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A New Hybrid Iterative Scheme for Approximating Fixed Points for Contraction Mappings in Banach Spaces

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Abstract

This study proposes a novel hybrid iterative scheme for approximating fixed points of contraction mappings called Picard-P iterative scheme, which is a combination of Picard and P iterative schemes. The efficiency of present iterative scheme is to provide faster convergence in contrast to several well-known iterative schemes. It is efficiently illustrated with the help of a numerical example followed by a graph. Some convergence and stability results for contraction mappings in the context of Banach spaces are established using the proposed scheme. Additionally, to support our claim, MATLAB programme is used to approximate fixed points for contraction mappings.

Keywords

Contraction Mappings, Fixed Points, Iterative Schemes, Banach Space, Strong Convergence Results.

1. Introduction

Many non-linear problems pertaining to engineering, integral equations, economics, differential equations, management, and game theory, among others, can be solved using fixed point theory. In this regard, Banach contraction theorem [3] is the first and most significant tool in the hands of authors for ensuring the existence and uniqueness of fixed points. However, once the existence of a fixed point in a mapping is confirmed, the challenge is to determine the value of that fixed point, and iterative schemes excel in this respect. The celebrated Banach Contraction Theorem uses the Picard [14] Iterative scheme to approximate the fixed point of

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contraction mappings. Following that several new iterative schemes were developed by researchers to approximate the fixed points of nonlinear mappings fulfilling various contractive conditions. Faster convergence is an essential element for considering one iterative scheme over another. Some of the well-known iterative schemes are: Picard [14], Mann [11], Ishikawa [8], Agarwal [2], Noor [12], Abbas [1], CR [5], Picard-Mann [9], Picard-S [6], Picard-SP [10]. Rhoades [15] examined the convergence rates of the Mann and Ishikawa iterative schemes in 1976 and concluded that the Mann iterative scheme converges faster than the Ishikawa iterative scheme for decreasing functions while the Ishikawa iterative scheme is more effective for increasing functions (see also [16]). Agarwal et al. [2] showed in 2007 that the Agarwal iterative scheme converges faster than the Mann iterative scheme and at the same rate as the Picard iterative scheme for contraction maps. In 2014 Gursoy and Karakaya [6] demonstrated using numerical examples that the Picard-S iterative scheme for contraction mappings converges faster than all Picard, Mann, Ishikawa, Noor, SP, CR, S, and some other iterations. In 2015, Sainuan [17] claimed that P-iteration converges faster than S-iteration for the class of continuous and non-decreasing functions, and he presented numerical examples to compare P-iteration to Ishikawa and S-iterations. Kumar and Chugh [10] illustrated in 2019 using a numerical example that the Picard-SP iterative scheme converges faster than the Picard, Krasnosel'skii, Mann, Ishikawa, and Picard-Krasnosel'skii iterative schemes for contraction mappings in Banach space.

Motivated by the above, we propose a novel iterative scheme called Picard-P hybrid iterative scheme in this study to approximate fixed points of contraction mappings. We provide some convergence results for contraction mappings in Banach spaces. We discuss the stability of our iterative scheme. We demonstrate that the suggested scheme outperforms various well-known and leading iterative schemes in terms of convergence speed. Furthermore, the effectiveness of the proposed scheme is illustrated using a numerical example followed by a graph. To back up our assertion, we utilise the MATLAB programme to approximate fixed points for contraction mappings.

2. Preliminaries

In this paper, the set of all positive integers is denoted by $\mathbb{N}$. Assume T represent a mapping on the nonempty subset of Banach space X called B. Fixed point is a point that remains constant under a given mapping. Let, $F_T = \{x \in B : Tx = x\}$, be the set of all fixed points of the mapping T on B. The mapping $T: B \to B$ is said to be a contraction if

$d(Tx, Ty) \leq \alpha\, d(x, y)$, for all $x,y \in B$ and $\alpha \in (0,1)$. (2.1)

Here are a few of the definitions and lemmas:

Throughout this section we have $m \in \mathbb{N}$ and $\{a_m\}$, $\{b_m\}$ and $\{c_m\}$ are sequences in [0,1], satisfying appropriate conditions.

Definition 2.1.[4] Let $\{x_m\}$ and $\{y_m\}$ be two real sequences converging to x and y, respectively. Then we can say that the sequence $\{x_m\}$ has faster convergence than $\{y_m\}$ if

$$\lim_{m\to\infty} \frac{\|x_m - x\|}{\|y_m - y\|} = 0.$$

Definition 2.2. [4] Assume that $\{u_m\}$ and $\{v_m\}$ are two fixed point iterative schemes converging to a certain fixed point q. If $\|u_m - q\| \leq x_m$ and $\|v_m - q\| \leq y_m$, for all $m \in \mathbb{N}$, where $\{x_m\}$ and $\{y_m\}$ are two sequences of positive numbers converging to zero. Then we say that $\{u_m\}$ converges faster than $\{v_m\}$ to q, if $\{x_m\}$ converges faster than $\{y_m\}$.

