Traces of the Riemann zeta function on the complex plane.

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Abstract.

The traces resulting on the complex plane from the zeta function of Riemann, can be divided into two parts.

In the first part of the traces it is essential to focus attention on the start and end points of the vectors; in the second part it is essential to focus attention on the two origins of particular polygonal spirals, which I call "pseudo-clothoid".

All traces start from the origin of the complex plane, the first part of the traces tends to move away from the origin; it develops in a convoluted way reaching variable distances.

The two parts behave like two arms, of a mechanism which makes them both rotate but independently and clockwise; the hinge from which the rotation of the second arm begins, is located at the junction point with the first.

The rotation of the two arms cyclically brings the free end of the second arm, to pass where the origin of the complex plane is located; but only under one condition does it intercept it.

The condition is that the two parts of the trace must compensate each other; in this article I highlight how the compensation takes place between the two parts of the trace.

Given a complex number (s) I call (a) the real part and (b) the coefficient of the imaginary part; then s=a+b*i.

The value of (b) is the engine of the rotations; only if a=1/2, the value of (b) is neutral with respect to the distances between the two origins, of the pseudo-clothoids.

Premise.

To carry out the research I am using PARI-GP for the calculations and a two-dimensional CAD to reconstruct the traces on the complex plane.

The CAD is not essential but allows you to measure inclinations and distances; the one I use provides results up to the eighth decimal place, with an indisputably superior calculation precision.

In this article I present three formulations, but I mainly analyze the traces resulting from the first one; are simpler to understand and I am convinced that the analysis of these traces is useful, for understanding the traces resulting from the other two formulations.
My preference for the first of the three formulations has ulterior motives which I will describe.

1. The Riemann Hypothesis.

In 1859 the great German mathematician Bernhard Riemann, in one of his papers he presented a new version of the zeta function; in doing so he claimed that all "non-trivial" zeros are complex numbers with the real part 1/2. The "non-trivial" zeros coincide with the origin of the complex plane.

2. The three formulations of the zeta function, which I studied.

$$\zeta(s) = \sum_{n\geq1} \frac{1}{n^s} \ , \qquad \Re(s) > 1 \qquad (1)$$

$$(2^{1-s} - 1)\zeta(s) = \sum_{n\geq1} (-1)^n \frac{1}{n^s} \ , \qquad \Re(s) > 0 \qquad (2)$$

$$\zeta(s) = \sum_{n\geq1} (-1)^n \frac{1}{n^s} \ , \qquad \Re(s) > 0 \qquad (3)$$

• The formulation (1) simply does the summation of 1/n^s, this is the basis of the Riemann zeta function; the first vector always ends in +1+0i.

Although it is known that for values of the real part of (s) less than 1 this formulation does not converge, I am convinced of the importance of the traces resulting from this formulation.

• The formulation (2) is defined as "the classical formula" of the function zeta of Riemann. Note that the $(2^{1-s} - 1)$ part occurs after the summation is complete; the intervention of this part rotates and resizes the trace resulting from the summation, based on the origin of the complex plane.

• I obtained the formulation (3) from the formulation (2) by removing the $(2^{1-s} - 1)$ part. In addition to formulation (1) only the part $(-1)^n$ remains.

By reversing the direction of the vector when (n) is odd, the part $(-1)^n$ makes convergence possible even for values of the real part of (s) lower than 1; the first vector always ends in -1+0i.

There can be no doubt that it is the origin of the complex plane, the basis with respect to which part $(2^{1-s} - 1)$ rotates and resizes the trace resulting from the summation, since it is from that point that the trace begins; consequently the "zeros" of formulation (3) and formulation (2) coincide.

Having said this, there can be no doubt that part $(2^{1-s} - 1)$ *has no influence on Riemann hypothesis; for this reason I do not thoroughly analyze the traces resulting from formulation (2).*

It must be recognized that formulation (2) performs magic, but I will talk about this later.

3. Introduction.

Despite their differences, the three formulations share a fundamental characteristic.

In the traces resulting on the complex plane from any of the three formulations, there is always at least one spiral with its origin; being composed of vectors, they are polygonal spirals which can be convergent or divergent.

Starting from b≥π for formulations (2) and (3), starting from b≥2π for formulation (1), the presence of an increasing number of particular spirals can be seen more and more evidently in the final part of the traces; their formation is largely linked to the periodicity of the angles.

From their resemblance to the clothoids I have called them pseudo-clothoids; like the clothoids each of them has two origins, which are shared with the previous pseudo-clothoid and with the following one.

Regardless of which of the three formulations the trace results; the closer you get to any of the origins, the more it becomes apparent that the midpoints of the vectors get closest to them.

What happens at the last origin depends on the value of (a), the value of (b) and the value of (n) of the vectors that reach it; the value of (n) is finite and calculable for formulation (1), for formulations (2) and (3) the value of (n) tends to infinity.
In reality, for finite values of (n), also for these two formulations the last origin is highlighted.

Wanting to be precise, the traces resulting from formulation (1), after the last origin form a diverging polygonal spiral which, however, has two particularities; maintains the same origin and diverges with a step which in the final part tends to zero.
In the traces resulting from formulations (2) and (3) the final part of the pseudo-clothoid converging towards the last origin, has a step that tends to zero.

With (s) being equal, for values of b≥2π and for any value of (s), the following can be observed.
Given the number of vectors needed to complete the last pseudo-clothoid, of a trace resulting from formulation (1); with the same number of vectors, in the traces resulting from formulations (2) and (3) we arrive at the same point which is always in the middle of the last pseudo-clothoid.

For formulations (1) and (3), the value of (a) determines the length of the vectors and the value of (b) the angle that the vectors form with the real axis, both in function of (n); only for the formulation (3) the direction of the vectors corresponding to (n) odd is inverted.

Formulation (2) adds to formulation (3) the $(2^{1-s}-1)$ part which, intervening at the end of the summation, rotates and resizes the trace with respect to the origin of the complex plane; this has the only effect of moving the points of convergence that do not occur on the origin of the complex plane.
The points of convergence (last origins) resulting from formulation (3) which do not occur on the origin of the complex plane, never correspond to the last origins resulting from the other two formulations.
Evidently it is part $(2^{1-s}-1)$ that performs the magic of making all the last origins of the traces resulting from formulation (3) coincide with those of the traces resulting from formulation (1).

I will point out that the last origin present in the traces resulting from formulation (1) always coincides with the last origin (commonly defined as the point of convergence) present in the traces resulting from formulation (2).

After this introduction, I describe in detail the reasons why Riemann hypothesis is true for the traces resulting from formulation (1); below I describe in detail the traces resulting from formulation (1) providing functions and command lines for PARI-GP, with examples and results.

After the analysis of the formulation (1) I perform the analysis of the formulation (3) providing functions and command lines for PARI-GP, with examples and results; the analysis is preceded by the reasons for which the Riemann hypothesis is also true for the traces resulting from the formulation (3).

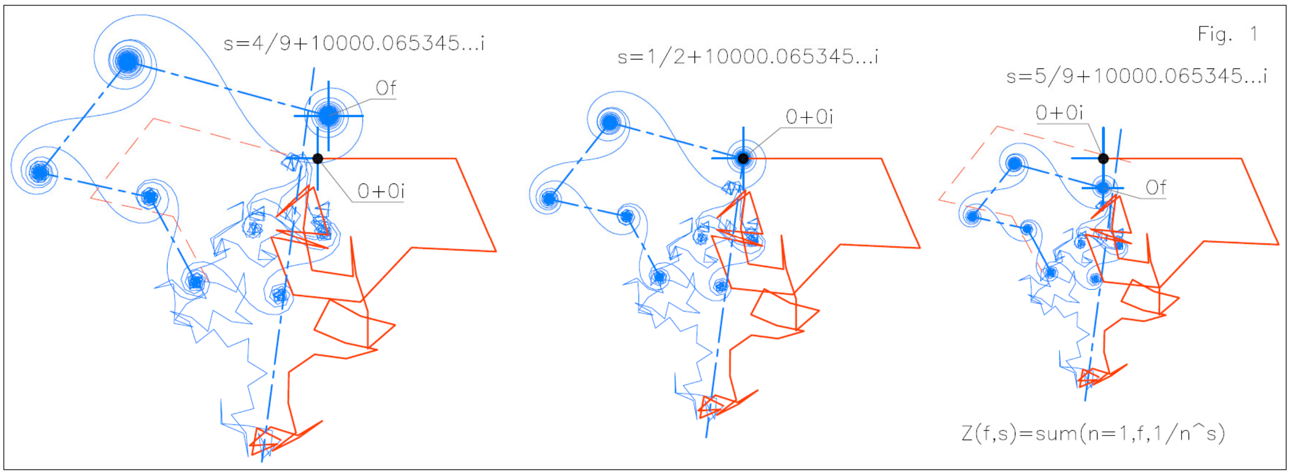
Now I want to show some images that may be interesting; I point out that the traces depicted in the following four images all derive from formulation (1).

Notes:

- Unless otherwise specified, all the traces resulting from the formulation (1) inserted in this article end with the last vector that converges on the last origin that I call (Of).
- The values of the convergence points inserted in Fig. 3 and Fig. 4, are calculated with the zeta(s) function of PARI-GP; I therefore correspond to the point of convergence of the traces resulting from formulation (2).

- For the traces resulting from formulation (1), if a=1/2 there is a perfect symmetry, between the start and end points of the vectors present in the first part of the trace and the two origins of the pseudo-clothoids present in the second part of the trace; this justifies the presence of an axis of symmetry doubly.
- Symmetry therefore concerns both angles and distances in these cases.
- In the traces resulting from any of the three formulations, for any value of (a) and for b≥2π, the orientations of the pseudo-clothoids are always symmetric, to the angles of the vectors corresponding in the first part of the trace; I think that this is enough to justify the axis of symmetry.
- The two different colors are intended to distinguish the two parts of the traces.

