t) = E_0(t)e^{-\sigma t}$ is a real analytic function and uses the fact that $E_0(t)$ is an **even** function of variable t and $E_0(t) > 0$ for $|t| < \infty$ (Details in Appendix A.6) and $E_0(t)$ is **strictly decreasing** function for $t > 0$ (Details in Section 5). These conditions may **not** be satisfied for many other functions including those which have non-trivial zeros away from the critical line and hence the new method may **not** be applicable to such functions.

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7.1. ***Acknowledgments***

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Appendix A. Properties of Fourier Transforms

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Appendix A.1. $E_p(t), h(t)$ ***are absolutely integrable functions and their Fourier Transforms are finite.***

The inverse Fourier Transform of the function $E_{p\omega}(\omega)$ is given by $E_p(t) = E_0(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{p\omega}(\omega)e^{i\omega t}d\omega$. In Eq. 15, we see that $E_0(t) = \sum_{n=1}^{\infty}(-1)^{n-1}(e^{-\pi\frac{n^2}{4}e^{-2t}} - e^{-\pi n^2 e^{-2t}})e^{-\frac{t}{2}} > 0$ and finite for all $-\infty < t < \infty$(Appendix A.6). Hence $E_p(t) = E_0(t)e^{-\sigma t} > 0$ and finite for all $-\infty < t < \infty$.

It is shown in Appendix A.4 that $E_0(t)$ has an asymptotic **exponential** fall-off rate of $o[e^{-|t|}]$ and hence $E_p(t) = E_0(t)e^{-\sigma t}$ has an asymptotic **exponential** fall-off rate of **at least** $o[e^{-(1-\sigma)|t|}]$, for $0 \leq |\sigma| < \frac{1}{2}$. Hence $E_p(t)$ goes to zero, at $t \to \pm\infty$ and we showed that $E_p(t) > 0$ and finite for all $-\infty < t < \infty$ in the last paragraph.(**Result 21**) Hence $E_{p\omega}(\omega) = \int_{-\infty}^{\infty} E_p(t)e^{-i\omega t}dt$, evaluated at $\omega = 0$ **cannot** be zero. Hence $E_{p\omega}(\omega)$ **does not have a zero** at $\omega = 0$ and hence $\omega_0 \neq 0$.

Given that $E_{p\omega}(\omega)$ is a holomorphic function for real ω, it is finite for real ω and also for $\omega = 0$. Hence $E_{p\omega}(0) = \int_{-\infty}^{\infty} E_p(t)dt$ is finite. Using Result 21, we can write $\int_{-\infty}^{\infty} |E_p(t)|dt$ is finite and $E_p(t)$ is an absolutely **integrable function** and its Fourier transform $E_{p\omega}(\omega)$ goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

Using the arguments in above paragraph, we replace σ in $E_p(t) = E_0(t)e^{-\sigma t}$ by 0 and 2σ respectively and see that $E_0(t)$ and $E_0(t)e^{-2\sigma t}$ are absolutely **integrable** functions and the integrals $\int_{-\infty}^{\infty} |E_0(t)|dt < \infty$ and $\int_{-\infty}^{\infty} |E_0(t)e^{-2\sigma t}|dt < \infty$.

Given that $E_p(t) = E_0(t)e^{-\sigma t}$ is an absolutely integrable function, its shifted versions are absolutely integrable and we see that $E_p^{'}(t, t_2) = e^{-\sigma t_2}E_p(t-t_2) - e^{\sigma t_2}E_p(t+t_2) = (E_0(t-t_2) - E_0(t+t_2))e^{-\sigma t}$ in Eq. 16 is an absolutely integrable function, for a finite shift of t_2. (We substitute $t - t_2 = \tau$ and $dt = d\tau$ and get $\int_{-\infty}^{\infty} |E_p(t-t_2)|dt = \int_{-\infty}^{\infty} |E_p(\tau)|d\tau$ and hence $E_p(t - t_2)$ is an absolutely integrable function, given that $E_p(t)$ is absolutely integrable. Same argument holds for $E_p(t + t_2)$.)

We see that $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ is an absolutely **integrable function** because $h(t) > 0$ for real t and $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} h(t)dt = [\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt]_{\omega=0} = [\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega}]_{\omega=0} = \frac{2}{\sigma}$, is finite for $0 < \sigma < \frac{1}{2}$ and its Fourier transform $H(\omega)$ converges and goes to zero as $\omega \to \pm\infty$, as per Riemann Lebesgue Lemma (link).

.

Appendix A.2. ***Convolution integral convergence***

Let us consider $h(t) = e^{\sigma t}u(-t) + e^{-\sigma t}u(t)$ whose first derivative given by $\frac{dh(t)}{dt} = \sigma e^{\sigma t}u(-t) -$

$\sigma e^{-\sigma t}u(t)$ and $A_0 = [\frac{dh(t)}{dt}]_{t=0+} - [\frac{dh(t)}{dt}]_{t=0-} = -2\sigma$ and hence $\frac{dh(t)}{dt}$ is **discontinuous** at $t = 0$, for $0 < \sigma < \frac{1}{2}$. The second derivative of $h(t)$ given by $h_2(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = -2\sigma$ and its Fourier transform $H_2(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $h(t)$ is obtained by integrating $h_2(t)$ twice and its Fourier transform $H(\omega)$ has a term $\frac{A_0}{(i\omega)^2}$ (link) and has a **fall off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ and $\int_{-\infty}^{\infty} H(\omega)d\omega$ converges.(**Result C.2**)

Let us consider the function $g(t, t_2, t_0) = f(t, t_2, t_0)e^{-\sigma t}u(-t) + f(t, t_2, t_0)e^{\sigma t}u(t)$ in Eq. 16 and its first derivative given by $\frac{dg(t,t_2,t_0)}{dt} = [-\sigma e^{-\sigma t} f(t, t_2, t_0) + e^{-\sigma t}\frac{df(t,t_2,t_0)}{dt}]u(-t) + [\sigma e^{\sigma t} f(t, t_2, t_0) + e^{\sigma t}\frac{df(t,t_2,t_0)}{dt}]u(t)$. We get $[\frac{dg(t,t_2,t_0)}{dt}]_{t=0-} = -\sigma f(0, t_2, t_0)+[\frac{df(t,t_2,t_0)}{dt}]_{t=0-}$ and $[\frac{dg(t,t_2,t_0)}{dt}]_{t=0+} = \sigma f(0, t_2, t_0)+ [\frac{df(t,t_2,t_0)}{dt}]_{t=0+}$(**Result C.2.1**).

We note that $f(t, t_2, t_0)$ is a differentiable function in Eq. 16 and get $[\frac{df(t,t_2,t_0)}{dt}]_{t=0+} = [\frac{df(t,t_2,t_0)}{dt}]_{t=0-}$ and get $[\frac{dg(t,t_2,t_0)}{dt}]_{t=0+} - [\frac{dg(t,t_2,t_0)}{dt}]_{t=0-} = 2\sigma f(0, t_2, t_0)$ using Result C.2.1. Hence $\frac{dg(t,t_2,t_0)}{dt}$ is **discontinuous** at $t = 0$, for $0 < \sigma < \frac{1}{2}$, if $f(0, t_2, t_0) \neq 0$.

We can see that the **first derivatives** of $g(t, t_2, t_0), h(t)$ are **discontinuous** at $t = 0$ and hence $G(\omega, t_2, t_0), H(\omega)$ have **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$, using Result C.2. Hence the convolution integral below converges to a finite value for real ω, for the case $f(0, t_2, t_0) \neq 0$.

$$F(\omega, t_2, t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega', t_2, t_0)H(\omega - \omega')d\omega' = \frac{1}{2\pi}[G(\omega, t_2, t_0) * H(\omega)] \tag{A.1}$$

If $f(0, t_2, t_0) = 0$, and if the N^{th} **derivative** of $g(t, t_2, t_0)$ is **discontinuous** at $t = 0$ where $N > 1$, we see that $G(\omega, t_2, t_0)$ has **fall-off rate** of $\frac{1}{\omega^{(N+1)}}$ as $|\omega| \to \infty$(Details in Appendix A.3). $G(\omega, t_2, t_0)$ has a minimum **fall-off rate** of $\frac{1}{\omega^2}$ as $|\omega| \to \infty$ for this case. Hence the convolution integral in Eq. A.1 converges to a finite value for real ω.

.

Appendix A.3. ***Fall off rate of Fourier Transform of functions***

Let us consider a real Fourier transformable function $P(t) = P_+(t)u(t) + P_-(t)u(-t)$ whose $(N - 1)^{th}$ **derivative is discontinuous** at $t = 0$. The $(N)^{th}$ derivative of $P(t)$ given by $P_N(t)$ has a Dirac delta function $A_0\delta(t)$ where $A_0 = [\frac{d^{N-1}P_+(t)}{dt^{N-1}} - \frac{d^{N-1}P_-(t)}{dt^{N-1}}]_{t=0}$ and its Fourier transform $P_{N\omega}(\omega)$ has a constant term A_0, corresponding to the Dirac delta function.

This means $P(t)$ is obtained by integrating $P_N(t)$, N times and its Fourier transform $P_\omega(\omega)$ has a term $\frac{A_0}{(i\omega)^N}$ (link) and has a **fall off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

We have shown that if the $(N - 1)^{th}$ **derivative** of the function $P(t)$ is **discontinuous** at $t = 0$ then its Fourier transform $P_\omega(\omega)$ has a **fall-off rate** of $\frac{1}{\omega^N}$ as $|\omega| \to \infty$.

.

Appendix A.4. ***Exponential Fall off rate of*** $x(t) = E_0(t)e^{-2\sigma t}$

Given that $E_0(t) = E_0(-t)$ (Appendix A.7), we write $E_0(t)$ in Eq. 15 as follows.

$$E_0(t) = \sum_{n=1}^{\infty}(-1)^{n-1}(e^{-\pi\frac{n^2}{4}e^{2t}} - e^{-\pi n^2 e^{2t}})e^{\frac{t}{2}} = \sum_{n=1}^{\infty}(-1)^{n-1}e^{-\pi\frac{n^2}{4}e^{2t}}(1 - e^{-\pi\frac{3n^2}{4}e^{2t}})e^{\frac{t}{2}} \tag{A.2}$$

We use Taylor series expansion around $t = 0$ for $e^{2t} = \sum_{r=0}^{\infty}\frac{(2t)^r}{!r}$, given that e^{2t} is an analytic function for real t.

$$E_0(t) = \sum_{n=1}^{\infty}(-1)^{n-1}e^{-\pi\frac{n^2}{4}(1+2t)}e^{-\pi\frac{n^2}{4}(\frac{(2t)^2}{!2}+\frac{(2t)^3}{!3}....)}(1 - e^{-\pi\frac{3n^2}{4}e^{2t}})e^{\frac{t}{2}} \tag{A.3}$$

We take the term $e^{-\frac{\pi}{2}t}e^{\frac{t}{2}} = e^{-1.0708t}$ out of the summation, corresponding to $n = 1$ and write Eq. A.3 as follows.

$$E_0(t) = e^{-\frac{\pi}{2}t}e^{\frac{t}{2}}\sum_{n=1}^{\infty}(-1)^{n-1}e^{-\pi\frac{n^2}{4}}e^{-\frac{\pi}{2}(n^2-1)t}e^{-\pi\frac{n^2}{4}(\frac{(2t)^2}{!2}+\frac{(2t)^3}{!3}....)}(1 - e^{-\pi\frac{3n^2}{4}e^{2t}}) \tag{A.4}$$

For $t > 0$, we see that the term corresponding to $n = 1$ in Eq. A.4 has an asymptotic fall-off rate of $o[e^{-t}]$. The terms corresponding to $n > 1$ have fall-off rates **higher** than $o[e^{-t}]$, due to the term $e^{-\frac{\pi}{2}(n^2-1)t}$.

Hence we see that $E_0(t)$ has an asymptotic fall-off rate of $o[e^{-t}]$, for $t > 0$. Given that $E_0(t) = E_0(-t)$(Appendix A.7), we see that $E_0(t)$ has an **exponential** asymptotic fall-off rate of $o[e^{-|t|}]$.

