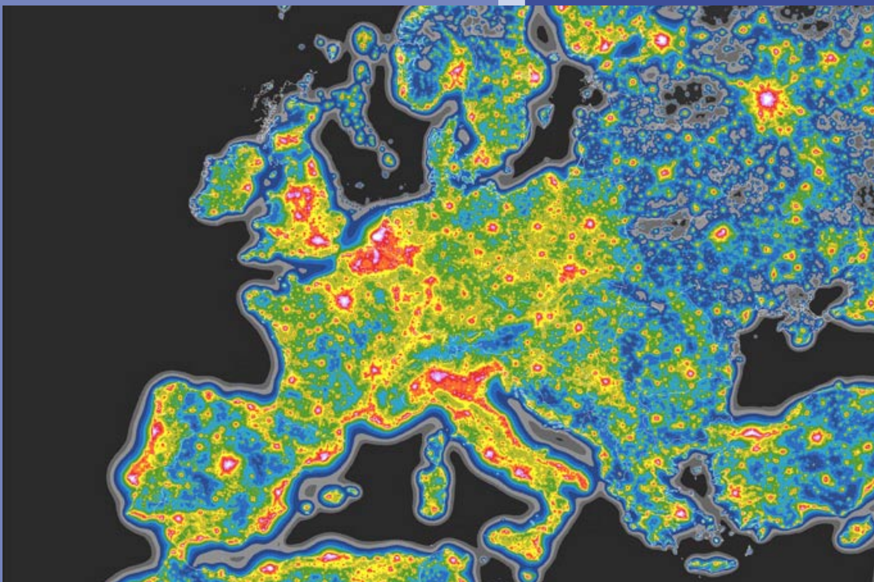
Problem Booklet
2021/22

Czech Astronomy Olympiad



Board of Organizers of the Czech Astronomy Olympiad

Prague, 2022

Czech Astronomy Olympiad

Problem Booklet 2021/22

Board of Organizers
of the Czech Astronomy Olympiad

Our partners

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Contents

Introduction

Czech Astronomy Olympiad is divided into four age categories AB, CD, EF and GH (from the oldest to the youngest). Each category is organized in three rounds. The first round takes place at school with its main objective to attract pupils to astronomy and motivate them for further work. In the second (regional) round, participants are asked to solve more complex problems, as well as to perform simple observations. The best participants proceed to the national rounds held in Opava and Prague in March and May.

Each problem presented in this booklet comes with its name and ID code containing information about the place of its original use in the Olympiad. For instance, "CD/R/2" denotes the second problem in the regional round of the CD category. Most problems have their answers shown in small print.

Majority of the competition problems are original work of the Czech AO organizers. Credits for the problems presented in this volume are as follows:

Jindřich Jelínek: CD/N/5; *Radka Křížová:* AB/N/4; *Pavel Kůs:* AB/R/2; *Václav Pavlík:* EF/R/1, EF/R/2, EF/N/1, EF/N/2; *Lukáš Supik:* AB/R/1; *Jiří Vala:* AB/N/3; *Jakub Vošmera:* CD/N/4, CD/N/6, CD/N/3, CD/R/2, CD/R/1, CD/N/1, CD/N/2, AB/N/1, AB/N/6, AB/N/2, AB/N/5, CD/N/7, AB/R/3a, AB/R/3b, AB/N/7

The reader certainly would not be able to enjoy the problems in their present form were it not for the careful reviews of *Petr Kulhánek*, *Ota Kéhar* and *Michal Švanda.*

Finally, we want to express our gratitude to the director of the Prague Observatory and Planetarium, *Jakub Rozehnal*, and the vice-dean of Faculty of Philosophy and Science of Silesian University in Opava, *Tomáš Gráf*, for kindly providing the venue for the national rounds. We also thank *Tomáš Prosecký* and *Lenka Soumarová* for helping make the Czech Astronomy Olympiad happen by providing administrative support.

Theoretical problems

Geometry, time and instrumentation

Setting stars CD/N/4

An observer (a turtle) at an unknown location on Earth sees a star with declination $\delta_1 \simeq 0°$ setting at 20:14 local time. At 21:32, the turtle astronomer notices that a star with the same right ascension, but with a declination of $\delta_2 \simeq 34°$ is setting. Determine the latitude ϕ of the observing site and draw a picture of the turtle looking up at the sky.

[$\arctan(\sin \Delta t / \tan \delta_2) \simeq 26.3°$]

Thickness of the atmosphere CD/N/6

In this problem, we will demonstrate how an effective thickness H of the Earth's atmosphere can be determined by measuring the sea-level refractive index n of the air and by determining the atmospheric refraction ρ at the horizon. We will model the Earth as an ideal sphere of radius R.

Based on various measurements and observations, we know that we can write $n = 1 + \nu$, where $0 < \nu \ll 1$, and also that $\rho \ll 1$ (expressed in radians). We will also introduce the dimensionless parameter $\chi \equiv H/R$ and assume that $\nu \ll \chi \ll 1$. You will find the following approximate relations useful

$$(1+x)^p \approx 1 + px\,,$$
$$\sin x \approx x\,,$$
$$\cos x \approx 1 - x^2/2\,.$$

These hold for $x \ll 1$ and for any p.

In the first part of the problem, we will assume a highly simplified model, where the Earth's atmosphere is replaced by a homogeneous spherical shell of thickness H and constant refractive index $n = 1 + \nu$. Consider an observer on the surface of the Earth and a light-ray that is tangent to the surface of the Earth.

a) Write down an exact expression for the sine of the incidence angle β of the light-ray, as it crosses the interface between the atmosphere and outer space. Express your result in terms of the parameter χ.
b) Write down an approximation of this expression when $\chi \ll 1$. You should keep only the terms which are linear in χ.
c) Express the angle β in terms of χ in an approximate form as a small deviation from $\pi/2$.
d) Determine the corresponding angle of refraction α in terms of χ and ν. Again, express your result in an approximate form as a small deviation from $\pi/2$. In particular, you should assume that $\nu \ll \chi$.
e) Write down an approximate expression for ρ in terms of χ and ν.
f) Using the numerical values $\nu \simeq 0.000\,293$, $\rho \simeq 34'$ and $R \simeq 6\,378\,\mathrm{km}$, determine the thickness H of the shell in km.

As you can see, results of this crude model are not very satisfactory. In the rest of the problem, we will therefore consider a somewhat more accurate model, where we will picture the atmosphere as a spherical shell of thickness H, where the refractive index decreases linearly from the sea-level value $n = 1 + \nu$ to the value $n = 1$ at the interface with outer space.

For the purpose of the calculation, we split the atmosphere into a large number N of concentric shells, each with thickness H/N, where in the k-th shell, the refractive index n_k reads $n_k = 1 + (N - k + 1)\nu/N$ for $k = 1, \ldots, N$. The value $k = N + 1$ corresponds to the outer space. By following similar steps as we did in the first part of the problem (i.e. using geometry of the triangle and Snell's law), we could show that the contribution ρ_k to the atmospheric refraction that arises at the interface between the k-th and $(k + 1)$-th shell can be approximated as

$$\rho_k \approx \frac{\nu}{\sqrt{2k\chi N}} \, .$$

The total atmospheric refraction ρ is then obtained by summing the partial contributions $\rho_1, \rho_2, \ldots, \rho_N$.

g) Use the above-described shell model to find a more accurate result for the effective thickness H of the atmosphere.

Hint: repeat the calculation for larger and larger values of N and try to guess the result for $N \to \infty$.

[a) $\sin\beta = 1/(1+\chi)$; b) $\sin\beta \approx 1 - \chi$; c) $\beta \approx \frac{\pi}{2} - \sqrt{2\chi}$; d) $\alpha \approx \frac{\pi}{2} - \sqrt{2\chi}[1 - \nu/(2\chi)]$; e) $\rho = \nu/\sqrt{2\chi}$; f) $H = \chi R \approx (\nu^2/2\rho^2)R \simeq 2.6\,\mathrm{km}$; g) $H \approx 2(\nu^2/\rho^2)R \simeq 10.3\,\mathrm{km}$]

Spherical aberration AB/R/2

Invention of the telescope, or rather its improvement, is intrinsically linked to the name of Galileo Galilei. On January 8, 2022, 380 years will pass since this great man walked the Earth for the last time, and it is on the occasion of this anniversary that we will study the so-called *spherical aberration*. This is a phenomenon which causes the lens not to have a well-defined focus[1] and famously arises when one considers a lens with a spherical interface (hence the name). As we will discover later in this problem, similar aberrations appear also for more general optical interfaces.