Fig. 1 represents three traces in which the value of (a) changes, while the value of (b) which remains unchanged is known to be a "zero" of the Zeta function.



In Fig. 1 and Fig. 2, when a=4/9 and a=5/9 I have inserted (dashed in brown) a mirrored copy of the first four vectors, this to show the parallelism with the axes of the last four pseudo-clothoids ; this comparison shows the effect that a≠1/2 has on pseudo-clothoids.

The real width of Fig. 2 is greater than the previous one, inserting it in the text it is more reduced, for this reason I remind you that the first vector always has length 1.

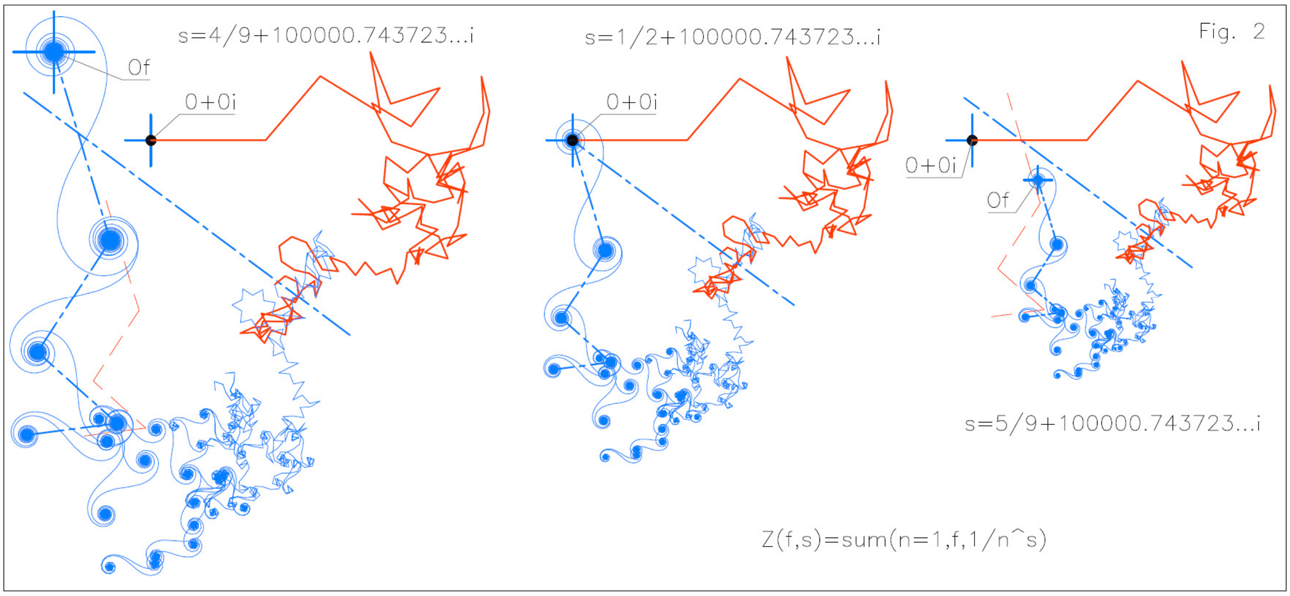


Fig. 2 is constructed in the same way as the previous one, with respect to which I have slightly increased the value of (b); the aim is to show that when a≠1/2 not only does there not exist the correspondence, of the distances between the two origins of the pseudo-clothoids and the length of the corresponding vectors in the first part of the trace, but the difference increases more and more with the increase of the value of (b).

For Fig. 3 and Fig. 4 that I have inserted in the following page, I have chosen (b) values such as to show opposite situations, to the traces represented in Fig. 1 and Fig. 2.

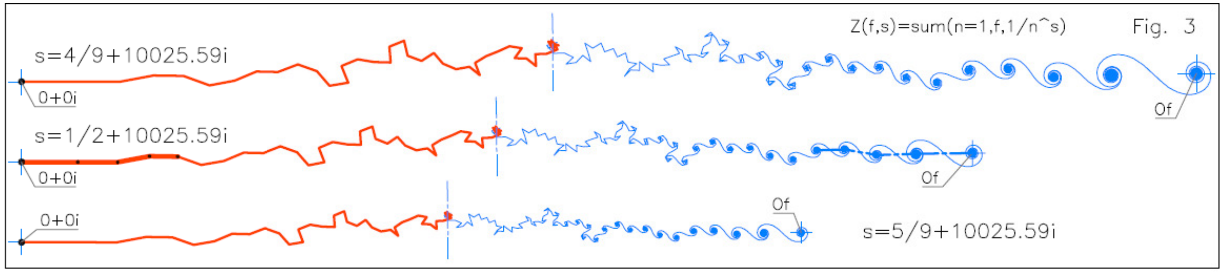
Also for Fig. 3 and Fig. 4 I have highlighted for a=1/2 (limiting myself to the first four so as not to make the images confused), the symmetry between the start and end points of the vectors present in the first part of the trace and the two origins of the pseudo-clothoids present in the second part.

For Fig. 3 and Fig. 4, if you want to graphically identify the position and inclination of the axis of symmetry, you can do it (if a=1/2) using the point of convergence indicated in the images; it is necessary to trace a segment from the origin of the complex plane to the point of convergence, after which it is sufficient to rotate the aforementioned segment by 90° with the center in its middle point.

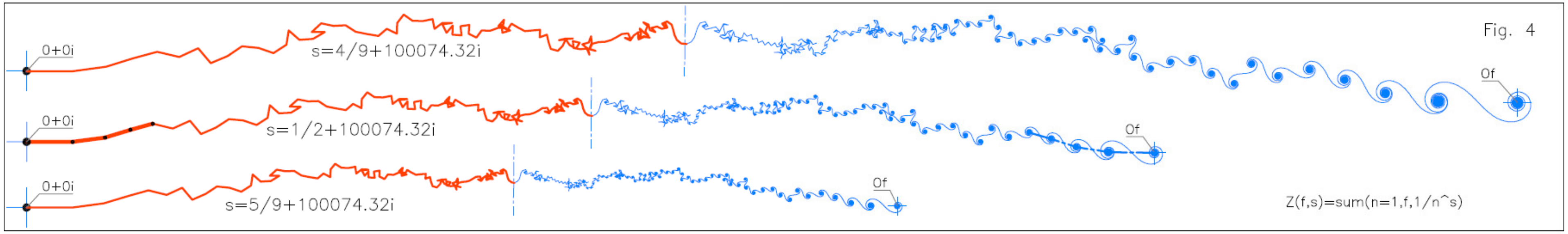
Recalling that I am referring to traces resulting from the formulation (1) for a=1/2, having traced the axis of symmetry it is possible to make a mirror copy, with respect to the axis of symmetry, of the vectors present in the first part of the trace; these are the vectors that go from the origin of the complex plane to the axis of symmetry.
The start and end points of the mirrored vectors identify the origins of the pseudo-clothoids, present in the second part of the trace.
Pseudo-clothoids do not form suddenly, initially using only one vector they tend to form from the next vector to what I call the "hinge"; the more the value of (b) increases, the more the part of the trace composed of these “embryos” of pseudo-clothoids lengthens, but also the number of the “finished” ones.
With greater evidence the greater the value of (b), the operation of mirroring shows after the “hinge” vector initially a weave regular; continuing towards the last pseudo-clothoid, it is highlighted that the simple weaving becomes a perfect symmetry, between the initial and endings points of the vectors present in the first part of the trace and the two origins of the pseudo-clothoids present in the second part of the trace.

In Fig. 4 the “hinge” vector is easily recognizable to be the only one intersected by the axis of symmetry.

With reference to the traces resulting from formulation (1) for a=1/2.
For large values of (b) they can be found in the first part of the trace figures similar to pseudo-clothoids.
Being present in the first part of the trace, I would say that these figures are not useful as a whole; the single vectors that compose them are useful, which correspond to the pseudo-clothoid “embryos”, present at the beginning of the second part of the trace.
The specularity between the two parts of the trace generates at the beginning of the second part of the trace a mirror copy of the same figures; this happens in the part where the “embryos” of pseudo-clothoids formed by a vector only.



zeta(4/9+10025.59*I)=20.875575054129614421302629431834641558 + 0.14394712464019861910635743664665682775*I
zeta(1/2+10025.59*I)=16.900400906635749063502159918611614548 + 0.16252322691469956117107095244179160737*I
zeta(5/9+10025.59*I)=13.857189639663578203369308929074034010 + 0.17096830669277548178752799838936978523*I



zeta(4/9+100074.32*I)=32.329400048954922154521977284833503779 − 0.68868638503030944036989896762095780201*I
zeta(1/2+100074.32*I)=24.466255804275120818399287427594376354 − 0.23915432432492810555908233572730219923*I
zeta(5/9+100074.32*I)=18.890485466976571220454142515299938536 + 0.033055717971441902071244131928451116899*I

4. For what reasons is Riemann hypothesis true for the traces resulting from formulation (1).

Note: My thesis on the equality of value between the last origin and the point of convergence must be accepted; those who do not agree with this thesis will find in section 7 the reasons why Riemann hypothesis is true for the traces resulting from formulation (3).