Similarly, $E_p(t) = E_0(t)e^{-\sigma t}$ has an asymptotic **exponential** fall-off rate of $o[e^{-0.5|t|}]$ (using $o[e^{-(1-\sigma)|t|}]$), for $0 \le |\sigma| < \frac{1}{2}$.

Similarly, $x(t) = E_0(t)e^{-2\sigma t}$ has an asymptotic **exponential** fall-off rate of $o[e^{-\delta|t|}]$ (using $o[e^{-(1-2\sigma)|t|}]$), for $0 \le |\sigma| < \frac{1}{2}$ and $\delta > 0$.

.

Appendix A.5. ***Absolutely integrable functions***

We see that a real function $y(t)$ which is finite for all t and has an asymptotic falloff rate of $O[\frac{1}{t^2}]$ is an absolutely integrable function, given that $\int_{-\infty}^{\infty}|y(t)|dt = \int_{-\infty}^{-T}|y(t)|dt + \int_{-T}^{T}|y(t)|dt + \int_{T}^{\infty}|y(t)|dt$ is finite, for non-zero and finite T, because when we integrate the integrand $|y(t)|$ with order $O[\frac{1}{t^2}]$, we get the result $O[\frac{1}{t}]$, which is finite at the limit $t = \pm T$ and the result $O[\frac{1}{t}]$ is zero at the limit $t \to \pm\infty$. If $y(t)$ has an exponential asymptotic falloff rate, when we integrate the integrand $|y(t)|$ with order $O[e^{-A|t|}]$ for real $A > 0$, we get the result $O[\frac{1}{A}e^{-A|t|}]$, which is finite at the limit $t = \pm T$ and the result is zero at the limit $t \to \pm\infty$ and hence $y(t)$ is an absolutely integrable function.

.

Appendix A.6. $E_0(t) > 0$ ***for*** $-\infty < t < \infty$

It is shown in this section that $E_0(t) > 0$ for $-\infty < t < \infty$. We take the term $e^{-\pi\frac{n^2}{4}e^{2t}}e^{\frac{t}{2}}$ out of the brackets in Eq. A.5 for $E_0(-t)$ and use $(n+1)^2 = n^2 + 2n + 1$ and rearrange the terms in the last line below.

$$
\begin{aligned}
E_0(-t) &= \sum_{n=1}^{\infty}(-1)^{n-1}(e^{-\pi\frac{n^2}{4}e^{2t}} - e^{-\pi n^2 e^{2t}})e^{\frac{t}{2}} \\
E_0(-t) &= \sum_{n=odd}^{\infty}(e^{-\pi\frac{n^2}{4}e^{2t}} - e^{-\pi n^2 e^{2t}} - e^{-\pi\frac{(n+1)^2}{4}e^{2t}} + e^{-\pi(n+1)^2 e^{2t}})e^{\frac{t}{2}} \\
E_0(-t) &= \sum_{n=odd}^{\infty} e^{-\pi\frac{n^2}{4}e^{2t}}e^{\frac{t}{2}}(1 - e^{-\pi\frac{3n^2}{4}e^{2t}} - e^{-\pi\frac{(2n+1)}{4}e^{2t}} + e^{-\pi\frac{3n^2}{4}e^{2t}}e^{-\pi(2n+1)e^{2t}})
\end{aligned}
\tag{A.5}
$$

We compute the **minimum** value of $E_0(-t)$ in Eq. A.5 for $0 \leq t < \infty$, by computing the minimum value of positive terms and maximum value of absolute value of negative terms. We ignore the last term $e^{-\pi\frac{3n^2}{4}e^{2t}}e^{-\pi(2n+1)e^{2t}} > 0$ because we want the minimum value of $E_0(-t)$.

The minimum value of the first term inside brackets in Eq. A.5 is $A_1 = 1$. The maximum value of the absolute value of the second term inside brackets $e^{-\pi\frac{3n^2}{4}e^{2t}}$ occurs at $n = 1$ and $t = 0$, given by $A_2 = e^{-\pi\frac{3}{4}}$. The maximum value of the absolute value of the third term $e^{-\pi\frac{(2n+1)}{4}e^{2t}}$ occurs at $n = 1$ and $t = 0$, given by $A_3 = e^{-\pi\frac{3}{4}}$. Hence the minimum value of the terms inside the brackets is given by $A_1 - A_2 - A_3 = 1 - 2e^{-\pi\frac{3}{4}} = 0.8104 > 0$ for all n and hence $E_0(-t) > 0$ for $0 \leq t < \infty$.

.

Appendix A.7. $E_0(t)$ ***is real and even***

We see that $E(\frac{1}{2} + i\omega) = E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ (**Result 13**) because $E(s) = E(1-s)$ in Eq. 5 and hence $E(\frac{1}{2} + i\omega) = E(\frac{1}{2} - i\omega)$ when evaluated at $s = \frac{1}{2} + i\omega$.

We take the Inverse Fourier transform of $E_{0\omega}(\omega)$ and use $E_{0\omega}(\omega) = E_{0\omega}(-\omega)$ from Result 13 and then substitute $\omega = -\omega^{'}$ in the integrand, as follows.

$$
\begin{aligned}
E_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega)e^{i\omega t}d\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(-\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{0\omega}(\omega^{'})e^{-i\omega^{'}t}d\omega^{'} = E_0(-t)
\end{aligned}
\tag{A.6}
$$

We see that $E_0(t)$ in Eq. 15 is real and $E_0(t)$ in Eq. A.6 is even and hence we have derived the result that $E_0(t)$ is a **real and even** function of variable t.

.

Appendix A.8. ***Exponential fall-off rate of Dirichlet Eta function***

The integrand in Eq. 13 given by $\sum_{n=1}^{\infty}(-1)^{n-1}e^{-\pi n^2 e^{-2t}}e^{-\frac{t}{2}}e^{-\sigma t}$ goes to zero with **exponential** fall-off rate, as $t \to -\infty$ because the term $e^{-\pi n^2 e^{-2t}}$ has a faster fall-off rate than the term $e^{-\frac{t}{2}}e^{-\sigma t}$.

The integrand in Eq. 13 given by $\sum_{n=1}^{\infty}(-1)^{n-1}e^{-\pi n^2 e^{-2t}}e^{-\frac{t}{2}}e^{-\sigma t}$ goes to zero with **exponential** fall-off rate, as $t \to +\infty$ because the term $\lim_{t\to\infty} e^{-\pi n^2 e^{-2t}} = 1 - 1 + 1 - 1... = \frac{1}{2}$ (Eq.1.2.7 in page 2) for each n and hence $\lim_{t\to\infty} \sum_{n=1}^{\infty}(-1)^{n-1}e^{-\pi n^2 e^{-2t}} = \frac{1}{2}$ and the term $\lim_{t\to\infty} e^{-\frac{t}{2}}e^{-\sigma t} = 0$ for $0 < \sigma < \frac{1}{2}$.

The above results also hold for **each** $n = 1, 2, ...$

.

Appendix A.9. ***Functional equation for Dirichlet Eta function***

We use the **functional equation** for Riemann's zeta function given by $\zeta(s) = \zeta(1-s)\Gamma(1-s)\sin(\frac{s\pi}{2})\pi^{(s-1)}2^s$ and use $\zeta(s) = \dfrac{\eta(s)}{1-2^{1-s}}$ and $s = \frac{1}{2} + \sigma + i\omega$ and $1 - s = \frac{1}{2} - \sigma - i\omega$.

$$\begin{aligned} \zeta(s) &= \zeta(1-s)\Gamma(1-s)\sin(\frac{s\pi}{2})\pi^{(s-1)}2^s \\ \frac{\eta(s)}{1-2^{1-s}} &= \frac{\eta(1-s)}{1-2^s}\Gamma(1-s)\sin(\frac{s\pi}{2})\pi^{(s-1)}2^s \end{aligned} \tag{A.7}$$

We use well known properties of Gamma function $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(s\pi)} = \frac{\pi}{2\sin(\frac{s\pi}{2})\cos(\frac{s\pi}{2})}$ in Eq. A.7 as follows. (link)

$$\frac{\eta(s)}{1-2^{1-s}} = \frac{\eta(1-s)}{1-2^s}\frac{\pi}{2\sin(\frac{s\pi}{2})\cos(\frac{s\pi}{2})\Gamma(s)}\sin(\frac{s\pi}{2})\pi^{(s-1)}2^s \tag{A.8}$$

We cancel the common term $\sin(\frac{s\pi}{2})$ in Eq. A.8 for $0 < Re[s] < 1$ and rearrange the terms as follows.

$$\eta(1-s) = \eta(s)\Gamma(s)\cos(\frac{s\pi}{2})\frac{(1-2^s)}{(1-2^{1-s})\pi^s 2^{s-1}} \tag{A.9}$$

In the modified functional equation in Eq. A.9 , we see that, **if** Dirichlet Eta function $\eta(s)$ has a zero in the region $0 < Re[s] < 1$ at $s = s_0$, **then** $\eta(s)$ also has a zero at $s = 1 - s_0$, due to the term $\eta(1-s)$, given that for $Re[s] > 0$, the gamma function is analytic in the complex plane (link).

.

Appendix B. Properties of Fourier Transforms Part 1

In this section, some well-known properties of Fourier transforms are re-derived.

.

Appendix B.1. ***Fourier transform of Real g(t)***

In this section, we show that the Fourier transform of a **real** function $g(t)$, given by $G(\omega) = G_R(\omega) + iG_I(\omega)$ has the properties given by $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$. We use the fact that $g(t)$ is real and $\cos(\omega t)$ is an **even** function of ω and $\sin(\omega t)$ is an **odd** function of ω below.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ G_R(\omega) &= \int_{-\infty}^{\infty} g(t)\cos(\omega t)dt = G_R(-\omega) \\ G_I(\omega) &= -\int_{-\infty}^{\infty} g(t)\sin(\omega t)dt = -G_I(-\omega) \end{aligned} \tag{B.1}$$

.

Appendix B.2. ***Even part of g(t) corresponds to real part of Fourier transform*** $G(\omega)$

In this section, we take the **even part** of real function $g(t)$, given by $g_{even}(t) = \frac{1}{2}[g(t) + g(-t)]$ and show that its Fourier transform is given by the **real part** of $G(\omega)$.

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega) \\ \int_{-\infty}^{\infty} g_{even}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t) + g(-t)]e^{-i\omega t}dt &= \frac{G(\omega)}{2} + \frac{1}{2}\int_{-\infty}^{\infty} g(-t)e^{-i\omega t}dt \end{aligned} \tag{B.2}$$

We substitute $t = -t$ in the second integral in Eq. B.2. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$. (Appendix B.1)

$$\begin{aligned} \int_{-\infty}^{\infty} g_{even}(t)e^{-i\omega t}dt &= \frac{G(\omega)}{2} + \frac{1}{2}\int_{-\infty}^{\infty} g(t)e^{i\omega t}dt = \frac{G(\omega)}{2} + \frac{G(-\omega)}{2} \\ = \frac{1}{2}[G_R(\omega) + iG_I(\omega) + G_R(-\omega) + iG_I(-\omega)] &= \frac{1}{2}[G_R(\omega) + iG_I(\omega) + G_R(\omega) - iG_I(\omega)] = G_R(\omega) \end{aligned} \tag{B.3}$$

.

Appendix B.3. ***Odd part of g(t) corresponds to imaginary part of Fourier transform*** $G(\omega)$

In this section, we take the **odd part** of real function $g(t)$, given by $g_{odd}(t) = \frac{1}{2}[g(t) - g(-t)]$ and show that its Fourier transform is given by the **imaginary part** of $G(\omega)$.

