

# Configuration Space

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## Abstract

A particular class of real manifolds (Hermitian spaces) naturally model smooth, possibly complex  $n$ -spaces. We show how to realize such a space as a restriction of a super-smooth stack using a compass. We also discuss the classical relationship between iterated loop spaces and the configuration space of a particle.

## 1 Background

### 1.1 Overview

We will trace our lineage back to, approximately, the early 1970s with works of Segal and McDuff, and to even earlier mathematical foundations in Boardman and Vogt, in the 1960s. Chapter one gives some preliminaries on tangent vector fields on smooth manifolds, and restrictions to the discrete case. Chapter two describes the configuration space of McDuff, with some criteria for tensorability. We also describe the E-spaces of Boardman and Vogt.

## 1.2 Imposing Discreteness on Smooth Spaces

Let  $\mathcal{C}_n^\infty$  be a smooth manifold of dimension  $d$ , possibly with corners, with or without boundary. One can restrict to a submanifold of the same dimension<sup>1</sup>:

$$C_n^\infty \rightarrow C_n^{fin}$$

so that the tangent vectors

$$\tan_{vec}(x) = (\tilde{v} \otimes \tilde{h})(x) \quad (1)$$

about the point  $x$  yield the following compass<sup>2</sup>:

$$\Omega_x^{k \sim \infty} = Comp_p$$

giving us  $x$  as the inf-pole and some discretized point  $k$  corresponding to a point “at infinity.” McDuff [1] used this discretization to model the creation and annihilation of antiparticle pairs, where

$$\lim_{\rightarrow} p = \sup(Comp_p) ; \quad \lim_{\leftarrow} = \inf(Comp_p)$$

Here, we choose to let  $k$  be any generic cardinal invariant within the compactly generated (presentable), smooth category  $SmFld$  of smooth fields. We have

$$(\lim_{n \rightarrow \infty} n) \twoheadrightarrow k$$

which “realizes” the smooth motion of a quasi-quantum as a particle in a Hermitian manifold

$$\mathbb{R}^4 \simeq \mathbb{C}^{\infty \dagger}$$

In [3], it was shown that if a boundary existed for  $\mathcal{C}_n^\infty$ , then it was unrealizable as a pullback locally within  $\mathbb{R}^4$ . We generalize this here to  $\mathbb{R}^d$  for any dimension.

## 1.3 Tangent vectors

**Axiom 1** *Let (eq. 1) be valid for any point  $p \in C_n^{fin}$ . Then, we say  $p$  obeys the “tangent space axiom.”*

**Proposition 1** *If  $p$  obeys the tangent space axiom, then  $Comp_p$  is stable.*

**Proof:** Since there exists a neighborhood  $\mathcal{U}(p)$  of  $p$ , and since the space is assumed to be Hausdorff, then there exists both a right and left limit of the directional derivative taken at  $p$ , lying inside  $\mathcal{U}(p)$

$$\forall p \in \text{spaces obeying Axiom 1} \quad \exists \lim_{\leftrightarrow} \vec{p} \in \mathcal{U}(p)$$

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<sup>1</sup>In doing so, one obtains a boundary,  $\partial C_n^{fin} \subset \text{int}(C_n^\infty)$

<sup>2</sup>See [2] for a brief introduction to compasses

### 1.3.1 Subcategories of $\square$

We denote the class of spaces obeying (Axiom 1) by  $\widehat{\square}$ , and let it be a full subcategory of  $\square$  (see [3] for more information). Objects of  $\widehat{\square}$  are spaces and morphisms are diffeotopies.

We can extend  $\widehat{\square}$  by taking its union with the class of pure potentials,  $\widehat{\square} \cap Pur$ , and denote this by  $\widehat{\square}_{Ext}$ . Since  $Pur$  is a smooth category [4],  $\widehat{\square}_{Ext}$  is also smooth (as can be easily demonstrated by the axiom of extension [5]). We then have the map

$$\mathcal{A} \in \widehat{\square} \xrightarrow{Mor} \mathcal{A}' \in \widehat{\square}_{Ext}$$

giving us a Morita equivalence between the underlying algebras of  $\widehat{\square}$  and its extension.<sup>3</sup>

The class of subcategories of  $\square$  is given by two pieces of data:

1. A full subcategory  $\widehat{\square}$  of  $\square$
2. An extension  $\widehat{\square}_{Ext}$  of  $\widehat{\square}$  into the union with  $Pur$