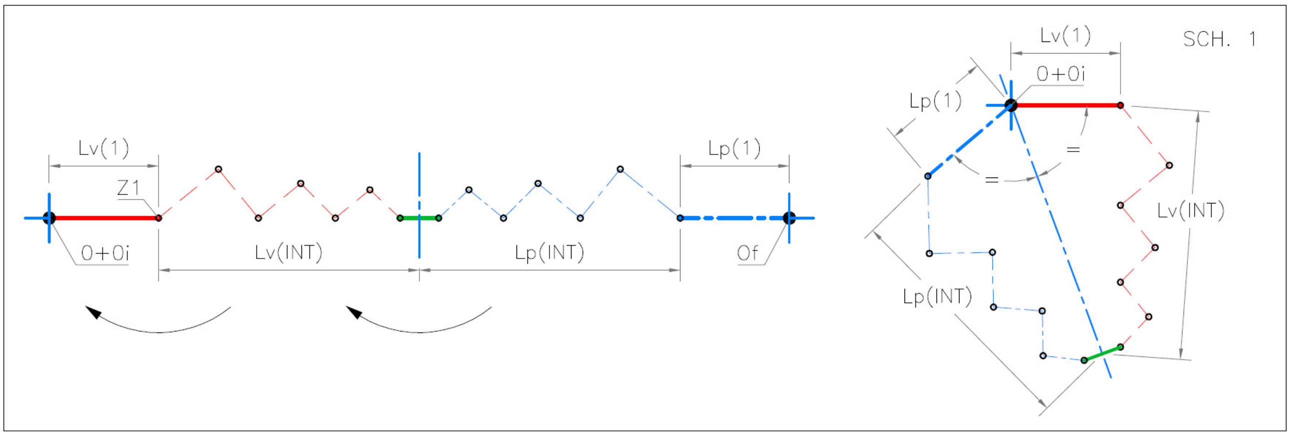
I still wish to highlight that the convergence points of the traces resulting from formulations (2) and (3), correspond to the second origin of the last pseudo-clothoid of these traces.

That said, I think it is right to attribute the importance it deserves to the last origin of the traces resulting from the formulation (1).

I recall that in this article I describe how to identify, with an accuracy that improves as the value of (b) increases, the origins of the last pseudo-clothoids; except only for the last origin of the traces resulting from formulations (2) and (3). Even if for particularly small values of (b), the precision is scarce, for values of (b) of the order of 30 billion, the precision with which it is possible to identify the latest origins reaches the sixteenth decimal place; the tendency to improve with the increase of (b) is evident.

I anticipate that the reasons that I will explain later for formulation (3) are substantially the same as those that I am about to highlight for formulation (1).

To make the reasons why Riemann hypothesis cannot be anything but true, I can help you with a schematic representation of two traces resulting from formulation (1).



The labels I used in SCH. 1 have the following meanings.
Lv(1) Length of the vector corresponding to n=1.
Lv(INT) Distance between the two extremes of the intermediate part, of the first part of the trace.
Lp(INT) Distance between the two extremes of the intermediate part, of the second part of the trace.
Lp(1) Distance between the two origins of the last pseudo-clothoid.

Between Lv(INT) and Lp(INT), colored green, is the hinge vector.

The pseudo-clothoids schematized in SCH. 1 correspond to a=1/2; therefore the origins, regardless of the value of (b), are always specular at the endpoints of the vectors present in the first part of the trace.

The value of (b) is the engine of the rotations; only if a=1/2 the value of (b) is neutral, with respect to the distances between the two origins of the pseudo-clothoids. In section 9 I explain why (b) results neutral.

Provided that a=1/2, the increase in (b) has only two direct and two indirect consequences.

The two direct consequences are:

- The increase of the value of (n) corresponding to the last vector that converges on (Of); therefore a greater number of vectors needed to reach (Of).
- The increase in the value of the angle that each single vector forms with the real axis; the actual result is a consequence of the periodicity of the angles.

The two indirect consequences are:

- A tendency to increase the number of vectors in the first part of the trace and a tendency to increase the number of pseudo-clothoids; this does not necessarily lead to a lengthening of Lv(INT) and Lp(INT) as angles and orientations also change.
- A clockwise rotation of the entire trace except the first vector; the hinge vector and the axis of symmetry also rotate together with the trace.

The magic of the traces deriving from the three formulations that I have analyzed consists in the fact that everything happens in a way that appears increasingly convoluted but respecting precise rules.

- In the traces resulting from any of the three formulations, for any value of (a) and for $b \geq 2\pi$, the orientations of the pseudo-clothoids are always symmetric, to the angles of the vectors corresponding in the first part of the trace; *this is the first reason why Riemann hypothesis can only be true.*
- In the traces resulting from formulation (1), if $a=1/2$ the distances between the two origins of the pseudo-clothoids are always equal, to the lengths of the corresponding vectors in the first part of the trace; *this is the second reason why Riemann hypothesis can only be true.*

The third reason why Riemann hypothesis can only be true is the following.

In the traces deriving from formulation (1), if $a \neq 1/2$ the distances between the two origins of the pseudo-clothoids are influenced by the value of (b), increasingly evident as the value of (b) increases; this influence is evident in two ways.

- If $a<1/2$ the distance between the two origins of the pseudo-clothoids is always greater than the length of the corresponding vectors, in the first part of the trace.
- If $a>1/2$ the distance between the two origins of the pseudo-clothoids is always smaller than the length of the corresponding vectors, in the first part of the trace.

At this point I explained the reasons why Riemann hypothesis can only be true for the traces resulting from formulation (1).

Now I want to add a new reason, to those already exposed in the introduction, to attribute the right value to formulation (1).

From what I have highlighted so far, there should be no doubt that, for the traces resulting from formulation (1), the last origins coincide with the origin of the complex plane, only when two conditions occur.

- $a=1/2$.
- The axis of symmetry passes through the origin of the complex plane.

I have discovered and I can prove it with concrete results, that in the traces resulting from formulation (1) provided that $a=1/2$ and $b \geq 2\pi$, the symmetry axis also passes in an always different but not random point of the hinge vector.

If a=1/2, given the value of (b) it is possible to calculate the value of (n) corresponding to the hinge vector, but also to establish in which point of the hinge vector, I can expect the axis of symmetry to pass.

Taking advantage of these two pieces of information, I can identify, with an accuracy that for now reaches the sixth decimal place, "zeros" whose integer part goes to 16 digits; this means "zeros" a few tens of thousands of times larger than those currently published.

The sixth decimal place is not much but it's a start.

Between b=1.000.000.000.000.000 and b=1.000.000.000.000.001 I have identified the following five "zeros"

b=1000000000000000.092053...   b=1000000000000000.555513...   b=1000000000000000.888060...
b=1000000000000000.262139...   b=1000000000000000.730076...

Unfortunately I fear that these zeros are currently not verifiable in a short time; always with a precision to the sixth decimal place, here are other "zeros" slightly larger than those I have found published.

Between b=30.610.047.000 and b=30.610.047.001 I have identified the following three "zeros"

b=30610047000.092975...   b=30610047000.509201...   b=30610047000.574744...

When and if the validity of my research is accepted, I will publish the formula for calculating the value of (n) relating to the hinge vector and I will describe the method with which I have been able to identify, in few hours and with a normal p.c. the "zeros" above.

At the moment I'm only available for a possible (seriously motivated) comparison on the number of zeros present between two consecutive integers, up to 17 digits.

5. The first tools I equipped myself with.

To get the necessary results from PARI-GP, I created a custom function for each of the three formulations; the basis is the sum(X=a, b, espr) function of PARI-GP.

Z(f,s)=sum(n=1,f,1/n^s)	For the formulation (1)
Z(f,s)=sum(n=1,f,(-1)^n*n^(-s))/(2^(1-s)-1)	For the formulation (2)
Z(f,s)=sum(n=1,f,(-1)^n*n^(-s))	For the formulation (3)

These functions provide the value of Z(f, s) for n=f, they are therefore a useful starting point but also need to be able to extract the intermediate points of interest.

The following command lines for PARI-GP are based on formulation (1) and get the values of Z(f, s), for (n) from (f1) to (f2).

```
Z(f,s)= sum(n=1,f,1/n^s)
a=1/2
b=100000
s=a+b*I
f1=
f2=
for(f=f1,f2, print(Z(f,s)))
```

As further help I can say that, to obtain the results in the form suitable for the Cartesian plane and therefore for a CAD software, the last line can for example become:

```
for(f=f1,f2, print(real(Z(f,s)) "," imag(Z(f,s))))
```

With these command lines it is therefore possible to obtain the data necessary to report on the complex plane any part of the trace resulting from that of the three formulations indicated in the first line.

6. Analysis of the traces resulting from the formulation (1).

The following two functions give for any vector occupying position (n); the first is the length ρ(n) of the vector, the second the angle θ(n) that the vector makes with the real axis.

Given (a) in fractional form, if (nr) is its numerator and (dr) its denominator; the length ρ(n) of each vector is related to (n) by function (4).

$$\rho(\mathrm{n}) = \frac{1}{\sqrt[dr]{n^{nr}}} \qquad (4)$$

Given the value of (b), the angle θ(n) that each vector forms with the real axis is related to (n) by the function (5).

$$\theta(\mathrm{n}) = \mathrm{b} \cdot \log(n) \qquad (5)$$

Notes for function (4) and (5):

- It is understood that (n) is the value that determines the point Z(n,s) of arrival of the vector, to which ρ(n) and θ(n) refer.
- log(n) is the natural logarithm of (n).
- For (b) positive, the growth of θ(n) is clockwise starting (zero) at 3.00.
- The resulting value for θ(n) is in radians.
- As the coefficient of the imaginary part of (s) and the value of (n) grow, the useful result of θ(n) is inevitably obtained net of the multiples of 2π.

6.1 Pseudo-clothoids and their origins.

I have already highlighted that the pseudo-clothoids of the second part of the trace gradually form starting from the hinge vector, their formation is due to the periodicity of the angles, but also to the trend of the curve θ(n) for a given value of (b).