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = G_R(\omega) + iG_I(\omega)$$
$$\int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2}[g(t) - g(-t)]e^{-i\omega t}dt = \frac{G(\omega)}{2} - \frac{1}{2}\int_{-\infty}^{\infty} g(-t)e^{-i\omega t}dt \tag{B.4}$$

We substitute $t = -t$ in the second integral in Eq. B.4. We use the fact that $G_R(-\omega) = G_R(\omega)$ and $G_I(-\omega) = -G_I(\omega)$ for a real function $g(t)$. (Appendix B.1)

$$\int_{-\infty}^{\infty} g_{odd}(t)e^{-i\omega t}dt = \frac{G(\omega)}{2} - \frac{1}{2}\int_{-\infty}^{\infty} g(t)e^{i\omega t}dt = \frac{G(\omega)}{2} - \frac{G(-\omega)}{2}$$
$$= \frac{1}{2}[G_R(\omega) + iG_I(\omega) - G_R(-\omega) - iG_I(-\omega)] = \frac{1}{2}[G_R(\omega) + iG_I(\omega) - G_R(\omega) + iG_I(\omega)] = iG_I(\omega) \tag{B.5}$$

.

Appendix B.4. ***Fourier transform of a real and even function*** $g(t)$

In this section, we show that the Fourier transform of a **real and even** function $g(t)$, given by $G(\omega)$ is also **real and even**. We use the fact that $\int_{-\infty}^{\infty} g(t)\sin\omega t dt = 0$ because $g(t)$ is even and the integrand is an **odd function** of variable t.

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} g(t)\cos\omega t dt - i\int_{-\infty}^{\infty} g(t)\sin\omega t dt$$
$$G(\omega) = \int_{-\infty}^{\infty} g(t)\cos\omega t dt \tag{B.6}$$

We see that $G(\omega) = \int_{-\infty}^{\infty} g(t)\cos\omega t dt$ is **real** function of ω, given that $g(t)$ and the integrand are real functions. We see that $G(\omega)$ is an **even** function of ω because $\cos\omega t$ is a **even** function of ω.

.

Appendix C. Details for Section 6

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Appendix C.1. $G_R(\omega, t_2, t_0)$ ***and*** $G_{R,2r}(\omega, t_2, t_0)$ ***are partially differentiable twice as a function of*** ω

$G_R(\omega, t_2, t_0)$ in Eq. 27 is copied below.

$$
\begin{aligned}
G_R(\omega,t_2,t_0) = e^{-2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos{(\omega\tau)}d\tau \\
+e^{2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\cos{(\omega\tau)}d\tau
\end{aligned}
\tag{C.1}
$$

We could then use $E_0^{'}(\tau,t_2) = (E_0(\tau-t_2) - E_0(\tau+t_2)$ (using Definition 1 in Section 2.1) and $E_{0n}^{'}(\tau,t_2) = E_0^{'}(-\tau,t_2) = -E_0^{'}(\tau,t_2)$ (using Definition 2 in Section 2.3 and Result 3.1 in Section 3). We see that $E_0(\tau)$ and its t_0 and t_2 shifted versions are analytic functions of τ, t_0 and t_2, given that the sum and product of exponential functions are analytic and hence infinitely differentiable.(**Result E.0**)

In Eq. C.1, $G_R(\omega,t_2,t_0)$ is partially differentiable at least twice with respect to ω and the integrals converge in Eq. C.1 and Eq. C.2 for $0 < \sigma < \frac{1}{2}$, because the terms $\tau^r E_0^{'}(\tau \pm t_0,t_2)e^{-2\sigma\tau}$ and $\tau^r E_{0n}^{'}(\tau \pm t_0,t_2) = -\tau^r E_0^{'}(\tau \pm t_0,t_2)$ have **exponential** asymptotic fall-off rate as $|\tau| \to \infty$, for $r \in W$ (Details in Appendix C.1.1). The integrands in Eq. C.1 and Eq. C.2 are analytic functions of variables ω and t_0, for a given t_2(using Result E.0 in Appendix C.1 and given that the terms $\cos{(\omega\tau)}, \sin{(\omega\tau)}$ and $e^{-2\sigma\tau}$ are analytic functions). The integrands have **exponential** asymptotic fall-off rate (Details in Appendix C.1.1) and absolutely integrable and we can find a suitable dominating function with exponential asymptotic fall-off rate which is absolutely integrable.(Details in Appendix C.2) Hence we can interchange the order of partial differentiation and integration in Eq. C.2 using theorem of differentiability of functions defined by Lebesgue integrals and theorem of dominated convergence, recursively as follows.(theorem)

$$
\begin{aligned}
\frac{\partial G_R(\omega,t_2,t_0)}{\partial\omega} &= -[e^{-2\sigma t_0}\int_{-\infty}^{0}\tau[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\sin{(\omega\tau)}d\tau \\
&+e^{2\sigma t_0}\int_{-\infty}^{0}\tau[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\sin{(\omega\tau)}d\tau] \\
\frac{\partial^2 G_R(\omega,t_2,t_0)}{\partial\omega^2} &= -[e^{-2\sigma t_0}\int_{-\infty}^{0}\tau^2[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos{(\omega\tau)}d\tau \\
&+e^{2\sigma t_0}\int_{-\infty}^{0}\tau^2[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\cos{(\omega\tau)}d\tau]
\end{aligned}
\tag{C.2}
$$

We can use the arguments in the above paras and derive the $(2r)^{th}$ derivative of $G_R(\omega,t_2,t_0)$, for $r \in W$, which is differentiable at least twice, as follows.

$$
\begin{aligned}
G_{R,2r}(\omega,t_2,t_0) = \frac{\partial^{2r} G_R(\omega,t_2,t_0)}{\partial\omega^{2r}} = (-1)^r[e^{-2\sigma t_0}\int_{-\infty}^{0}\tau^{2r}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos{(\omega\tau)}d\tau \\
+e^{2\sigma t_0}\int_{-\infty}^{0}\tau^{2r}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\cos{(\omega\tau)}d\tau]
\end{aligned}
\tag{C.3}
$$

We can prove Eq. C.3 using induction. We use Eq. C.3 as Induction Hypothesis. We take the second derivative of Eq. C.3 and we interchange the order of differentiation and integration, using the arguments used to derive Eq. C.2 as follows.

$$\begin{aligned}
\frac{\partial^{2r+1} G_R(\omega, t_2, t_0)}{\partial \omega^{2r+1}} &= (-1)^{r+1}[e^{-2\sigma t_0} \int_{-\infty}^{0} \tau^{2r+1}[E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)] \sin(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^{0} \tau^{2r+1}[E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)] \sin(\omega\tau) d\tau] \\
\frac{\partial^{2r+2} G_R(\omega, t_2, t_0)}{\partial \omega^{2r+2}} &= (-1)^{r+1}[e^{-2\sigma t_0} \int_{-\infty}^{0} \tau^{2r+2}[E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)] \cos(\omega\tau) d\tau \\
&\quad + e^{2\sigma t_0} \int_{-\infty}^{0} \tau^{2r+2}[E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)] \cos(\omega\tau) d\tau]
\end{aligned} \tag{C.4}$$

We see that the second equation in Eq. C.4 is the same as the equation obtained by setting $r = r + 1$ in Eq. C.3. Thus we have proved Eq. C.3 using mathematical induction.

.

Appendix C.1.1. ***Exponential Fall off rate of*** $B(t) = t^r E_0^{'}(t \pm t_0, t_2)e^{-2\sigma t}$ ***for*** $r \in W$

In this section, it is shown that the term $B(t) = t^r E_0^{'}(t \pm t_0, t_2)e^{-2\sigma t}$ has exponential asymptotic fall-off rate as $|t| \rightarrow \infty$, for $r \in W$ where $E_0^{'}(t, t_2) = E_0(t - t_2) - E_0(t + t_2)$. Hence $B(t) = t^r e^{-2\sigma t}[E_0(t - t_2 \pm t_0) - E_0(t + t_2 \pm t_0)]$ (**Result E.1.1**).

We consider $C(t) = t^r e^{-2\sigma t} E_0(t - t_a)$ for real t_a. We see that $C(t + t_a) = (t + t_a)^r e^{-2\sigma t} e^{-2\sigma t_a} E_0(t)$. We see that $E_0(t)e^{-2\sigma t}$ is an absolutely integrable function, for $0 \leq |\sigma| < \frac{1}{2}$ given that it has exponential fall-off rates as $|t| \rightarrow \infty$. (Details in Appendix A.4 and Appendix A.5).

Hence $C(t + t_a) = (t + t_a)^r e^{-2\sigma t_a} E_0(t)e^{-2\sigma t}$ also has exponential fall-off rates as $|t| \rightarrow \infty$, for $r \in W$ and finite t_a and is an absolutely integrable function.

Hence $C(t) = t^r e^{-2\sigma t} E_0(t - t_a)$ has exponential fall-off rates as $|t| \rightarrow \infty$, for finite t_a and is an absolutely integrable function. We set $t_a = t_2 \pm t_0$ and $t_a = -t_2 \pm t_0$ and see that $B(t)$ in Result E.1.1, has **exponential fall-off rates** as $|t| \rightarrow \infty$, for finite t_2, t_0 and is an absolutely integrable function.

.

Appendix C.2. ***Dominating function***

We consider $x(t) = E_0(t)e^{-2\sigma t}$ which has asymptotic exponential fall-off rate of $o[e^{-0.5|t|}]$.(Details in Appendix A.4) We see that $x(t + t_a)$ also has the same asymptotic exponential fall-off rate, for finite shift of $t_a = t_2 + t_0$ and $y(t, t_a) = t^r x(t + t_a)e^{2\sigma t_a}$ also has the same asymptotic exponential fall-off rate, for $r \in W$. We consider the intervals $0 < t_0 \leq t_{0_{max}}$, $0 < t_2 \leq t_{2_{max}}$ and $0 < t_a \leq t_{a_{max}}$ where $t_{0_{max}}, t_{2_{max}}, t_{a_{max}}$ are finite.

We consider $t_d >> t_{a_{max}}$ where $y(t, t_a) = t^r x(t + t_a)e^{2\sigma t_a}$ falls off at the rate of $o[e^{0.5t}]$ for $t << -t_d$. We consider $f(t, t_a, \omega) = y(t, t_a)\cos(\omega t)$ and we get $\frac{\partial f(t, t_a, \omega)}{\partial \omega} = -t y(t, t_a) \sin(\omega t)$ which falls off at the rate of $o[e^{0.5t}]$ for $t << -t_d$. Let $f_{max} > 0$ be the maximum value of $|\frac{\partial f(t,t_a,\omega)}{\partial \omega}|$ in the interval $-\infty < t < \infty$.

We can find a suitable **dominating function** $D(t) = e^{-K|t|} f_{max} e^{Kt_d} > 0$ with a fall off rate of $O[e^{-K|t|}]$ where $0 < K < 0.5$ and hence $D(t)$ has a slower fall off rate than $\frac{\partial f(t,t_a,\omega)}{\partial \omega}$ and $D(t) = f_{max}$ at $t = -t_d$ and hence $D(t) > |\frac{\partial f(t,t_a,\omega)}{\partial \omega}|$ for $-\infty < t \leq 0$ and hence $|\frac{\partial f(t,t_a,\omega)}{\partial \omega}| \leq D(t)$ in the interval $(-\infty, 0]$ and $\int_{-\infty}^{0} |D(t)| dt = \int_{-\infty}^{0} e^{Kt} f_{max} e^{Kt_d} dt = \frac{1}{K} f_{max} e^{Kt_d} [e^{Kt}]_{-\infty}^{0} = \frac{1}{K} f_{max} e^{Kt_d}$ is finite.(**Result E.2.1**)

The first term in Eq. C.2 given by $B(t) = t^r E_0^{'}(t+t_0, t_2) e^{-2\sigma t} = t^r e^{-2\sigma t}[E_0(t-t_2+t_0) - E_0(t+t_2+t_0)]$ using Result E.1.1 in Appendix C.1.1. We set $t_a = t_2 + t_0$ and $t_b = t_2 - t_0$ and get $B(t) = t^r e^{-2\sigma t}[E_0(t-t_b) - E_0(t+t_a)]$. Hence $y(t,t_a) = t^r x(t+t_a) e^{2\sigma t_a} = t^r E_0(t+t_a) e^{-2\sigma t}$ in the second para, corresponds to the second term in $B(t)$ and Result E.2.1 holds for this term. The first term in $B(t)$ is obtained by replacing t_a by $-t_b$ and Result E.2.1 holds for this term and hence for $B(t)$. We see that Result E.2.1 holds for the other 3 terms in Eq. C.2 using arguments in above paragraphs and replacing t_0 by $-t_0$ and setting $\sigma = 0$ as needed.