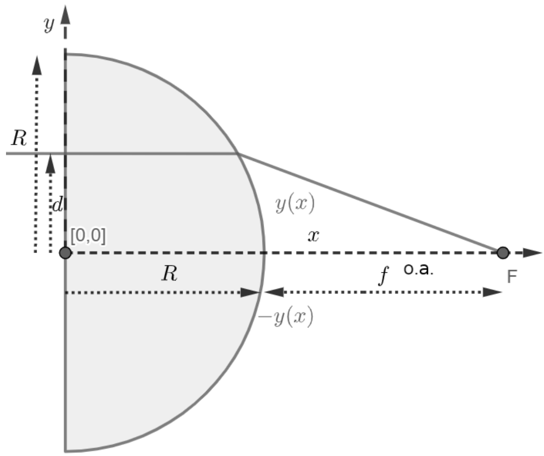


Figure 1: Refraction of light-rays on a lens with rotational symmetry.

We will first consider lenses of the type depicted in Figure 1. We can see that the lens has two smooth optical interfaces: first, a planar disk of radius R and, second, a smooth convex surface of revolution (such as a part of a sphere, paraboloid or similar). We will consider light-rays which are incident in the direction perpendicular to the planar interface and investigate how they get refracted as they pass through the lens. Because of the rotational symmetry, such light-rays are always planar curves. As a result, the situation can be discussed in any plane that contains the symmetry axis, such as the xy plane shown in Figure 1. The convex interface is then described by a certain function $y(x)$, as indicated in the same figure.

[1] In particular, it is not related to the colour of the light: the position of the focus will be smeared even for monochromatic light.

Due to rotational symmetry, all light-rays which were initially propagating parallel to the symmetry axis at a fixed distance d will intersect the axis at a single point F (focus) whose position will generally depend on d. We will denote by f the distance of F from the point where the convex optical interface intersects the symmetry axis and refer to it as focal length.

Let us first assume that the convex optical interface is a *half-sphere*, namely

$$y(x) = \sqrt{R^2 - x^2}\,, \qquad x \in [0, R]\,.$$

a) Determine the focal length f as a function of d for general R, n. Find its extremal values, $f_{\min}$ and $f_{\max}$, as we vary d.

Second, assuming instead a *parabolic* interface, we take it to be described by the function

$$y(x) = \sqrt{R^2 - Rx}\,, \qquad x \in [0, R]\,.$$

b) Determine the range of the refractive index n for which total internal reflection can occur in the lens. Your result should not depend on the radius R.

c) Assume that the lens has refractive index n which falls into the range found in the previous part. Find $f_{\min}$ and $f_{\max}$ for such a lens.

Hint: to tackle part b), you may find it useful to construct a line tangent to the graph of the function $y(x)$. This is described by the function $h(x) = kx+q$ where, for the parabola described above, k can be found as

$$k = -\frac{1}{2}\frac{R}{\sqrt{R^2 - Rx}}\,.$$

Finally, let us consider a homogeneous sphere of refractive index n and radius R, illuminated by a beam of parallel light-rays, see Figure 2.

d) Find f explicitly as a function of the angle of incidence α. You should find that

$$f + R = \frac{R}{2}\frac{1}{\cos\alpha - \frac{1}{n^2}\sin\alpha\sin(2\alpha) - \frac{1}{n}\cos(2\alpha)\sqrt{1 - \frac{1}{n^2}\sin^2\alpha}}\,.$$

e) Determine $f_{\min}$ and $f_{\max}$.

[a) $f = R\big[1/\big(\sqrt{1 - d^2/R^2} - \sqrt{1/n^2 - d^2/R^2}\big) - 1\big]$, $f_{\min} = R(n/\sqrt{n^2-1} - 1)$, $f_{\max} = R/(n-1)$; b) $\sqrt{5}/2 \le n < \infty$; c) $f_{\min} = R/[4(n^2-1)]$, $f_{\max} = (R/2)[1/(n-1)]$; e) $f_{\min} = (R/2)[n/(n-1)]$, $f_{\max} = (R/2)(n^2/\sqrt{n^2-1})$]

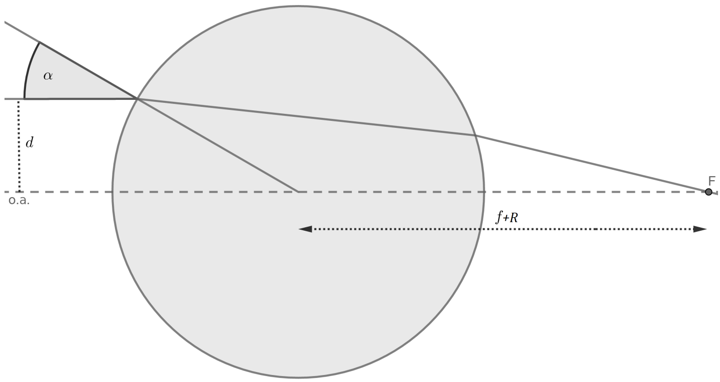


Figure 2: Refraction of light-rays on a sphere.

Volcano eruption — AB/N/3

On 15 January 2022, the underwater volcano Hunga Tonga-Hunga Ha'apai erupted in the South Pacific. The consequences of this eruption were reflected in the atmospheric pressure measurements at the CHMI station in Prague-Libuš, where the first pressure wave was registered on the same day at 19:29 local time. The second pressure wave came from the opposite direction exactly 6 hours later, i.e. at 01:29 the following morning. Assume that the wave propagates at the speed of sound $c_\mathrm{s} \simeq 343\,\mathrm{m\,s^{-1}}$.

a) Find the distance d of the Hunga Tonga-Hunga Ha'apai volcano from Prague.

Let us further assume that the azimuth $A \simeq 33.8°$ (measured from the north) of the direction from where the first shock wave came has been determined. The geographical coordinates of the Prague-Libuš station are $\phi_\mathrm{P} \simeq 50.0077°$ N and $\lambda_\mathrm{P} \simeq 14.4467°$ E.

b) Determine the geographical coordinates $(\phi_\mathrm{s}, \lambda_\mathrm{s})$ of the volcano.

[a) 16 330 km; b) $(-20.3°, 175.4°)$]

Solar system

Spinlaunch EF/R/1

“Spinlaunch” is the name of a new project which is aimed at putting satellites into orbit at a lower cost. Currently, only a prototype has been built that is capable of sending projectiles on suborbital trajectories. Instead of the chemical energy of rocket fuel, this concept uses kinetic energy of a rotating arm to propel a satellite (hidden inside the projectile) into space. In its final form, the launch mechanism will take the shape of a cylinder with a diameter of 100 m, in which an arm with a radius of 45.0 m will rotate in a vacuum at up to 450 rpm (see Figure 3).

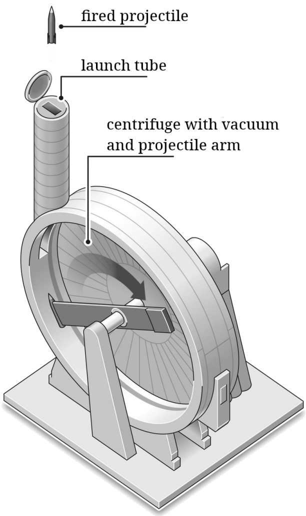


Figure 3: Illustration for the problem EF/R/1.

a) How fast does the projectile move when we place it at the end of the arm? Express your result in m/s, rounded to the nearest ten.
b) Suppose the projectile has a mass of 100 kg. What is the force needed to keep it on the rotating arm before it is launched? Express the value in newtons and round it to 3 significant figures.
c) Is the speed of the projectile sufficient for the satellite to be placed in orbit around the Earth? Briefly explain.
d) Assume that the gravitational field around the Earth were homogeneous,

namely that the gravitational acceleration $g \simeq 9.81\,\mathrm{m/s^2}$ does not change with altitude above the surface. Find the maximum altitude (in kilometers, rounded to the nearest integer) reached by the projectile if it were sent vertically upwards.

e) In reality, the gravitational force field of the Earth is central and the gravitational acceleration *does vary* with the altitude. Find the maximum altitude in this case.