Which correspond to the set  $\square(\mathcal{A}, \mathcal{A}') = \square_{\mathcal{A}}$ .<sup>4</sup> We can show that this set is actually a poset by imposing an arbitrary relationship  $\mathcal{R}$  on  $\square_{\mathcal{A}}$  such that the generalized cocycle condition holds. That is to say, for a partial flag variety:

$$\widehat{\square} \simeq \mathcal{A} \subset \widehat{\square}_{Ext} \simeq \mathcal{A}' \subset \widehat{\square}_{Ext_{Ext}} \simeq \mathcal{A}''$$

we obtain the transitive relationship  $\mathcal{A}\mathcal{R}\mathcal{A}'\mathcal{R}\mathcal{A}''$ , which corresponds to the extension of the Morita equivalence class  $\mathcal{A}/Mor$  of the algebra  $\mathcal{A}$  to encompass boundaries beyond the limit  $k$  of a generic sequence of operators acting on geometric realization of the algebra.

**Example 1** Let  $\mathbb{C}$  be a complex space and  $\mathcal{A}$  its underlying algebra. The Riemann sphere,  $\mathbb{C} \cup \{\infty\}$  extends the algebra of this space to a new algebra  $\mathcal{A}'$ .

According to [6]<sup>5</sup>, in order for our theory to be quantum, we must allow for tensoring of manifolds:

$$\mathbb{M}_{\mathcal{A}} \otimes \mathbb{M}_{\mathcal{A}'} \xrightarrow{\simeq} \mathbb{M}_{\mathcal{A} \sqcup \mathcal{A}'} \quad (2)$$

$$\mathcal{U}(p) \otimes \mathcal{U}(p') \xrightarrow{\simeq} \mathcal{U}(p \amalg p') \quad (3)$$

<sup>3</sup>See [6] for more information.

<sup>4</sup>See [7, sect. 2b]

<sup>5</sup>The notation used by Segal differed from ours. Very elegantly, he wrote  $\mathcal{O}_x = \lim_{\leftarrow} \mathcal{H}\partial D$ . The vacuum expectation value,  $\Theta_k$ , is taken by tensoring over all  $\mathcal{O}_{x_k}$

## 2 Configuration space

Here we will discuss the configuration space of McDuff.

Let  $E_M$  be a bundle over a manifold  $M$ ; McDuff showed that the homotopy type of the bundle is equivalent to the homotopy type of a configuration space  $\tilde{C}^\pm$  of some set of particles  $\mathcal{P}^\pm$  which may have a positive or negative charge. In his model, all particles had pairwise separation  $\geq 2\varepsilon$ , and only particles of the same parameters could annihilate one another. McDuff proved this fact by invoking quasifibrations on a disc centered about some particle  $p$ .

Here, we add the following ingredient: every particle  $p^\pm$  has an associated truth value  $\tau(p^\pm)$  in the structure sheaf  $\mathcal{O}_X$  of the particle. The bijection

$$\tau(p^\pm) \leftrightarrow \dot{a}(p)$$

corresponds to a referential instantiation by an agent at a particular modal frame, corresponding physically to either the existence or non-existence of the particle in a position at a time  $t=0$ . That position is determined by the variable  $\theta$ , which determines the anisotropy between the absolute frame generated by  $\mathbb{T} \sqsubset \mathcal{O}_X$ . This is called a “state,” and is given by a bijective map of algebras  $\mathbb{T}_A \xleftarrow{MorExt} \mathcal{O}_{X_{A'}}.$

We can more succinctly summarize the results of McDuff’s wonderful treatment of configuration spaces if we introduce a canonical fiber bundle,  $\Gamma_\Delta$ , over  $\mathcal{O}_A$  by letting each  $\delta_i$ -small neighborhood about  $p$  take its fibers in  $\Gamma_\Delta$ . In this way, we derive the structure sheaf of the particle by

$$\mathcal{O}_X = Hom(\mathcal{O}_A, fib(\mathcal{O}_A)) \simeq \Gamma_\Delta$$

### 2.1 Symmetric Product

Let  $\otimes^m$  be the  $m$ -fold symmetric product. For every neighborhood  $\mathcal{U}(p^\pm)$ , we having an incoming connection,  $in_{\Gamma_\Delta}$ , and an outbound connection  $out_{\Gamma_\Delta}$ .

