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Energy Conservation in General Relativity and Flat Rotation Curves of Spiral Galaxies

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Abstract

This article shows that the vanishing covariant divergence of the energy-momentum tensor of matter is a conservation law. In addition, it is pointed out why energy-momentum pseudo-tensors of the gravitational field cannot represent the energy density of the gravitational field, but, apart from a factor, is described by the Einstein tensor. The necessarily existing conservation law of total energy, momentum and stress in general relativity is derived, thereby solving the cosmological constant problem as well as that of dark energy and dark matter. In Newton's theory of gravitation, it is the modified Poisson equation that approximately meets the requirement of conservation of total energy. A simple but fairly accurate model, consistent with observations, is presented, solving the modified Poisson equation to fit the calculated rotation curves to the observed speeds in spiral galaxies being composed of several components: the central region of the bulge, the bulge, the disk and dark matter, the latter of which is exactly described by this model.

Keywords

general relativity, matter tensor, energy-momentum tensor of the gravitational field, energy-momentum pseudo-tensor of the gravitational field, total energy-momentum tensor, cosmological constant, conservation law, dark energy, dark matter, modified Newtonian gravitation, spiral galaxy, rotation curve, method of weighted least squares.

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Contents

1. Matter tensor

By definition, the energy-momentum tensor of matter,

$$T^{\mu\nu} = T^{\mu\nu}_{(\mathrm{pf})} + T^{\mu\nu}_{(\mathrm{em})} + \dots , \tag{1}$$

also called the energy-momentum tensor or simply the matter tensor, consists of all kinds of matter-energy, but does *not* comprise the energy of the gravitational field [1, 2]. Herein,

$$T^{\mu\nu}_{(\mathrm{pf})} = \left(\varrho + \frac{P}{c^2}\right) u^\mu u^\nu + P g^{\mu\nu} \tag{2}$$

is the energy-momentum tensor of a perfect fluid and

$$T^{\mu\nu}_{(\mathrm{em})} = \frac{1}{\mu_0}\left(F^{\mu\alpha} F^{\nu}{}_{\alpha} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right) \tag{3}$$

is the energy-momentum tensor of the electromagnetic field [3].

In special relativity, i.e. in an inertial frame as well as in a local inertial frame, the vanishing partial divergence of the matter tensor,

$$\partial_\nu T^{\mu\nu} = 0\,, \tag{4}$$

demonstrates that the sum of all kinds of matter-energy are conserved [1, 4]. However, Landau and Lifshitz state in § 96 of Ref. [2] regarding the vanishing covariant divergence of the matter tensor, i.e. Eq. (96.1),

$$T^k_{i;\,k} = \frac{1}{\sqrt{-g}}\frac{\partial(T^k_i\sqrt{-g})}{\partial x^k} - \frac{1}{2}\frac{\partial g_{kl}}{\partial x^i}T^{kl} = 0\,, \tag{5}$$

that "in this form, however, this equation does not generally express any conservation law whatever. Because the integral

$$\int T^k_i\sqrt{-g}\;dS_k \tag{6}$$

is conserved only if the condition

$$\frac{\partial(\sqrt{-g}\,T^k_i)}{\partial x^k} = 0 \tag{7}$$

is fulfilled, and not (96.1)." Corresponding arguments can also be found on page 27 of Ref. [3] and in § 126 of Ref. [5].

In contrast, to paraphrase Lavenda in Sec. 3.2 of Ref. [6], Eq. (7) is not covariant and therefore in general not a condition for a conservation law, so that the latter in general is not represented by the integral (6).

In Eq. (20.31) of Ref. [1], Fließbach has applied the principle of covariance on the vanishing partial divergence of the matter tensor in a local inertial frame,

$$\partial_\nu T^{\mu\nu} = 0 \quad\longrightarrow\quad \nabla_\nu T^{\mu\nu} = 0\,, \tag{8}$$

representing a conservation law.

By performing an infinitesimal transformation of the dynamical variables in the change of the matter action given by Eq. (12.2.2),

$$\delta I_M = \frac{1}{2}\int d^4x\,\sqrt{g(x)}\;T^{\mu\nu}(x)\,\delta g_{\mu\nu}(x)\,, \tag{9}$$

Weinberg proves in Sec. 12.3 of Ref. [7] that the vanishing covariant divergence of the matter tensor (5) represents a conservation law. He emphasizes that "*the energy-momentum tensor defined by Eq.* (12.2.2) *is conserved* (*in the covariant sense*) *if and only if the matter action is a scalar*. Also, with I_M a scalar, (12.2.2) shows immediately that $T^{\mu\nu}$ is a symmetric *tensor*, so this definition of the energy-momentum tensor has all the properties for which one could wish. This proof, that general covariance implies energy-momentum conservation, has an exact analog in the proof that gauge invariance implies charge conservation."