Definition 2.3. [7] Let $\{s_m\}$ represent any random sequence in B. The iterative scheme u_{m+1} = f (T, u_m), converging to fixed point q, is therefore said to be T-stable or stable with regard to T, if for $\epsilon_m = \|s_{m+1} - f(T, s_m)\|$, m = 0,1,2,3, ... , we have

$$\lim_{m\to\infty} \epsilon_m = \lim_{m\to\infty} s_m = q.$$

Lemma 2.4. [1] Let $\{t_m\}$ be a sequence of positive real numbers which satisfies:

$$t_{m+1} \leq (1 - s_m)\, t_m.$$

If $\{s_m\} \subset (0,1)$ and $\sum_{m=1}^{\infty} s_m = \infty$, then $\lim_{m\to\infty} t_m = 0$.

Lemma2.5. [19] Let $\{\alpha_m\}_{m=0}^{\infty}$ and $\{\beta_m\}_{m=0}^{\infty}$ are non-negative real sequences satisfying the following inequality:

$$\alpha_{m+1} \leq (1 - \gamma_m)\, \alpha_m + \beta_m,$$

where $\gamma_m \in (0,1)$ for all $m \in \mathbb{N}$, $\sum_{m=0}^{\infty} \gamma_m = \infty$ and $\frac{\beta_m}{\gamma_m} \to 0$ as $m \to \infty$. Then $\lim_{m\to\infty} \alpha_m = 0$.

Some pre-existing Iterative schemes are listed below.

Picard[14] established the iterative scheme specified by $\{x_m\}$ in 1890 as:

$$\begin{cases} x_0 \in B, \\ x_{m+1} = Tx_m. \end{cases} \quad (2.2)$$

Mann[11] established the one-step iterative scheme specified by $\{s_m\}$ in 1953 as:

$$\begin{cases} s_0 \in B, \\ s_{m+1} = (1 - c_m)s_m + c_m Ts_m, \end{cases} \tag{2.3}$$

Ishikawa[8] established the first two-step iterative scheme specified by $\{y_m\}$ in 1974 as:

$$\begin{cases} y_0 \in B, \\ z_m = (1 - b_m)y_m + b_m Ty_m, \\ y_{m+1} = (1 - c_m)y_m + c_m Tz_m, \end{cases} \tag{2.4}$$

Khan[9] established the two step Picard-Mann hybrid iterative scheme specified by $\{y_m\}$ in 2013 as:

$$\begin{cases} y_0 \in B, \\ z_m = (1 - b_m)y_m + b_m Ty_m, \\ y_{m+1} = Tz_m, \end{cases} \tag{2.5}$$

Gursoy and Karakaya[6] established the three step Picard-S hybrid iterative scheme specified by $\{v_m\}$ in 2014 as:

$$\begin{cases} v_0 \in B, \\ t_m = (1 - a_m)v_m + a_m Tv_m, \\ s_m = (1 - b_m)Tv_m + b_m Tt_m, \\ v_{m+1} = Ts_m, \end{cases} \tag{2.6}$$

Sainuan[17] established the three step P iterative scheme specified by $\{u_m\}$ in 2015 as:

$$\begin{cases} u_0 \in B, \\ z_m = (1 - a_m)u_m + a_m Tu_m, \\ y_m = (1 - b_m)z_m + b_m Tz_m, \\ u_{m+1} = (1 - c_m)Tz_m + c_m Ty_m, \end{cases} \tag{2.7}$$

N. Kumar and R. Chugh[10] established the Picard-SP iterative scheme specified by $\{t_m\}$ in 2019 as:

$$\begin{cases} t_0 \in B, \\ y_m = (1 - a_m)t_m + a_m Tt_m, \\ z_m = (1 - b_m)y_m + b_m Ty_m, \\ x_m = (1 - c_m)z_m + c_m Tz_m, \\ t_{m+1} = Tx_m. \end{cases} \tag{2.8}$$

3. Main Results :

We now introduce a new hybrid iterative scheme "Picard-P iterative scheme" as follows:

$$\begin{cases} r_0 \in B, \\ t_m = (1 - a_m)r_m + a_m Tr_m, \\ s_m = (1 - b_m)t_m + b_m Tt_m, \\ u_m = (1 - c_m)Tt_m + c_m Ts_m, \\ r_{m+1} = Tu_m \end{cases} . \tag{3.1}$$

where $\{a_m\}$, $\{b_m\}$ and $\{c_m\}$ are sequences in [0,1], satisfying appropriate conditions.