[a) 2120 m/s; b) 9.99 MN; c) No; d) 229 km; e) 238 km]

Influence of planets EF/R/2

In the school round, we calculated the force of attraction between a physicist and the planet Saturn. That calculation was done under the assumption that all bodies involved can be replaced with point particles, thus neglecting the *tidal force*. In fact, this is a force exerted on an extended object which arises as a consequence of the inhomogeneous nature of a force field in which the object is placed.

Figure 4: The Earth-Moon system (not to scale).

a) Consider the situation which is shown in Figure 4. Assume that the distance between the Earth and the Moon (that is, between the points A and M) is $d_{\mathsf{AM}} \simeq 3.84 \times 10^8\,\mathrm{m}$ and that the radius of the Earth is equal to its equatorial radius. Calculate the distances of the points A through E (which are collectively denoted as z) from the point M, as well as the magnitude of the gravitational acceleration

$$a_z = \frac{GM_{\mathrm{M}}}{d_{z\mathsf{M}}^2}\,,$$

due to the Moon at the respective points on the Earth.

b) In Figure 4, use arrows mark the gravitational acceleration due to the Moon for each of the points A through E. The length of the arrows should be directly proportional to the magnitude of the acceleration.

c) Find the relative acceleration at the points B through E compared to acceleration which is felt at the point A. Mark it with arrows in Figure 4.
e) What is the most noticeable effect of tidal forces due to the Moon on the Earth? How many times a day can such a phenomenon be observed at a single location on the Earth?

Stereo solar eclipse EF/N/1

The pair of STEREO probes orbit the Sun along trajectories which are similar to that of the Earth, with STEREO-A orbiting somewhat closer to the Sun and STEREO-B a little further away. The orbital direction of the probes is the same as that of the Earth, but as each probe orbits with a slightly different period, STEREO-A races a little ahead of the Earth while STEREO-B is lagging behind. The orbits are shown in Figure 5.

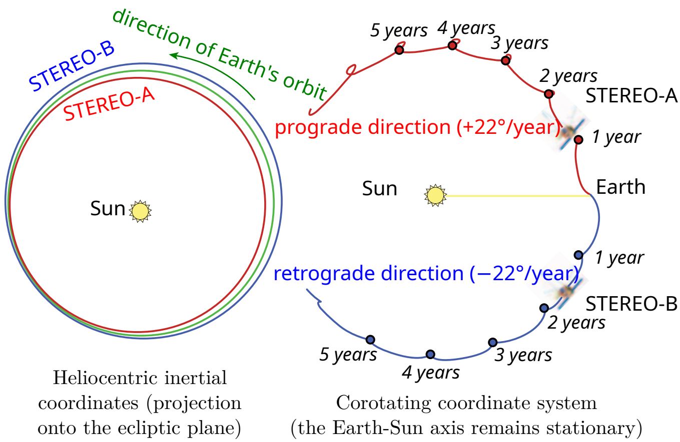


Figure 5: Orbits and angular speeds of the STEREO probes in two different reference frames: heliocentric intertial frame on the left and the corotating frame on the right.

a) How long does it take for each probe to make one complete orbit around the Sun? Give your answer in years to three significant figures.
b) Find the time (with a precision to the nearest day) which elapses from the launch until the probes reach a configuration where they find themselves exactly on the opposite side of the Sun that the Earth.

On February 25, 2007, one of the probes captured an image of the Moon (see Figure 6) as it passed over the solar disk.

c) Determine which of the two probes captured the image.
d) Let us assume that at the time at which the image was taken, the spacecraft, the Moon, the Earth and the Sun were all in the same plane. Decide in which of the 4 phases (New Moon/ First Quarter / Full Moon / Last Quarter) the Moon was. Explain briefly.
e) Find the distance between the Moon and the spacecraft at the time the picture was taken. Give the result in km.

Hint: you may find it helpful to make a sketch of the situation.

[a) 0.942 yr for STEREO-A, 1.07 yr for STEREO-B; b) 2988 d; c) STEREO-B; d) first quarter; e) 1.7 milions of km]

Figure 6: Transit of the Moon across the Sun as observed by one of the two STEREO probes. The image is in inverted colors – the Moon appears as the white disk passing over the disk of the Sun. Direction to the north agrees with the upward direction on the page. Credits: NASA.

Comet tail CD/R/2

Wilhelm Tempel was undoubtedly one of the most prolific comet hunters (he was involved in 21 discoveries). In this problem, we will take a look at how one would go about determining the shape of a comet's dust tail. Since this is generally a very complicated calculation, we will focus on the simplified situation where the comet is orbiting the Sun along a circular orbit and the radiation pressure force acting on the dust grains is exactly equal to the gravitational force due to the Sun. Let us denote the luminosity of the Sun by L_S.

a) Determine the flux $\Phi(r)$, that is, the total power incident per unit area, at a distance r from the Sun. Express the result in terms of r and L_S.

For the sake of simplicity, let us assume that the tail of a comet is made out of dust grains taking the shape of small spheres with radius R and density ρ. Material of these spheres is assumed to be perfectly conductive and to perfectly absorb all incident radiation.

b) Find the force $F_{\mathrm{rad}}(r)$ acting on one dust grain located at a distance r from the Sun. Express your result in terms of r, R, L_S and c. You should neglect all phenomena associated with the grains having generally non-zero velocity v (that is, you should work in the regime where $v/c \ll 1$).

Hint: write down the law of conservation of momentum for the process of absorption and re-emission of a single photon by a dust grain.

To simplify our analysis, in the following we will assume that the only gravitational attraction which the dust grains feel is that due to the Sun.

c) Verify that this is a reasonable assumption: consider a typical comet with a density $\rho_c \simeq 0.5\,\mathrm{g\,cm^{-3}}$, radius $R_c = 5\,\mathrm{km}$ and assume that the tail forms at distances of the order of 1 au from the Sun.
d) Hence show that the grain moves in a central force field with a centripetal acceleration of

$$g_{\mathrm{eff}}(r) = \frac{\gamma G M_S}{r^2}\,.$$

Express the parameter γ in terms of R, ρ, G, M_S, L_S and c.
e) Calculate the critical value R_0 of the radius of each grain for which we get $\gamma = 0$. Consider that $\rho \approx \rho_c$.

In the rest of the problem, we will assume that *all* dust grains have radius R_0 and that the comet orbits the Sun in a circular orbit with radius $a_c \simeq 1\,\mathrm{au}$. In such a heavily simplified setting, we will be able to find an exact prescription for the shape of the comet's tail.

f) Determine the shape of the trajectories of the individual grains in the heliocentric reference frame. Also, find their heliocentric velocities v.

Notwithstanding, these trajectories cannot be plainly identified with the curve giving the shape of the tail. This is because the source of the dust grain emission (the comet) changes its location with time. Let us introduce a Cartesian coordinate system $[x, y]$ in the comet's orbital plane, in which the comet orbits

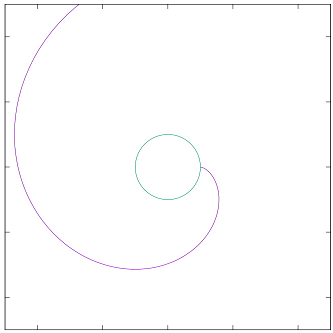


Figure 7: Involute of a unit circle.

along a trajectory $[x_{\mathrm{c}}(t), y_{\mathrm{c}}(t)]$ with the parametric prescription

$$x_{\mathrm{c}}(t) = a_{\mathrm{c}} \cos \omega_{\mathrm{c}} t\,,$$
$$y_{\mathrm{c}}(t) = a_{\mathrm{c}} \sin \omega_{\mathrm{c}} t\,,$$

where ω_{c} is the angular frequency of the comet's orbit.

g) Determine the trajectory $[x(t;\tau), y(t;\tau)]$ of a dust grain that was released by the comet at time τ.

h) Write down a parametric prescription $[x_{\mathrm{t}}(\tau;t_0), y_{\mathrm{t}}(\tau;t_0)]$ for the shape of the tail at some fixed timeframe $t = t_0$. As a parameter of the curve, use the time τ at which the respective section of the tail was released by the comet (where $-\infty \leqslant \tau \leqslant t_0$). Write down the name of the curve.