Therefore, one must conclude that a conservation law in general relativity is formed by a tensor, the covariant divergence of which vanishes [8]. Hence, the matter tensor is conserved in general relativity contrary to the wording of Landau and Lifshitz in § 96 of Ref. [2] that "in this form, however, this equation does not generally express any conservation law whatever." But they actually relate this statement to conservation of *total* energy by adding that "this is related to the fact that in a gravitational field the four-momentum of the matter alone must not be conserved, but rather the four-momentum of matter plus gravitational field; the latter is not included in the expression for T^k_i." This is also why total energy equals matter-energy plus energy of the gravitational field.

2. Completed field equations

Straumann states in Sec. 3.4 of Ref. [3] that "a general conservation law for energy and momentum does *not* exist in GR. This has been disturbing to many people, but one simply has to get used to this fact. There is no 'energy-momentum tensor for the gravitational field'. Independently of any formal arguments, Einstein's equivalence principle tells us directly that there is no way to localize the energy of the gravitational field: The 'gravitational field' (the connection $\Gamma^\mu_{\ \alpha\beta}$) can be locally transformed away. But if there is no field, there is locally no energy and no momentum." Corresponding arguments can also be found in §20.4 of Ref. [4].

In Newton's theory of gravitation, the energy density of the gravitational field amounts to

$$\varepsilon_{\rm gf}(\mathbf{r}) = -\frac{[\boldsymbol{\nabla}\Phi(\mathbf{r})]^2}{8\pi G}\,. \tag{10}$$

However, in the Poisson equation,

$$\Delta\Phi(\mathbf{r}) = 4\pi G\varrho(\mathbf{r})\,, \tag{11}$$

there does not appear the energy density of the gravitational field, but only the mass distribution on the right-hand side as a source of gravitation. The Poisson equation (11) is only an approximation of general relativity in the limit of weak gravitational fields. However, all kinds of energy have to be taken into account to satisfy the requirements of a precise theory of gravitation. Not only for this reason the energy-momentum *tensor* of the gravitational field must appear in Einstein's field equations, but also to satisfy the correspondence principle because of the existence and the localizability of the energy density of the gravitational field in Newton's theory of gravitation [8], see Eq. (10). Moreover, our universe would be a chaos if there were no conservation of total energy.

By introducing the energy-momentum tensor of the gravitational field,

$$A_{im} = -\kappa^{-1}G_{im}\,, \tag{12}$$

where $\kappa = 8\pi G/c^4$ and G_{im} is the Einstein tensor, Einstein's field equations

$$G_{im} = \kappa T_{im} \tag{13}$$

can be rearranged to get Levi-Civita's field equations,

$$T_{im} + A_{im} = 0\,, \tag{14}$$

the latter of which have been criticized by Einstein in §6 of Ref. [9]. He rightly objects that "in (14) the components of the *total energy* vanish everywhere. The equations (14), for example, do not exclude the possibility ... that a material system dissolves into just nothing without leaving any trace. Because the total energy in (14) ... is zero from the beginning: the conservation of this energy value does not demand the continued existence of the system in any form." However, this physical shortcoming can be remedied by a simple modification: One just needs to introduce the *non-zero* total energy-momentum tensor L_{im} on the right-hand side of Eq. (14), so that the completed Levi-Civita field equations read [8]

$$T_{im} + A_{im} = L_{im}\,. \tag{15}$$

This modification highlights that Einstein's field equations (13) must be incomplete and moreover violate the conservation law of total energy. Nonetheless, they can be utilized as a good approximation in cases where the total energy-momentum tensor L_{im} does not play a significant role.