The θ(n) curve has an initial part that tends to grow rapidly (tending to vertical), followed by an almost equally rapid slowdown, in turn followed by growth tending to zero.

For b≤2π the part tending vertical is practically non-existent, as the value of (b) increases, the initial part tends more and more to the vertical; is this part of the curve θ(n) in combination with the periodicity of the angles, which generates the pseudo-clothoids.

It is no coincidence that the formation of pseudo-clothoids corresponds only to the initial part of the curve θ(n); the length of the vectors is also fundamental, it is no coincidence that they are initially formed from a single vector and are perfected by increasing the number of vectors that compose them.

There is a particular harmony between θ(n) and the periodicity of the angles; from this harmony derives the connection of the pseudo-clothoids with the relationship between the value of (b) and the multiples of (π).

Further on I present three functions which, exploiting this link, allow to calculate the values of (n) relating to the final points, of the first, of the last and of the medium vector of any pseudo-clothoid, *belonging to the second part of the trace*; in this article I analyze the last pseudo-clothoids, the best formed.

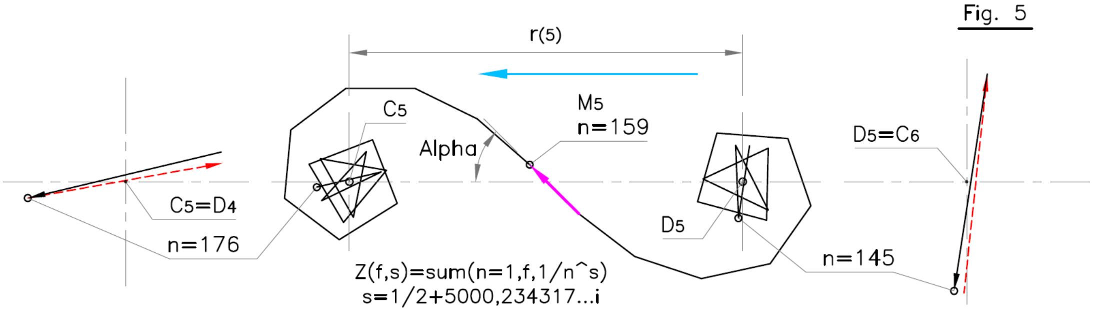
With reference to Fig. 5, the first vector of a pseudo-clothoid is the first that diverges from (D_p) leaving it to its right; the last vector of a pseudo-clothoid is the last one that converges on (C_p) or (C_p=Of) finding it to its left.

(D_p) is the first origin of the pseudo-clothoid, (C_p) or (C_1=Of) is the second origin.

(p) indicates the position of the pseudo-clothoid, p=1 indicates the last one.

Fig. 5 is extracted from a trace generated by formulation (1) for s=1/2+5000.234317...i, n=143÷177 and is the fifth counting from the last, therefore p=5.

The two vectors dotted in red in the two enlargements to the sides are, on the right, the last vector of the previous pseudo-clothoid and on the left, the first vector of the following one.



Divergence from (D_p) is always clockwise and convergence on (C_p) or (C_1=Of) is always counterclockwise.

As the last origin approaches and the value of (b) increases, the last vector that converges and the first that diverges tend to become parallel.

Note: I call "axis" with reference to a pseudo-clothoid, a segment joining the two origins.

The angle (Alpha) that the vector ending in (M_p) makes with the axis of the pseudo-clothoid tends to $\pi/4$ radians as the value of (b) increases.

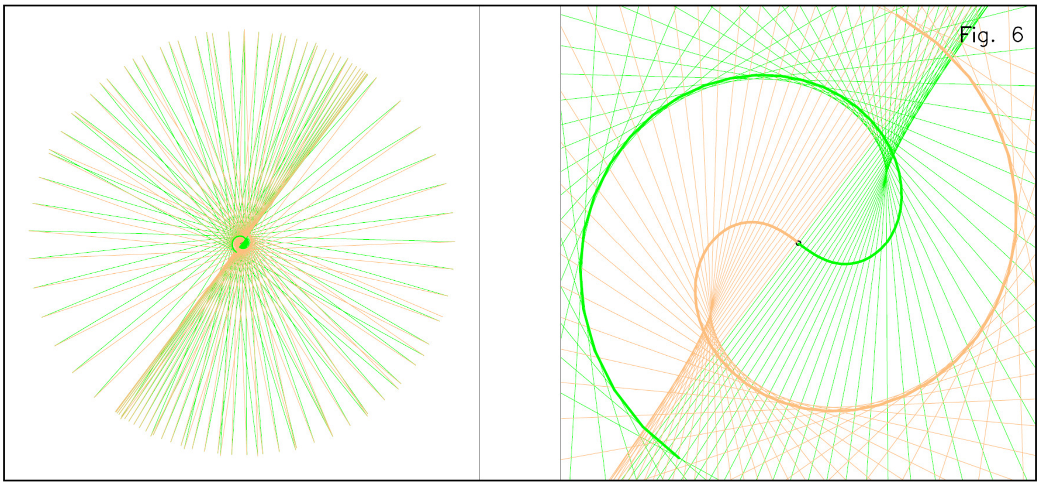
When I calculate the value of (n) corresponding to the first or last vector of a pseudo-clothoid, I want to get an integer value; for this reason the result of the calculation must be rounded.

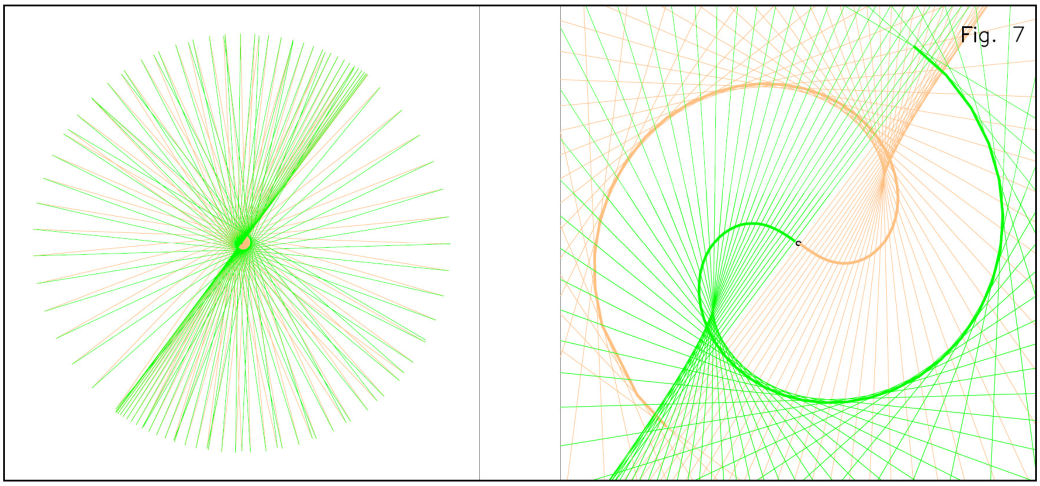
It can be noted that the closest proximity to the origins of the pseudo-clothoids is always disputed by the midpoints of the last vector that converges there and of the first vector that diverges from the same origin.

If not rounded, the result of the above calculation indicates which of the two vectors has its midpoint closest to the origin.

If the decimal part of the result of the calculation in question is less than 0.5, the closest to the origin is the midpoint of the last vector that converges, on the contrary, it is the midpoint of the first that diverges.

Figs. 6 and 7 show the trajectories of the midpoints of the vectors towards and from the origins; the traces used concern the point (Of) resulting from the formulation (1) for s=1/2+10000.065345...i.





Both images in Fig. 6 and Fig. 7 are an enlargement of the trace, those on the right are a further enlargement of the central part of the images on the left.

Fig. 6 uses the last hundred vectors (n=3083÷3183) that converge on (Of), Fig. 7 uses the first hundred vectors (n=3183÷3283) that diverge from (Of); I added a small circle to indicate the origin and two spirals connecting the midpoints of the vectors.

It can be seen that the spirals traced in the two images, by connecting the midpoints of the vectors, are almost superimposable.

For both figures, in brown are the vectors for which their final point corresponds to (n) even, in green are the vectors for which their final point corresponds to (n) odd.

What happens is that the angle between two consecutive vectors tightens more and more around the origin until it reaches the inversion and subsequent enlargement of the angle.

Note: The same behavior of the vectors is found in the origins that precede (Of), but also in the origins that precede the last one in the traces resulting from formulations (2) and (3).

- Three functions that allow to calculate, for a given pseudo-clothoid, the values of (n) corresponding to the first, last and intermediate vector; valid for formulation (1).

The value of (n) corresponds to the end point of the vector and (p) is the position of the pseudo-clothoid; p=1 corresponds to the last pseudo-clothoid.

Last vector (rounding to the previous integer) $$n(p) = \frac{b}{\pi \cdot (2 \cdot p - 1)} \quad (6)$$

Point (M_p) (rounding to the nearest integer) $$n(p) = \frac{b}{2 \cdot \pi \cdot p} \quad (7)$$

First vector (rounding to the next integer) $$n(p) = \frac{b}{\pi \cdot (2 \cdot p + 1)} \quad (8)$$

From functions (6), (7) and (8) it is evident that the divisor of (b), for p=1 is: π, 2π, 3π, for p=2 is 3π, 4π, 5π, and so on.

- Command lines for PARI-GP useful for identifying the origins of pseudo-clothoids, in the traces resulting from formulation (1).

```
a=
b=
p=
s=a+b*I
x1=b/(Pi*(2*p-1))
x2=truncate(b/(Pi*(2*p-1)))
Z(f,s)=sum(n=1,f,1/n^s)
if((x1-x2) < 0.5, f1=truncate(b/(Pi*(2*p-1)))-1, f1=truncate(b/(Pi*(2*p-1))))
f2=f1+1
z1=Z(f1,s)
z2=Z(f2,s)
Cp=(real(z1)+real(z2))/2+((imag(z1)+imag(z2))/2)*I
```

The requested data are (a), (b) and (p).

