INTEGERS: ELECTRONIC JOURNAL OF COMBINATORIAL NUMBER THEORY 6 (2006), #A40

ON THE PRODUCT OF LOG-CONCAVE POLYNOMIALS

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Received: 8/21/06, Revised: 11/23/06, Accepted: 11/29/06, Published: 12/20/06

Abstract

A real polynomial is called log-concave if its coefficients form a log-concave sequence. We give a new elementary proof of the fact that a product of log-concave polynomials with nonnegative coefficients and no internal zero coefficients is again log-concave. In addition, we show that if the coefficients of the polynomial $\prod_{m\in M}(x+m)$ form a monotone sequence where M is a finite multiset of positive real numbers, so do the coefficients of $\prod_{r\in N}(x+r)$ for any submultiset $N \subset M$.

1. Introduction

A polynomial with real coefficients $f(x) = \sum_i a_i x^i$ is *log-concave* if we have $a_i^2 - a_{i-1}a_{i+1} \geq 0$ for all $0 < i < \deg(f)$. Also, $f(x)$ is said to have *no internal zero coefficients* if there do not exist integers $i < j < k$ such that $a_i \neq 0$, $a_j = 0$, and $a_k \neq 0$. The following theorem concerning the product of log-concave polynomials is well-known (e.g., refer to Theorem 1.2 of Chapter 8 in [1]).

Theorem 1. *Let $f(x)$ and $g(x)$ be log-concave polynomials with nonnegative coefficients and no internal zero coefficients. Then the product $f(x)g(x)$ is also log-concave.*

We will give a new elementary proof of this theorem in Section 2. Refer to [3] for an advanced linear algebraic proof, and [2] for another elementary proof. We also refer the reader to [4] for a proof via the LC-positivity of a constant triangle of nonnegative numbers.

Recall that a *multiset* is a set where repetitions of elements are allowed. If M is a finite multiset of positive numbers, then it follows from Theorem 1 that the polynomial $\prod_{m\in M}(x+m)$ is log-concave. It is easy to see that every log-concave sequence $a_0, \ldots, a_n$ is unimodal, i.e., $a_0 \leq \cdots \leq a_k \geq \cdots \geq a_n$ for some k. An interesting special case of a unimodal sequence is a monotone sequence, i.e., a sequence that is weakly increasing or decreasing. We show in Section 3 that if the coefficients of $\prod_{m\in M}(x+m)$ form a monotone sequence, then the coefficients of $\prod_{r \in N}(x+r)$ for any submultiset $N \subset M$ also form a monotone sequence. At the end of the paper, we will briefly discuss how this interesting property can be used to define a simplicial complex for any multiset of positive integers as the vertex set.

2. A New Proof of Theorem 1

In this section, every sequence is assumed to be an infinite sequence whose terms are indexed by the set of integers. A sequence (a_k) of real numbers is *log-concave* if $a_k^2 \geq a_{k-1}a_{k+1}$ for every integer k. The sequence (a_k) has *no internal zeros* if there do not exist integers $i < j < k$ such that $a_i \neq 0$, $a_j = 0$, and $a_k \neq 0$. The following lemma characterizes a log-concave sequence of nonnegative real numbers with no internal zeros.

Lemma 2. *A sequence of nonnegative real numbers (a_k) with no internal zeros is log-concave if and only if $a_i a_j \geq a_{i-1}a_{j+1}$ for all integers $i \leq j$.*

Proof. The sufficiency is clear. We will prove the necessity by induction on $j - i$. Since (a_k) is nonnegative and has no internal zeros, it suffices to prove the inequality for integers $i \leq j$ such that $a_i > 0$ and $a_j > 0$. Now, the case $j - i = 0$ is clear. Let $j - i > 0$. Since (a_k) is log-concave, we have $a_i^2 a_j^2 \geq a_{i-1}a_{i+1}a_{j-1}a_{j+1}$. For $j - i = 1$, we have $a_{i+1}a_{j-1} = a_j a_i$, and the desired inequality follows. For $j - i \geq 2$, we have $a_{i+1}a_{j-1} \geq a_i a_j$ by induction, and the desired inequality follows. □

Given a sequence $\mathbf{a} = (a_k)$ of real numbers, define $\Sigma^n \mathbf{a}$ to be the sequence given by shifting every term of $\mathbf{a}$ *right* by n terms, i.e., $\Sigma^n \mathbf{a} = (a'_k)$ where $a'_i = a_{i-n}$ for all i. We will write $\Sigma \mathbf{a}$ for $\Sigma^1 \mathbf{a}$. Given another sequence $\mathbf{b} = (b_k)$ of real numbers, we define $\mathbf{a} * \mathbf{b} := \sum_i a_i b_i$. For each pair of integers (i, j) such that $i \leq j$ define $\mathbf{a}_{ij} = (a_i a_j - a_{i-1}a_{j+1})$ and $\mathbf{b}_{ij} = (b_i b_j - b_{i-1}b_{j+1})$.