Here we prove that the sequence generated by the proposed Picard-P iterative scheme converges to a fixed point of the mapping T faster than other existing iterative schemes. We proved some convergence results for contraction mappings in Banach spaces. We discuss the stability of our iterative scheme. We demonstrated a numerical example to show that Picard-P iterative scheme converges faster than Picard, P, Picard-Mann, Picard-S and Picard-SP iterative schemes.

Theorem 3.1: Let C be a nonempty closed and convex subset of a Banach space X, let T: C → C be a contraction mapping. Let $\{r_m\}$ be a sequence generated by iterative scheme (3.1) with real sequences $\{a_m\}$, $\{b_m\}$ and $\{c_m\}$ in [0,1] satisfying $\sum_{m=0}^{\infty} c_m = \infty$. Then $\{r_m\}$ converges strongly to a unique fixed point of T.

Proof: The well known Banach contraction principle guarantees the existence and uniqueness of a fixed point p. We shall show that $r_m \to p$ for $m \to \infty$. From iterative scheme (3.1), we have

$$\begin{aligned} \|t_m - p\| &= \|(1 - a_m) r_m + a_m T r_m - p\| \\ &= \|(1 - a_m) r_m + a_m T r_m - (1 - a_m + a_m) p\| \\ &\leq (1 - a_m)\|r_m - p\| + a_m\|T r_m - p\| \\ &\leq (1 - a_m)\|r_m - p\| + a_m \alpha \|r_m - p\| \\ &= (1 - a_m(1 - \alpha))\|r_m - p\| . \end{aligned} \tag{3.2}$$

Similarly,

$$\begin{aligned} \|s_m - p\| &= \|(1 - b_m) t_m + b_m T t_m - p\| \\ &\leq (1 - b_m)\|t_m - p\| + b_m\|T t_m - p\| \\ &\leq (1 - b_m)\|t_m - p\| + b_m \alpha \|t_m - p\| \\ &\leq \big(1 - b_m(1 - \alpha)\big)\big(1 - a_m(1 - \alpha)\big)\|r_m - p\|. \end{aligned} \tag{3.3}$$

$$\begin{aligned} \|u_m - p\| &= \|(1 - c_m) T t_m + c_m T s_m - p\| \\ &\leq (1 - c_m)\|T t_m - p\| + c_m\|T s_m - p\| \\ &\leq (1 - c_m)\alpha\|t_m - p\| + c_m \alpha \|s_m - p\| \\ &\leq (1 - c_m)\alpha(1 - a_m(1 - \alpha))\|r_m - p\| + c_m \alpha (1 - b_m(1 - \alpha))(1 - a_m(1 - \alpha))\|r_m - p\| \\ &= \alpha(1 - a_m(1 - \alpha))(1 - b_m c_m(1 - \alpha))\|r_m - p\|. \end{aligned} \tag{3.4}$$

$$\begin{aligned} \|r_{m+1} - p\| &= \|T u_m - p\| \\ &\leq \alpha\|u_m - p\| \end{aligned}$$

$$\leq \alpha^2(1 - a_m(1-\alpha))(1 - b_m c_m(1-\alpha))\|r_m - p\|$$

$$\leq \alpha^2(1 - b_m c_m(1-\alpha))\|r_m - p\| \quad (3.5)$$

By using the fact that $\left(1 - a_m(1-\alpha)\right) < 1$ and $\left(1 - b_m(1-\alpha)\right) < 1$ for $\alpha \in (0,1)$ and $\{a_m\}$ and $\{b_m\}$ in [0,1]. From (3.5) we have

$$\begin{cases} \|r_{m+1} - p\| \leq \alpha^2\left(1 - b_m c_m(1-\alpha)\right)\|r_m - p\| \\ \|r_m - p\| \leq \alpha^2\left(1 - b_{m-1}c_{m-1}(1-\alpha)\right)\|r_{m-1} - p\| \\ \|r_{m-1} - p\| \leq \alpha^2\left(1 - b_{m-2}c_{m-2}(1-\alpha)\right)\|r_{m-2} - p\| \\ \vdots \\ \|r_1 - p\| \leq \alpha^2(1 - b_0 c_0(1-\alpha))\|r_0 - p\| \end{cases} \quad (3.6)$$

Now, from (3.6) we can easily find

$$\|r_{m+1} - p\| \leq \alpha^{2(m+1)}\|r_0 - p\| \prod_{h=0}^{m}\left(1 - b_h c_h(1-\alpha)\right) \quad (3.7)$$

where $(1 - b_h c_h(1-\alpha)) \in (0,1)$ because $\alpha \in (0,1)$ and $b_h, c_h \in [0,1]$, for all $m \in \mathbb{N}$. As we know that $1-t \leq e^{-t}$ for all $t \in [0,1]$, so from (3.7), we get

$$\|r_{m+1} - p\| \leq \frac{\|r_0 - p\|\alpha^{2(m+1)}}{e^{(1-\alpha)\sum_{s=0}^{m} b_h c_h}} \quad (3.8)$$

Taking limits both sides, inequality (3.8) yields $\lim_{m\to\infty}\|r_m - p\| = 0$, i.e $r_m \to p$ for $m \to \infty$ as required.

