INTEGERS: ELECTRONIC JOURNAL OF COMBINATORIAL NUMBER THEORY 7 (2007), #A56

MIXED SUMS OF SQUARES AND TRIANGULAR NUMBERS (II)

Song Guo
Department of Mathematics, Huaiyin Teachers College, Huaian 223001, P. R. China
guosong77@sohu.com

Hao Pan
Department of Mathematics, Shanghai Jiaotong University, Shanghai 200240, P. R. China
haopan79@yahoo.com.cn

Zhi-Wei Sun[1]
Department of Mathematics, Nanjing University, Nanjing 210093, P. R. China
zwsun@nju.edu.cn

Received: 4/16/07, Revised: 11/12/07, Accepted: 12/6/07, Published: 12/20/07

Abstract

For $x \in \mathbb{Z}$, let t_x denote the triangular number $x(x+1)/2$. Following a recent work of Z. W. Sun, we show that every natural number can be written in any of the following forms with $x, y, z \in \mathbb{Z}$:

$$x^2 + 3y^2 + t_z,\ x^2 + 3t_y + t_z,\ x^2 + 6t_y + t_z,\ 3x^2 + 2t_y + t_z,\ 4x^2 + 2t_y + t_z.$$

This confirms a conjecture of Sun.

1. Introduction

In 1916 S. Ramanujan [6] found all those positive integers a, b, c, d such that every natural number can be written in the form $ax^2 + by^2 + cz^2 + dw^2$ with $x, y, z, w \in \mathbb{Z}$.

Let a, b, c be positive integers with $a \leqslant b \leqslant c$. In 2005 L. Panaitopol [5] showed that any positive odd integer can be written as $ax^2 + by^2 + cz^2$ with $x, y, z \in \mathbb{Z}$, if and only if the vector (a, b, c) is $(1, 1, 2)$ or $(1, 2, 3)$ or $(1, 2, 4)$.

[1]This author is responsible for communications, and supported by the National Science Fund (Grant No. 10425103) for Distinguished Young Scholars in China. Homepage: http://math.nju.edu.cn/∼zwsun

As usual, for any $x \in \mathbb{Z}$ we call $t_x = x(x+1)/2$ a *triangular number*. In 1862 J. Liouville (cf. L. E. Dickson [1, p. 23]) determined those positive integers a, b, c for which any natural number can be written as $at_x + bt_y + ct_z$ with $x, y, z \in \mathbb{Z}$.

Let $n \in \mathbb{N} = \{0, 1, 2, \dots\}$. As observed by L. Euler (cf. [1, p. 11]), the fact that $8n + 1$ is a sum of three squares (of integers) implies that n can be expressed as a sum of two squares and a triangular number. According to [1, p. 24], E. Lionnet stated, and V. A. Lebesgue [3] and M. S. Réalis [7] showed that n is also a sum of two triangular numbers and a square. In 2006 this was re-proved by H. M. Farkas [2] via the theory of theta functions.

In [8] Z. W. Sun investigated mixed sums of squares and triangular numbers systematically, and he mainly proved the following result.

Theorem 1 (Sun [8])**.** (i) *Any natural number is a sum of an even square and two triangular numbers, and each positive integer is a sum of a triangular number plus* $x^2 + y^2$ *for some* $x, y \in \mathbb{Z}$ *with* $x \not\equiv y \pmod 2$ *or* $x = y > 0$.

(ii) *Let* a, b, c *be positive integers with* $a \leqslant b$. *If every* $n \in \mathbb{N}$ *can be written as* $ax^2 + by^2 + ct_z$ *with* $x, y, z \in \mathbb{Z}$, *then* (a, b, c) *is among the following vectors:*

$$(1,1,1),\ (1,1,2),\ (1,2,1),\ (1,2,2),\ (1,2,4),$$
$$(1,3,1),\ (1,4,1),\ (1,4,2),\ (1,8,1),\ (2,2,1).$$

(iii) *Let* a, b, c *be positive integers with* $b \geqslant c$. *If every* $n \in \mathbb{N}$ *can be written as* $ax^2 + bt_y + ct_z$ *with* $x, y, z \in \mathbb{Z}$, *then* (a, b, c) *is among the following vectors:*

$$(1,1,1),\ (1,2,1),\ (1,2,2),\ (1,3,1),\ (1,4,1),\ (1,4,2),\ (1,5,2),$$
$$(1,6,1),\ (1,8,1),\ (2,1,1),\ (2,2,1),\ (2,4,1),\ (3,2,1),\ (4,1,1),\ (4,2,1).$$

Sun also reduced the converses of (ii) and (iii) to Conjectures 1 and 2 of [8]. In this paper we prove his second conjecture, namely we establish the following theorem.