According to Lovelock's theorem – see e.g. Sec. 3.2.2 in Ref. [3], in particular Theorem 3.1 and Eq. (3.51) – Einstein's field equations in their maximum possible modified form read

$$G_{im} = \kappa T_{im} - \Lambda g_{im}\,, \tag{16}$$

which are Einstein's field equations with the cosmological constant Λ. They can be rearranged to get the completed Levi-Civita field equations (15), where

$$L_{im} = \kappa^{-1}\Lambda g_{im}\,. \tag{17}$$

2.1 Cosmological constant

By replacing the cosmological constant Λ with a universal constant λ in Eq. (16) and then taking the trace gives

$$\kappa T + R = 4\lambda\,, \tag{18}$$

which can be differentiated to obtain

$$\frac{\partial}{\partial x^k}(\kappa T + R) = 0\,. \tag{19}$$

The general solution to these differential equations reads

$$\kappa T + R = 4\Lambda\,, \tag{20}$$

which equals to the trace of Eq. (16), where Λ, in contrast to λ, is a constant of integration and is therefore a *parameter* and no universal constant [10] reflecting the fact that different gravitational systems hold different total energy densities [8]. Thus, there exists a different metric ds^2 with a different *non-zero* Λ for each gravitational system. This finding solves the cosmological constant problem and is not in conflict with Lovelock's theorem. However, it is questionable whether the term "cosmological constant" is still appropriate, but one has got used to this designation.

The value of the cosmological constant Λ regarding the metric ds^2 of each gravitational system has to be determined by observations because it is initially unknown. The cosmological constant Λ regarding the metric ds^2 of our solar system must be tiny, otherwise the computed angle of the perihelion shift of Mercury would not match the observed one. Moreover, the cosmological constant Λ has no effect on the angle of light deflection.

With these new findings, Einstein's condition for emptiness, $G_{im} = 0$, is outdated and must now read

$$G_{im} = -\Lambda g_{im} \tag{21}$$

to satisfy the requirement of conservation of total energy. In fact, "empty" space-time is not really empty because it consists of the energy of the gravitational field, which is why it is more appropriate to designate it *matter-free* instead of "empty" space-time. In matter-free space-time, $T_{im} = 0$ and consequently in this case, the total energy density equals the energy density of the gravitational field.

Relating the cosmological constant to the energy density of the vacuum leads to a huge mismatch between its theoretical and observed value, giving rise to the cosmological constant problem. It is important to classify general relativity as a classical and not as a quantum theory. Hence, the cosmological constant must not be related to the energy density of the vacuum, since the latter only occurs in a quantum theory. Apart from a factor, the cosmological constant corresponds to a scalar curvature of space-time,

$$\Lambda = \frac{R}{4}\,, \tag{22}$$

which can be easily recognized from Eq. (20) in matter-free space-time.

The *modified* Poisson equation,

$$\Delta\Phi(\mathbf{r}) = 4\pi G\varrho(\mathbf{r}) - \Lambda c^2\,, \tag{23}$$

that is obtained from Einstein's field equations *with* the cosmological constant (16) in the limit of weak gravitational fields, approximately meets the requirement of conservation of total energy in Newton's theory of gravitation in contrast to Eq. (11). Nonetheless, the latter can be utilized as a good approximation in cases where the cosmological constant Λ does not play a significant role.

2.2 Energy-momentum tensor of the gravitational field

The field equations (15) are a tensor equation that must comprise the energy density of the gravitational field in form of a *tensor*, which applies to Eq. (12). Additionally, this tensor has the unit of measurement of an energy density required to represent a tensor of any kind of energy.

The component

$$g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right) \tag{24}$$

of the metric tensor of the Schwarzschild metric comprises the Newtonian gravitational potential

$$\Phi(r) = -\frac{GM}{r}\,, \tag{25}$$

which is why the metric tensor g_{im} is a quantity that has to be assigned to the gravitational field and hence to its energy density. It is of great importance to recognize that in the mixed-tensor representation of the field equations (15),

$$T_i^k + A_i^k = L_i^k\,, \tag{26a}$$
$$A_i^k = -\,\kappa^{-1} G_i^k\,, \tag{26b}$$
$$L_i^k = \kappa^{-1} \Lambda\, \delta_i^k\,, \tag{26c}$$

all metric tensors and their first two derivatives therein appear only in the Einstein tensor G_i^k, so that solely the tensor A_i^k is eligible for representing the energy density of the gravitational field [8]. The energy-momentum tensor of the gravitational field (26b) comprises terms with Christoffel symbols squared. This is in conformance with Newton's theory of gravitation, in which the analogous expression $[\boldsymbol{\nabla}\Phi(\mathbf{r})]^2$ occurs in the energy density of the gravitational field (10).