As $t_{0_{max}}, t_{2_{max}}, t_{a_{max}}$ increase to a larger and larger **finite value** without bounds, we consider larger intervals $0 < t_0 \leq t_{0_{max}}$, $0 < t_2 \leq t_{2_{max}}$ and $0 < t_a \leq t_{a_{max}}$ and f_{max} and t_d also increase correspondingly and the results in above paragraphs are valid in these intervals.

Similarly, we consider $f(t,t_a,\omega) = y(t,t_a)\cos(\omega t) = t^r E_0(t+t_a) e^{-2\sigma t} \cos(\omega t) = t^r E_0(t+t_0+t_2) e^{-2\sigma t} \cos(\omega t)$ and we see that $\frac{\partial f(t,t_a,\omega)}{\partial t_0}$ and $\frac{\partial f(t,t_a,\omega)}{\partial t_2}$ which fall off at the rate of $o[e^{0.5t}]$ for $t << -t_d$, using Eq. C.8 and $E_0(t) = E_0(-t)$ and due to the term $e^{-\pi n^2 e^{-2t}}$ and we can use arguments in above paragraphs to get a result similar to Result E.2.1 for the terms in Eq. C.5 and Eq. C.15. We can use these arguments to get a result similar to Result E.2.1 for the second derivative terms $\frac{\partial^2 f(t,t_a,\omega)}{\partial t_0^2}$ and $\frac{\partial^2 f(t,t_a,\omega)}{\partial t_2^2}$ in Eq. C.10 and Eq. C.19.
.

Appendix C.3. $G_{R,2r}(\omega, t_2, t_0)$ ***are partially differentiable twice as a function of*** t_0***,*** $r \in W$

In Eq. C.3, $G_{R,2r}(\omega, t_2, t_0)$ is partially differentiable at least twice as a function of t_0 and the integrals converge in Eq. C.5 and Eq. C.10 shown as follows. The integrands in the equation for $G_{R,2r}(\omega, t_2, t_0)$ in Eq. C.5 are absolutely integrable because the terms $\tau^{2r} E_0^{'}(\tau \pm t_0, t_2) e^{-2\sigma\tau}$ and $\tau^{2r} E_{0n}^{'}(\tau \pm t_0, t_2) = -\tau^{2r} E_0^{'}(\tau \pm t_0, t_2)$ have **exponential** asymptotic fall-off rate as $|\tau| \to \infty$, for $r \in W$ (Details in Appendix C.1.1). The integrands in Eq. C.5 are absolutely integrable and are analytic functions of variables ω and t_0, for a given t_2 (using Result E.0 in Appendix C.1). The integrands have **exponential** asymptotic fall-off rate(Details in Appendix C.1.1) and we can find a suitable dominating function with exponential asymptotic fall-off rate which is absolutely integrable.(Details in Appendix C.2) Hence we can interchange the order of partial differentiation and integration in Eq. C.5 using theorem of differentiability of functions defined by Lebesgue integrals and theorem of dominated convergence as follows. (theorem)

$$
\begin{aligned}
G_{R,2r}(\omega,t_2,t_0) &= e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos{(\omega\tau)}d\tau \\
&\quad + e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\cos{(\omega\tau)}d\tau \\
\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial t_0} &= -2\sigma e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos{(\omega\tau)}d\tau \\
&\quad + e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}\frac{\partial(E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2))}{\partial t_0}\cos{(\omega\tau)}d\tau \\
&\quad + 2\sigma e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\cos{(\omega\tau)}d\tau \\
&\quad + e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}\frac{\partial(E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2))}{\partial t_0}\cos{(\omega\tau)}d\tau
\end{aligned}
\tag{C.5}
$$

We show that the integrals in Eq. C.5 converge, as follows. We see that $E_0^{'}(\tau+t_0,t_2) = E_0(\tau+t_0-t_2) - E_0(\tau+t_0+t_2)$ and $E_{0n}^{'}(\tau-t_0,t_2) = -E_0^{'}(\tau-t_0,t_2) = E_0(\tau-t_0+t_2) - E_0(\tau-t_0-t_2)$ (using Definition 1 in Section 2.1 and Result 3.1 in Section 3).We see that the first and third integrals in the equation for $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial t_0}$ in Eq. C.5 converge because the terms $\tau^{2r}E_0^{'}(\tau \pm t_0,t_2)e^{-2\sigma\tau}$ and $\tau^{2r}E_{0n}^{'}(\tau \pm t_0,t_2) = -\tau^{2r}E_0^{'}(\tau \pm t_0,t_2)$ have exponential asymptotic fall-off rate as $|\tau| \to \infty$(Details in Appendix C.1.1).

We consider the integrand in the second integral in the equation for $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial t_0}$ in Eq. C.5 first and use the results in the above paragraph.

$$
\begin{aligned}
\frac{\partial(E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2))}{\partial t_0} &= \frac{\partial(E_0(\tau+t_0-t_2)e^{-2\sigma\tau} - E_0(\tau+t_0+t_2)e^{-2\sigma\tau})}{\partial t_0} \\
&\quad + \frac{\partial(E_0(\tau-t_0+t_2) - E_0(\tau-t_0-t_2))}{\partial t_0}
\end{aligned}
\tag{C.6}
$$

We consider the term $E_0(\tau+t_0+t_2)$ first in Eq. C.6 and can show that the integrals converge in Eq. C.5, as follows. We take the factor of 2 out of the summation in $E_0(\tau)$ in Eq. **??** copied below.

$$
\begin{aligned}
E_0(\tau) &= 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4\tau} - 3\pi n^2 e^{2\tau}]e^{-\pi n^2 e^{2\tau}} e^{\frac{\tau}{2}} \\
E_0(\tau+t_2+t_0) &= 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)}]e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}}
\end{aligned}
\tag{C.7}
$$

We can show that $\frac{\partial}{\partial t_0}E_0(\tau+t_2+t_0) = \frac{\partial}{\partial \tau}E_0(\tau+t_2+t_0)$ as follows, given that the equation for $E_0(\tau+t_2+t_0)$ in Eq. C.7 has terms of the form $e^{\tau+t_0}$ and the equation is **invariant** if we interchange the variables τ and t_0. (**Result E.3.A**)

$$\begin{aligned}\frac{\partial}{\partial t_0}E_0(\tau+t_2+t_0) &= 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau}e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)} \\ &\quad +(\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)})(2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})] \\ \frac{\partial}{\partial \tau}E_0(\tau+t_2+t_0) &= 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2\tau}e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} [8\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 6\pi n^2 e^{2\tau} e^{2(t_2+t_0)} \\ &\quad +(\frac{1}{2} - 2\pi n^2 e^{2\tau} e^{2(t_2+t_0)})(2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)})]\end{aligned} \tag{C.8}$$

We can replace t_0 by $t_0^{'} = -t_0$ in Eq. C.7 and see that $\frac{\partial}{\partial t_0^{'}}E_0(\tau+t_2+t_0^{'}) = \frac{\partial}{\partial \tau}E_0(\tau+t_2+t_0^{'})$ (**Result E.3.E**) given that the equation is invariant if we interchange τ and $t_0^{'}$. Given that $\frac{\partial}{\partial t_0^{'}} = \frac{\partial}{\partial t_0}\frac{dt_0}{dt_0^{'}} = -\frac{\partial}{\partial t_0}$, we substitute it in Result E.3.E and get $\frac{\partial}{\partial t_0}E_0(\tau+t_2-t_0) = -\frac{\partial}{\partial \tau}E_0(\tau+t_2-t_0)$.(**Result E.3.B**)

We can write the term in the second integral in the equation for $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial t_0}$ in Eq. C.5, corresponding to the term $E_0(\tau+t_0+t_2)e^{-2\sigma\tau}$ in Eq. C.6, using Result E.3.A, as follows. We use the fact that $\int_{-\infty}^{0}\frac{dA(\tau)}{d\tau}B(\tau)d\tau = \int_{-\infty}^{0}\frac{d(A(\tau)B(\tau))}{d\tau}d\tau - \int_{-\infty}^{0}A(\tau)\frac{dB(\tau)}{d\tau}d\tau$.

$$\begin{aligned}\int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0))}{\partial t_0}\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)d\tau &= \int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0)}{\partial \tau}\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)d\tau \\ = \int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau))}{\partial \tau}d\tau &- \int_{-\infty}^{0}E_0(\tau+t_2+t_0)\frac{\partial(\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)}{\partial\tau}d\tau \\ = [E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)]_{-\infty}^{0} &+ \omega\int_{-\infty}^{0}E_0(\tau+t_2+t_0))\tau^{2r}e^{-2\sigma\tau}\sin(\omega\tau)d\tau \\ +2\sigma\int_{-\infty}^{0}E_0(\tau+t_2+t_0))\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)d\tau &- 2r\int_{-\infty}^{0}E_0(\tau+t_2+t_0))\tau^{2r-1}e^{-2\sigma\tau}\cos(\omega\tau)d\tau\end{aligned} \tag{C.9}$$

We see that the integrals in Eq. C.9 converge because the integrands are absolutely integrable because the terms $E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\sin(\omega\tau)$, $E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)$ and $E_0(\tau+t_2+t_0))\tau^{2r-1}e^{-2\sigma\tau}\cos(\omega\tau)$ have exponential asymptotic fall-off rate as $|\tau|\to\infty$(Details in Appendix C.1.1). The term $[E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)]_{-\infty}^{0}$ is finite, given that $\tau^{2r}E_0(\tau)e^{-2\sigma\tau}$ and its shifted versions go to zero as $t\to-\infty$(Details in Appendix A.4). Hence the integral $\int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau})}{\partial t_0}\cos(\omega\tau)d\tau$ in Eq. C.9 and in Eq. C.5 corresponding to the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ in Eq. C.6, converges.

We set $\sigma=0$ and $t_0=-t_0$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ and see that the integral $\int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2-t_0))}{\partial t_0}\tau^{2r}\cos(\omega\tau)d\tau$ in Eq. C.5 corresponding to the term $E_0(\tau+t_2-t_0)$ in Eq. C.6 also converges, using Result E.3.B and the procedure used in Eq. C.7 to Eq. C.9.

We set $t_2=-t_2$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ in Eq. C.7 to Eq. C.9 and see that the integral $\int_{-\infty}^{0}\frac{\partial(E_0(\tau-t_2+t_0)e^{-2\sigma\tau})}{\partial t_0}\tau^{2r}\cos(\omega\tau)d\tau$ in Eq. C.5 corresponding to the term $E_0(\tau-t_2+t_0)e^{-2\sigma\tau}$ in

Eq. C.6 also converges.

We set $t_2 = -t_2$, $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ and see that the integral $\int_{-\infty}^{0} \frac{\partial(E_0(\tau-t_2-t_0))}{\partial t_0}\tau^{2r}\cos(\omega\tau)d\tau$ in Eq. C.5 corresponding to the term $E_0(\tau - t_2 - t_0)$ in Eq. C.6 also converges, using Result E.3.B and the procedure used in Eq. C.7 to Eq. C.9. Hence the second integral in the equation for $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial t_0}$ in Eq. C.5, also converges.

We can see that the last integral in Eq. C.5 converges, by setting $t_0 = -t_0$ in Eq. C.6 and using Result E.3.B and using the procedure in Eq. C.7 to Eq. C.9. Hence all the integrals in Eq. C.5 converge.

.