Hint: it might help you to think about the problem of unwinding the end of a very thin thread from a cylindrical spool.

i) Make a plot of the curve found in the previous part.

[a) $\Phi(r) = L_{\mathrm{S}}/(4\pi r^2)$; b) $F_{\mathrm{rad}}(r) = L_{\mathrm{S}} R^2/(4cr^2)$; c) Sun exerts at least 10× greater force on the grain than the comet; d) $\gamma = 1 - 3L_{\mathrm{S}}/(16\pi G M_{\mathrm{S}} \rho R c)$; e) $R_0 = 3L_{\mathrm{S}}/(16\pi G M_{\mathrm{S}} \rho c) \simeq 1\,\mu\mathrm{m}$; f) $v = \sqrt{GM_{\mathrm{S}}/1\,\mathrm{au}} \simeq 30\,\mathrm{km\,s^{-1}}$, the grains move along rays tangent to the comet's orbit;

g) $x(t;\tau) = -(t-\tau)\omega_{\mathrm{c}} a_{\mathrm{c}} \sin \omega_{\mathrm{c}}\tau + a_{\mathrm{c}} \cos \omega_{\mathrm{c}}\tau\,, y(t;\tau) = (t-\tau)\omega_{\mathrm{c}} a_{\mathrm{c}} \cos \omega_{\mathrm{c}}\tau + a_{\mathrm{c}} \sin \omega_{\mathrm{c}}\tau$ (straight line when τ is fixed and t parametrizes the curve);

h) $x_{\mathrm{t}}(\tau;t_0) = -(t_0-\tau)\omega_{\mathrm{c}} a_{\mathrm{c}} \sin \omega_{\mathrm{c}}\tau + a_{\mathrm{c}} \cos \omega_{\mathrm{c}}\tau\,, y_{\mathrm{t}}(\tau;t_0) = (t_0-\tau)\omega_{\mathrm{c}} a_{\mathrm{c}} \cos \omega_{\mathrm{c}}\tau + a_{\mathrm{c}} \sin \omega_{\mathrm{c}}\tau$ (involute when t_0 is fixed and τ parametrizes the curve);

i) Figure 7]

Comet motion CD/N/3

A comet crosses the Earth's orbit around the Sun at an angle $\phi \simeq 60°$ with a speed of $u = \sqrt{2GM_\odot/r_\mathrm{E}}$, where r_E is the radius of the Earth's orbit. We define ϕ as the angle between the tangents to the orbits of the Earth and the comet at the point where the orbits intersect. Determine the distance of the comet from the Sun at perihelion in astronomical units.

Hint: you should find it useful that the quantities

$$l = rv\sin\alpha\,,$$
$$h = \frac{1}{2}v^2 - \frac{GM_\odot}{r}\,,$$

are constants of the comet's motion. Here we denote by v the instantaneous speed of the comet in its orbit, by r the distance of the comet from the Sun, and by α the angle between the comet's velocity and its position vector.

[$\frac{1}{4}r_\mathrm{E} \simeq 0.25\,\mathrm{au}$]

Cooling the Earth CD/N/5

The objective of this problem will be to investigate a way of reducing the average temperature on the Earth's surface.

a) Write down the total power P of the radiation from the Sun that is absorbed by the Earth. You should express your result in terms of the luminosity $L_\odot$ of the Sun, the distance a of the Earth from the Sun, the radius R of the Earth, and the albedo A of the Earth.
b) Assuming thermodynamic equilibrium, write down a relation for the equilibrium temperature T of the Earth.
c) Find the ratio $(T + \Delta T)/T$, where ΔT is the change in temperature that would correspond to a change Δa in the distance of the Earth from the Sun.
d) Provided that $\Delta a/a \ll 1$, express the ratio of $\Delta a/a$ approximately as a multiple of $\Delta T/T$.

Hint: For $x \ll 1$, $(1 + x)^p \approx 1 + px$ holds for any p.

e) Find the change in Earth's distance from the Sun (in au) that would result in an average cooling of the Earth's surface by $\Delta T \simeq 1\,°\mathrm{C}$. Find the corresponding change in Earth's orbital period (in years). Take the average temperature of the Earth's surface to be $T \simeq 14\,°\mathrm{C}$.

In the remainder of the problem, we will consider an orbital maneuver that would transfer the Earth from a circular orbit of radius a to a circular orbit of radius $a + \Delta a$. This can be achieved by using a Hohmann ellipse that has a periapsis at distance a and an apoapsis at distance $a + \Delta a$ from the Sun.

f) Find the velocity Δv_1 that must be supplied to the Earth in the direction of its orbit for it to start orbiting along the Hohmann ellipse.

g) Find the velocity Δv_2 that must be given to the Earth in the apoapsis of the Hohmann ellipse in order for it to make a transition to a circular orbit of radius $a + \Delta a$.

h) What would be the total kinetic energy per unit mass that one has to supply to the Earth during such a maneuver? Compare this with the corresponding change in the total mechanical energy $-GM_\odot/(2a)$.

[a) $P = \pi R^2(1-A)L_\odot/(4\pi a^2)$; b) $T = [(1-A)L_\odot/(16\pi\sigma a^2)]^{1/4}$;
c) $(T + \Delta T)/T = \sqrt{a/(a+\Delta a)}$; d) $\Delta a/a \approx -2\Delta T/T$;
e) $\Delta P/P \approx 3\Delta a/2a \approx -3\Delta T/T \Rightarrow \Delta a \simeq 0.007\,\text{au}$;
f) $\Delta v_1/v \approx -\Delta T/2T \Rightarrow \Delta v_1 \simeq 50\,\text{m}\,\text{s}^{-1}$; g) $\Delta v_2/v \approx -\Delta T/2T \Rightarrow \Delta v_2 \simeq 50\,\text{m}\,\text{s}^{-1}$; h) the total kinetic energy supplied is equal to the change in total mechanical energy]

Space station AB/R/1

At the end of Interstellar (2014), the lead characters find themselves on a space station that has saved humanity from extinction. Suppose the station has the shape of a cylinder with radius $r \simeq 100\,\text{m}$ and length $l \simeq 200\,\text{m}$. The station is located in free space, rotating along its axis of symmetry, making its inhabitants feel the same gravity as if they were on the Earth.

a) Find the angular speed ω at which the station has to rotate. Find the corresponding period T of the station's rotation about its axis.

In the film, we could see kids playing baseball on a playground located at the station. At one point, after the ball was hit, it flew all the way through the station and broke the roof window of a house that stood roughly above the playground. In the following parts, we will analyze trajectories of point particles moving inside the station. For the sake of simplicity, we will neglect air drag.

b) Find the velocity which the children have to give to the ball (specify its magnitude and direction) so that from the point of view of an observer inside the station, the ball appears as orbiting just above the surface of the station along a circular trajectory.

Considering in more detail the setup described in the previous part, the ball repeatedly returns to the player who hit it. Notice that the corresponding circular trajectory is not the only one that returns the ball to the player who hit it. In fact, there are an infinite number of trajectories with this property. A natural parameter that characterizes each one of them is the time it takes for the ball to return to the place of its launch for the first time.

c) Find the velocities which the player has to impart on the ball in order for the ball to return to him in the time $t_1 = T/2$ and $t_2 = 3T/2$. Express your results in terms of r and T.
d) Find the velocity which the player must give to the ball for it to return to him in a general time t.
e) An astronaut inhabiting the station made his way from the perimeter of the station up a ladder all the way to the axis of the station to check the lighting. Find the work he had to do assuming that his mass is 75 kg.