The difference between (x1) and (x2) determines which of the two vectors has its midpoint closest to the origin.

(z1) and (z2) are the extremes of the vector used and (C_p) is the midpoint which I assume as the origin $C_p=D_{p-1}$.

The customized function Z(f,s) has the drawback of starting over from 1 at each increment of the value of (n), for values of (b) which involve long times I use the following variant of the command lines.

I somehow solved the problem by calculating a value (Base) just smaller than (z1).

```
a=
b=
p=
s=a+b*I
x1=b/(Pi*(2*p-1))
x2=truncate(b/(Pi*(2*p-1)))
Z(f,s)=sum(n=1,f,1/n^s)
if((x1-x2) < 0.5, f1=truncate(b/(Pi*(2*p-1)))-1, f1=truncate(b/(Pi*(2*p-1))))
f2=f1+1
fb1=f1-2
fb2=f1-1
Base=Z(fb1,s)
Z(f,s)=Base+sum(n=fb2,f,1/n^s)
z1=Z(f1,s)
z2=Z(f2,s)
Cp=(real(z1)+real(z2))/2+((imag(z1)+imag(z2))/2)*I
```

- Distance between the two origins of a pseudo-clothoid; function valid for formulation (1).

This function, for the traces resulting from formulation (1) and only if a=1/2, allows to calculate the distance between the two origins of a pseudo-clothoid.

$$\mathrm{r(p)} = \frac{1}{\sqrt{p}} \qquad (9)$$

I derived it from function (4) by replacing (n) with (p), however I point out that only the square root is expected in this function; in function (4) the root is square only if a=1/2.

In another way it is possible to calculate r(p) when a≠1/2; in this case I use the following command lines for PARI-GP.

With these command lines starting from the resulting value of r(p) for a=1/2, the distance between the two origins of the pseudo-clothoid is obtained, given the values of (b), (p) and a≠1/2.

```
b=
p=
nr=
dr=
medio=b/(2*Pi*p)         \\ In this case, rounding would be counterproductive.
rBase=1/sqrt(p)
rho1=1/sqrt(medio)
rho2=1/sqrtn(medio^nr,dr)
rpb=(rho2/rho1)*rBase
```

Note: I called (rpb) the distance between the two origins of the pseudo-clothoids calculated in this way, to distinguish it from the one that can be calculated when a=1/2; when I use this method below (rpb) it becomes for example (r1b) if the result refers to the last pseudo-clothoid.

As for function (4), (nr) and (dr) used in the penultimate line, are the numerator and denominator of (a).

- Inclination and position of the axis of symmetry.

In the traces resulting from any of the three formulations, the axis of symmetry always divides the angle between the axis of the last pseudo-clothoid and the first vector into two equal parts; also divides the angles between the axes of the other pseudo-clothoids and the corresponding vectors into two equal parts.

If a=1/2, and the last origin (Of) does not coincide with the origin of the complex plane, the axis is perpendicular to a segment which joins (Of) with the origin of the complex plane; the intersection point corresponds to the center line of the same segment.

If a=1/2and the last origin (Of) coincides with the complex plane, the axis passes through the origin of the plane.

If a≠1/2, the axis of symmetry should be located at the intersection of the axis extensions of the last pseudo-clothoid and the first vector.

For the traces resulting from formulation (1), when the last origin does not coincide with the origin of the complex plane, I have chosen to position the symmetry axis so that it crosses the hinge vector.

To calculate the inclination of the axis of symmetry it is sufficient to know the inclination of the last pseudo-clothoid.

For values of (b) higher than a few tens of millions and with reference to the last pseudo clothoid, the angle (Alpha) that I indicated in Fig. 5 is already very close to π/4 and therefore can be used.

For lower values of (b) and in cases in which (Of) coincides with the origin of the complex plane, the method that I find to be the most accurate is to use the PARI-GP zeta(s) function, to locate two points of convergence very close to and straddling the origin of the complex plane; therefore I determine the inclination of the axis of symmetry for these two cases, and I use a value halfway between the two.

The reason why I described how I determine the inclination of the axis of symmetry, for relatively small values of (b) is the following.

To show how the precision, with which I determine the origins of the pseudo-clothoids, increases as the value of (b) increases, in the tests that follow I use increasing values of (b), starting from b=10000.065345... up to b=30384448732.364055...

When the value of (b) is large enough, I use the following command lines for PARI-GP with which I calculate (in decimal degrees) the inclination of the symmetry axis which I call Gamma; as for the vectors the zero is at 3.00 and for (b) positive the growth is hourly.

```
b=
p=1
n(p)=round(b/(2*Pi*p))
ThetaM(p)=b*log(n(p))
ThetaMp=ThetaM(p)*180/Pi-(truncate((ThetaM(p)*180/Pi)/360)*360)
Beta=ThetaMp-45
Gamma=(Beta-180)/2 \\ It can even be Gamma=(Beta+180)/2
```

As data only (b) is required, in these command lines n(p) has the same role that in function (5) it is played by (n); n(p) is calculated using function (7).

The angle ThetaMp is obtained by transforming ThetaM(p) into decimal degrees and removing the multiples of 360°.

6.2 Verification of the method of calculating the origins of pseudo-clothoids.

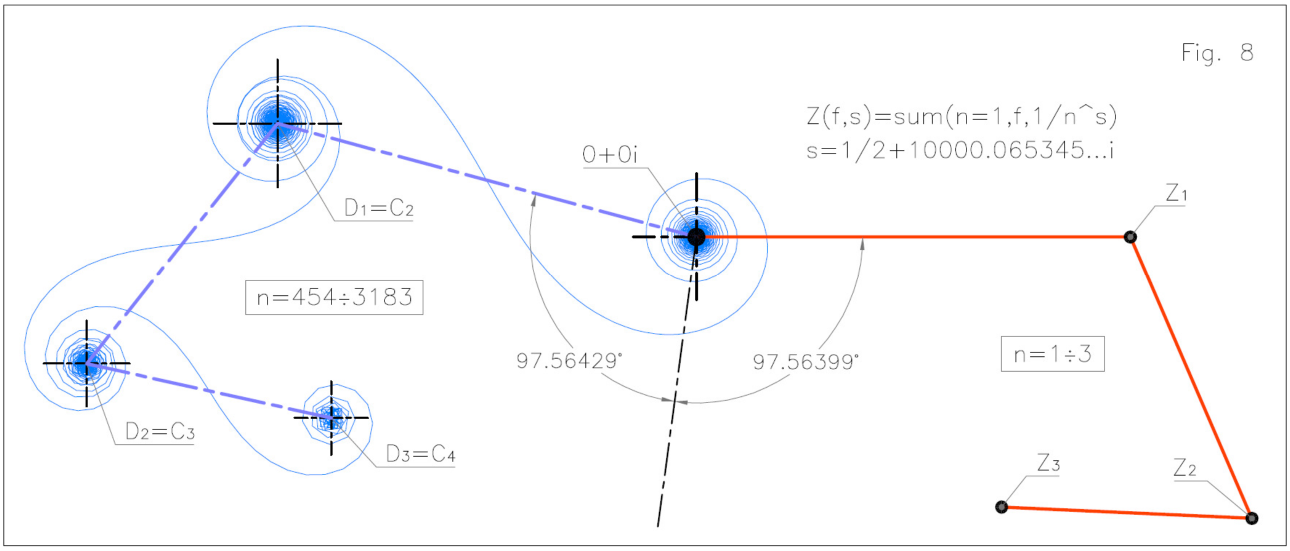
Below I present a series of images, followed by the values I have calculated for the points highlighted in the images, the aim is to provide concrete results of what I have described.

With these checks, in addition to the two symmetries, I want to show the improvement of the precision with which I determine the position of the last origin (Of); starting from a precision corresponding to the eighth decimal place, I arrive at a precision corresponding to the sixteenth decimal place.

When larger "zeros" are available, you can continue with the check.