The vanishing of the covariant divergence of the matter tensor, $T^k_{i;\,k} = 0$, means that the matter tensor is conserved in general relativity. Accordingly, the vanishing of the covariant divergence of the energy-momentum tensor of the gravitational field, $A^k_{i;\,k} = 0$, means that the energy-momentum tensor of the gravitational field is conserved. Consequently, matter-energy is *not* converted into energy of the gravitational field and vice versa.

The energy-momentum tensor of the gravitational field given by Eq. (26b) does not vanish in a local inertial frame, because it contains non-vanishing terms with second derivatives of the metric tensor, otherwise the total energy-momentum tensor shown in Eq. (26c) would vanish [8]. In this regard, Einstein rightly objects in §6 of Ref. [9] that "the conservation of this energy value does not demand the continued existence of the system in any form." Consequently, there would be no free fall because then "a material system dissolves into just nothing without leaving any trace." In fact, this finding rules out the statement of Straumann in Sec. 3.4 of Ref. [3] that "if there is no field, there is locally no energy and no momentum" and that of Misner, Thorne and Wheeler in §20.4 of Ref. [4] that "no local gravitational field means no 'local gravitational energy-momentum'."

This is also the reason why energy-momentum pseudo-tensors of the gravitational field cannot describe the energy density of the gravitational field, because they are no tensors as they vanish in a local inertial frame. Nevertheless, they are used to represent the energy density of the gravitational field and to form a "conservation law". Misner, Thorne and Wheeler justifiably state in §20.4 of Ref. [4] regarding energy-momentum pseudo-tensors of the gravitational field that "there is no unique formula for it, but a multitude of quite distinct formulas. The two cited are only two among an infinity." However, contrary to their statements in §20.4 of Ref. [4], Eq. (26b) demonstrates the necessarily existing unique formula for local gravitational energy-momentum. Thereby, it is localizable and neither in conflict with nor forbidden by the equivalence principle.

In contrast to Newton's theory of gravitation, in general relativity, the energy of the gravitational field is stored as curvature of space-time, which is why it is *not* equivalent to the gravitational potential energy that, apart from a sign, equals the gravitational binding energy [8].

3. Conservation law of total energy

The *total* energy-momentum tensor (26c) is *conserved*, which is why its covariant divergence vanishes,

$$L^k_{i;\,k} = \nabla_k (T_i^k + A_i^k) = 0\,. \tag{27}$$

By taking advantage of the special property of the Kronecker tensor,

$$\delta^k_{i;\,k} = \delta^k_{i,\,k} = 0\,, \tag{28}$$

this conservation law of total energy can be simplified,

$$L^k_{i;\,k} = L^k_{i,\,k} = \frac{\partial (T_i^k + A_i^k)}{\partial x^k} = 0\,. \tag{29}$$

One can go even further and take the derivative instead of the divergence because

$$\delta^k_{i;\,j} = \delta^k_{i,\,j} = 0\,. \tag{30}$$

Thus, not only the divergences but also the derivatives vanish, so that

$$L^k_{i;\,j} = L^k_{i,\,j} = \frac{\partial (T_i^k + A_i^k)}{\partial x^j} = 0\,, \tag{31}$$

which shows the conservation law of total energy, momentum and stress in general relativity in its differential form [8].

One can consider a closed region with volume V. The volume integration over $L^k_{i,\,0}$ in Eq. (31) results in

$$\frac{\partial}{\partial t}\int_V dV\, L_i^k = \frac{\partial}{\partial t}\int_V dV\,(T_i^k + A_i^k) = 0\,, \tag{32}$$

whereby the conserved total energy, momentum and stress within the closed region with volume V are given by

$$E_i^k = \kappa^{-1}\Lambda\, V \delta_i^k = \int_V dV\, L_i^k = \int_V dV\,(T_i^k + A_i^k) = \text{constant}\,, \tag{33}$$

which can only be obtained in the mixed-tensor representation [8].

4. Dark energy and dark matter

The cosmological constant regarding the Friedmann-Lemaître-Robertson-Walker metric of the universe is proportional to its total energy density,

$$L_0^0 = \kappa^{-1} \Lambda\, \delta_0^0 \,. \tag{34}$$

This explains the phenomenon of dark energy [8]. The cosmos is expanding at an accelerated rate, since its $\Lambda > 0$, so also $L_0^0 > 0$.