Theorem 3.2. Let C be a nonempty closed and convex subset of a Banach space X, let T: C → C be a contraction mapping. Let $\{r_m\}$ be a sequence generated by iterative process (3.1) with real sequences $\{a_m\}, \{b_m\}$ and $\{c_m\}$ in [0,1] satisfying $\sum_{m=0}^{\infty} b_m a_m = \infty$. Then the iterative scheme is then stable with regard to T.

Proof. Assume $\{v_m\} \subset X$ to be any random sequence in C and a sequence $r_{m+1} = f(T, r_m)$ formed by (3.1) converges to a unique fixed point p (by above result 3.1) and $\epsilon_m = \|v_{m+1} - f(T, v_m)\|$.

We will prove that $\lim_{n\to\infty} \epsilon_m = 0 \Leftrightarrow \lim_{n\to\infty} v_m = p$.

Let $\lim_{n\to\infty} \epsilon_m = 0$. By using (3.5), we get

$$\|v_{m+1} - p\| \leq \|v_{m+1} - f(T, v_m)\| + \|f(T, v_m) - p\|$$

$$= \epsilon_m + \left\|T\left[\begin{matrix}(1-c_m)[(1-b_m)T\{(1-a_m)v_m + a_m T v_m\}] + \\ c_m T[(1-b_m)\{(1-a_m)v_m + a_m T v_m\} + b_m T\{(1-a_m)v_m + a_m T v_m\}]\end{matrix}\right] - p\right\|$$

$$\leq \epsilon_m + \alpha^2(1 - b_m c_m(1-\alpha))\,\|v_m - p\|.$$

Define $\alpha_m = \|v_m - p\|$, $\beta_m = b_m c_m(1 - \alpha) \in (0,1)$ and $\gamma_m = \epsilon_m = 0$, which implies that $\frac{\gamma_m}{\beta_m} \to 0$ as $m \to \infty$. Then by lemma 2.5, we get $\lim_{m\to\infty} \alpha_m = 0$ i.e., $\lim_{m\to\infty} v_m = p$.

Conversely,

Let $\lim_{m\to\infty} v_m = p$, we have

$$\begin{aligned}\epsilon_m &= \|v_{m+1} - f(T, v_m)\| \\ &\le \|v_{m+1} - p\| + \|f(T, v_m) - p\| \\ &\le \|v_{m+1} - p\| + \alpha^2\big(1 - b_m c_m(1 - \alpha)\big)\|v_m - p\|\end{aligned}$$

This implies that $\lim_{m\to\infty} \epsilon_m = 0$.

Hence our Picard-P hybrid iteration scheme (3.1) is T stable.

Theorem 3.3. Let C be a nonempty closed and convex subset of a Banach space X, let T: C → C be a contraction mapping such that each of the iterative schemes: Picard(2.2), P(2.7), Picard-Mann (2.5), Picard-S(2.6), Picard-SP(2.8) and Picard-P(3.1) converges to the same fixed point p of T, where $\{a_m\}_{m=0}^{\infty}$, $\{b_m\}_{m=0}^{\infty}$ and $\{c_m\}_{m=0}^{\infty}$ are real sequences in [0,1] such that $0 < \mu \le a_m, b_m, c_m < 1$, for all $m \in \mathbb{N}$ and for some μ. Then the Picard-P hybrid iterative scheme (3.1) surpasses all other iterative schemes in rate of convergence.