Let us now consider the issue of spinning the station up starting from rest. Assuming that the station orbits in the vicinity of the Sun, its solar panels may provide just enough electricity for this to be possible. However, in any case, it has to economize on its mass. Therefore, it is proposed that ion engines should be used for its propulsion due to their comparably high fuel flow rate. Suppose the ion engines of the future accelerate singly-ionized atoms of xenon (with relative atomic mass $A_\mathrm{r} = 131$) in an electric field with a voltage of 10 kV. You can think of an ion engine as a plate capacitor, where the xenon is ionized at the positive electrode and accelerated towards the negative electrode, which is made of thin wires so that the ions can pass behind it and then move freely. (In reality, the departing ions need to be neutralized with electrons, but here we choose to neglect all processes taking place outside the capacitor.) The outflow rate of xenon from the engine is $10\,\mathrm{mg\,s^{-1}}$.

f) Find the speed of the xenon atoms as they fly out of the ion engine.
g) Determine the resulting thrust force.

200 pairs of ion engines located on the perimeter of the station are used to spin the station up. The engines in each pair are located at antipodal points on the perimeter of the station and their exhaust gases are directed tangentially to the cylindric hull, perpendicular to the axis of the cylinder and in opposite directions to each other. For the following calculation, assume that the mass of the station $M = 200\,000$ tons is concentrated mainly in the station hull, and that the mass of fuel m required to spin the station up is negligible relative to the mass of the station.

h) Find the time in which the station reaches the speed of rotation found in the first part of this problem, assuming that it starts from rest.

[a) $0.313\,\mathrm{s^{-1}}$, 0.1 s; b) $31.3\,\mathrm{m\,s^{-1}}$ against the sense in which the space station rotates; c) tangential $v_{\mathrm{t},1} = v_{\mathrm{t},2} = \omega r$, transverse $v_{\perp,1} = 4r/T$, $v_{\perp,2} = 4r/(3T)$; d) $v_\mathrm{t} = \sqrt{gr} - (r/t)\sin(t\sqrt{g/r})$, $v_\perp = (r/t)[1 - \cos(t\sqrt{g/r})]$; e) 36.8 kJ; f) $121\,\mathrm{km\,s^{-1}}$; g) 1.21 N, h) 0.41 y]

Eternal sunset — AB/N/4

Find how the eccentricity of the Earth's orbit would have to change (while fixing the value of the semi-major axis and the rotation period) in order for the Sun to start moving from west to east in the sky at some point. Assume that the current orbit of the Earth is exactly circular. Furthermore, determine the distance r_p of the Earth from the Sun at the perihelion of the new orbit in multiples solar radii.

Hint: write the resulting eccentricity e in the form $e = 1 - \eta$, where you can expect $\eta \ll 1$.

[$e \simeq 0.975$, $r_\mathrm{p} \simeq 5.3 R_\odot$]

Stellar astronomy and radiation

Disks of stars — CD/R/1

In this problem, we will analyze what conditions need to be met in order for the disks of individual stars to be resolved. We will consider a simple telescope with an aperture of diameter D, which is equipped with a detector with maximum sensitivity in the center of the optical band (effective wavelength $\lambda \simeq 550\,\mathrm{nm}$). A disk of a star will be considered as resolved whenever any two antipodal points on the limb satisfy the so-called Rayleigh criterion. That is, whenever the angular diameter of the disk is at least equal to the radius $\alpha_\mathrm{min} = 1.22\lambda/D$ of the Airy disk. For the sake of completeness, let us note that nowadays, much more advanced techniques for high-resolution imaging (such as interferometry) are available.

a) Determine the value of α_min for the GTC telescope (the largest telescope at the time) whose mirror has diameter $D_\mathrm{max} \simeq 10.4\,\mathrm{m}$.

Consider a star with radius R and temperature T, which is located at a distance d from the Earth. Let α denote the angular radius of the disk of the star.

b) Express the luminosity L of the star in terms of R and T.
c) Express the flux Φ coming from the star in terms of T and α.

The brightest stars in our sky reach apparent magnitudes of approximately 0 mag. In the first part of the problem, we also found that the angular radii of the stellar disks that we can resolve with current telescopes are limited from below by the value of $\alpha_\mathrm{min}/2$. Thus, your results in part c) should show that there is a maximum effective temperature T_max of the stars whose disk we can resolve.

d) Determine the value of T_{max} and identify the spectral types of stars for which we are able to resolve their disk using current telescopes (choose from the sequence O, B, A, F, G, K, M).

Hint: compare with the flux coming from the Sun and use Pogson's equation.

[a) $\alpha_{\text{min}} \simeq 0.013$"; b) $L = 4\pi R^2 \sigma T^4$; c) $\Phi = \sigma\alpha^2 T^4$; d) $T_{\text{max}} \simeq 4\,600\,\text{K}$, we resolve spectral types K and M]

Radius of Sirius CD/N/1

Sirius (α CMa), the brightest star in the Earth's sky, has apparent magnitude $m_{\text{v}} \simeq -1.47\,\text{mag}$. By measuring its spectrum, its peak wavelength can be found as $\lambda_{\text{max}} \simeq 290\,\text{nm}$. It can also be determined that it is an A1-type star for which the bolometric correction is $\text{BC} \simeq -0.08\,\text{mag}$. Finally, we can measure its parallax as $\pi \simeq 0.374$". Use this data to calculate the radius R of Sirius, which you should model as an absolutely black body. Express your result in multiples of solar radii.

[$R = \sqrt{L/(4\pi\sigma T^4)} = \sqrt{(10^{-0.4(m_{\text{v}}+5+5\log\pi+\text{BC}-4.75)} L_\odot/4\pi\sigma)(\lambda_{\text{max}}/b)^4} \simeq 1.75 R_\odot$]

Binary systems, clusters and exoplanets

A new Earth? EF/N/2

As the number of exoplanet discoveries grows, so does the chance that one of them will host life. One such candidate is a planet from the TRAPPIST-1 system – a system of seven planets located 12 pc away orbiting an extremely cold red dwarf. Parameters of these planets are shown in Table 1.

Table 1: Parameters of planets in the TRAPPIST-1 system. Here a denotes the semi-major axis, T is the orbital period, R is the radius.

planet	$\frac{a}{\text{au}}$	$\frac{T}{\text{d}}$	$\frac{R}{R_\oplus}$
TRAPPIST-1b	1.15×10^{-2}	1.511	1.121
TRAPPIST-1c	1.58×10^{-2}	2.422	1.095
TRAPPIST-1d	2.23×10^{-2}	4.050	0.784
TRAPPIST-1e	2.93×10^{-2}	6.099	0.910
TRAPPIST-1f	3.85×10^{-2}	9.206	1.046
TRAPPIST-1g	4.69×10^{-2}	12.354	1.148
TRAPPIST-1h	6.19×10^{-2}	18.778	0.773

Assume that the exoplanets are spherical and that the star always illuminates the entire hemisphere facing the star. Thus, each exoplanet receives a fraction of the stellar radiation that depends on the ratio of the area of its cross-section to the surface of the sphere with a radius which is equal to the distance of the exoplanet from the star.

a) Find the area of the exoplanet's cross-section in terms of its radius R.

In addition, each exoplanet radiates away power in the form of thermal radiation from its entire surface.

b) Find the surface area through which the exoplanet radiates in terms of R.

The basic condition for the existence of life as we know it is water. If an exoplanet orbited too close to a star, all water would evaporate. If the exoplanet orbited too far away, all water would freeze. To determine which exoplanet lies in the so-called habitable zone, we will assume that the exoplanet emits the same power as it receives from its parent star, that is

$$S_{\mathrm{v}}\sigma T^4 = \frac{S_{\mathrm{p}}}{4\pi d^2}\alpha L\,.$$

On the left hand side: S_{v} is the area through which the exoplanet emits, σ is the Stefan-Boltzmann constant, and T is the effective temperature of the exoplanet (assumed to be uniform over the exoplanet's surface).

On the right hand side: S_{p} is the area of the cross-section of the exoplanet, the parameter α determines how much of the incident radiation from the star the exoplanet absorbs (assume 70 %, that is $\alpha \simeq 0.70$), L is the luminosity of the star (for TRAPPIST-1 this is $L \simeq 2.127 \times 10^{23}$ W) and finally d is the distance of the exoplanet from the star.

c) Find d in terms of α, L, T and σ.
d) Determine the distance from the star at which the effective temperature of an exoplanet would correspond to the *freezing* point of water. Express your result in astronomical units.
e) Determine the distance from the star at which the effective temperature of an exoplanet would correspond to the *boiling* point of water (assuming a pressure of 1 atm). Express your result in astronomical units.
f) Identify the exoplanet(s) in the TRAPPIST-1 which lie within the habitable zone.
g) Finally, by determining its mass, check that TRAPPIST-1 is indeed a dwarf star.