In Newton's theory of gravitation, the energy density of the gravitational field in matter-free space outside of a star is always negative, $\varepsilon_{\mathrm{gf}} < 0$, see Eq. (10). Consequently in general relativity, the energy density of the gravitational field in matter-free space-time regarding the metric ds^2 of a star must be negative as well, i.e. $A_0^0 < 0$, so that by using Eqs. (26a) and (26c) one easily recognizes that $\Lambda < 0$, which explains the phenomenon of dark matter [8].

Either the "mass" density of dark matter,

$$\varrho_{\mathrm{dm}} = \frac{\Lambda}{\kappa c^2} \qquad (\Lambda < 0)\,, \tag{35}$$

or that of dark energy,

$$\varrho_{\mathrm{de}} = \frac{\Lambda}{\kappa c^2} \qquad (\Lambda > 0)\,, \tag{36}$$

is homogeneously distributed in space-time regarding the metric ds^2 of the respective gravitational systems. Because Eq. (22) applies in matter-free space-time, apart from a factor, the "mass" density of dark matter in fact is nothing else than a constant, negative scalar curvature of space-time, while that of dark energy is nothing else than a constant, positive scalar curvature of space-time which both arise due to conservation of total energy regarding the metric ds^2 of the respective gravitational systems [8]. But also very faint baryonic matter and, if present, exotic particles contribute to dark matter, however are taken into account in the matter tensor.

5. Model of spiral galaxies

In this section, a simple but fairly accurate model, consistent with observations, is presented, solving the modified Poisson equation (23) to fit the calculated rotation curves to the observed speeds in spiral galaxies being composed of several components [11, 12]: the central region of the bulge (bh), the bulge (b), the disk (d) and dark matter (dm). The gravitational potential of a spiral galaxy can be subdivided into the part of baryonic and that of dark matter,

$$\Phi(r) = \Phi_{\mathrm{bar}}(r) + \Phi_{\mathrm{dm}}(r)\,. \tag{37}$$

The centripetal acceleration on a test particle is given by

$$g(r) = \frac{\partial \Phi}{\partial r} = g_{\mathrm{bar}}(r) + g_{\mathrm{dm}}(r)\,, \tag{38}$$

from which the radial acceleration relation

$$g = g(g_{\mathrm{bar}}) \tag{39}$$

can be determined. The squared circular speed of a test particle is obtained by

$$v^2(r) = r\frac{\partial \Phi}{\partial r} = v_{\mathrm{bar}}^2(r) + v_{\mathrm{dm}}^2(r)\,, \tag{40}$$

the square root of which is the computed rotation curve, that can be fitted to the n observed speeds $v_{\mathrm{obs}}(r_i)$ and their respective uncertainties $\sigma_{v_{\mathrm{obs}}}(r_i)$ by using the method of weighted least squares [13],

$$\chi^2 = \sum_{i=1}^{n} \left[\frac{v_{\mathrm{obs}}(r_i) - v(r_i)}{\sigma_{v_{\mathrm{obs}}}(r_i)}\right]^2 , \tag{41a}$$

$$\frac{\partial \chi^2}{\partial M_{\mathrm{bh}}} = \frac{\partial \chi^2}{\partial M_{\mathrm{b}}} = \frac{\partial \chi^2}{\partial M_{\mathrm{d}}} = \frac{\partial \chi^2}{\partial R_{\mathrm{b}}} = \frac{\partial \chi^2}{\partial R_{\mathrm{d}}} = \frac{\partial \chi^2}{\partial \Lambda} = 0\,, \tag{41b}$$

where the quantities according to which differentiation is made are the model parameters that are specified below.

5.1 Baryonic matter components

The baryonic mass M_{bh} in the central region of the bulge is considered to be point-like and probably containing a supermassive black hole (bh). The baryonic matter in the bulge is considered to be exponentially and isotropically distributed,

$$\varrho_{\mathrm{b}}(r) = \varrho_0 \exp\left(-\frac{r}{R_{\mathrm{b}}}\right)\,, \tag{42}$$

where ϱ_0 is the central mass density of the bulge and R_{b} is the bulge scale length. The mass of the exponential bulge inside the sphere with radius r is given by

$$\begin{aligned} M_\varrho(r) &= 4\pi \int_0^r dr'\, r'^2 \varrho_{\mathrm{b}}(r') \\ &= M_{\mathrm{b}}\left[1 - \exp\left(-\frac{r}{R_{\mathrm{b}}}\right)\left(1 + \frac{r}{R_{\mathrm{b}}} + \frac{r^2}{2R_{\mathrm{b}}^2}\right)\right]\,, \end{aligned} \tag{43}$$