Proof: Assume that for mapping T, p represents its fixed point. Utilising Picard iterative scheme (2.2) with (2.1), we accomplish

$$\begin{aligned}\|x_{m+1} - p\| &= \|Tx_m - p\| \\ &\le \alpha\|x_m - p\| \\ &\vdots \\ &\le \alpha^m\|x_1 - p\| \qquad (3.9)\end{aligned}$$

Suppose

$$g_m = \alpha^m\|x_1 - p\|. \qquad (3.10)$$

Utilising P iterative scheme (2.7) with (2.1), we accomplish

$$\begin{aligned}\|u_{m+1} - p\| &= \|(1 - c_m)(Tz_m - p) + c_m(Ty_m - p)\| \\ &\le (1 - c_m)\alpha\|z_m - p\| + c_m\alpha\|y_m - p\| \\ &= (1 - c_m)\alpha\|z_m - p\| + c_m\alpha\|(1 - b_m)(z_m - p) + b_m(Tz_m - p)\| \\ &\le (1 - c_m)\alpha\|z_m - p\| + c_m\alpha[(1 - b_m)\|z_m - p\| + b_m\alpha\|z_m - p\|] \\ &= \alpha(1 - (1 - \alpha)b_m c_m)\|z_m - p\| \\ &= \alpha(1 - (1 - \alpha)b_m c_m)[(1 - a_m)\|u_m - p\| + a_m\|Tu_m - p\|]\end{aligned}$$

$$\leq \alpha(1-(1-\alpha)b_m c_m)[(1-a_m)\|u_m - p\| + a_m\alpha\|u_m - p\|]$$
$$= \alpha(1-(1-\alpha)b_m c_m)(1-(1-\alpha)a_m)\|u_m - p\|$$
$$\leq [\alpha(1-(1-\alpha)\mu^2)][(1-(1-\alpha)\mu)]\|u_m - p\|$$
$$\vdots$$
$$\leq [\alpha(1-(1-\alpha)\mu^2)]^m[1-(1-\alpha)\mu]^m\|u_1 - p\| \quad (3.11)$$

Let

$$h_m = [1-(1-\alpha)\mu]^m[\alpha(1-(1-\alpha)\mu^2)]^m\|u_1 - p\| \quad (3.12)$$

Utilising Picard-Mann iterative scheme (2.5) with (2.1), we accomplish

$$\|y_{m+1} - p\| = \|Tz_m - p\|$$
$$\leq \alpha\|z_m - p\|$$
$$= \alpha\|(1-b_m)(y_m - p) + b_m(Ty_m - p)\|$$
$$\leq \alpha[(1-b_m)\|y_m - p\| + b_m\alpha\|y_m - p\|]$$
$$= \alpha(1-(1-\alpha)b_m)\|y_m - p\|$$
$$\leq \alpha(1-(1-\alpha)\mu)\|y_m - p\|$$
$$\vdots$$
$$\leq [\alpha(1-(1-\alpha)\mu)]^m\|y_1 - p\| \quad (3.13)$$

Let

$$j_m = [\alpha(1-(1-\alpha)\mu)]^m\|y_1 - p\| \quad (3.14)$$

Utilising Picard-SP iterative scheme (2.8) with (2.1), we accomplish

$$\|t_{m+1} - p\| = \|Tx_m - p\|$$
$$\leq \alpha\|x_m - p\|$$
$$= \alpha\|(1-c_m)(z_m - p) + c_m(Tz_m - p)\|$$
$$\leq \alpha(1-c_m)\|z_m - p\| + c_m\alpha\|z_m - p\|$$
$$= [\alpha(1-(1-\alpha)c_m)]\|z_m - p\|$$
$$= [\alpha(1-(1-\alpha)c_m)]\|(1-b_m)(y_m - p) + b_m(Ty_m - p)\|$$
$$\leq [\alpha(1-(1-\alpha)c_m)][(1-b_m)\|y_m - p\| + b_m\alpha\|y_m - p\|]$$
$$= [\alpha(1-(1-\alpha)c_m)](1-(1-\alpha)b_m)\|y_m - p\|$$
$$= [\alpha(1-(1-\alpha)c_m)](1-(1-\alpha)b_m)\|(1-a_m)(t_m - p) + a_m(Tt_m - p)\|$$
$$\leq [\alpha(1-(1-\alpha)c_m)](1-(1-\alpha)b_m)[(1-a_m)\|t_m - p\| + a_m\alpha\|t_m - p\|]$$
$$= [\alpha(1-(1-\alpha)c_m)](1-(1-\alpha)b_m)(1-(1-\alpha)a_m)\|t_m - p\|]$$
$$\leq [\alpha(1-(1-\alpha)\mu)](1-(1-\alpha)\mu)(1-(1-\alpha)\mu)\|t_m - p\|$$
$$\leq [\alpha(1-(1-\alpha)\mu)][(1-(1-\alpha)\mu)]^2\|t_m - p\|$$

$$\vdots$$

$$\leq [\alpha(1-(1-\alpha)\mu)]^{m}[(1-(1-\alpha)\mu)]^{2m}\|t_1-p\| \tag{3.15}$$

Let

$$k_m = [\alpha(1-(1-\alpha)\mu)]^{m}[(1-(1-\alpha)\mu)]^{2m}\|t_1-p\| \tag{3.16}$$