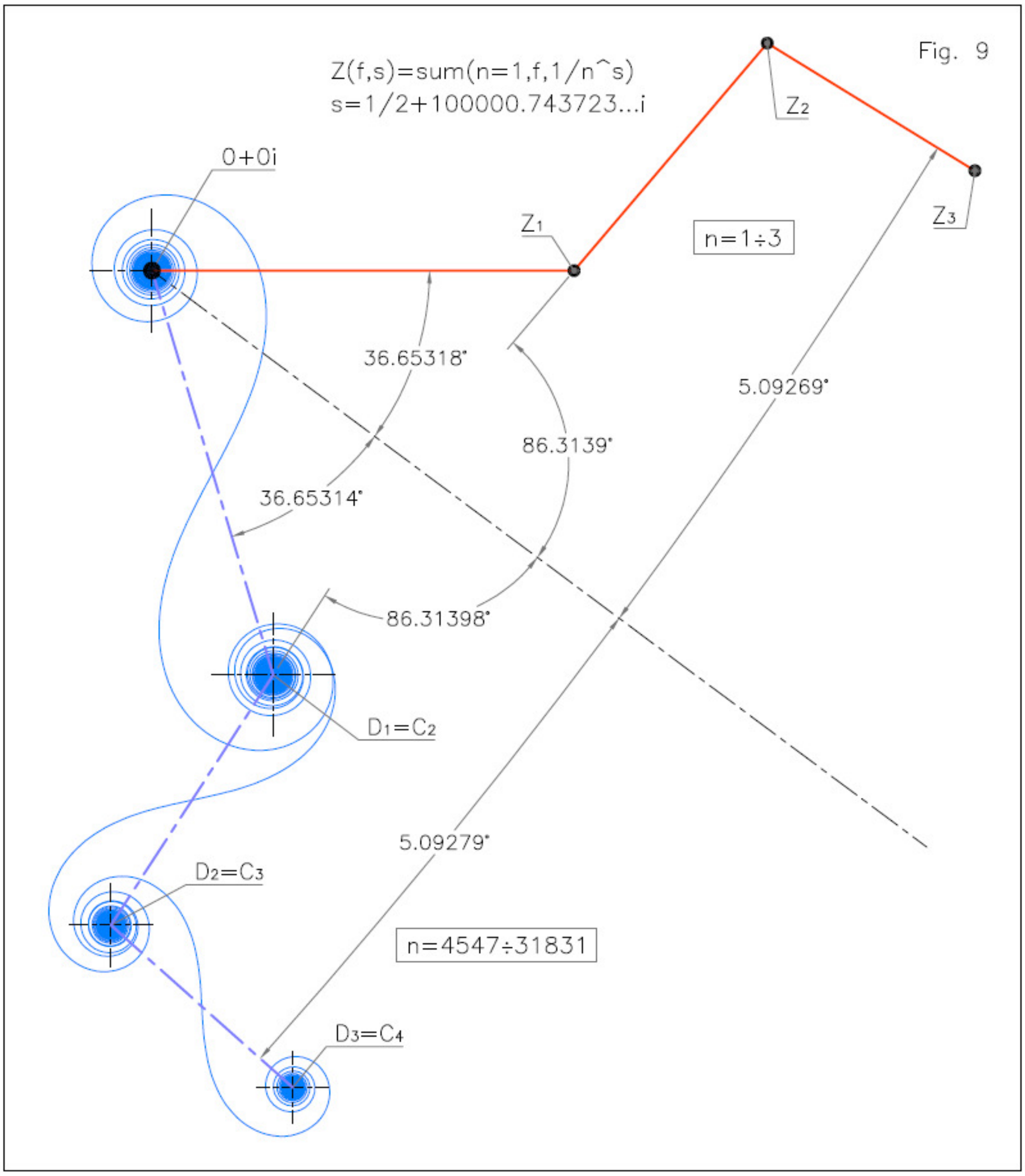
To create Fig. 8 I used function (6) to extract the last three pseudo-clothoids from the trace generated by formulation (1), by s=1/2+10000.065345...i.

In the image I inserted the axes of the pseudo clothoids, the first three vectors, the symmetry axis, the elevations of the angles and the initial and final value of (n) relating to the last three pseudo-clothoids.



These are the values I calculated for the indicated points.
Z1=1+0*I
Z2=1.2799713876695531665433480503409392319-0.64931966094242423850894089152269112084*I
Z3=0.70320991548829788391519000600998527088-0.62325172008311542221422620564855517131*I
ρ1=1/sqrt(1)=1
ρ2=1/sqrt(2)=0.70710678118654752440084436210484903928
ρ3=1/sqrt(3)=0.57735026918962576450914878050195745565
C1=Of=0.0000000016568636028954821850763582973272989791-0.00000087072899473011627738866111851860217961*I
D1=C2=-0.96534547532809193980548449762428039177+0.26098055150477956082855336392248307520*I
D2=C3=-1.4049884397114748890022772162087235663-0.29281653390599964384102604500552021983*I
D3=C4=-0.84163696637935761995323916729650411235-0.41807765046838787848281749156583641923*I
r1=1.0000015963419633007658231096434332396
r2=0.70709062215622857589281981925590547340
r3=0.57710937423327603288955743178562614204

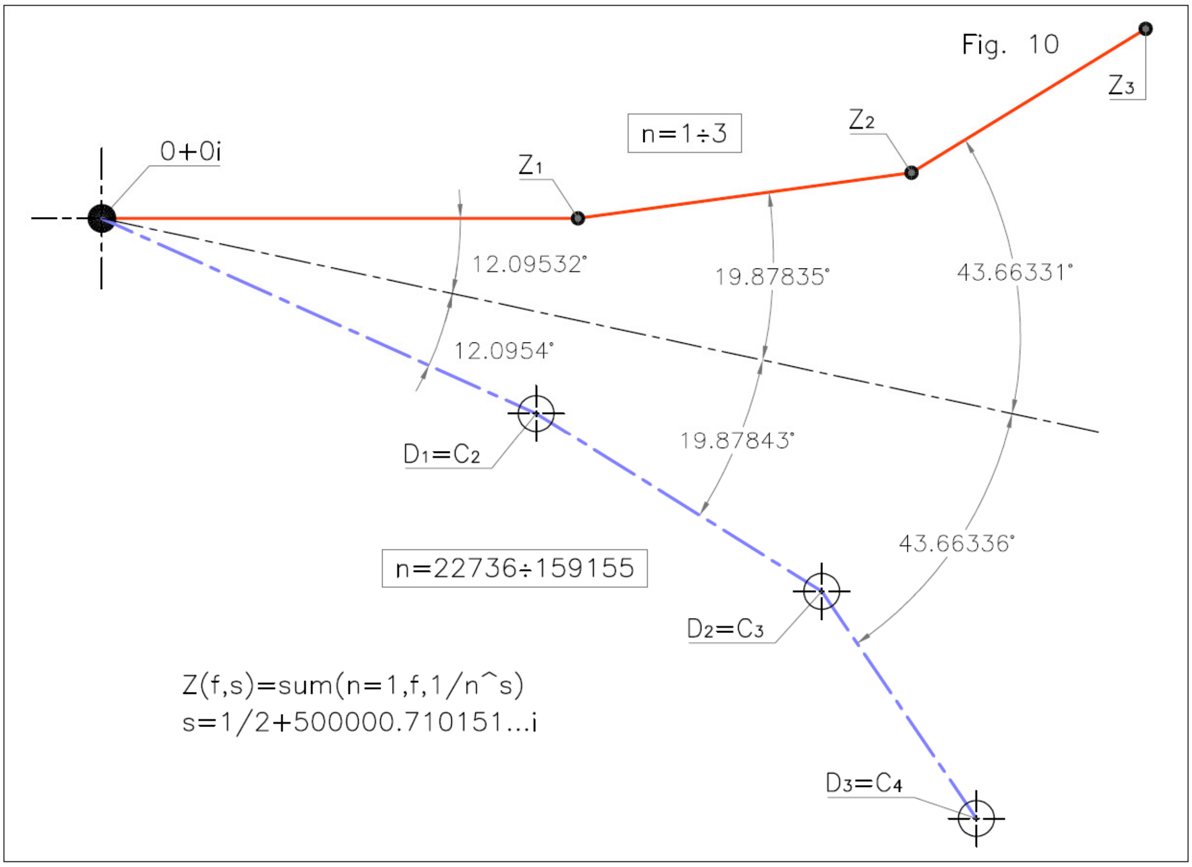
Fig. 9 is constructed like Fig. 8 and presents the same information; in (s) only the value of (b) changes.
s=1/2+100000.743723...i



These are the values I calculated for the indicated points.
Z1=1+0*I
Z2=1.4577191028839244668317458483658435732+0.53897423208826539301305203002727329262*I
Z3=1.9496723521738081328435222385295941515+0.23678996074632590722272502990810512370*I
ρ1=1/sqrt(1)=1
ρ2=1/sqrt(2)=0.70710678118654752440084436210484903928
ρ3=1/sqrt(3)=0.57735026918962576450914878050195745565
C1=Of=-0.0000000370401101004665927656685630672274266696+0.000000009152152648327514216743410886362451O277*I
D1=C2=0.28725465486692208776447065621645959543-0.95785380271380953561492621185996559978*I
D2=C3-0.097522918532458818642275418447890088235-1.5511033368811415036496594340538736238*I
D3=C4=0.33324555464323082245826597416800008388-1.9355245403148147732998157747450029426*I
r1=0.99999959146429558010392354593870939209
r2=0.70710592613898544602740375870939454761
r3=0.57735702916956718654887616656953995788

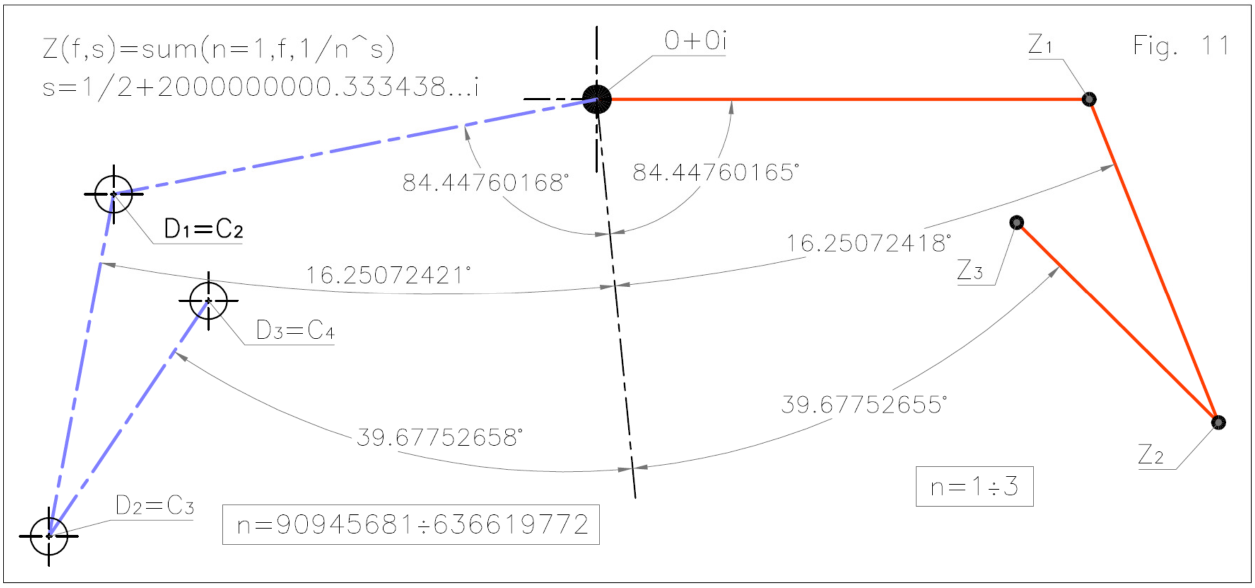
Note: Due to the significant increase in the number of vectors, deriving from the values of (b) that I use to make the next images, from now on I will draw only the axes of the pseudo-clothoids.