where

$$M_{\mathrm{b}} = 8\pi \varrho_0 R_{\mathrm{b}}^3 \tag{44}$$

is the total baryonic mass of the bulge. The disk of a spiral galaxy can considered to be infinitesimally thin showing an exponentially mass distribution of baryonic matter [14, 15], so that its surface mass distribution is given by

$$\Sigma(r) = \Sigma_0 \exp\left(-\frac{r}{R_{\mathrm{d}}}\right)\,, \tag{45}$$

where Σ_0 is the central surface mass density of the disk and R_{d} is the disk scale length. The baryonic mass of the disk inside the radius r amounts to [15]

$$M_\Sigma(r) = 2\pi \int_0^r dr'\, r'\Sigma(r') = M_\mathrm{d}\left[1 - \exp\left(-\frac{r}{R_\mathrm{d}}\right)\left(1 + \frac{r}{R_\mathrm{d}}\right)\right], \tag{46}$$

where

$$M_\mathrm{d} = 2\pi\Sigma_0 R_\mathrm{d}^2 \tag{47}$$

is the total baryonic mass of the disk. Hence, the total baryonic mass of the spiral galaxy amounts to

$$M_\mathrm{bar} = M_\mathrm{bh} + M_\mathrm{b} + M_\mathrm{d}\,. \tag{48}$$

The gravitational potential in the disk of a spiral galaxy generated by baryonic matter is composed of those ones of the respective baryonic matter components [11, 12, 14, 15],

$$\Phi_\mathrm{bar}(r) = \Phi_\mathrm{bh}(r) + \Phi_\mathrm{b}(r) + \Phi_\mathrm{d}(r)\,, \tag{49}$$

where

$$\Phi_\mathrm{bh}(r) = -\frac{GM_\mathrm{bh}}{r}\,, \tag{50a}$$

$$\Phi_\mathrm{b}(r) = -\frac{GM_\varrho}{r} - 4\pi G \int_r^\infty dr'\, r'\varrho_\mathrm{b}(r') = -\frac{GM_\varrho}{r} - 4\pi G\varrho_\mathrm{b}R_\mathrm{b}^2\left(1 + \frac{r}{R_\mathrm{b}}\right), \tag{50b}$$

$$\Phi_\mathrm{d}(r) = -\pi G\Sigma_0 r[I_0(y)K_1(y) - I_1(y)K_0(y)] \tag{50c}$$

and

$$y = \frac{r}{2R_\mathrm{d}}\,. \tag{51}$$

At this point it should be noted that Eq. (50c) is only valid in the galactic disk. The centripetal acceleration on a test particle in the disk caused by baryonic matter is obtained by

$$g_\mathrm{bar}(r) = \frac{\partial\Phi_\mathrm{bar}}{\partial r}\,. \tag{52}$$

The corresponding squared circular speed amounts to

$$v_\mathrm{bar}^2(r) = r\frac{\partial\Phi_\mathrm{bar}}{\partial r} = v_\mathrm{bh}^2(r) + v_\mathrm{b}^2(r) + v_\mathrm{d}^2(r)\,, \tag{53}$$

where the contributions from the respective components of a spiral galaxy read [11, 14, 15]

$$v_\mathrm{bh}^2(r) = r\frac{\partial\Phi_\mathrm{bh}}{\partial r} = \frac{GM_\mathrm{bh}}{r}\,, \tag{54a}$$

$$v_\mathrm{b}^2(r) = r\frac{\partial\Phi_\mathrm{b}}{\partial r} = \frac{GM_\varrho}{r}\,, \tag{54b}$$

$$v_\mathrm{d}^2(r) = r\frac{\partial\Phi_\mathrm{d}}{\partial r} = 4\pi G\Sigma_0 R_\mathrm{d} y^2 \times [I_0(y)K_0(y) - I_1(y)K_1(y)]\,. \tag{54c}$$

