Special Relativity and the Limiting Speed of Light

Francesco R. Ruggeri Hanwell, N.B. Sept. 25, 2023

We argue that the limiting speed of light in a vacuum for a particle with rest mass, i.e. c, may be seen from the unitarity of the Lorentz transformation, which in turn follows from a time reversal undoing the transformation. One may derive the Lorentz transformation for a particle with rest mass mo at x=0 at t by considering what would be seen from a frame moving at a constant -v. The time reversal of this frame, i.e. a frame moving at v, would undo the speed. Thus L(v)L(-v) = Identity. This is equivalent to the unitarity condition, which in turn leads to a (1, -1) type metric (i.e. brings in the 1-vv/bb factor, where b is a constant unknown velocity. It seems that if v exceed b, then free particle energy and momentum become imaginary. The case of v=b, is equivalent to mo=0 or E=pb, but light satisfies E=pc so one may associate b with c.

Although unitarity and L(v)L(-v)=identity are convenient for showing that E and p become imaginary if v>c, we argue that the limitation on speed already appears if one considers speed as appearing when an object at rest is viewed from a moving frame. At rest, the particle has x=0, t, but in the moving frame x'/t'=v. Thus, x,t and x',t' cannot be the same. One may then suggest a transformation linking them which introduces v (so that x'/t'=v). The transformation, however, must be dimensionless meaning one is forced to use x/b. This already imposes a limit on speed.

We also consider the case of special relativity applying to a photon. In such a case, the velocity v in -Et+ p dot r, i.e. r/t vector, is not that of the photon, but rather describes a moving mirror or interface, in the case of refraction. In such a case, one has: -Et+p dot r= t|p| { -E/|p| + unit vector dot dr/dt). Here dr/dt is the speed of the moving mirror/interface and E/|p| is the speed of the photon. We argue that if dr/dt exceeds the speed of light, one again obtains contradictions in equations. Here the unit vector is in the direction of the p momentum vector.

Deriving the Lorentz Transformation for a Particle with Rest Mass

Consider a particle with rest mass mo at x=0 at time t. As seen from a frame moving at -v vector, the particle has a velocity of v. At first, one may think |v| may take on any positive value, say to infinite. One may note, however, that if one first views the particle from a frame moving at -v and then views it after this by a frame moving at v, the particle is motionless, i.e.

L(v)L(-v) = identity ((1))

((1)) places a constraint on the transformation.

Mo at x=0 at t is seen to be a particle moving at v by a frame moving at -v. Thus x'/t'=v or:

$$\begin{vmatrix} x' \\ t' \end{vmatrix} = \begin{vmatrix} A(v) & (v/b)f(v) \\ B(v) & f(v) \end{vmatrix} \begin{vmatrix} x \\ bt \end{vmatrix} \quad ((2))$$

Here A(v) and B(v) are unknowns.

We set b=1 for convenience at this point, but note that one needs to have proper units, i.e. the matrix elements should be dimensionless and x and bt must have the same units.
Here it is assumed that f(v) is even in v so that the sign is carried by the factor v in vf(v).

One requires that L(v) L(-v) = identity where L(v) is the matrix in ((2)). This yields:

A(v)A(-v) - vfB(v) = 1   A(v)B(-v) + fB(v)=0   B(-v)vf +ff=0   B(-v)vf+ff=1 ((3))

In the third equation of ((3)), ff is positive, so for a positive v and f, B(-v) must be negative, so we assume B(v) is odd in B(v) i.e. B(v) = v g(v) where g is even in v. Then one has;
g(v)=1/sqrt(1-vv/bb) where b is a constant with units of velocity. The fourth equation of ((3)) yields: -vv gf+ ff=1 with a solution g=f. Finally, A=f. Thus, the Lorentz transformation is:

|f(v) vf(v) | with f(v) = 1/sqrt(1-vv/bb) with b being an unknown constant with units of velocity
|-vf(v) f(v)| ((4))

The Lorentz matrix ((4)) has been derived for a particle with rest mass (and hence a rest frame) through considerations on x,t. What about mo? As seen from the moving frame, the particle is in motion and so, from Newtonian mechanics, should have momentum and energy. One may assume a transformation N(v) such that N(v)N(-v) = identity. One may note, however, that momentum in Newtonian mechanics is mo v, so the (1,2) element of N(v) should be v q(v) where q(v) is an even function of v. This suggests that ((4)) may, in fact, also be the transformation matrix for (p,E). To test this, one may see that:

E= f(v) mo C = moC/ sqrt(1-vv/bb) where C and b are constants. Taking the low v limit with respect to b yields: E = moC (1+.5 vv/bb). To be consistent with Newtonian mechanics, C=bb.