Utilising Picard-S iterative scheme (2.6) with (2.1), we accomplish

$$\|v_{m+1}-p\| = \|Ts_m-p\|$$
$$\leq \alpha\|s_m-p\|$$
$$= \alpha\|(1-b_m)(Tv_m-p)+b_m(Tt_m-p)\|$$
$$\leq \alpha^2\big((1-b_m)\|v_m-p\| + b_m\|t_m-p\|\big)$$
$$= \alpha^2\big((1-b_m)\|v_m-p\| + b_m\|(1-a_m)(v_m-p)+a_m(Tv_m-p)\|\big)$$
$$\leq \alpha^2\big((1-b_m)\|v_m-p\| + b_m(1-a_m)\|v_m-p\| + a_m b_m\alpha\|v_m-p\|\big)$$
$$= \alpha^2(1-(1-\alpha)b_m c_m)\|v_m-p\|$$
$$\leq [\alpha^2(1-(1-\alpha)\mu^2)]\|v_m-p\|$$
$$\vdots$$
$$\leq [\alpha^2(1-(1-\alpha)\mu^2)]^{m}\|v_1-p\| \tag{3.17}$$

Let

$$i_m = [\alpha^2(1-(1-\alpha)\mu^2)]^{m}\|v_1-p\| \tag{3.18}$$

Utilising Picard – P iterative scheme (3.1) with (2.1), we accomplish

$$\|r_{m+1}-p\| = \|Tu_m-p\|$$
$$\leq \alpha\|u_m-p\|$$
$$= \alpha\|(1-c_m)(Tt_m-p)+c_m(Ts_m-p)\|$$
$$\leq \alpha^2[(1-c_m)\|t_m-p\| + c_m\|s_m-p\|]$$
$$= \alpha^2[(1-c_m)\|t_m-p\| + c_m\|(1-b_m)(t_m-p)+b_m(Tt_m-p)\|]$$
$$\leq \alpha^2[(1-c_m)\|t_m-p\| + c_m(1-b_m)\|t_m-p\| + b_m c_m\alpha\|t_m-p\|]$$
$$= \alpha^2(1-(1-\alpha)b_m c_m)\|t_m-p\|$$
$$= \alpha^2(1-(1-\alpha)b_m c_m)\|(1-a_m)(r_m-p)+a_m(Tr_m-p)\|$$
$$\leq \alpha^2(1-(1-\alpha)b_m c_m)[(1-a_m)\|r_m-p\| + \alpha a_m\|r_m-p\|]$$
$$= \alpha^2(1-(1-\alpha)b_m c_m)(1-(1-\alpha)a_m)\|r_m-p\|$$
$$\leq [\alpha^2(1-(1-\alpha)\mu^2)][(1-(1-\alpha)\mu)]\|r_m-p\|$$
$$\vdots$$
$$\leq [\alpha^2(1-(1-\alpha)\mu^2)]^{m}[(1-(1-\alpha)\mu)]^{m}\|r_1-p\| \tag{3.19}$$

Let

$$l_m = [\alpha^2(1-(1-\alpha)\mu^2)]^m[(1-(1-\alpha)\mu)]^m\|r_1-p\| \tag{3.20}$$

Now, we find the rate of convergence of our iterative scheme (3.1) as

a) By using (3.10) and (3.20), we have

$$\frac{l_m}{g_m} = \frac{[\alpha^2(1-(1-\alpha)\mu^2)]^m[(1-(1-\alpha)\mu)]^m\|r_1-p\|}{\alpha^m\|x_1-p\|}$$

$$= [\alpha(1-(1-\alpha)\mu^2)]^m[(1-(1-\alpha)\mu)]^m\frac{\|r_1-p\|}{\|x_1-p\|} \to 0 \text{ as } m \to \infty \tag{3.21}$$

Thus $\{r_m\}$ converges faster than $\{x_m\}$ to p. That is, the Picard-P iterative scheme converges faster than Picard Iterative scheme.

Similarly,

b) By using (3.12) and (3.20), we have

$$\frac{l_m}{h_m} = \frac{[\alpha^2(1-(1-\alpha)\mu^2)]^m[(1-(1-\alpha)\mu)]^m\|r_1-p\|}{[\alpha(1-(1-\alpha)\mu)]^m[(1-(1-\alpha)\mu^2)]^m\|u_1-p\|}$$

$$= \alpha^m\frac{\|r_1-p\|}{\|u_1-p\|} \to 0 \text{ as } m \to \infty \tag{3.22}$$

Thus $\{r_m\}$ converges faster than $\{u_m\}$ to p. That is, the Picard-P iterative scheme converges faster than P Iterative scheme.