Lemma 3. *Let $\mathbf{a} = (a_k)$ and $\mathbf{b} = (b_k)$ be sequences of real numbers. Suppose only finitely many terms in $\mathbf{a}$ are nonzero. Then we have*

$$(\mathbf{a} * \mathbf{b})^2 - (\mathbf{a} * \Sigma \mathbf{b})(\Sigma \mathbf{a} * \mathbf{b}) = \sum_{i \leq j} \mathbf{a}_{ij}\mathbf{b}_{ij},$$

where the sum is over all pairs of integers (i, j) such that $i \leq j$. In particular, if both $\mathbf{a}$ and $\mathbf{b}$ are log-concave then $(\mathbf{a} * \mathbf{b})^2 \geq (\mathbf{a} * \Sigma \mathbf{b})(\Sigma \mathbf{a} * \mathbf{b})$.

Proof. Given any sequences (x_i) and (y_j), we have

$$\Big(\sum_i x_i\Big)\Big(\sum_j y_j\Big) = \sum_{i \leq j}(x_i y_j + x_j y_{i-1}) = \sum_{i \leq j}(x_i y_j + x_{j+1} y_{i-1}) + \sum_i x_i y_{i-1},$$

assuming that every sum in this equation exists. Applying this equation to the sums $\mathbf{a} * \mathbf{b} = \sum_i a_i b_i$, $(\mathbf{a} * \Sigma \mathbf{b}) = \sum_i a_i b_{i-1}$, and $\Sigma \mathbf{a} * \mathbf{b} = \sum_j a_j b_{j+1}$, which are well-defined because all but finitely many terms of (a_k) are zero, we get

$$(\mathbf{a} * \mathbf{b})^2 = \sum_{i \le j} (a_i a_j b_i b_j + a_{i-1} a_{j+1} b_{i-1} b_{j+1}) + \sum_i (a_i a_{i-1} b_i b_{i-1}) \quad \text{and}$$

$$(\mathbf{a} * \Sigma\mathbf{b})(\Sigma\mathbf{a} * \mathbf{b}) = \sum_{i \le j} (a_i a_j b_{i-1} b_{j+1} + a_{i-1} a_{j+1} b_j b_i) + \sum_i (a_i a_{i-1} b_{i-1} b_i).$$

From these equations, it follows easily that

$$(\mathbf{a} * \mathbf{b})^2 - (\mathbf{a} * \Sigma\mathbf{b})(\Sigma\mathbf{a} * \mathbf{b}) = \sum_{i \le j} (a_i a_j - a_{i-1} a_{j+1}), (b_i b_j - b_{i-1} b_{j+1})$$

as desired. The second statement of the lemma follows from this and Lemma 2. □

Proof of Theorem 1. Let $f(x) = \sum_i a_i x^i$ and $g(x) = \sum_j b_i x^i$ be as in Theorem 1. If we define $a_i = 0$ and $b_j = 0$ when $i \notin [0, \deg(f)]$ and when $j \notin [0, \deg(g)]$, respectively, then the infinite sequences $\mathbf{a} = (a_i)$ and $\mathbf{b} = (b_j)$ are log-concave. Let the sequence $\mathbf{d} = (d_i)$ be the "mirror image" of $\mathbf{b}$ about b_0, i.e., $d_{-i} = b_i$ for all i. Note that $\mathbf{d}$ is still log-concave, hence so is $\Sigma^i\mathbf{d}$ for all i. Let $f(x)g(x) = \sum_i c_i x^i$. Then Theorem 1 follows from Lemma 3 together with the facts $c_i = \mathbf{a} * \Sigma^i\mathbf{d}$, $c_{i-1} = \Sigma\mathbf{a} * \Sigma^i\mathbf{d}$, and $c_{i+1} = \mathbf{a} * \Sigma^{i+1}\mathbf{d}$. □

3. Monotone Polynomials

Recall that a *multiset* of cardinality n is a set $M = \{m_1, m_2, \ldots, m_n\}$, where repetitions of elements are allowed. For each $0 \le k \le n$, let $s_k(M)$ denote the coefficient of x^{n-k} in the polynomial $f(x) = \prod_{1 \le i \le n}(x + m_i)$. (For $M = \emptyset$ we define $s_0(M) = 1$.) Define $s_k(M) = 0$ for $k \notin [0, n]$. Now, the following identity is easy to check: for any $m \in M$ and k,

$$s_k(M) = m s_{k-1}(M \setminus m) + s_k(M \setminus m).$$

Note that when M is a finite multiset of positive real numbers, the sequence $(s_k(M))$ is log-concave by Theorem 1. Moreover, $(s_k(M))$ is unimodal because every log-concave sequence is unimodal.