[a) $S_{\mathrm{p}} = \pi R^2$; b) $S_{\mathrm{v}} = 4\pi R^2$; c) $d = \sqrt{\alpha L/(16\pi\sigma T^4)}$; d) 2.05×10^{-2} au; e) 1.10×10^{-2} au; f) b and c; g) $0.0898\,M_\odot$]

Spectroscopic binary star CD/N/2

Astronomers have found that for a particular star, the Hα line splits symmetrically into two components with a period $P \simeq 1.34\,\mathrm{d}$. The maximum displacement of each component from the mean position $\lambda_0 \simeq 656.28\,\mathrm{nm}$ is $\Delta\lambda \simeq 0.31\,\mathrm{nm}$ and the distance of the components is a harmonic function of time. Calculate the total mass of the system in units of the solar mass, assuming that the binary's orbital plane is viewed exactly edge-on.

[$M = (\Delta\lambda/\lambda)^3(4Pc^3/\pi G) \simeq 3.14 M_\odot$]

Earth from afar AB/N/1

At what maximum angular distance from a star with magnitude $m = 4.0\,\mathrm{mag}$ can we observe a planet that receives the same amount of radiation as the Earth?

[$0.15''$]

Lagrange point L2 AB/N/6

In this problem we will consider the motion of a satellite of mass μ in the vicinity of the Lagrange point L2 of the system of two bodies of masses M and m. For simplicity, we will consider the hierarchy of masses $M \gg m \gg \mu$. That is, introducing the parameter $\lambda = m/M$, we can assume that $\lambda \ll 1$.

a) Assuming that they are separated by a constant distance a, write down a formula for the orbital angular frequency ω of the bodies M and m around their center of gravity. Express your result in terms of G, M, a and the parameter λ.

b) Determine the distances R and r of the bodies M and m from the center of gravity. Express your results in terms of a and λ.

Lagrange points are defined as the locations where the total gravitational force acting on a test particle μ in the non-inertial reference frame corotating with the system is equal to the centrifugal force. The Lagrange point L2 is located on the line joining the two bodies M and m behind the less massive body m. Let us denote the distance of the point L2 from the body m as d and define $\delta = d/a$. In the case $M \gg m$, we can assume that $d \ll a$.

c) Express the centrifugal acceleration $a_{\mathrm{o,L2}}$ acting on the test particle located at L2 in terms of G, M, a, λ and δ. Neglect the terms where we multiply small quantities between themselves.

d) Express the total gravitational acceleration $a_{\mathrm{g,L2}}$ acting on the test particle μ located at L2 in terms of G, M, a, λ and δ. To simplify the contributions due to the heavier of the two bodies, you should use the binomial approximation $(1+x)^p \approx 1+px$, which holds for $x \ll 1$.

e) By invoking the equilibrium of the total gravitational and centrifugal forces acting on a (stationary) test particle μ located at L2, express the parameter δ as a function of λ: in particular, show that $\delta = (\lambda/3)^{1/3}$.

f) Determine the values which the parameter δ takes for the binary systems Sun-Earth and Earth-Moon.

g) For both of these binary systems, determine also the value of the distance d of the point L2 from the less massive body.

h) Find the angular radii α_E and α_M of the Earth and Moon as they would be seen by an observer located at the L2 point of the Earth-Moon system.

If we were to displace a test particle from L2 a little, we would find that in the direction of the line joining M and m, it would tend to move away from L2. On the other hand, displacing the particle in the other two directions, it would tend to return back to L2. The analysis of trajectories around L2 is further complicated by the fact that the Coriolis force acts on moving bodies in non-inertial reference frames. A suitable choice of initial conditions can yield orbits that remain close to L2: for small displacements, these trajectories take the form of three-dimensional Lissajous patterns, while for larger displacements, we would obtain, for example, the so-called halo orbit, along which the recently-launched JWST moves.

In the rest of the problem, we will consider a specialized situation where the test particle is displaced from the point L2 by a small distance $h \ll d \ll a$ in the direction perpendicular to the orbital plane of the bodies M and m: in such cases, the Coriolis force can be seen to be always absent. Introducing the parameter $\chi = h/d$, we can assume that $\chi \ll 1$.

i) Determine the total acceleration a_h of the test particle displaced in the direction perpendicular to the orbital plane of the bodies M and m. Using the condition $\chi \ll 1$, express your result in the form $-\Omega^2 h$, where the angular frequency Ω should be expressed as a multiple of ω.

Consider a satellite that sits at rest at point L2 of the Earth-Moon system.

j) Find the minimum value V of the speed which we must give to the satellite in the direction perpendicular to the plane of the Moon's orbit around the Earth in order for it to be able to communicate with any place on the Earth's surface, at least at a single point in time.

[a) $\omega = \sqrt{G(M+m)/a^3} = \sqrt{GM(1+\lambda)/a^3}$; b) $R = [m/(M+m)]a = [\lambda/(1+\lambda)]a$, $r = [M/(M+m)]a = [1/(1+\lambda)]a$; c) $a_{\mathrm{o,L2}} \approx (GM/a^2)(1+\delta)$; d) $a_{\mathrm{g,L2}} \approx (GM/a^2)(1-2\delta) + (GM/a^2)(\lambda/\delta^2)$; f) Sun–Earth: 0.01, Earth–Moon 0.16; g) Sun–Earth: 1.50×10^6 km, Earth–Moon: 61 600 km; h) $\alpha_\mathrm{M} \simeq 3.2°$, $\alpha_\mathrm{E} \simeq 1.6°$; i) $\Omega = 2\omega$; j) $11\,\mathrm{m\,s^{-1}}$]

Cosmology and relativity

Black radiation AB/N/2

Determine the peak-wavelength $\lambda_{\max}$ of the Hawking radiation in multiples of the Schwarzschild radius R_S. The temperature of the Hawking radiation can be found as $T_\mathrm{H} = \hbar c^3/(8\pi G M k_\mathrm{B})$, where M is the mass of the black hole.

[15.902...]

Breakthrough starshot AB/N/5

In this problem, we will explore feasibility of interstellar travel using laser propulsion. Specifically, we will consider a very powerful monochromatic laser with total power $P \simeq 100\,\mathrm{GW}$ and wavelength $\lambda_0 \simeq 650\,\mathrm{nm}$ located in orbit around the Earth. We will assume that the entire power of this laser is directed into a very light (rest mass $m \simeq 10\,\mathrm{g}$) and perfectly reflective sail. We will see that by doing so, one can attempt to accelerate the sail to a non-trivial fraction of the speed of light.

a) Determine the total number ν_0 of photons incident on the sail per unit time when the sail is at rest relative to the laser source.
b) How does this result change if the sail is moving at speed v away from the source? Denote the corresponding number of photons per unit time as ν.
c) For $v \ll c$, find the change in momentum of the sail per unit time (that is, the force F acting on it) in terms of P, c, and v.
d) Using your non-relativistic result for the force, determine the time $t(V)$ it would take to accelerate the sail to a final velocity of $V = 0.1c$. Express your result in seconds.

In the rest of the problem, we will work relativistically. In particular, we will distinguish between the time measured in the system associated with the laser source and the time that would be measured by an observer connected to the sail (proper time). It will be convenient to introduce the kinematic parameters of the sail

$$\alpha = \frac{p}{mc}\,, \qquad \beta = \frac{v}{c}\,, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}\,,$$

where p denotes the (relativistic) momentum of the sail. Since $p = mv\gamma$, we have the relation $\alpha = \beta\gamma$. For the relativistic energy of the sail, we then have the relation

$$E = \sqrt{p^2c^2 + m^2c^4} = mc^2\sqrt{1+\alpha^2} = mc^2\gamma\,.$$

We also introduce the photon kinematic parameter $\delta = h/(mc\lambda_0)$.