Theorem 2. *Every* $n \in \mathbb{N}$ *can be expressed in any of the following forms with* $x, y, z \in \mathbb{Z}$:

$$x^2 + 3y^2 + t_z,\ x^2 + 3t_y + t_z,\ x^2 + 6t_y + t_z,\ 3x^2 + 2t_y + t_z,\ 4x^2 + 2t_y + t_z.$$

2. Proof of Theorem 2

The following theorem is well-known (cf. [4, pp. 17-23]).

Gauss-Legendre Theorem. *A natural number can be written as a sum of three squares of integers if and only if it is not of the form* $4^k(8l+7)$ *with* $k, l \in \mathbb{N}$.

We also need an identity of Jacobi which can be verified directly.

Jacobi's Identity. *We have*

$$3(x^2+y^2+z^2) = (x+y+z)^2 + 2\left(\frac{x+y-2z}{2}\right)^2 + 6\left(\frac{x-y}{2}\right)^2.$$

Proof of Theorem 2. (i) By the Gauss-Legendre theorem, $8n+3 = x^2+y^2+z^2$ for some $x, y, z \in \mathbb{Z}$. Clearly, each of x, y, z is congruent to 1 or -1 modulo 4. Without any loss of generality, we simply let $x \equiv y \equiv z \equiv 1 \pmod 4$. Two of x, y, z are congruent modulo 8, say, $x \equiv y \pmod 8$. Set

$$x_0 = \frac{x-y}{8}, \quad y_0 = \frac{x+y-2}{4} \quad \text{and} \quad z_0 = \frac{z-1}{2}.$$

Then

$$8n+3 = 2\left(\frac{x-y}{2}\right)^2 + 2\left(\frac{x+y}{2}\right)^2 + z^2 = 2(4x_0)^2 + 2(2y_0+1)^2 + (2z_0+1)^2$$

and hence $n = 4x_0^2 + 2t_{y_0} + t_{z_0}$.

(ii) By the Gauss-Legendre theorem, $12(4n+2) = x^2+y^2+z^2$ for some $x, y, z \in \mathbb{Z}$. As $x^2+y^2+z^2 \equiv 0 \pmod 3$, we can choose suitable $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}$ such that $\varepsilon_1 x \equiv \varepsilon_2 y \equiv \varepsilon_3 z \equiv 0$ or $1 \pmod 3$. Therefore we may simply let $x \equiv y \equiv z \pmod 3$. Since $x^2+y^2+z^2 \equiv 8 \pmod{16}$, x, y, z are all even and exactly one of them is divisible by 4. Suppose that $x \equiv y+2 \equiv z+2 \equiv 0 \pmod 4$. It is easy to see that

$$x+y+z \equiv 0 \pmod{12}, \; x+y-2z \equiv 6 \pmod{12}, \; x-y \equiv 6 \pmod{12}.$$

Set

$$x_0 = \frac{x+y+z}{12}, \quad y_0 = \frac{x+y-2z-6}{12} \quad \text{and} \quad z_0 = \frac{x-y-6}{12}.$$

By Jacobi's identity,

$$\begin{aligned} 36(4n+2) =& 3(x^2+y^2+z^2) \\ =& (12x_0)^2 + 2(6y_0+3)^2 + 6(6z_0+3)^2 \\ =& 144x_0^2 + 72y_0(y_0+1) + 18 + 216z_0(z_0+1) + 54. \end{aligned}$$

It follows that $n = x_0^2 + t_{y_0} + 3t_{z_0}$.