Fig. 10 concerns a trace resulting from the formulation (1) for s=1/2+ 500000.710151…i



These are the values that I have calculated for the indicated points.
Z1=1+0*I
Z2=1.7005929088419630417295185492240888243+0.0957579034876853578635892246165251321 58*I
Z3=2.1925066171244968265845137993760351633+0.3980065377890823025989589199577555602 4*I
ρ1=1/sqrt(1)=1
ρ2=1/sqrt(2)=0.7071067811865475244008443621048490392 8
ρ3=1/sqrt(3)=0.5773502691896257645091487805019574556 5
C1=Of=-0.0000000027792154992385275301971029483330326177+0.000000000727352149747625967257732912522078314 94*I
D1=C2=0.9121864236762500916359146632064519067 5-0.4097753485673244177656202308722463219 3*I
D2=C3=1.5120186073328475138105606507384839647-0.7842101288669172535425254634407157817 2*I
D3=C4=1.8368822136889766260247599874444268071-1.2614917527751333713604144910546480167*I
r1=0.9999999567495824261484295977976409664 3
r2=0.7071068188387426198704614662974294205 5
r3=0.5773509428893087610529471151900224790 0

Fig. 11 concerns a trace resulting from the formulation (1) for s=1/2+ 2000000000.333438…i



These are the values that I have calculated for the indicated points.

Z1=1+0*I

Z2=1.2626324947565415001462498888292029605-0.6565243123433854992668644777840575746 6*I

Z3=0.8527492172945229950937196782609413422 5-0.2499175831930586150724199454699965551 1*I

ρ1=1/sqrt(1)=1

ρ2=1/sqrt(2)=0.70710678118654752440084436210484903928

ρ3=1/sqrt(3)=0.57735026918962576450914878050195745565

C1=Of=-2.1018598982344225278283959074756982889 E-14 +1.2461516514710634043133623699640240876 E-14*I

D1=C2=-0.98127654307805967228352981464418392877-0.19260411730945407260045854945432846159*I

D2=C3=-1.1125423639618321346981979271755096935-0.88742012480068279941117074486178930331*I

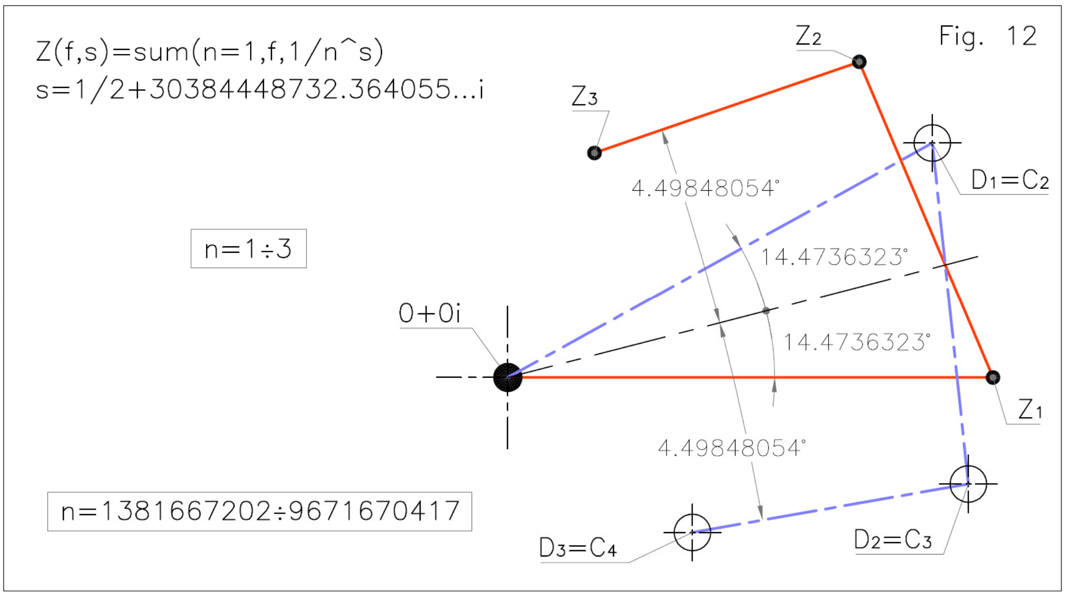
D3=C4=-0.78864764854755573619411916254460757751-0.40948127237240694681287429965890406009*I

r1=0.99999999999987229790352673921371217883

r2=0.70710678118537503454176497881172460539

r3=0.57735026918998863498291843250261601856

Fig. 12 concerns a trace resulting from the formulation (1) for s=1/2+ 30384448732.364055...i



These are the values that I have calculated for the indicated points.
Z1=1+0*I
Z2=0.72451936892479093834331224742288776835 + 0.65123760786859089336627404113201726582*I
Z3=0.17853254170072032741347865413625756921 + 0.46353646820469274016619620261863722781*I
ρ1=1/sqrt(1)=1
ρ2=1/sqrt(2)=0.70710678118654752440084436210484903928
ρ3=1/sqrt(3)=0.57735026918962576450914878050195745565
C1=Of=6.3551578879780175364007735182949533852 E-16 -5.5733444776495422447224959486958064976 E-17*I
D1=C2=0.87506555844477223355329140922329002172+0.48400440950856523581229024026018467892*I
D2=C3=0.94920382001841059201382652512671962206-0.21920503167580122000271295809032617014*I
D3=C4=0.38058137288296335873836931283654881749-0.31921426098517925149637462220624440284*I
r1=0.99999999999999747271997531593516918605
r2=0.70710678118654057191979551222158032576
r3=0.57735026918964906754505400912423534885

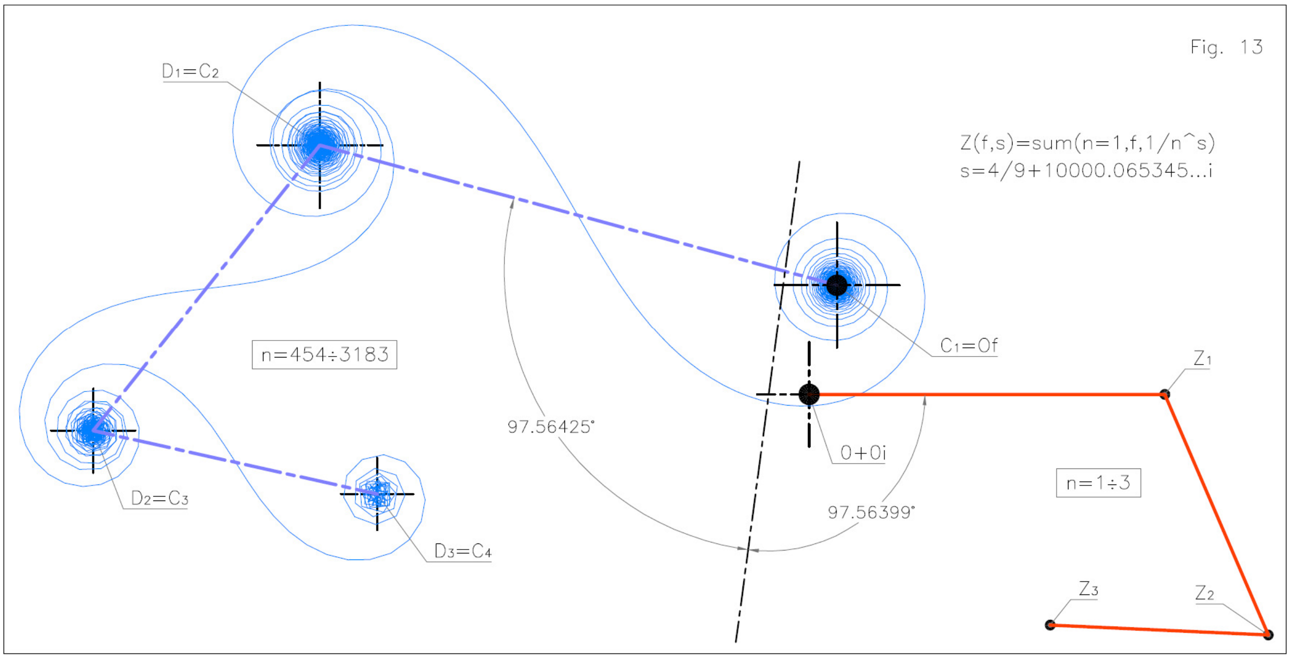
- What changes if a≠1/2.

If (a) is different from 1/2 the distances between the two origins of the pseudo-clothoids are no longer only a function of (p), but are influenced by the value of (b); this influence increases as (b) increases.

With reference to the Riemann hypothesis I remind again that even when a≠1/2, the symmetry remains between the orientations of the pseudo-clothoids and the angles of the corresponding vectors in the first part of the trace.

Always using traces deriving from formulation (1) for the next images I change the value of (a); for a comparison I use the same values of (b) already used for Fig. 8 and Fig. 9, for (a) I use 4/9 and 5/9.