5.2 Dark matter component

From the contribution of dark matter in the modified Poisson equation (23),

$$\Delta\Phi_\mathrm{dm} = \frac{1}{r}\frac{\partial^2(r\Phi_\mathrm{dm})}{\partial r^2} = -\Lambda c^2 = -8\pi G\varrho_\mathrm{dm}\,, \tag{55}$$

the gravitational potential generated by dark matter is obtained [12]

$$\Phi_\mathrm{dm}(r) = -\frac{\Lambda c^2 r^2}{6} = -\frac{4}{3}\pi G r^2\varrho_\mathrm{dm} = -\frac{GM_\mathrm{dm}}{r}\,, \tag{56}$$

where

$$M_\mathrm{dm}(r) = \frac{\Lambda c^2 r^3}{6G} = \frac{4}{3}\pi r^3\varrho_\mathrm{dm} \le 0 \tag{57}$$

is the amount of dark matter inside the sphere with radius r. The centripetal acceleration on a test particle caused by dark matter is given by

$$g_\mathrm{dm}(r) = \frac{\partial\Phi_\mathrm{dm}}{\partial r} = -\frac{\Lambda c^2 r}{3} = -\frac{8}{3}\pi G r\varrho_\mathrm{dm} = -\frac{2GM_\mathrm{dm}}{r^2}\,. \tag{58}$$

The corresponding squared circular speed amounts to

$$v_\mathrm{dm}^2(r) = r\frac{\partial\Phi_\mathrm{dm}}{\partial r} = -\frac{\Lambda c^2 r^2}{3} = -\frac{8}{3}\pi G r^2\varrho_\mathrm{dm} = -\frac{2GM_\mathrm{dm}}{r}\,. \tag{59}$$

6. Results and discussion

In Fig. 1, the calculated rotation curves are shown which have been fitted to the H I/Hα observational data of a few spiral galaxies from SPARC [16] using the model shown in Sec. 5, while refraining from utilizing the mass models of Ref. [16]. The reason for this is that the latter have been established by using the data of the surface photometry from SPARC which has been made at just one wavelength, namely at 3.6 microns in the infrared portion of the electromagnetic spectrum. However, it is well-known that spiral galaxies become bluer with increasing distances from the galactic center [17]. This means that matter in the outer regions, which rather emits at shorter wavelengths, may not be detected to the same extent at 3.6 microns as in the central region of a spiral galaxy. Not only for this reason, the author utilizes the model given in Sec. 5, but also because therein the exact formulas of the dark matter component are demonstrated which is not in case of the mass models of Ref. [16].

For the computations of the model in Sec. 5, the author has made use of the spreadsheet program "LibreOffice Calc" and its included tool "Solver" by specifying reasonable search intervals for the six model parameters for solving Eqs. (41). The computed rotation curves fit the observed speeds remarkably well, see Fig. 1. Flat and even increasing rotation curves can be described by the simple model in Sec. 5. The values of the fitted model parameters of the spiral galaxies under consideration as well as their total baryonic mass are given in Tab. 1. These results differ from those listed in Tab. 1 of Ref. [12], because in contrast to the present work, the method of unweighted least squares has been used in Ref. [12].

7. Conclusions and outlook

The vanishing covariant divergence of the matter tensor demonstrates that the matter tensor is conserved in general relativity. The conserved energy-momentum tensor of the gravitational field is given by Eq. (26b). Einstein's field equations with the cosmological constant Λ as a parameter satisfy the requirement of conservation of total energy, momentum and stress.

The rotation curves of spiral galaxies can be described remarkably well by using the model in Sec. (5) comprising the exact formulas for the dark matter component and solving the *modified* Poisson equation (23), the latter of which approximately meets the requirement of conservation of total energy in Newton's theory of gravitation. Each spiral galaxy has different model parameters, so that there is a lot of work to do in future, because there are many spiral galaxies the computed rotation curves have to be fitted to.

An extension of general relativity is the Einstein-Cartan theory which takes torsion and spin into consideration [18]. This causes a *repulsive* gravitational interaction within matter preventing the formation of singularities and thereby explaining the inflation of the early universe. However, the Einstein-Cartan theory only plays a significant role at huge mass densities, while in matter-free space-time, there is no difference between Einstein-Cartan theory and general relativity. The cosmological term has to be taken into account to satisfy the requirement of conservation of total energy, momentum and stress in the Einstein-Cartan theory.