**Axiom 2 (Looping)** *Given a collection of neighborhoods  $\sum_{i=0}^n \mathcal{U}_i(p^\pm)$ , the  $m$ -fold symmetric product of incoming connection yields an outbound connection about a fixed point. Formally:*

$$in_{\Gamma_\Delta}(\mathcal{U}_i(p^\pm) \otimes^m \mathcal{U}_i(p^\pm)) = out_{\Gamma_\Delta}(\ast) \quad (4)$$

This gives us a fairly nice agreement with the vision of Boardman and Vogt of configuration spaces as iterated loop spaces. See, for instance, [7] and [8]. We obtain the following exact sequence:

$$\Gamma_\Delta : in_{\Gamma_\Delta} \rightarrow in_{\Gamma_\Delta} \rightarrow \dots \rightarrow out_{\Gamma_\Delta};$$

the sequence is long whence the symmetric product is taken about a smooth space, and short whence this space is discrete. We make the identification

$$Ho(\mathcal{U}_i(p^\pm)) = Ho(\mathbb{R}^n)$$

by letting the l.h.s. be equal to  $\mathcal{P}$ , and letting the r.h.s. be equal to  $\tilde{C}^\pm$ .

## 2.2 Superposition

We denote the superposition of all the particles in  $\mathcal{P}$  by  $\Psi_{\mathcal{P}}(\heartsuit)$ . This notation is due to O. Hancock, and is a very succinct representation of the “pure space” over a particle. Always, when such a superposition is considered, it is either over  $Pur$ , or over  $\hat{\square}_{Ext}$ . That is to say:

$$\Psi_{\mathcal{P}}(\heartsuit) \sim Pur \sqcap \vec{p}^\pm$$

In some sense, the wordline of a particle may be considered as a classical analogue of the structure sheaf  $\mathcal{O}_{Pur}$  of the particle over a probabilistic space. For technical reasons, we let the probability space be metrizable, and denote its metric by  $\mu$ .

The realization of a quantum is denoted by:

$$\hat{q} \star_\mu \mathcal{O}_{Pur} \longrightarrow q ; \quad q \in \mathbb{R}^n$$

Or as a *closure*:

$$\overline{(\hat{q}, \hat{q}')} ; \quad \hat{q} \mathcal{R} \hat{q}'$$

Zanthius<sup>6</sup> proposed that something similar to a quasi-quantum ( $\hat{q}$ ) may exist as a sort of “strange attractor,” where high probabilities of attraction converge (presumably asymptotically) to the instantiation of gravitational effects. This elegantly seems to unify the principles of relativity with quantum effects in a phenomenologically consistent manner.

However, one seems to be missing something from this theory. Namely, the underlying topological stack on which quantum gravitational phenomena emerge is left anonymous. Here, we propose that  $\mathbb{R}^n$  be a sufficient general manifold for realizing this stack. We must include the following caveat - the realization of the *stack*  $\mathcal{A}$  is not an ordinary (concrete) realization, but a *projective realization* onto the topology in which quanta actually emerge.

### 2.2.1 Quasi-quanta

Quasi-quanta were first invoked by P. Emmerson in [9] and were expanded upon in [3]. To provide a brief summary, a quasi-quantum,  $\hat{q}$ , has an a-priori existence which is not yet tied to an existential quantification. The type-dependent inclusion,  $\in^\bullet$ , in a subclass of  $Man$  gives us an existential quantification

$$\exists^\bullet p^\pm \in Man$$

where  $Man$  is the category of manifolds. We restrict to the category of real manifolds in the case where an n-tuple of quasi-quanta is realized as a physical quantum. The realization of quasi-quanta is given by a basepoint preserving

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<sup>6</sup>Real name unknown; private conversation

homomorphism  $S^n \times \text{Simpl} \longrightarrow S^n$ , which preserves the endpoints of a pre-determined interval. This interval is given by

$$(-\infty, 0] \times [0, \infty) \longrightarrow (-\infty, 0, \infty)$$

In our case<sup>7</sup>, and in Segal's case, a specialization  $\infty \rightsquigarrow k$  can be made to a discrete cardinal  $k$ . This assignment of an infinitary ideal to a discrete cardinal approximates a certain locally constant section of the smooth manifold  $\mathcal{C}_n^\infty$  at a place. This will be referred to as the *truncation* of  $\mathcal{C}_n^\infty$  with respect to a metric  $\mu$ .

I shall argue that truncation acts effectively as a form of quantization to instantiate action, as projected by the absolute frame  $\mathcal{A}$  to the real manifold  $\mathbb{R}^n$ . This is *Emmerson's thesis on quasi-quanta*.