Similarly,

c) By using (3.14) and (3.20),we have

$$\frac{l_m}{j_m} = \frac{[\alpha^2(1-(1-\alpha)\mu^2)]^m[(1-(1-\alpha)\mu)]^m\|r_1-p\|}{[\alpha(1-(1-\alpha)\mu)]^m\|y_1-p\|}$$

$$= [\alpha(1-(1-\alpha)\mu^2)]^m\frac{\|r_1-p\|}{\|y_1-p\|} \to 0 \text{ as } m \to \infty \tag{3.23}$$

Thus $\{r_m\}$ converges faster than $\{y_m\}$ to p. That is, the Picard-P iterative scheme converges faster than Picard-Mann Iterative scheme.

Similarly,

d) By using (3.16) and (3.20), we have

$$\frac{l_m}{k_m} = \frac{[\alpha^2(1-(1-\alpha)\mu^2)]^m[(1-(1-\alpha)\mu)]^m\|r_1-p\|}{[\alpha(1-(1-\alpha)\mu)]^m[(1-(1-\alpha)\mu)]^{2m}\|t_1-p\|}$$

$$= \frac{\alpha^m[(1-(1-\alpha)\mu^2)]^m}{[(1-(1-\alpha)\mu)]^{2m}}\frac{\|r_1-p\|}{\|t_1-p\|} \to 0 \text{ as } m \to \infty \tag{3.24}$$

Thus $\{r_m\}$ converges faster than $\{t_m\}$ to p. That is, the Picard-P iterative scheme converges faster than Picard-SP Iterative scheme.

Similarly,

e) By using (3.18) and (3.20), we have

$$\frac{l_m}{i_m} = \frac{[\alpha^2(1-(1-\alpha)\mu^2)]^m[(1-(1-\alpha)\mu)]^m\|r_1-p\|}{[\alpha^2(1-(1-\alpha)\mu^2)]^m\|v_1-p\|}$$

$$= [(1-(1-\alpha)\mu)]^{m}\frac{\|r_1-p\|}{\|v_1-p\|} \to 0 \text{ as } m \to \infty \quad (3.25)$$

Thus $\{r_m\}$ converges faster than $\{v_m\}$ to p. That is, the Picard-P iterative scheme converges faster than Picard-S Iterative scheme. This completes the proof.

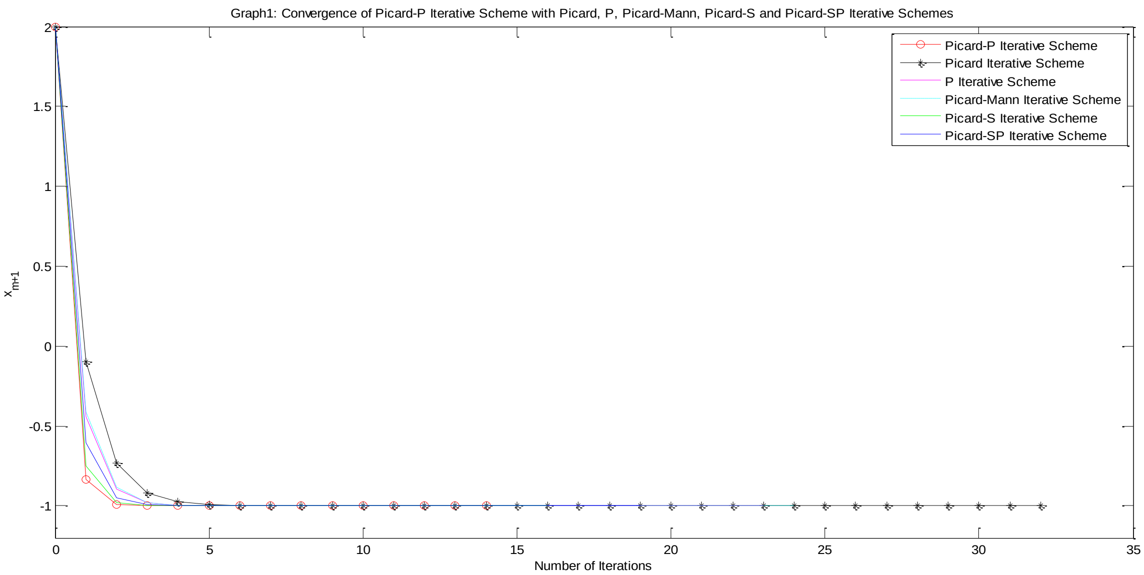
Now we will illustrate numerically the convergence of Picard-P hybrid iteration scheme with other iterative schemes like Picard, P, Picard-Mann, Picard-SP and Picard-S. MATLAB software is used for generating the comparison table and graph.