Appendix C.3.1. ***Second Partial Derivative of*** $G_{R,2r}(\omega, t_2, t_0)$ ***with respect to*** t_0

The second partial derivative of $G_{R,2r}(\omega, t_2, t_0)$ with respect to t_0 is given by $\frac{\partial^2 G_{R,2r}(\omega,t_2,t_0)}{\partial t_0^2} = \frac{\partial}{\partial t_0}\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial t_0}$ as follows. We use the result in Eq. C.5 and the fact that the integrands are absolutely integrable using the results in Appendix C.3 and are analytic functions of variables ω and t_0 for a given t_2 (using Result E.0 in Appendix C.1). The integrands have **exponential** asymptotic fall-off rate (Details in Appendix C.1.1) and we can find a suitable dominating function with exponential asymptotic fall-off rate which is absolutely integrable.(Details in Appendix C.2) Hence we can interchange the order of partial differentiation and integration in Eq. C.10 using theorem of differentiability of functions defined by Lebesgue integrals and theorem of dominated convergence as follows. (theorem)

$$\begin{aligned}
\frac{\partial^2 G_{R,2r}(\omega,t_2,t_0)}{\partial t_0^2} &= 4\sigma^2 e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega\tau)d\tau \\
&-4\sigma e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}\frac{\partial(E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2))}{\partial t_0}\cos(\omega\tau)d\tau \\
&+e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}\frac{\partial^2(E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2))}{\partial t_0^2}\cos(\omega\tau)d\tau \\
&+4\sigma^2 e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\cos(\omega\tau)d\tau \\
&+4\sigma e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}\frac{\partial(E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2))}{\partial t_0}\cos(\omega\tau)d\tau \\
&+e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}\frac{\partial^2(E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2))}{\partial t_0^2}\cos(\omega\tau)d\tau
\end{aligned} \tag{C.10}$$

The first two integrals and fourth and fifth integrals in Eq. C.10 are the same as the integrals in the equation for $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial t_0}$ in Eq. C.5 and have been shown to converge in Appendix C.3. We will show that the third and sixth integrals in Eq. C.10 converge, as follows.

We consider the integrand in the third integral in Eq. C.10 first. We see that $E_0^{'}(\tau + t_0, t_2) = E_0(\tau+t_0-t_2) - E_0(\tau+t_0+t_2)$ and $E_{0n}^{'}(\tau-t_0,t_2) = -E_0^{'}(\tau-t_0,t_2) = E_0(\tau-t_0+t_2) - E_0(\tau-t_0-t_2)$ (using Definition 1 in Section 2.1 and Result 3.1 in Section 3). We write an equation similar to Eq. C.6.

$$\begin{aligned}\frac{\partial^2(E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2))}{\partial t_0^2} &= \frac{\partial^2(E_0(\tau+t_0-t_2)e^{-2\sigma\tau}-E_0(\tau+t_0+t_2)e^{-2\sigma\tau})}{\partial t_0^2}\\ &\quad+\frac{\partial^2(E_0(\tau-t_0+t_2)-E_0(\tau-t_0-t_2))}{\partial t_0^2}\end{aligned} \tag{C.11}$$

We consider the term $E_0(\tau+t_0+t_2)$ first in Eq. C.11 and copy Eq. C.7 below.

$$\begin{aligned} E_0(\tau) &= 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4\tau}-3\pi n^2e^{2\tau}]e^{-\pi n^2e^{2\tau}}e^{\frac{\tau}{2}}\\ E_0(\tau+t_2+t_0) &= 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4\tau}e^{4(t_2+t_0)}-3\pi n^2e^{2\tau}e^{2(t_2+t_0)}]e^{-\pi n^2e^{2\tau}e^{2(t_2+t_0)}}e^{\frac{\tau}{2}}e^{\frac{(t_2+t_0)}{2}}\end{aligned} \tag{C.12}$$

We can see that $\frac{\partial^2}{\partial t_0^2}E_0(\tau+t_2+t_0)=\frac{\partial^2}{\partial\tau^2}E_0(\tau+t_2+t_0)$, given that the equation has terms of the form $e^{\tau+t_0}$ and the equation **is invariant** if we interchange the variables τ and t_0.(**Result E.3.1.A'**)

We can replace t_0 by $t_0^{'}=-t_0$ in Eq. C.12 and see that $\frac{\partial^2}{\partial(t_0^{'})^2}E_0(\tau+t_2+t_0^{'})=\frac{\partial^2}{\partial\tau^2}E_0(\tau+t_2+t_0^{'})$ (**Result E.3.1.E'**) given that the equation has terms of the form $e^{\tau+t_0^{'}}$ and the equation **is invariant** if we interchange the variables τ and $t_0^{'}$.

Given that $\frac{\partial}{\partial t_0}=\frac{\partial}{\partial t_0^{'}}\frac{\partial t_0^{'}}{\partial t_0}=-\frac{\partial}{\partial t_0^{'}}$, we get $\frac{\partial^2}{\partial t_0^2}=\frac{\partial}{\partial t_0}(\frac{\partial}{\partial t_0})=-\frac{\partial}{\partial t_0}(\frac{\partial}{\partial t_0^{'}})=\frac{\partial}{\partial t_0^{'}}(\frac{\partial}{\partial t_0^{'}})=\frac{\partial^2}{\partial(t_0^{'})^2}$, we substitute it in Result E.3.1.E' and get $\frac{\partial^2}{\partial t_0^2}E_0(\tau+t_2-t_0)=\frac{\partial^2}{\partial\tau^2}E_0(\tau+t_2-t_0)$.(**Result E.3.1.B'**)

We can write the term in the third integral in Eq. C.10, corresponding to the term $E_0(\tau+t_0+t_2)e^{-2\sigma\tau}$ in Eq. C.11, using Result $E.3.1.A'$, as follows. We use the fact that $\int_{-\infty}^{0}\frac{dA(\tau)}{d\tau}B(\tau)d\tau=\int_{-\infty}^{0}\frac{d(A(\tau)B(\tau))}{d\tau}d\tau-\int_{-\infty}^{0}A(\tau)\frac{dB(\tau)}{d\tau}d\tau$.

$$\begin{aligned}\int_{-\infty}^{0}\frac{\partial^2(E_0(\tau+t_2+t_0))}{\partial t_0^2}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau &= \int_{-\infty}^{0}\frac{\partial^2(E_0(\tau+t_2+t_0))}{\partial\tau^2}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau\\ =\int_{-\infty}^{0}\frac{\partial(\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)})}{\partial\tau}d\tau &- \int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\frac{\partial(\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)})}{\partial\tau}d\tau\\ =[\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}]_{-\infty}^{0} &+ \omega\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)}d\tau\\ +2\sigma\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau &- 2r\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r-1}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau\end{aligned} \tag{C.13}$$

We see that the integrals $\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau$ and $\int_{-\infty}^{0}\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r-1}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau$ in Eq. C.13 converge, using Eq. C.9 in the previous subsection. We see the term $[\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}]_{-\infty}^{0}$

also converges, given that $E_0(\tau) = E_0(-\tau)$ and $E_0(\tau + t_2 + t_0) = E_0(-\tau - t_2 - t_0)$ and we consider $\frac{\partial E_0(\tau+t_2+t_0)}{\partial \tau}\tau^{2r}e^{-2\sigma\tau} = \frac{\partial E_0(-\tau-t_2-t_0)}{\partial \tau}\tau^{2r}e^{-2\sigma\tau}$ using Eq. C.8 and see that the term $e^{-\pi n^2 e^{-2\tau}}$ goes to zero faster than the rising term $\tau^{2r}e^{-2\sigma\tau}e^{-6\tau}e^{-\frac{\tau}{2}}$, as $\tau \to -\infty$. (**Result E.3.1.1**)

It is shown below that the term $\int_{-\infty}^{0} \frac{\partial E_0(\tau+t_2+t_0)}{\partial \tau}\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)}d\tau$ in Eq. C.13 also converges.

$$
\begin{aligned}
&\int_{-\infty}^{0} \frac{\partial (E_0(\tau+t_2+t_0))}{\partial \tau}\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)}d\tau \\
&= \int_{-\infty}^{0} \frac{\partial (E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)})}{\partial \tau}d\tau - \int_{-\infty}^{0} E_0(\tau+t_2+t_0)\frac{\partial (\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)}}{\partial \tau}d\tau \\
&= [E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)}]_{-\infty}^{0} - \omega\int_{-\infty}^{0} E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau \\
&+2\sigma\int_{-\infty}^{0} E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)}d\tau - 2r\int_{-\infty}^{0} E_0(\tau+t_2+t_0)\tau^{2r-1}e^{-2\sigma\tau}\sin{(\omega\tau)}d\tau
\end{aligned}
\tag{C.14}
$$

We see that the integrals in Eq. C.14 converge because the integrands are absolutely integrable because the terms $E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)}$, $E_0(\tau+t_2+t_0)\tau^{2r-1}e^{-2\sigma\tau}\sin{(\omega\tau)}$ and $E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}$ have exponential asymptotic fall-off rate as $|\tau| \to \infty$(Details in Appendix C.1.1). The term $[E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)}]_{-\infty}^{0}$ is finite, given that $\tau^{2r}E_0(\tau)e^{-2\sigma\tau}$ and its shifted versions go to zero as $t \to -\infty$(Details in Appendix A.4). Hence the integral $\int_{-\infty}^{0} \frac{\partial^2 (E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau})}{\partial t_0^2}\cos{(\omega\tau)}d\tau$ in Eq. C.13 and in Eq. C.10 corresponding to the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ in Eq. C.11, also converges.

We set $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ and see that the integral $\int_{-\infty}^{0} \frac{\partial^2 (E_0(\tau+t_2-t_0))}{\partial t_0^2}\tau^{2r}\cos{(\omega\tau)}d\tau$ in Eq. C.10 corresponding to the term $E_0(\tau+t_2-t_0)$ in Eq. C.11 also converges, using Result $E.3.1.B'$ and the procedure used in Eq. C.12 to Eq. C.14.

We set $t_2 = -t_2$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ in Eq. C.12 to Eq. C.14 and see that the integral $\int_{-\infty}^{0} \frac{\partial^2 (E_0(\tau-t_2+t_0)\tau^{2r}e^{-2\sigma\tau})}{\partial t_0^2}\cos{(\omega\tau)}d\tau$ in Eq. C.10 corresponding to the term $E_0(\tau-t_2+t_0)e^{-2\sigma\tau}$ in Eq. C.11 also converges.

We set $t_2 = -t_2$, $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ and see that the integral $\int_{-\infty}^{0} \frac{\partial^2 (E_0(\tau-t_2-t_0))}{\partial t_0^2}\tau^{2r}\cos{(\omega\tau)}d\tau$ in Eq. C.10 corresponding to the term $E_0(\tau-t_2-t_0)$ in Eq. C.11 also converges, using Result $E.3.1.B'$ and the procedure used in Eq. C.12 to Eq. C.14. Hence the third integral in Eq. C.10, also converges.

We can see that the sixth integral in Eq. C.10 converges, by setting $t_0 = -t_0$ in Eq. C.11 to Eq. C.14 and using Result $E.3.1.B'$ and the procedure used in Eq. C.12 to Eq. C.14. Hence all the integrals in Eq. C.10 converge.

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Appendix C.4. $G_{R,2r}(\omega, t_2, t_0)$ ***is partially differentiable twice as a function of*** t_2 ***for*** $r \in W$

In Eq. C.3, $G_{R,2r}(\omega, t_2, t_0)$ is partially differentiable at least twice as a function of t_2 and the integrals converge in Eq. C.15 and Eq. C.19 shown as follows. The integrands in the equation for

$G_{R,2r}(\omega, t_2, t_0)$ in Eq. C.15 are absolutely integrable because the terms $\tau^{2r}E_0^{'}(\tau \pm t_0, t_2)e^{-2\sigma\tau}$ and $\tau^{2r}E_{0n}^{'}(\tau \pm t_0, t_2) = -\tau^{2r}E_0^{'}(\tau \pm t_0, t_2)$ have **exponential** asymptotic fall-off rate as $|\tau| \to \infty$(Details in Appendix C.1.1). The integrands are analytic functions of variables ω and t_2, for a given t_0 (using Result E.0 in Appendix C.1). The integrands have **exponential** asymptotic fall-off rate (Details in Appendix C.1.1) and we can find a suitable dominating function with exponential asymptotic fall-off rate which is absolutely integrable.(Details in Appendix C.2) Hence we can interchange the order of partial differentiation and integration in Eq. C.15 using theorem of differentiability of functions defined by Lebesgue integrals and theorem of dominated convergence as follows. (theorem)

$$\begin{aligned} G_{R,2r}(\omega, t_2, t_0) &= e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}[E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2)] \cos{(\omega\tau)} d\tau \\ &+ e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r}[E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2)] \cos{(\omega\tau)} d\tau \\ \frac{\partial G_{R,2r}(\omega, t_2, t_0)}{\partial t_2} &= e^{-2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r} \frac{\partial(E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2))}{\partial t_2} \cos{(\omega\tau)} d\tau \\ &+ e^{2\sigma t_0}(-1)^r \int_{-\infty}^{0} \tau^{2r} \frac{\partial(E_0^{'}(\tau - t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau + t_0, t_2))}{\partial t_2} \cos{(\omega\tau)} d\tau \end{aligned} \tag{C.15}$$

We use the procedure outlined in Eq. C.6 to Eq. C.9, with t_0 replaced by t_2 and show that all the integrals in Eq. C.15 converge, as follows.