Theorem 4. *Let M be a finite multiset of cardinality n consisting of positive real numbers. Suppose we have $s_0(M) \le \cdots \le s_k(M) \ge \cdots \ge s_n(M)$ for some k. Then for any $m \in M$, we also have*
(a) $s_0(M \setminus m) \le s_1(M \setminus m) \le \cdots \le s_{k-1}(M \setminus m)$, and
(b) $s_k(M \setminus m) \ge s_{k+1}(M \setminus m) \ge \cdots \ge s_{n-1}(M \setminus m)$.

Proof. Fix $m \in M$. For notational simplicity, we will denote $s_i = s_i(M)$ and $u_j = s_j(M \setminus m)$. To prove (a), we will show that if we have $s_t \le s_{t+1}$ for any $t \ge 1$, then we also have $u_{t-1} \le u_t$. Suppose otherwise. Then, we must have $s_{t+1}/s_t \ge 1$ and $u_t/u_{t-1} < 1$ for some t. Now, let $g(x) = (u_t x + u_{t+1})/(u_{t-1} x + u_t)$. Since every u_j is positive, it follows that $g(x)$ and its first derivative $g'(x)$ are well-defined for $x > 0$. Also, we have $u_t^2 - u_{t-1}u_{t+1} \geq 0$ because the sequence (u_j) is log-concave. Using this fact, one can show that $g'(x) \geq 0$, i.e., $g(x)$ is weakly increasing for $x > 0$. It follows that $g(x) \leq (u_t/u_{t-1}) < 1$ for all $x > 0$. In particular, we must have $g(m) = (u_t m + u_{t+1})/(u_{t-1}m + u_t) = s_{t+1}(M)/s_t(M) < 1$, which is a contradiction. This proves (a).

For part (b), we will show that if $s_t \geq s_{t+1}$ for any t, then we also have $u_t \geq u_{t+1}$. Again, suppose otherwise. Then we must have $s_t/s_{t+1} \geq 1$ and $u_t/u_{t+1} < 1$ for some t. Now, let $h(x) = (u_{t-1}x + u_t)/(u_t x + u_{t+1})$. Using the log-concavity of (u_j) as in the proof of (a), one can show that $h(x)$ is weakly decreasing for $x \geq 0$. From this, we have $s_t/s_{t+1} = h(m) \leq h(0) = u_t/u_{t+1} < 1$, which is a contradiction. This proves (b). □

If the sequence $(s_k(M) : 0 \leq k \leq n)$ is monotone increasing, i.e., $s_0(M) \leq s_1(M) \leq \cdots \leq s_n(M)$, then part (a) of Theorem 4 implies that we also have $s_0(N) \leq s_1(N) \leq \cdots \leq s_{|N|}(N)$ for any submultiset $N \subset M$. By part (b), similar result holds when $(s_k(M) : 0 \leq k \leq n)$ is monotone decreasing. Hence we have

Corollary 5. *Let M be a finite multiset of positive real numbers. Suppose the coefficients of the polynomial $\prod_{m\in M}(x+m)$ are either monotone increasing or monotone decreasing. Then, for any submultiset $N \subset M$, the coefficients of $\prod_{r\in N}(x+r)$ are either monotone increasing or monotone decreasing also, respectively.* □

Finally, we briefly discuss a topological implication of this corollary. Let V be a (finite) multiset of positive *integers*. Let Δ be the collection of all finite submultisets $M \subset V$ such that the sequence $(s_k(M) : 0 \leq k \leq |M|)$ is monotone increasing. (This sequence cannot be monotone decreasing when $|M| > 1$.) Then by the remarks before Corollary 5, Δ satisfies the following property: if $M \in \Delta$ and $N \subset M$, then $N \in \Delta$. Clearly, we have $\{v\} \in \Delta$ for all $v \in V$, and we have $\phi \in \Delta$. Hence, Δ is a simplicial complex with the vertex set V. (Refer to any text in algebraic topology for the definition of a simplicial complex.)

4. Acknowledgments

The author would like to thank the referee for helpful comments and suggestions.

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