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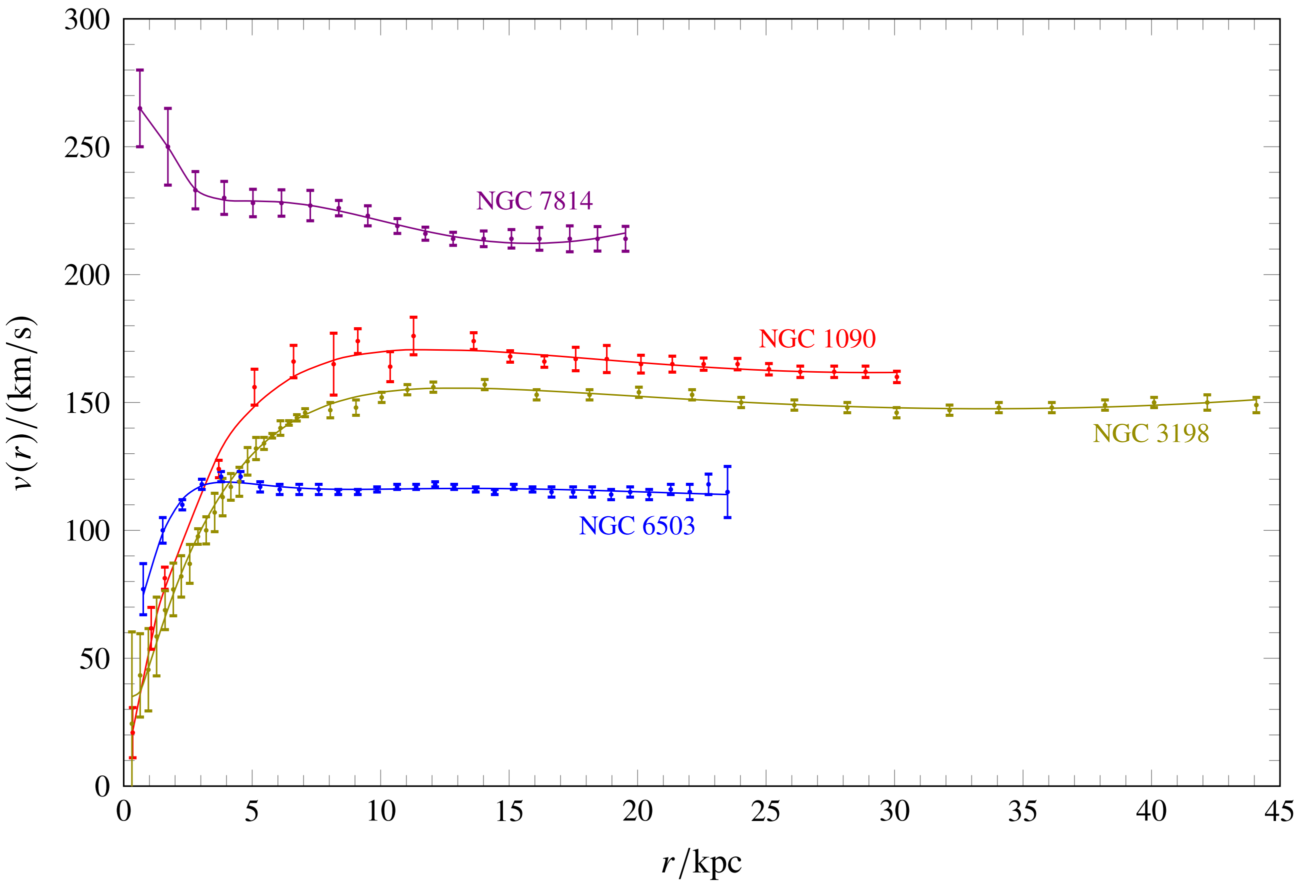

Figure 1. The observed speeds in a few spiral galaxies from SPARC are shown by marks with error bars. The fitted rotation curves are depicted by the respective solid lines. The fitted model parameters of the spiral galaxies and their respective total baryonic masses are listed in Tab. 1.

Table 1. This table lists the fitted model parameters of the spiral galaxies considered in Fig. 1 and their respective total baryonic masses.

spiral galaxy	M_{bh} (10^9 $\mathrm{M}_\odot$)	M_{b} (10^9 $\mathrm{M}_\odot$)	M_{d} (10^9 $\mathrm{M}_\odot$)	M_{bar} (10^9 $\mathrm{M}_\odot$)	R_{b} (kpc)	R_{d} (kpc)	Λ (10^{-48} m^{-2})
NGC 1090	0.0	37.3107	72.2153	109.526	2.1227	7.3978	−0.3301
NGC 3198	0.0726	44.5548	67.5261	112.1534	2.5049	9.6868	−0.1905
NGC 6503	0.2867	11.6626	42.2537	54.203	0.8919	7.663	−0.1511
NGC 7814	4.0504	17.2821	74.4207	95.7532	0.3088	3.4627	−2.0907