We see that $E_0^{'}(\tau + t_0, t_2) = E_0(\tau + t_0 - t_2) - E_0(\tau + t_0 + t_2)$ and $E_{0n}^{'}(\tau - t_0, t_2) = -E_0^{'}(\tau - t_0, t_2) = E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2)$ (using Definition 1 in Section 2.1 and Result 3.1 in Section 3). We consider the integrand in the first integral in the equation for $\frac{\partial G_{R,2r}(\omega, t_2, t_0)}{\partial t_2}$ in Eq. C.15 first.

$$\begin{aligned} \frac{\partial(E_0^{'}(\tau + t_0, t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau - t_0, t_2))}{\partial t_2} &= \frac{\partial(E_0(\tau + t_0 - t_2)e^{-2\sigma\tau} - E_0(\tau + t_0 + t_2)e^{-2\sigma\tau})}{\partial t_2} \\ &+ \frac{\partial(E_0(\tau - t_0 + t_2) - E_0(\tau - t_0 - t_2))}{\partial t_2} \end{aligned} \tag{C.16}$$

We consider the term $E_0(\tau + t_0 + t_2)$ first and can show that the integrals converge in Eq. C.15, as follows. We copy Eq. C.7 below.

$$\begin{aligned} E_0(\tau) &= 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4\tau} - 3\pi n^2 e^{2\tau}]e^{-\pi n^2 e^{2\tau}} e^{\frac{\tau}{2}} \\ E_0(\tau + t_2 + t_0) &= 2\sum_{n=1}^{\infty}[2\pi^2 n^4 e^{4\tau} e^{4(t_2+t_0)} - 3\pi n^2 e^{2\tau} e^{2(t_2+t_0)}]e^{-\pi n^2 e^{2\tau} e^{2(t_2+t_0)}} e^{\frac{\tau}{2}} e^{\frac{(t_2+t_0)}{2}} \end{aligned} \tag{C.17}$$

We see that $\frac{\partial}{\partial t_2}E_0(\tau + t_2 + t_0) = \frac{\partial}{\partial \tau}E_0(\tau + t_2 + t_0)$ given that the equation has terms of the form $e^{\tau + t_2}$ and hence the equation is invariant if we interchange τ and t_2.(**Result E.4.C**)

We can replace t_2 by $t_2^{'} = -t_2$ in Eq. C.17 and see that $\frac{\partial}{\partial t_2^{'}}E_0(\tau+t_2^{'}+t_0) = \frac{\partial}{\partial \tau}E_0(\tau+t_2^{'}+t_0)$ given that the equation is invariant if we interchange τ and $t_2^{'}$(**Result E.4.F**). Given that $\frac{\partial}{\partial t_2^{'}} = \frac{\partial}{\partial t_2}\frac{dt_2}{dt_2^{'}} = -\frac{\partial}{\partial t_2}$, we use it in Result E.4.F and we get $\frac{\partial}{\partial t_2}E_0(\tau-t_2+t_0) = -\frac{\partial}{\partial \tau}E_0(\tau-t_2+t_0)$.(**Result E.4.D**)

We consider the term in the first integral in the equation for $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial t_2}$ in Eq. C.15 , corresponding to the term $E_0(\tau+t_0+t_2)e^{-2\sigma\tau}$ in Eq. C.16, as follows, using Result E.4.C. We use the fact that $\int_{-\infty}^{0}\frac{dA(\tau)}{d\tau}B(\tau)d\tau = \int_{-\infty}^{0}\frac{d(A(\tau)B(\tau))}{d\tau}d\tau - \int_{-\infty}^{0}A(\tau)\frac{dB(\tau)}{d\tau}d\tau$.

$$\int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0))}{\partial t_2}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau = \int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0))}{\partial \tau}\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau$$
$$= \int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)})}{\partial\tau}d\tau - \int_{-\infty}^{0}E_0(\tau+t_2+t_0)\frac{\partial(\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}}{\partial\tau}d\tau$$
$$= [E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}]_{-\infty}^{0} + \omega\int_{-\infty}^{0}E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)}d\tau$$
$$+2\sigma\int_{-\infty}^{0}E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau - 2r\int_{-\infty}^{0}E_0(\tau+t_2+t_0)\tau^{2r-1}e^{-2\sigma\tau}\cos{(\omega\tau)}d\tau \tag{C.18}$$

We see that the integrals in Eq. C.18 converge because the integrands are absolutely integrable because the terms $E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\sin{(\omega\tau)}$, $E_0(\tau+t_2+t_0)\tau^{2r-1}e^{-2\sigma\tau}\cos{(\omega\tau)}$ and $E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}$ have exponential asymptotic fall-off rate as $|\tau| \to \infty$(Details in Appendix C.1.1). The term $[E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau}\cos{(\omega\tau)}]_{-\infty}^{0}$ is finite, given that $\tau^{2r}E_0(\tau)e^{-2\sigma\tau}$ and its shifted versions go to zero as $t \to -\infty$(Details in Appendix A.4). Hence the integral $\int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2+t_0)e^{-2\sigma\tau})}{\partial t_2}\tau^{2r}\cos{(\omega\tau)}d\tau$ in Eq. C.18 and Eq. C.15 corresponding to the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ in Eq. C.16 also converges.

We set $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ and use the procedure in Eq. C.17 to Eq. C.18 and see that the integral $\int_{-\infty}^{0}\frac{\partial(E_0(\tau+t_2-t_0))}{\partial t_2}\tau^{2r}\cos{(\omega\tau)}d\tau$ in Eq. C.15 corresponding to the term $E_0(\tau+t_2-t_0)$ in Eq. C.16 also converges.

We set $t_2 = -t_2$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ and use the procedure in Eq. C.17 to Eq. C.18 and see that the integral $\int_{-\infty}^{0}\frac{\partial(E_0(\tau-t_2+t_0)e^{-2\sigma\tau})}{\partial t_2}\tau^{2r}\cos{(\omega\tau)}d\tau$ in Eq. C.15 corresponding to the term $E_0(\tau-t_2+t_0)e^{-2\sigma\tau}$ in Eq. C.16 also converges, using Result E.4.D.

We $t_2 = -t_2$, $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau+t_2+t_0)e^{-2\sigma\tau}$ and use the procedure in Eq. C.17 to Eq. C.18 and see that the integral $\int_{-\infty}^{0}\frac{\partial(E_0(\tau-t_2-t_0))}{\partial t_2}\tau^{2r}\cos{(\omega\tau)}d\tau$ in Eq. C.15 corresponding to the term $E_0(\tau-t_2-t_0)$ in Eq. C.16 also converges, using Result E.4.D. Hence the first integral in the equation for $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial t_2}$ in Eq. C.15 also converges.

We can see that the last integral in Eq. C.15 converges, by setting $t_0 = -t_0$ in Eq. C.18. Hence all the integrals in Eq. C.15 converge.

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Appendix C.4.1. ***Second Partial Derivative of*** $G_{R,2r}(\omega,t_2,t_0)$ ***with respect to*** t_2 ***for*** $r \in W$

The second partial derivative of $G_{R,2r}(\omega,t_2,t_0)$ with respect to t_2 is given by $\frac{\partial^2 G_{R,2r}(\omega,t_2,t_0)}{\partial t_2^2} =$

$\frac{\partial}{\partial t_2}\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial t_2}$ as follows. We use the result in Eq. C.15 and the fact that the integrands are absolutely integrable using the results in Appendix C.4 and the integrands are analytic functions of variables ω and t_2 for a given t_0 (using Result E.0 in Appendix C.1). The integrands have **exponential** asymptotic fall-off rate(Details in Appendix C.1.1) and we can find a suitable dominating function with exponential asymptotic fall-off rate which is absolutely integrable.(Details in Appendix C.2) Hence we can interchange the order of partial differentiation and integration in Eq. C.19 using theorem of differentiability of functions defined by Lebesgue integrals and theorem of dominated convergence as follows. (theorem)

$$\begin{aligned}\frac{\partial^2 G_{R,2r}(\omega,t_2,t_0)}{\partial t_2^2} &= e^{-2\sigma t_0}(-1)^r\int_{-\infty}^{0}\tau^{2r}\frac{\partial^2(E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2))}{\partial t_2^2}\cos(\omega\tau)d\tau\\ &+e^{2\sigma t_0}(-1)^r\int_{-\infty}^{0}\tau^{2r}\frac{\partial^2(E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau+t_0,t_2))}{\partial t_2^2}\cos(\omega\tau)d\tau\end{aligned} \tag{C.19}$$

We consider the first integral in Eq. C.19 and using $E_0^{'}(\tau+t_0,t_2)=E_0(\tau+t_0-t_2)-E_0(\tau+t_0+t_2)$ and $E_{0n}^{'}(\tau-t_0,t_2)=-E_0^{'}(\tau-t_0,t_2)=E_0(\tau-t_0+t_2)-E_0(\tau-t_0-t_2)$(using Definition 1 in Section 2.1 and Result 3.1 in Section 3), we write an equation similar to Eq. C.16.

$$\begin{aligned}\frac{\partial^2(E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2))}{\partial t_2^2} &= \frac{\partial^2(E_0(\tau+t_0-t_2)e^{-2\sigma\tau}-E_0(\tau+t_0+t_2)e^{-2\sigma\tau})}{\partial t_2^2}\\ &+\frac{\partial^2(E_0(\tau-t_0+t_2)-E_0(\tau-t_0-t_2))}{\partial t_2^2}\end{aligned} \tag{C.20}$$

We consider the term $E_0(\tau+t_0+t_2)$ first in Eq. C.20 as follows. We copy Eq. C.7 below.

$$\begin{aligned}E_0(\tau) &= 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4\tau}-3\pi n^2e^{2\tau}]e^{-\pi n^2e^{2\tau}}e^{\frac{\tau}{2}}\\ E_0(\tau+t_2+t_0) &= 2\sum_{n=1}^{\infty}[2\pi^2n^4e^{4\tau}e^{4(t_2+t_0)}-3\pi n^2e^{2\tau}e^{2(t_2+t_0)}]e^{-\pi n^2e^{2\tau}e^{2(t_2+t_0)}}e^{\frac{\tau}{2}}e^{\frac{(t_2+t_0)}{2}}\end{aligned} \tag{C.21}$$

We can see that $\frac{\partial^2}{\partial t_2^2}E_0(\tau+t_2+t_0)=\frac{\partial^2}{\partial\tau^2}E_0(\tau+t_2+t_0)$, given that the equation has terms of the form $e^{\tau+t_2}$ and the equation **is invariant** if we interchange the variables τ and t_2.(**Result E.4.1.C'**)

We can replace t_2 by $t_2^{'}=-t_2$ in Eq. C.21 and see that $\frac{\partial^2}{\partial(t_2^{'})^2}E_0(\tau+t_2^{'}+t_0)=\frac{\partial^2}{\partial\tau^2}E_0(\tau+t_2^{'}+t_0)$ (**Result E.4.1.F'**) given that the equation has terms of the form $e^{\tau+t_2^{'}}$ and the equation **is invariant** if we interchange the variables τ and $t_2^{'}$.