Consider the collision of a single photon of wavelength λ_0 with the sail (of rest mass m) as it moves at speed v in the reference frame of the source. The sail is moving away from the photon source. After the collision, the photon (with a new wavelength λ_0') is reflected back towards the source, while the sail will move with an updated speed v'.

e) Write down the law of conservation of relativistic momentum and energy for the system before and after the collision in the reference frame of the photon source. Express the momenta and energies of the sail and the photon before and after the collision using their respective parameters α, δ and α', δ'.

f) Express $\alpha' - \alpha$ as a function of α and δ. Simplify the final result by assuming that the momenta of individual photons are always very small compared to the momentum of the sail (i.e. $\delta \ll \alpha$). In particular, show that

$$\alpha' - \alpha = 2\delta \frac{\sqrt{1+\alpha^2}}{\sqrt{1+\alpha^2}+\alpha} .$$

g) Write down the change in the sail parameter α per unit time measured in the laser reference frame (that is, the force f measured in the laser reference frame normalized by the factor mc). Also give the approximate dependence of $f(\alpha)$ for $\alpha \gg 1$ (ultrarelativistic regime).

h) Repeat the calculation from the previous question for the case where time is measured in a sail reference frame. Denote the resulting force as $\phi(\alpha)$.

Suppose that we want to accelerate the sail to the (ultra-relativistic) final speed $V \simeq 0.999c$.

i) Find the corresponding value of the parameter $\alpha(V)$.

j) Determine the times t and τ required to accelerate the sail to this speed in the laser frame and the sail frame, respectively. Express the results in days.

Hint: The area under the graph of the parabola $y = x^2$ from $x = 0$ to $x = a$ can be calculated as $(1/3)a^3$.

[a) $P\lambda_0/hc \simeq 3 \times 10^{29}\,\mathrm{s}^{-1}$; b) $(P\lambda_0/hc)(1 - v/c)$; c) $(2P/c)(1 - 2v/c)$; d) 495 s; e) conservation of momentum: $\alpha + \delta = \alpha' - \delta'$, conservation of energy: $\sqrt{1+\alpha^2} + \delta = \sqrt{1+\alpha'^2} + \delta'$;
g) $f(\alpha) = (2P/mc^2)[1/(1 + 2\alpha^2 + 2\alpha\sqrt{1+\alpha^2})] \approx (P/2mc^2)\alpha^{-2}$ for $\alpha \gg 1$;
h) $\phi(\alpha) = (2P/mc^2)[\sqrt{1+\alpha^2}/(1 + 2\alpha^2 + 2\alpha\sqrt{1+\alpha^2})] \approx P/(2mc^2)\alpha^{-1}$ for $\alpha \gg 1$;
i) $22.3 \gg 1$; j) $t(V) \approx \frac{2}{3}(mc^2/P)\alpha(V)^3 \simeq 775\,\mathrm{d}$, $\tau(V) \approx (mc^2/P)\alpha(V)^2 \simeq 52\,\mathrm{d}$]

Practical problems

The law of gravitation CD/N/7

Table 2 gives some data for a number of selected planets of the Solar System. Assume that the planets orbit the Sun along circular trajectories.

a) Plot the values of $\log T$ against the values of $\log r$.

b) Using the graph from the previous part, determine the value of the numerical parameter p in the expression of the Newton's law of gravitation

$$a_{\mathrm{g}} = \frac{GM}{r^p} \, ,$$

where a_{g} is the gravitational acceleration at distance r from the gravitating body. Also determine the value of the parameter GM for Solar System.

Table 2: Data for Problem CD/N/7 (r is the radius of a circular orbit and T is the orbital period).

Planet	$\frac{r}{\mathrm{au}}$	$\frac{T}{\mathrm{y}}$
Mercury	0.39	0.24
Venus	0.72	0.61
Earth	1.00	1.00
Mars	1.52	1.88
Jupiter	5.20	11.86
Saturn	9.55	29.46
Uranus	19.20	84.01

[a) plot with a linear dependence; b) $p \simeq 2$, $GM \simeq 1.3 \times 10^{20}\,\mathrm{m^3\,s^{-2}}$]

Limiting magnitude AB/R/3a

Use at least two different methods to determine the limiting magnitude at zenith for your chosen observing site. In your solution, thoroughly document the location and time of the observation, the phase of the Moon, and the details of your methodology. Compare your results for the different methods. Be sure to thoroughly adapt your eyes to the darkness before each observation!

Exoplanets orbiting a pulsar AB/R/3b

The race to detect the first planet outside the Solar System (that is, an *exoplanet*), which peaked some 30 years ago, had unexpected winners: the radio astronomers Aleksander Wolszczan and Dale Frail, who, on January 9, 1992, announced to the world the discovery of two planets orbiting the pulsar PSR B1257+12. In this problem, you will try to put yourself in their shoes and, based on the data from observations of the changes in the period of the incoming pulses over time, you will not only infer the presence of exoplanets, but also calculate some of their parameters.

For this purpose, you will use suitable software designed to fit a sum of harmonic functions (sines) to irregularly sampled time series. We leave the final choice up to your discretion, but we recommend the following options.

- `Period04`: uses discrete Fourier transform and the method of successive frequency subtraction. Available at `http://www.period04.net/`.
- `SparSpec`: uses the sparse representation method to predict the frequencies contained in a time series. Available at `http://userpages.irap.omp.eu/~hcarfantan/SparSpec1.4/SparSpec_html.html`.
- *Phase Dispersion Minimization – PDM*. Available in the form of Python scripts from various sources on the Internet.

In Table 3, you can find data from measurements of the changes in the period of pulses coming from PSR B1257+12 over time. These data have been corrected for the annual variation in the relative radial velocity due to the Earth orbiting the Sun. In the following questions, we will assume that the observed variations in the pulsar period are due to presence of two exoplanets orbiting this pulsar along circular orbits.

a) Using the software of your choice, analyse the attached data series and determine the values of the significant frequencies that occur in the series. Discuss your choice and your criterion for evaluating significance. Plot the measured data as a function of time. Furthermore, model the changes in the period as a series of sine waves with the frequencies which you specify. Plot this fit over the actual measurements and discuss its quality.
b) Determine the mean period $\overline{P}$ of the incoming pulses.
c) Determine the first two dominant frequencies f_1, f_2 contributing to the fit. Determine the corresponding orbital periods T_1, T_2 of the exoplanets orbiting the pulsar (in days).

d) Determine also the amplitudes ΔP_1, ΔP_2 of the changes in the pulsation period changes due to the presence of each of the two exoplanets.

The estimated mass of the pulsar PSR B1257+12 is $M_* \simeq 1.4 M_\odot$.

e) For both exoplanets, calculate the radii a_p of their orbits around the pulsar in astronomical units.
f) For each exoplanet, calculate also its mass parameter $M_\mathrm{p} \sin i$, where M_p is the mass of the exoplanet and i is the angle between the normal to the plane of its orbit and the direction towards the observer. Give your results in units of the mass of the Earth $M_\oplus$.

In problem f), be sure to discuss the possible effect of the radial motion of the center of mass of the exoplanetary system relative to the Earth.

Table 3: Data from the observations of the pulsar PSR B1257+12: measurement epoch (in years) and difference of the measured pulse period (in nanoseconds) from the value $P_0 = 6\,218\,530\,\mathrm{ns}$. Source: Wolszczan A., Frail D. A.: *A planetary system around the millisecond pulsar PSR1257+12*, Nature, **355**, 145-147 (1992).

