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A Note on Trigonometric Interpolation and the Discrete Fourier Transform

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Abstract:

The concept of Fourier Series is widely used in several Engineering problems like Wave Equations, Heat Equations, Laplace Equations, Signal Processing and much more. The concept of Discrete Fourier Transforms is the equivalent of continuous Fourier Transforms (DFT) for signals transmitted at finite number of points. The interpolation process allows us to manipulate the values of a discrete data in between the given input values. The process of interpolation is usually done with finite differences methods using forward, backward, central operators. We can also make interpolation process by using trigonometric functions. The Fourier series is a classic example for trigonometric interpolation where we use sine and cosine functions with different harmonics and try to express the given function as a linear combination of such harmonics. In this paper by considering equidistant data points, we show how the DFT allows us to construct the corresponding trigonometric interpolation.

Keywords: *Trigonometric interpolation, Uniform sampling, Discrete Fourier Transform.*

DFT of Uniform Sampling

Here we consider a uniform sampling of $f(x)$ with the following $N = 2n$ equidistant data points:

$$x_j = -\pi + j\,\frac{2\pi}{N}, \qquad f(x_j) = f_j, \qquad j = 0, 1, \dots, N-1, \tag{1}$$

Then in this region, the corresponding function accepts the trigonometric approximation (Lanczos, 1938; 1966; 1988; 1996):

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n-1} [a_k\, Cos(kx) + b_k\, Sin(kx)] + \frac{1}{2}a_n\, Cos(nx), \tag{2}$$

such that:

$$a_j = \frac{(-1)^j}{n} \sum_{k=0}^{N-1} f(x_k)\ Cos\left(kj\ \frac{\pi}{n}\right),$$
$$0 \le j \le n\,, \qquad (3)$$

$$b_r = \frac{(-1)^r}{n} \sum_{k=1}^{N-1} f(x_k)\ Sin\left(kr\ \frac{\pi}{n}\right),$$
$$1 \le r \le n-1\,.$$

It is interesting to note that this trigonometric interpolation doesn't have the instability of Lagrangian interpolation of equidistant data points (Lanczos, 1988; 1996).

The coefficients (harmonics) in the Fourier – Like series expansion (2) can be calculated directly from the expressions (3). However, here we will determine them using the Discrete Fourier Transform (DFT). In fact, we shall indicate the essential procedure.

First we obtain the values:

$$\hat{f}_k = [-\exp\left(\frac{ik\pi}{n}\right)]^{n-1} f_k\,, \qquad k = 0, 1, \dots, N-1, \qquad (4)$$

We now implement the DFT for the sequence $\{\hat{f}_0, \dots, \hat{f}_{N-1}\}$, that is, we construct the quantities $\{F_0, \dots, F_{N-1}\}$ via the relation (Briggs & Henson, 1995; Heat, 1997):

$$F_k = \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \hat{f}_r \exp\left(-\frac{ikr\pi}{n}\right), \qquad 0 \le k \le N-1, \qquad (5)$$

Then the coefficients of the expansion (2) are given by:

$$a_0 = (-1)^{n-1}\sqrt{\frac{2}{n}} F_{n-1}\,, a_n = -\sqrt{\frac{2}{n}} F_{N-1}\,, \qquad (6)$$
$$a_r = \frac{(-1)^{n-1+r}}{\sqrt{N}} (F_{n-1-r} + F_{n-1+r}), b_r = i\,\frac{(-1)^{n+r}}{\sqrt{N}} (F_{n-1-r} - F_{n-1+r}), \quad 1 \le r \le n-1.$$

Thus equation (6) provides the DFT with uniform sampling gives a trigonometric interpolation for equidistant data points. The values F_k can be determined with (5) or through the Fast Fourier Transform technique (Duhamel & Vetterli, 1990; Van Loan, 1992), the FFT is merely an efficient means of computing the DFT, by conveniently choosing $N = 2^m$.

Illustration

As an example, for $N = 4,\ n = 2$:

$$\hat{f}_0 = -f_0\,, \quad \hat{f}_1 = -i\,f_1\,, \quad \hat{f}_2 = f_2\,, \quad \hat{f}_3 = i\,f_3\,, F_0 = \frac{1}{2}(\hat{f}_0 + \hat{f}_1 + \hat{f}_2 + \hat{f}_3),$$

$$F_1 = \tfrac{1}{2}[\hat{f}_0 - \hat{f}_2 - i(\hat{f}_1 - \hat{f}_3)], \quad F_2 = \tfrac{1}{2}[\hat{f}_0 + \hat{f}_2 - \hat{f}_1 - \hat{f}_3], \quad F_3 = \tfrac{1}{2}[\hat{f}_0 - \hat{f}_2 + i(\hat{f}_1 - \hat{f}_3)], \qquad (7)$$

$$a_0 = -F_1 = \frac{1}{2}[f_0 + f_1 + f_2 + f_3], \quad a_1 = \frac{1}{2}(F_2 + F_0) = \frac{1}{2}(f_2 - f_0),$$

$$a_2 = -F_3 = \frac{1}{2}(f_0 + f_2 - f_1 - f_3), \quad b_1 = \frac{i}{2}(F_2 - F_0) = \frac{1}{2}(f_3 - f_1),$$

These expressions in (7) are in agreement with the values in (3) obtained with respect to trigonometric interpolation.

Inverse Transforms

The inversion of (6) will provide the relations shown below.

$$F_{n-1} = (-1)^{n-1} \sqrt{\frac{n}{2}} a_0 \, , F_{N-1} = - \sqrt{\frac{n}{2}} a_n \, , \quad (8)$$

$$F_{n-1-r} = \frac{(-1)^{n-1+r}}{2} \sqrt{N} (a_r + i\, b_r), F_{n-1+r} = \frac{(-1)^{n-1+r}}{2} \sqrt{N} (a_r - i\, b_r) = F^*_{n-1-r},$$

For $1 \leq r \leq n-1$,which give us the DFT if we know the corresponding harmonic interpolation (2). Thus we see the relationship between the trigonometric interpolation of equidistant data points and the Discrete Fourier Transform.

Conclusion

In this paper, we have obtained relations connecting Trigonometric Interpolation process with Discrete Fourier Transform (DFT) technique. It is well known that the harmonics of Trigonometric Interpolation can be easily calculated with respect to functions which are continuous or at least piecewise continuous in the given domain, it is very easy to do so for discrete data points. In this paper, by considering equidistant data points, we have obtained the harmonics as provided in (6). To obtain the original equidistant data points, we employ Inverse Transforms and (8) provides such values upon retrieving the original data points. Thus this short note connects the concepts of interpolation using trigonometric functions with Discrete Fourier Transforms (DFT). Further research can be done by considering unequal data points and try to obtain expressions to (6) and (8) of this paper. This would provide a more general insight between these two concepts. These connections have wide range of applications in Digital Signal Processing and Image Processing Techniques. Hence, more methods we know the more useful it would be to process the data.

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