Given that $\frac{\partial}{\partial t_2}=\frac{\partial}{\partial t_2^{'}}\frac{\partial t_2^{'}}{\partial t_2}=-\frac{\partial}{\partial t_2^{'}}$, we get $\frac{\partial^2}{\partial t_2^2}=\frac{\partial}{\partial t_2}(\frac{\partial}{\partial t_2})=-\frac{\partial}{\partial t_2}(\frac{\partial}{\partial t_2^{'}})=\frac{\partial}{\partial t_2^{'}}(\frac{\partial}{\partial t_2^{'}})=\frac{\partial^2}{\partial(t_2^{'})^2}$, we substitute it in Result E.4.1.F' and get $\frac{\partial^2}{\partial t_2^2}E_0(\tau-t_2+t_0)=\frac{\partial^2}{\partial\tau^2}E_0(\tau-t_2+t_0)$.(**Result E.4.1.D'**)

We can write the term in the first integral in Eq. C.19 corresponding to the term $E_0(\tau + t_0 + t_2)e^{-2\sigma\tau}$ in Eq. C.20, using Result E.4.1.C', as follows. We use the fact that $\int_{-\infty}^{0} \frac{dA(\tau)}{d\tau}B(\tau)d\tau = \int_{-\infty}^{0} \frac{d(A(\tau)B(\tau))}{d\tau}d\tau - \int_{-\infty}^{0} A(\tau)\frac{dB(\tau)}{d\tau}d\tau$.

$$\int_{-\infty}^{0} \frac{\partial^2(E_0(\tau + t_2 + t_0))}{\partial t_2^2}\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)d\tau = \int_{-\infty}^{0} \frac{\partial^2(E_0(\tau + t_2 + t_0))}{\partial \tau^2}\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)d\tau$$
$$= \int_{-\infty}^{0} \frac{\partial(\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau))}{\partial\tau}d\tau - \int_{-\infty}^{0} \frac{\partial E_0(\tau + t_2 + t_0)}{\partial\tau}\frac{\partial(\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)}{\partial\tau}d\tau$$
$$= [\frac{\partial E_0(\tau + t_2 + t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)]_{-\infty}^{0} + \omega\int_{-\infty}^{0} \frac{\partial E_0(\tau + t_2 + t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\sin(\omega\tau)d\tau$$
$$+2\sigma\int_{-\infty}^{0} \frac{\partial E_0(\tau + t_2 + t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)d\tau - 2r\int_{-\infty}^{0} \frac{\partial E_0(\tau + t_2 + t_0)}{\partial\tau}\tau^{2r-1}e^{-2\sigma\tau}\cos(\omega\tau)d\tau \tag{C.22}$$

We see that the integrals $\int_{-\infty}^{0} \frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)d\tau$ and $\int_{-\infty}^{0} \frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r-1}e^{-2\sigma\tau}\cos(\omega\tau)d\tau$ in Eq. C.22 converge, using Eq. C.18 in the previous subsection. We see the term $[\frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\cos(\omega\tau)]_{-\infty}^{0}$ also converges, using Result E.3.1.1 in Section Appendix C.3.1. It is shown in Eq. C.14 that the remaining term $\int_{-\infty}^{0} \frac{\partial E_0(\tau+t_2+t_0)}{\partial\tau}\tau^{2r}e^{-2\sigma\tau}\sin(\omega\tau)d\tau$ also converges.

We see that the integrals in Eq. C.22 converge and hence the integral $\int_{-\infty}^{0} \frac{\partial^2(E_0(\tau+t_2+t_0)\tau^{2r}e^{-2\sigma\tau})}{\partial t_2^2}\cos(\omega\tau)d\tau$ in Eq. C.19 corresponding to the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ in Eq. C.20 also converges.

We set $\sigma = 0$ and $t_0 = -t_0$ in Eq. C.22 and see that the integral $\int_{-\infty}^{0} \frac{\partial^2(E_0(\tau+t_2-t_0))}{\partial t_2^2}\tau^{2r}\cos(\omega\tau)d\tau$ in Eq. C.19 corresponding to the term $E_0(\tau + t_2 - t_0)$ in Eq. C.20 also converges.

We set $t_2 = -t_2$ in the term $E_0(\tau + t_0 + t_2)e^{-2\sigma\tau}$ and use the procedure in Eq. C.21 to Eq. C.22 and see that the integral $\int_{-\infty}^{0} \frac{\partial^2(E_0(\tau+t_0-t_2)e^{-2\sigma\tau})}{\partial t_2^2}\tau^{2r}\cos(\omega\tau)d\tau$ in Eq. C.19 corresponding to the term $E_0(\tau - t_2 + t_0)e^{-2\sigma\tau}$ in Eq. C.20 converges, using Result E.4.1.D'.

We set $t_2 = -t_2$, $\sigma = 0$ and $t_0 = -t_0$ in the term $E_0(\tau + t_2 + t_0)e^{-2\sigma\tau}$ and use the procedure in Eq. C.21 to Eq. C.22 and Result E.4.1.D' and see that the integral $\int_{-\infty}^{0} \frac{\partial^2(E_0(\tau-t_0-t_2))}{\partial t_2^2}\tau^{2r}\cos(\omega\tau)d\tau$ in Eq. C.19 corresponding to the term $E_0(\tau - t_2 - t_0)$ in Eq. C.20 also converges. Hence the first integral in Eq. C.19, also converges.

We can see that the second integral in Eq. C.19 converge, by setting $t_0 = -t_0$ in Eq. C.20 to Eq. C.22 . Hence all the integrals in Eq. C.19 converge.

.

Appendix C.5. ***Zero Crossings in $G_{R,2r}(\omega, t_2, t_0)$ move continuously as a function of t_0, for a given t_2, for $r \in W$.***

Result E.5.1: It is shown in **Lemma 1** in Section 2.1 that $G_R(\omega, t_2, t_0) = 0$ at $\omega = \omega_z(t_2, t_0)$ where it crosses the zero line to the opposite sign, if Statement 1 is true. It is shown in Appendix C.7) that $G_{R,2r}(\omega, t_2, t_0) = 0$ and $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial\omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$, for some value of $r \in W$ where

$(2r+1)$ is the highest order of the zero of $G_R(\omega, t_2, t_0)$ at $\omega = \omega_z(t_2, t_0)$. (example plot)

We use **Implicit Function Theorem** for the two dimensional case (link and link). Given that $G_{R,2r}(\omega, t_2, t_0)$ is partially differentiable with respect to ω and t_0, for a given value of t_2, with continuous partial derivatives (Details in Appendix C.1 and Appendix C.3) and given that $G_{R,2r}(\omega, t_2, t_0) = 0$ at $\omega = \omega_z(t_2, t_0)$ and $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial \omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$, for some value of $r \in W$ where $(2r+1)$ is the highest order of the zero of $G_R(\omega, t_2, t_0)$ at $\omega = \omega_z(t_2, t_0)$ (using Result E.5.1 in this section and using Appendix C.7), we see that $\omega_z(t_2, t_0)$ is a differentiable function of t_0, for $0 < t_0 < \infty$, for each value of t_2 in the interval $0 < t_2 < \infty$.

Hence $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 for $0 < t_0 < \infty$, for each value of t_2 in the interval $0 < t_2 < \infty$.

- It is shown in Appendix C.4 that $G_{R,2r}(\omega, t_2, t_0)$ is partially differentiable at least twice with respect to t_2. We can use the procedure in previous paras and Implicit Function Theorem and show that $\omega_z(t_2, t_0)$ is a **continuous** function of t_2, for $0 < t_2 < \infty$, for each value of t_0 in the interval $0 < t_0 < \infty$.

.

Appendix C.6. ***Zero Crossings in*** $G_{R,2r}(\omega, t_2, t_0)$ ***move continuously as a function of*** t_0 ***and*** t_2***, for*** $r \in W$

We use **Implicit Function Theorem** for the three dimensional case $\Re^3$ (link and Theorem 3.2.1 in page 36). Given that $G_{R,2r}(\omega, t_2, t_0)$ is partially differentiable with respect to ω and t_0 and t_2, with continuous partial derivatives, for $r \in W$ (Details in Appendix C.1, Appendix C.3 and Appendix C.4) and given that $G_{R,2r}(\omega, t_2, t_0) = 0$ at $\omega = \omega_z(t_2, t_0)$ and $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial \omega} \neq 0$ at $\omega = \omega_z(t_2, t_0)$, for some value of $r \in W$ where $(2r+1)$ is the highest order of the zero of $G_R(\omega, t_2, t_0)$ at $\omega = \omega_z(t_2, t_0)$ (using Result E.5.1 in Appendix C.5 and using Appendix C.7), we see that $\omega_z(t_2, t_0)$ is a differentiable function of t_0 and t_2, for $0 < t_0 < \infty$ and $0 < t_2 < \infty$.

Hence $\omega_z(t_2, t_0)$ is a **continuous** function of t_0 and t_2, for $0 < t_0 < \infty$ and $0 < t_2 < \infty$.

.

Appendix C.7. ***Order of the zero in*** $G_R(\omega, t_2, t_0)$ ***is finite.***

It is shown in this section that, **if** $G_R(\omega, t_2, t_0) = 0$ at $\omega = \pm\omega_z(t_2, t_0)$ to satisfy Statement 1, for each fixed choice of positive $t_0, t_2 \in \Re$, **then** $G_{R,2r}(\omega, t_2, t_0) = \frac{\partial^{2r} G_R(\omega,t_2,t_0)}{\partial \omega^{2r}} = 0$ at $\omega = \pm\omega_z(t_2, t_0)$ and $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial \omega} = \frac{\partial^{2r+1} G_R(\omega,t_2,t_0)}{\partial \omega^{2r+1}} \neq 0$ at $\omega = \pm\omega_z(t_2, t_0)$ for some value of $r \in W$ (element of set of whole numbers including zero) and $(2r+1)$ is the highest order of the zero of $G_R(\omega, t_2, t_0)$ at $\omega = \pm\omega_z(t_2, t_0)$ which is finite.

This is shown using Proof by Contradiction by assuming the **opposite** case that $\frac{\partial^{2r} G_R(\omega,t_2,t_0)}{\partial \omega^{2r}} = 0$ and $\frac{\partial^{2r+1} G_R(\omega,t_2,t_0)}{\partial \omega^{2r+1}} = 0$ at $\omega = \omega_z(t_2, t_0)$, for $r = 0, 1, ...,$ as $r \to \infty$ (**Statement D**) and show that it leads to a **contradiction**.