Example 3.4. Consider $T: C \to C$ is given by $Tr = \frac{3r-7}{10}$ $\forall\ r \in C$, where $C = [0,2] \subseteq X = \mathbb{R}$. Take $a_m = 0.5$ $b_m = 0.5$ $c_m = 0.3$ $\forall\ m \in \mathbb{N}$ starting with $x_1 = 2$. In comparison to the other Picard, P, Picard-Mann, Picard-S and Picard-SP iterative schemes, the Picard-P iterative scheme (3.1) converges more quickly, as seen in the following table 3.5 and graph 3.6:

Table 3.5. Comparison table of Picard, P, Picard-Mann, Picard-S, Picard-SP and Picard-P iterative schemes

S.No	Picard	P	Picard-Mann	Picard-S	Picard-SP	Picard-P
1	2.00000000000000000	2.00000000000000000	2.00000000000000000	2.00000000000000000	2.00000000000000000	2.00000000000000000
2	-0.10000000000000001	-0.43956999999999996	-0.41500000000000004	-0.74890000000000001	-0.60255099999999995	-0.83187099999999992
3	-0.72999999999999998	-0.89530607169999987	-0.88592499999999996	-0.97898293000000014	-0.94734476413300006	-0.99057754645299989
4	-0.91899999999999993	-0.98044212725427693	-0.97775537499999987	-0.99824087124100003	-0.99302407638663226	-0.99947193743586560
5	-0.97570000000000001	-0.99634639379237133	-0.99566229812499996	-0.99985276092287168	-0.99907580871193036	-0.99997040578971830
6	-0.99270999999999998	-0.99931746982435277	-0.99915414813437509	-0.99998767608924444	-0.99987756036558262	-0.99999834145167321
7	-0.99781300000000006	-0.99987249653788735	-0.99983505888620317	-0.99999896848866976	-0.99998377882991352	-0.99999990704997610
8	-0.99934390000000006	-0.99997618107824271	-0.99996783648280962	-0.99999991366250163	-0.99999785097072347	-0.99999999479080182
9	-0.99980317000000007	-0.99999555038722643	-0.99999372811414788	-0.99999999277355145	-0.99999971529015430	-0.99999999970806086
10	-0.99994095099999991	-0.99999916876783779	-0.99999877698225892	-0.99999999939514628	-0.99999996228078558	-0.99999999998363887
11	-0.99998228530000000	-0.99999984471751968	-0.99999976151154046	-0.99999999994937383	-0.99999999500284531	-0.99999999999908307
12	-0.99999468558999993	-0.99999997099167981	-0.99999995349475035	-0.99999999999576272	-0.99999999933796191	-0.99999999999994849
13	-0.99999840567699994	-0.99999999458095568	-0.99999999093147629	-0.99999999999964539	-0.99999999991229116	-0.99999999999999711
14	-0.99999952170309991	-0.99999999898766823	-0.99999999823163788	-0.99999999999997036	-0.99999999998838018	-0.99999999999999978
15	-0.99999985651092993	-0.99999999981088616	-0.99999999965516939	-0.99999999999999756	-0.99999999999846056	**-1.00000000000000000**
16	-0.99999995695327892	-0.99999999996467159	-0.99999999993275812	-0.99999999999999978	-0.99999999999979605	-1.00000000000000000

17	-0.99999998708598370	-0.99999999999340017	-0.99999999998688782	**-1.00000000000000000**	-0.99999999999997302	-1.00000000000000000
18	-0.99999999612579527	-0.99999999999876710	-0.99999999999744316	-1.00000000000000000	-0.99999999999999645	-1.00000000000000000
19	-0.99999999883773860	-0.99999999999976963	-0.99999999999950140	-1.00000000000000000	-0.99999999999999944	-1.00000000000000000
20	-0.99999999965132158	-0.99999999999995692	-0.99999999999990286	-1.00000000000000000	**-1.00000000000000000**	-1.00000000000000000
21	-0.99999999989539634	-0.99999999999999201	-0.99999999999998113	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000
22	-0.99999999996861888	-0.99999999999999845	-0.99999999999999645	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000
23	-0.99999999999058564	-0.99999999999999967	-0.99999999999999933	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000
24	-0.99999999999717559	**-1.00000000000000000**	-0.99999999999999978	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000
25	-0.99999999999915268	-1.00000000000000000	**-1.00000000000000000**	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000
26	-0.99999999999974576	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000
27	-0.99999999999992362	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000
28	-0.99999999999997713	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000
29	-0.99999999999999323	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000
30	-0.99999999999999789	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000
31	-0.99999999999999933	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000
32	-0.99999999999999978	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000
33	**-1.00000000000000000**	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000	-1.00000000000000000



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