Epoch	$\frac{P-P_0}{\mathrm{ns}}$	Epoch	$\frac{P-P_0}{\mathrm{ns}}$	Epoch	$\frac{P-P_0}{\mathrm{ns}}$
1990.54	1.92427	1991.30	1.93559	1991.41	1.93133
1990.55	1.92182	1991.30	1.93404	1991.56	1.94910
1990.58	1.92112	1991.31	1.93629	1991.60	1.93909
1990.60	1.92843	1991.31	1.93599	1991.66	1.92247
1990.64	1.93399	1991.35	1.94480	1991.68	1.92563
1990.71	1.92823	1991.37	1.94270	1991.69	1.92708
1990.79	1.93999	1991.38	1.94029	1991.70	1.93038
1990.85	1.93444	1991.38	1.94089	1991.71	1.93479
1990.89	1.91832	1991.38	1.94164	1991.73	1.93884
1990.92	1.91852	1991.38	1.94204	1991.77	1.93939
1990.96	1.93173	1991.39	1.93634	1991.81	1.93419
1990.98	1.94104	1991.39	1.93789	1991.83	1.93243
1991.21	1.93994	1991.39	1.93939	1991.85	1.93639
1991.22	1.93954	1991.40	1.93444		

[b) $6\,218\,531.935\,\mathrm{ns}$; c) $f_1 \simeq 5.50\,\mathrm{yr}^{-1}$, $f_2 \simeq 3.69\,\mathrm{yr}^{-1}$, $T_1 \simeq 66.7\,\mathrm{d}$ a $T_2 \simeq 98.2\,\mathrm{d}$; d) $\Delta P_1 \simeq 9.66 \times 10^{-3}\,\mathrm{ns}$, $\Delta P_2 \simeq 7.19 \times 10^{-3}\,\mathrm{ns}$; e) 0.36 au, 0.47 au; f) $3.7 M_\oplus$, $3.1 M_\oplus$]

GK Persei — AB/N/7

The figure on the following page shows two images of the nova GK Per and the ejected material around it. The images were taken on January 29, 2004 and December 12, 2011. In both images, we can see a large number of easy-to-identify blobs. First, you should choose 10 points on the image where you will measure the angular radius ρ of the nebula.

a) Find the average value of ρ and its uncertainty. Perform the measurements for both images.
b) Perform an initial estimate of the observed expansion speed ω of the nebula (in $\mathrm{arcsec\,y^{-1}}$).

Let us now assume that the blobs form a thin shell around the star.

c) Choose 10 blobs evenly distributed across all possible angular distances α from the star and determine their proper motion μ.
d) Plot the proper motion of the blob as a function of its distance from the star. Assess the correctness of the hypothesis that the shell is spherical.
e) Use linear regression to find the value of ω with greater accuracy.

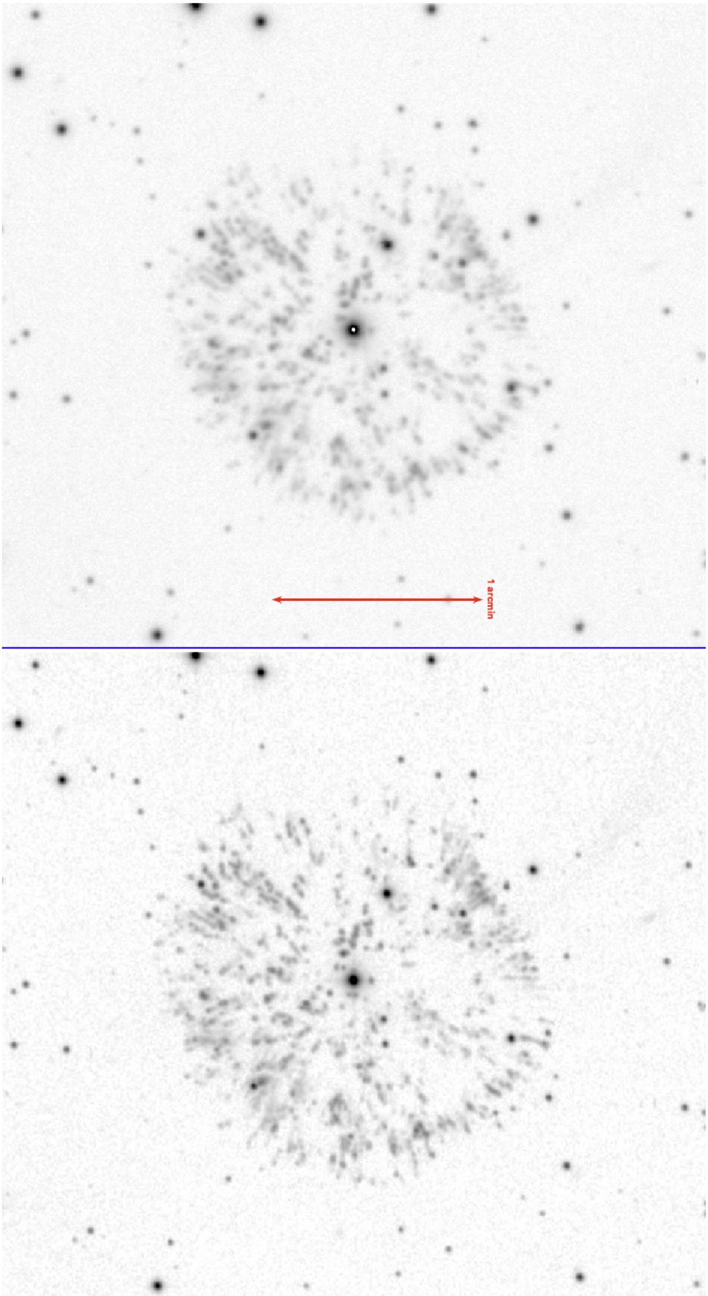
You are also provided with data from radial velocity measurements of the blobs within 10 arcsec of the central star (Table 4). These velocities are already corrected for the motion of the Earth around the Sun. We also know that the radial velocity of GK Per relative to the Sun is $v_{\mathrm{rad},*} \simeq 6\,\mathrm{km\,s^{-1}}$.

Table 4: Angular coordinates (x, y) of selected blobs (relative to GK Persei) and their measured radial velocities.

$\frac{x}{\mathrm{arcsec}}$	$\frac{y}{\mathrm{arcsec}}$	$\frac{v_{\mathrm{rad}}}{\mathrm{km\,s^{-1}}}$	$\frac{x}{\mathrm{arcsec}}$	$\frac{y}{\mathrm{arcsec}}$	$\frac{v_{\mathrm{rad}}}{\mathrm{km\,s^{-1}}}$
0.21	4.72	853	6.8	-7.49	1004
0.21	4.72	964	-9.67	-0.07	816
2.25	-9.25	-777	-8.51	3.51	-881
3.69	-0.88	-849	-7.28	-2.41	-780
4.11	0.38	-749	-3.12	-4.02	-834
4.11	0.38	-845	-3.11	-2.73	-811

f) Determine the value of the expansion speed v of the nebula.
g) Determine the distance d of GK Persei from the Sun.

[a) (0.90 ± 0.07) arcmin; b) $(0.40 \pm 0.07)\,\mathrm{arcsec\,y^{-1}}$; e) $(0.48 \pm 0.01)\,\mathrm{arcsec\,y^{-1}}$; f) $(860 \pm 90)\,\mathrm{km\,s^{-1}}$; g) (3800 ± 400) pc]


1 arcmin

20 years since the first law on light pollution

On 1 June 2002, the world's first law on light pollution was adopted. The Czech Republic was the first country to do so. At the time, an article was published in *Nature* that began with the sentence: "From now on, the sky over the Czech Republic will be more starry."

Dark sky areas

Several dark sky areas, i.e. areas with low light pollution, have already been created in the Czech Republic on the initiative of volunteers in cooperation with regional parks. They are used for observing the night sky and raising awareness. Czech Dark Sky Areas are often the centre of observing events organised by the Czech Astronomical Society.

Jizera Dark Sky Park (established in 2009)

The Jizera Dark Sky Park (abbreviated JOTO) is the first international dark sky park in the world. It is located in an almost uninhabited part of the Jizera Mountains, half on the Czech and half on the Polish side, and covers an area of 75 km^2. On the Czech side, it stretches from the village of Jizerka to Smrk Mountain, while in Poland it continues along the High Jizera Ridge and includes the Great Jizera Meadow and the town of Orle.

Beskydy Dark Sky Park (established in 2013)

The Beskydy Dark Sky Park (BOTO) is the second international dark sky park in the world. It is located in the Moravian-Silesian Beskydy Mountains, with its centre near the village of Staré Hamry. Approximately two thirds of the area is in the Czech Republic and one third in Slovakia and it covers a total area of 308 km^2.

Manětín Dark Sky Park (established in 2014)

The Manětín Dark Sky Park (MOTO) is the first dark sky park whose entire area is located only on Czech territory. It extends over the territory of ten municipalities and covers an area of 346 km^2. It has a relatively well-preserved nocturnal environment and starry sky.

International aspects of this issue are covered by the IDA (International Dark-Sky Association), see also www.darksky.org

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