$G_R(\omega, t_2, t_0)$ in Eq. 27 is copied below. It is shown in **Lemma 1** in Section 2.1 that $G_R(\omega, t_2, t_0) = 0$ at $\omega = \omega_z(t_2, t_0)$ where it crosses the zero line to the opposite sign, **if** Statement 1 is true.

$$\begin{aligned}G_R(\omega,t_2,t_0) = e^{-2\sigma t_0}&\int_{-\infty}^{0}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega\tau)d\tau \\ +e^{2\sigma t_0}&\int_{-\infty}^{0}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\cos(\omega\tau)d\tau\end{aligned} \tag{C.23}$$

We compute the $(2r)^{th}$ and $(2r+1)^{th}$ derivative of $G_R(\omega,t_2,t_0)$ and copy Eq. C.3 and Eq. C.4 below.

$$\begin{aligned}\frac{\partial^{2r}G_R(\omega,t_2,t_0)}{\partial\omega^{2r}} &= (-1)^r[e^{-2\sigma t_0}\int_{-\infty}^{0}\tau^{2r}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega\tau)d\tau \\ &\quad +e^{2\sigma t_0}\int_{-\infty}^{0}\tau^{2r}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\cos(\omega\tau)d\tau] \\ \frac{\partial^{2r+1}G_R(\omega,t_2,t_0)}{\partial\omega^{2r+1}} &= (-1)^{r+1}[e^{-2\sigma t_0}\int_{-\infty}^{0}\tau^{2r+1}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\sin(\omega\tau)d\tau \\ &\quad +e^{2\sigma t_0}\int_{-\infty}^{0}\tau^{2r+1}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\sin(\omega\tau)d\tau]\end{aligned} \tag{C.24}$$

We compute $C(\omega,t_2,t_0) = \sum_{r=0}^{\infty}\frac{(\delta\omega)^{2r}}{!(2r)}\frac{\partial^{2r}G_R(\omega,t_2,t_0)}{\partial\omega^{2r}}$ and $S(\omega,t_2,t_0) = \sum_{r=0}^{\infty}\frac{(\delta\omega)^{2r+1}}{!(2r+1)}\frac{\partial^{2r+1}G_R(\omega,t_2,t_0)}{\partial\omega^{2r+1}}$ below, using Eq. C.24.

$$\begin{aligned}C(\omega,t_2,t_0) &= \sum_{r=0}^{\infty}(-1)^r\frac{(\delta\omega)^{2r}}{!(2r)}[e^{-2\sigma t_0}\int_{-\infty}^{0}\tau^{2r}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos(\omega\tau)d\tau \\ &\quad +e^{2\sigma t_0}\int_{-\infty}^{0}\tau^{2r}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\cos(\omega\tau)d\tau] \\ S(\omega,t_2,t_0) &= \sum_{r=0}^{\infty}(-1)^{r+1}\frac{(\delta\omega)^{2r+1}}{!(2r+1)}[e^{-2\sigma t_0}\int_{-\infty}^{0}\tau^{2r+1}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\sin(\omega\tau)d\tau \\ &\quad +e^{2\sigma t_0}\int_{-\infty}^{0}\tau^{2r+1}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\sin(\omega\tau)d\tau]\end{aligned} \tag{C.25}$$

We can interchange the order of integration and summation in Eq. C.25 using Fubini's theorem given that the integrands in Eq. C.25 before the interchange and the integrands in Eq. C.26 and Eq. C.27 after the interchange are absolutely integrable as shown in Section Appendix C.1 and hence the integrals in Eq. C.26 and Eq. C.27 for $C(\omega,t_2,t_0)$ and $S(\omega,t_2,t_0)$ converge and equal the corresponding expressions in Eq. C.25 and we write as follows. (link)

$$
\begin{aligned}
C(\omega,t_2,t_0) &= e^{-2\sigma t_0}\int_{-\infty}^{0}[\sum_{r=0}^{\infty}(-1)^r\frac{(\delta\omega)^{2r}}{!(2r)}\tau^{2r}][E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2)]\cos{(\omega\tau)}d\tau\\
&+e^{2\sigma t_0}\int_{-\infty}^{0}[\sum_{r=0}^{\infty}(-1)^r\frac{(\delta\omega)^{2r}}{!(2r)}\tau^{2r}][E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau+t_0,t_2)]\cos{(\omega\tau)}d\tau\\
S(\omega,t_2,t_0) &= e^{-2\sigma t_0}\int_{-\infty}^{0}[\sum_{r=0}^{\infty}(-1)^{r+1}\frac{(\delta\omega)^{2r+1}}{!(2r+1)}\tau^{2r+1}][E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2)]\sin{(\omega\tau)}d\tau\\
&+e^{2\sigma t_0}\int_{-\infty}^{0}[\sum_{r=0}^{\infty}(-1)^{r+1}\frac{(\delta\omega)^{2r+1}}{!(2r+1)}\tau^{2r+1}][E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau+t_0,t_2)]\sin{(\omega\tau)}d\tau
\end{aligned}
\tag{C.26}
$$

We use $\sum_{r=0}^{\infty}(-1)^r\frac{(\delta\omega)^{2r}}{!(2r)}\tau^{2r}=\cos{((\delta\omega)\tau)}$ and $\sum_{r=0}^{\infty}(-1)^{r+1}\frac{(\delta\omega)^{2r+1}}{!(2r+1)}\tau^{2r+1}=-\sin{((\delta\omega)\tau)}$ and write Eq. C.26 as follows. The integrands in Eq. C.27 are absolutely integrable using the arguments in Section Appendix C.1. Hence interchanging the order of integration and summation in Eq. C.25 is justified.

$$
\begin{aligned}
C(\omega,t_2,t_0) &= e^{-2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2)]\cos{(\omega\tau)}\cos{((\delta\omega)\tau)}d\tau\\
&+e^{2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau+t_0,t_2)]\cos{(\omega\tau)}\cos{((\delta\omega)\tau)}d\tau\\
S(\omega,t_2,t_0) &= -[e^{-2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2)]\sin{(\omega\tau)}\sin{((\delta\omega)\tau)}d\tau\\
&+e^{2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau+t_0,t_2)]\sin{(\omega\tau)}\sin{((\delta\omega)\tau)}d\tau]
\end{aligned}
\tag{C.27}
$$

We compute $C_S(\omega,t_2,t_0)=C(\omega,t_2,t_0)+S(\omega,t_2,t_0)$ as follows, using the identity $\cos{((\omega+\delta\omega)\tau)}=\cos{(\omega\tau)}\cos{((\delta\omega)\tau)}-\sin{(\omega\tau)}\sin{((\delta\omega)\tau)}$.

$$
\begin{aligned}
C_S(\omega,t_2,t_0) &= e^{-2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau-t_0,t_2)]\cos{((\omega+\delta\omega)\tau)}d\tau\\
&+e^{2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau}+E_{0n}^{'}(\tau+t_0,t_2)]\cos{(\omega\tau)}\cos{((\omega+\delta\omega)\tau)}d\tau
\end{aligned}
\tag{C.28}
$$

If Statement D is true, **then** $\frac{\partial^{2r}G_R(\omega,t_2,t_0)}{\partial\omega^{2r}}=0$ and $\frac{\partial^{2r+1}G_R(\omega,t_2,t_0)}{\partial\omega^{2r+1}}=0$ at $\omega=\omega_z(t_2,t_0)$ in Eq. C.24 and $C(\omega,t_2,t_0)=\sum_{r=0}^{\infty}\frac{(\delta\omega)^{2r}}{!(2r)}\frac{\partial^{2r}G_R(\omega,t_2,t_0)}{\partial\omega^{2r}}=0$ and $S(\omega,t_2,t_0)=\sum_{r=0}^{\infty}\frac{(\delta\omega)^{2r+1}}{!(2r+1)}\frac{\partial^{2r+1}G_R(\omega,t_2,t_0)}{\partial\omega^{2r+1}}=0$ at $\omega=\omega_z(t_2,t_0)$ in Eq. C.25 and hence $C_S(\omega,t_2,t_0)=C(\omega,t_2,t_0)+S(\omega,t_2,t_0)=0$ at $\omega=\omega_z(t_2,t_0)$ in Eq. C.28.

$$C_S(\omega_z(t_2,t_0),t_2,t_0) = e^{-2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau+t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau-t_0,t_2)]\cos((\omega_z(t_2,t_0)+\delta\omega)\tau)d\tau$$
$$+e^{2\sigma t_0}\int_{-\infty}^{0}[E_0^{'}(\tau-t_0,t_2)e^{-2\sigma\tau} + E_{0n}^{'}(\tau+t_0,t_2)]\cos((\omega_z(t_2,t_0)+\delta\omega)\tau)d\tau = 0 \tag{C.29}$$

Eq. C.29 is similar to Eq. 28 in Section 2.4 with $\cos(\omega_z(t_2,t_0)\tau)$ replaced by $\cos((\omega_z(t_2,t_0)+\delta\omega)\tau)$.

Eq. C.29 holds for real $\delta\omega$ as $\delta\omega \to 0$, **if** Statement D is true. This **contradicts** Result 2.1.5 in Section 2.1 which requires $G_R(\omega,t_2,t_0) = 0$ at $\omega = \omega_z(t_2,t_0)$ where it crosses the zero line to the opposite sign, to satisfy Statement 1.

Hence we see that, **if** Statement 1 is true, **then** Statement D is false and hence there exists **at least one finite** $s \in W$ (element of set of whole numbers including zero) for which the $(s)^{th}$ derivative of $G_R(\omega,t_2,t_0)$ given by $G_{R,s}(\omega,t_2,t_0) = \frac{\partial^s G_R(\omega,t_2,t_0)}{\partial\omega^s} \neq 0$ at $\omega = \omega_z(t_2,t_0)$, where $s = 2r$ is even or $s = 2r+1$ is odd, for $r \in W$, for each fixed positive $t_0, t_2 \in \Re$.

We choose the **minimum** value of $s \in W$, for which $G_{R,s}(\omega,t_2,t_0) = \frac{\partial^s G_R(\omega,t_2,t_0)}{\partial\omega^s} \neq 0$ at $\omega = \omega_z(t_2,t_0)$and hence $G_{R,s-1}(\omega,t_2,t_0) = \frac{\partial^{s-1} G_R(\omega,t_2,t_0)}{\partial\omega^{s-1}} = 0$ at $\omega = \omega_z(t_2,t_0)$ (**Result 4.9**). It is shown in Result 4.9.a in the paras below that the case of $s = 2r$ is ruled out and hence $s = 2r+1$ is the order of the zero of $G_R(\omega,t_2,t_0)$ at $\omega = \omega_z(t_2,t_0)$ and hence $G_{R,2r+1}(\omega,t_2,t_0) = \frac{\partial^{2r+1} G_R(\omega,t_2,t_0)}{\partial\omega^{2r+1}} \neq 0$ at $\omega = \omega_z(t_2,t_0)$ and $G_{R,2r}(\omega,t_2,t_0) = \frac{\partial^{2r} G_R(\omega,t_2,t_0)}{\partial\omega^{2r}} = 0$ at $\omega = \omega_z(t_2,t_0)$, using Result 4.9. Hence $s = 2r+1$ is the order of the zero of $G_R(\omega,t_2,t_0)$ at $\omega = \omega_z(t_2,t_0)$, the order of this zero is **finite**.

Hence we can write $G_R(\omega,t_2,t_0) = (\omega_z(t_2,t_0)^2 - \omega^2)^{2r+1}N'(\omega,t_2,t_0)$, for $r \in W$, where $N'(\omega,t_2,t_0) \neq 0$ at $\omega = \pm\omega_z(t_2,t_0)$, for each fixed positive $t_0, t_2 \in \Re$ and $(2r+1)$ is the highest order of the zero at $\omega = \omega_z(t_2,t_0)$ which is finite. It is noted that $\omega_z(t_2,t_0)$ represents the **zero crossing** in $G_R(\omega,t_2,t_0)$, for each fixed positive $t_0, t_2 \in \Re$. It is noted that $N'(\omega,t_2,t_0)$ may or may not be zero at $\omega \neq \pm\omega_z(t_2,t_0)$ and we **do not** claim otherwise.

The case of $(\omega_z(t_2,t_0)^2 - \omega^2)^{2r}$ is **ruled out** because $G_R(\omega,t_2,t_0)$ changes sign at $\omega = \pm\omega_z(t_2,t_0)$ and $N'(\omega,t_2,t_0) \neq 0$ does not change sign at $\omega = \pm\omega_z(t_2,t_0)$ and $(\omega_z(t_2,t_0)^2 - \omega^2)^{2r} \geq 0$ for real ω and does not change sign at $\omega = \pm\omega_z(t_2,t_0)$.(**Result 4.9.a**)

We have shown that, **if** $G_R(\omega,t_2,t_0) = 0$ at $\omega = \pm\omega_z(t_2,t_0)$ to satisfy Statement 1, for each fixed choice of positive $t_0, t_2 \in \Re$, **then** $G_{R,2r}(\omega,t_2,t_0) = \frac{\partial^{2r} G_R(\omega,t_2,t_0)}{\partial\omega^{2r}} = 0$ at $\omega = \pm\omega_z(t_2,t_0)$ and $\frac{\partial G_{R,2r}(\omega,t_2,t_0)}{\partial\omega} = \frac{\partial^{2r+1} G_R(\omega,t_2,t_0)}{\partial\omega^{2r+1}} \neq 0$ at $\omega = \pm\omega_z(t_2,t_0)$ for some value of $r \in W$ and $(2r+1)$ is the highest order of the zero of $G_R(\omega,t_2,t_0)$ at $\omega = \pm\omega_z(t_2,t_0)$ which is finite.