One may notice immediately that b is an upper limit on the magnitude of v This means that v cannot have an infinite value. What, however, is b?

If one considers the Lorentz invariant   -EE + pbpb = - momo bbbb ((5)), then:

E= mobb/ sqt(1-vv/bb) and p= mo v / sqrt(1-vv/bb) ((6))

If v>b, then both energy and momentum are imaginary, which is unphysical.
One may also notice that for mo=0, ((5)) yields E=pb. This equation, however, mimics that of light (i.e. a wave with E=frequency and p proportional to 1/wavelength). Thus one may suggest that b = c the speed of light in a vacuum.

At first one would have assumed that v could take on any value, but unitarity following L(v)L(-v) = identity prevents this. It seems that more than unitarity, however, is causing this result. It was assumed that x’/t’ =v which is why the (1,2) element of the 2x2 transformation matrix is written as (v/b)f(v). The matrix elements must be dimensionless as x and t carry the units, actually x, bt. In other words, b, the limiting velocity appears when one insists that x’/t’ = v. The transformation must introduce v and so it must appear as v/b in the matrix in order to be dimensionless. The notion that velocity appears for a particle at rest as viewed by a frame moving at constant v, imposes the notion of maximum speed b because one must link x,t values in the rest frame to what is seen in the moving. If one were simply to say that in the moving frame the particle moves with v and is linked with an x' and t', there is no issue of a transformation or a limiting velocity b. The question then is: Are x and t in the rest frame the same as those experienced in the moving frame? It seems this cannot be because x'/t'=v, while this does not hold in the rest frame, where x=0 with t. Thus relating x and t to (x',t') in the two frames forces the transformation which forces the limiting velocity b. In other words, if there are constraints on what x' and t' may be (in terms of x,t), then these same constraints affect the velocity which may be seen from the moving frame. This, however, is minus the velocity of the particle and so restricts the physical velocity which it may attain.

Photon Considerations

Consider the case of a photon reflecting from a moving mirror. A photon has a speed c as seen from any frame, but -Et + p dot r still holds. We argue that for:

A = -Et + p dot r = t |p| (-E/|p| + unit vector dot r/t) ((7))

where the unit vector is in the direction of p and E/|p|=c and r/t vector is the velocity of the moving mirror, one needs to show that r/t cannot be larger than c. Here reflection from the moving mirror takes place in a vacuum.

We have argued in previous notes that A/t is a constant for the incident and reflected photons.

Unit vector dot r/t= v cos(AA) for the incident photon and -vcos(BB) for the reflected, where AA and BB are the incident and reflected angles measured from the y-axis, for a mirror lying along the x and moving along the y-axis.

The incident and reflected photons have |p1| and |p2| if the mirror is moving. Using conservation of momentum along the surface of the mirror yields:

|p1|sin(AA) = |p2|sin(BB) ((8))

Thus: A/t = A/t implies: sin(AA)/sin(BB) = (c-v cos(AA)) / (c+vcos(BB)) ((9))

AA and BB are between 0 and 90 degrees and so are positive. vcos(AA) and vcos(BB) are less than the values of v. If c>v, then ((9)) is guaranteed to be positive. In the limit AA->0, BB->0, one still needs the ratio sin(AA) / sin(BB) to be positive so v cannot be larger than c. c-v cos(AA) is interpreted as a relative velocity along the ray. If vcos(AA) were to become larger than c, (same argument for BB), then the incident ray would suddenly be moving in the opposite direction (i.e. away from the mirror) which is unphysical. Thus, moving mirror and reflection seems to also confirm that one cannot have a mirror move faster than c in a vacuum.

Conclusion

In conclusion, we try to investigate why there should be a limit on the speed of a particle with rest mass. We conclude that a particle at rest receives a speed if viewed from a moving frame. At first, this suggests that any v is possible because there is no constraint. The problem is that one needs an x’ and t’ value such that x’/t’=v. This begs the question: is x’ and t’ the same as in the rest frame? It is not, because one has x=0 and t for the rest frame. Thus, there must be a transformation on (x,t) to (x’,t’) which brings in a v. This v, however, must be dimensionless and so one has v/b where b is some limiting speed. Thus it is the constraint linking x,t to x’,’t with x’/t’=v which forces a limitation on maximum speed. We also consider how unitarity of the transformation leads to imaginary E,p for v>c and how considerations of light reflecting from a moving mirror break down in v>c.