This thesis is essentially metaphysical, as it takes some objects (i.e., the "energy numbers") to be a-priori to others, such as the reals. The real numbers are obtained by the restriction  $\mathbb{E}|_{\infty \rightarrow k}$ . We obtain not just one map,  $\mathbb{E} \longrightarrow \mathbb{R}$ , but a whole slough, via  $\text{hom}(\mathbb{E}, \mathbb{R})$ . In this way, it makes sense to define a sort of generalized connection between the two, and even more abstractly, a generalized connection  $\Gamma_{Ext}$  between a ring and its overring. In order to define such a construction, we assign an index set  $\mathcal{I}$  to each ring under the operation  $\star$ . We then have

$$\mathcal{I}_\star(\mathcal{R}_{ng}) : \mathcal{R}_{ng} \longrightarrow \mathcal{R}_{ng}|_{\tau \in \mathcal{O}_X}$$

giving us the faction-level correspondence between the elements of the ring and their localized (refracted) truth values. This corresponds to the canonical *operator product expansion*:

$$A(x)B(y) = \sum_i c_i(x-y)C_i(y)$$

where  $y$  is a point,  $A$  and  $B$  are operator-valued fields,  $C_i$  are operator-valued fields, and  $c_i$  are analytic functions over  $O \setminus \{y\}$ . The sums are convergent in the operator topology within  $O \setminus \{y\}$ .

**Example 2** Let  $\text{Strat}_{\mathcal{M}}$  be a stratified manifold, and let  $\mathcal{EP} : y \longrightarrow \xi$  be an exit path. Then, we have the equivalence

$$A(x)B(y) \sim G(y, \xi)$$

if  $x=y$ .

## 2.3 E-spaces

We kindly refer the reader to [7, theorem A] for the masterful articulation of Boardman and Vogt. Stated verbatim,

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<sup>7</sup>c.f. [10]

**Theorem A.** A CW-complex  $X$  admits an E-space structure with  $\pi_0(X)$  a group if and only if it is an infinite loop space. Every E-space  $X$  has a ‘‘classifying space’’  $BX$ , which is again an E-space.

**Definition 1** A ‘‘homotopy-everything  $H$  space’’ ( $E$ -space) is an  $H$ -space in which all coherence conditions hold.

An E-space  $\mathcal{E}$  has quotient uniformity for  $Y$  relative to a functor  $f$ , which has been denoted by Himmelberg [12] as  $f(\mathfrak{U})$ , where

$$f = X \times X \rightarrow Y \times Y$$

For suitably chosen bases, we have the homotopy groups  $\pi_0(X)$  and  $\pi_0(Y)$ , and also  $\text{hom}(X, Y)$ . In an E-space, this hom-set commutes under the group operation. A remark made at the end of the paper introducing quotient uniformities suggested that, for distinct timelike equivalent particles  $(p \sim p')/\in \mathbb{R}$ , there ought to be a distinct neighborhoods

$$(\mathcal{U}(p) \setminus p') \sim (\mathcal{U}(p') \setminus p)$$

This implies that for sufficiently small tangent vectors  $(\text{tan}_{vec}(p, p')) < 1$ , there exists a connection

$$(p, p') \xrightarrow{\Gamma_\Delta} \mathcal{U}(p, p'),$$

and a larger neighborhood

$$\sum_i \mathcal{U}_i(p, p') = \bigcap_i p_i \sim (p, p') \in^{\mathbb{R}} \mathbb{M}^n$$

which covers both of the two. The localization procedure represents finding the real part of a complex equation, but goes much further, in that it can express the similar behavior of overrings in general to extend their algebraic parts via the inclusion of some transcendental element, which forces the new members of the overring. These members are bijective onto some set of numbers, which can, in principle, be arbitrarily extended to inordinately large sizes.

The choice of E-spaces as models of configuration spaces were described beautifully as early as 1973.

### 3 Appendix A

**Axiom 3**

$$\text{Cov}(\hat{q}_i) = \bigcap_i p_i \text{ iff } p_i = \hat{q}_i$$

This gives us:

$$\mu(p_i) \stackrel{*}{=} \mu(\hat{q}_i)$$

so the metrics (stalks of  $\mathcal{O}_{\mu(\dot{x})}$ ) agree at a locally constant point  $\dot{x} = xyz$  on a real manifold.

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