(iii) Let $\varepsilon \in \{0, 1, 3\}$. By the Gauss-Legendre theorem, $24n + 3 + 6\varepsilon = x^2 + y^2 + z^2$ for some $x, y, z \in \mathbb{Z}$. As $3 \mid x^2 + y^2 + z^2$, without loss of generality we may assume that $x \equiv y \equiv z \equiv 0$ or $1 \pmod 3$. Applying Jacobi's identity, we obtain that

$$72n + 9 + 18\varepsilon = (x + y + z)^2 + 2\left(\frac{x + y - 2z}{2}\right)^2 + 6\left(\frac{x - y}{2}\right)^2.$$

Recall that $x^2 + y^2 + z^2 \equiv 3 + 6\varepsilon \pmod 8$. If $\varepsilon = 0$, then x, y, z are odd, and two of them are congruent modulo 4, say, $x \equiv y \pmod 4$. In the case $\varepsilon = 1$, we may suppose that $x \equiv y \equiv z - 1 \equiv 0 \pmod 2$ and $x \equiv y \pmod 4$. When $\varepsilon = 3$, we may assume that $x \equiv 2 \pmod 4$, $4 \mid y$ and $2 \nmid z$. Clearly,

$$x + y + z \equiv 3 \pmod 6, \quad x + y - 2z \equiv \begin{cases} 0 \pmod{12} & \text{if } \varepsilon = 0, 3, \\ 6 \pmod{12} & \text{if } \varepsilon = 1, \end{cases}$$

and

$$x - y \equiv \begin{cases} 0 \pmod{12} & \text{if } \varepsilon = 0, 1, \\ 6 \pmod{12} & \text{if } \varepsilon = 3. \end{cases}$$

Set

$$x_0 = \begin{cases} (x + y - 2z)/12 & \text{if } \varepsilon = 0, \\ (x - y)/12 & \text{if } \varepsilon = 1, \\ (x + y - 2z)/12 & \text{if } \varepsilon = 3, \end{cases}$$

$$y_0 = \begin{cases} (x - y)/12 & \text{if } \varepsilon = 0, \\ (x + y - 2z - 6)/12 & \text{if } \varepsilon = 1, \\ (x - y - 6)/12 & \text{if } \varepsilon = 3, \end{cases}$$

and $z_0 = (x + y + z - 3)/6$. By the above,

$$\begin{aligned} 72n + 9 + 18\varepsilon &= \begin{cases} (6z_0 + 3)^2 + 2(6x_0)^2 + 6(6y_0)^2 & \text{if } \varepsilon = 0, \\ (6z_0 + 3)^2 + 2(6y_0 + 3)^2 + 6(6x_0)^2 & \text{if } \varepsilon = 1, \\ (6z_0 + 3)^2 + 2(6x_0)^2 + 6(6y_0 + 3)^2 & \text{if } \varepsilon = 3, \end{cases} \\ &= \begin{cases} 72x_0^2 + 216y_0^2 + 36z_0(z_0 + 1) + 9 & \text{if } \varepsilon = 0, \\ 216x_0^2 + 72y_0(y_0 + 1) + 36z_0(z_0 + 1) + 27 & \text{if } \varepsilon = 1, \\ 72x_0^2 + 216y_0(y_0 + 1) + 36z_0(z_0 + 1) + 63 & \text{if } \varepsilon = 3. \end{cases} \end{aligned}$$

It follows that

$$n = \begin{cases} x_0^2 + 3y_0^2 + t_{z_0} & \text{if } \varepsilon = 0, \\ 3x_0^2 + 2t_{y_0} + t_{z_0} & \text{if } \varepsilon = 1, \\ x_0^2 + 6t_{y_0} + t_{z_0} & \text{if } \varepsilon = 3. \end{cases}$$

Combining (i)–(iii) we have completed our proof of Theorem 2. $\square$

References

1. L. E. Dickson, *History of the Theory of Numbers,* Vol. II, AMS Chelsea Publ., 1999.

2. H. M. Farkas, *Sums of squares and triangular numbers*, Online J. Anal. Combin. **1** (2006), #1, 11 pp. (electronic).

3. V. A. Lebesgue, *Questions 1059,1060,1061 (Lionnet)*, Nouv. Ann. Math. **11** (1872), 516–519.

4. M. B. Nathanson, *Additive Number Theory: the Classical Bases*, Graduate Texts in Math. 164, Springer, New York, 1996.

5. L. Panaitopol, *On the representation of natural numbers as sums of squares*, Amer. Math. Monthly **102** (2005), 168–171.

6. S. Ramanujan, *On the expression of a number in the form* $ax^2 + by^2 + cz^2 + du^2$, Proc. Camb. Philo. Soc. **19** (1916), 11–21.

7. M. S. Réalis, *Scolies pour un théoreme d'arithmétique*, Nouv. Ann. Math. **12** (1873), 212–217.

8. Z. W. Sun, *Mixed sums of squares and triangular numbers*, Acta Arith. **127** (2007), 103–113.