Fig. 13 concerns a trace resulting from the formulation (1) for s=4/9+10000.065345...i.

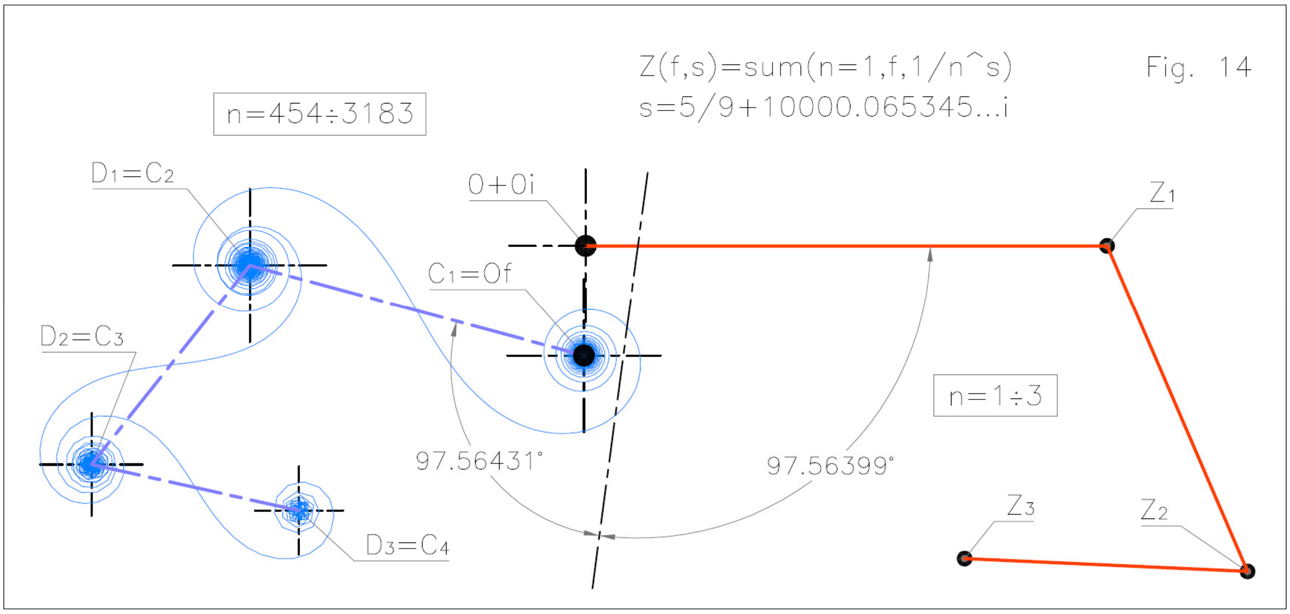


For a comparison, when a≠1/2 I add the result given by the zeta(s) function of PARI-GP; I recall that this result corresponds to the point of convergence for formulation (2).

These are the values I calculated for the indicated points.
Z1=1+0*I
Z2=1.2909628476605210084326503566901269838-0.67481144827828279017947706088493074162*I

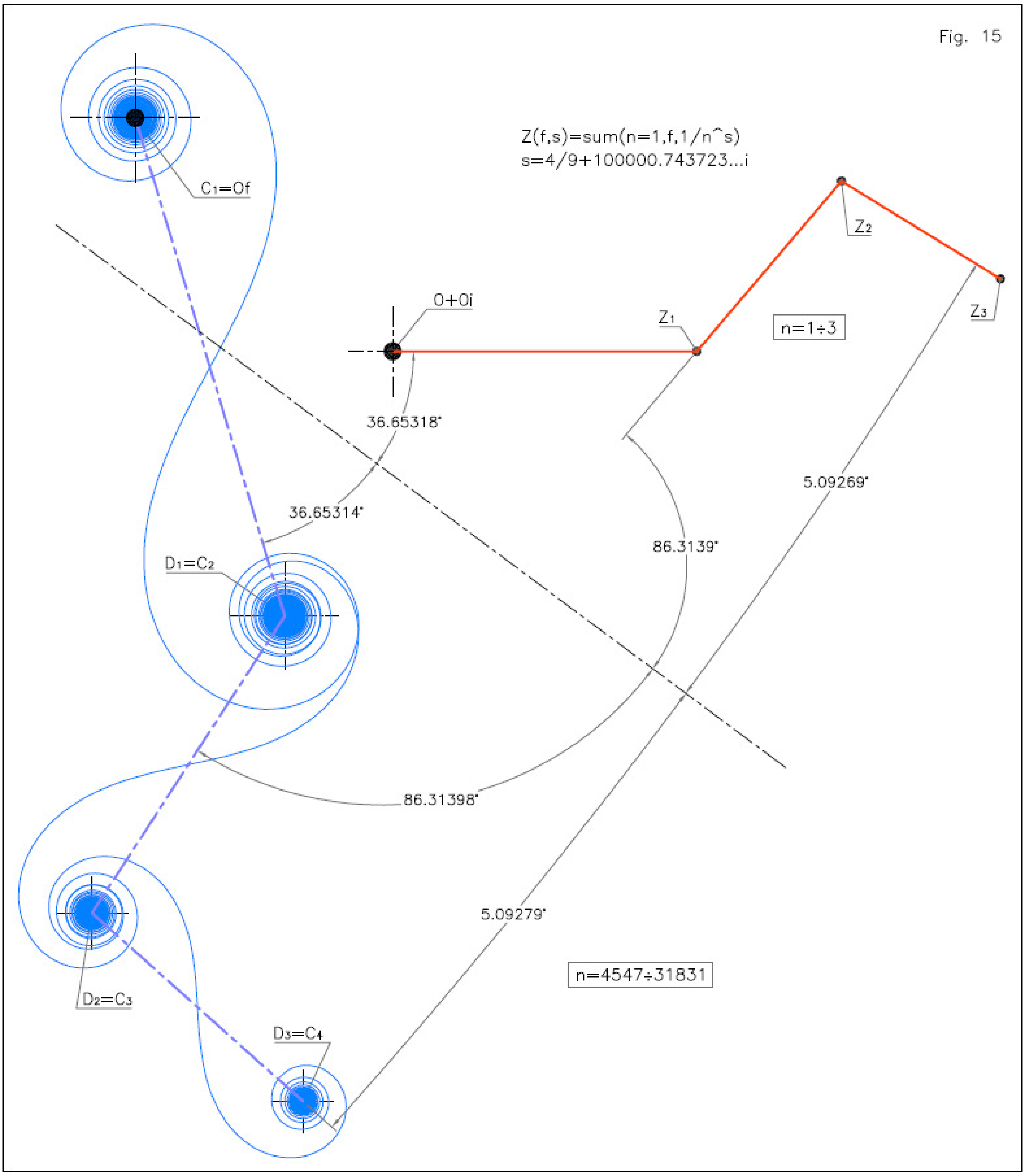
Z3=0.67790285161715506965661567094157789038-0.64710291972552217333701989568955777750*I
ρ1=1/sqrtn(1^4, 9)=1
ρ2=1/sqrtn(2^4, 9)=0.73486724613779942569210434909166210696
ρ3=1/sqrtn(3^4, 9)=0.61368584903291603508168767420564100200
C1=Of=0.078069391208961165037678213752753167089+0.30790622669024558836850157627847854343*I
D1=C2=-1.3759224096351823485055318274443961847+0.70099228607907868934278563988713532281*I
D2=C3=-2.0130940193707840466725240741463213824-0.10162010026651917911461168374890386250*I
D3=C4=-1.2148183879979694909518895943439788094-0.27911318374263457926107836694686998779*I
r1b=1.5061875514552878274959500761833515671
r2b=1.0248024792037427678003416878700458546
r3b=0.81810997914151621963050311883954803174
zeta(4/9+4/9+10000.065345...*I)
=0.078069316066729723355353601968168174916 + 0.30790749273808201448467851295757120616*I

Fig. 14 concerns a trace resulting from the formulation (1) for s=5/9+10000.065345...i.



These are the values I have calculated for the indicated points.
Z1=1+0*I
Z2=1.2693951428639756038395399121092363909-0.62479085552282484482008672991793951506*I
Z3=0.72678300385701611782810984212626430215-0.60026636518944265781730853817891941494*I
ρ1=1/sqrtn(1^5, 9)=1
ρ2=1/sqrtn(2^5, 9)=0.68039500008718848212784018766240654944
ρ3=1/sqrtn(3^5, 9)=0.54316607407294877966782901238146633121
C1=Of=-0.0033147435877998243112414904661196068522-0.21087158976642897434059217789423631076*I
D1=C2=-0.64423442472625189608026483754734672657-0.037598519000937387673623507097398156760*I
D2=C3=-0.94758417293476067586836421007323799333-0.41971481855275176196918706134508491323*I
D3=C4=-0.55002112542708837076036069739916681463-0.50811446247038464179998399611821705095*I
r1b=0.66392794113441831924094269421957823048
r2b=0.48789889773538899506180447578843385952
r3b=0.40744318225175140476662413531467566286
zeta(5/9+5/9+10000.065345...*I)
=-0.0033147150371539603997005834610779758630 - 0.21087099394740857764934465843944062796*I

Fig. 15 concerns a trace resulting from the formulation (1) for s=4/9+ 100000.743723...i.



These are the values I have calculated for the indicated points.

Z1=1+0*I

Z2=1.4756888006031370415901178246828021585+0.56013394329115782793117731500934304380*I

Z3=1.9986031622760265011684423638054623296+0.23893168355520023256719449799635452800*I

ρ1=1/sqrtn(1^4, 9)=1

ρ2=1/sqrtn(2^4, 9)=0.73486724613779942569210434909166210696

ρ3=1/sqrtn(3^4, 9)=0.61368584903291603508168767420564100200

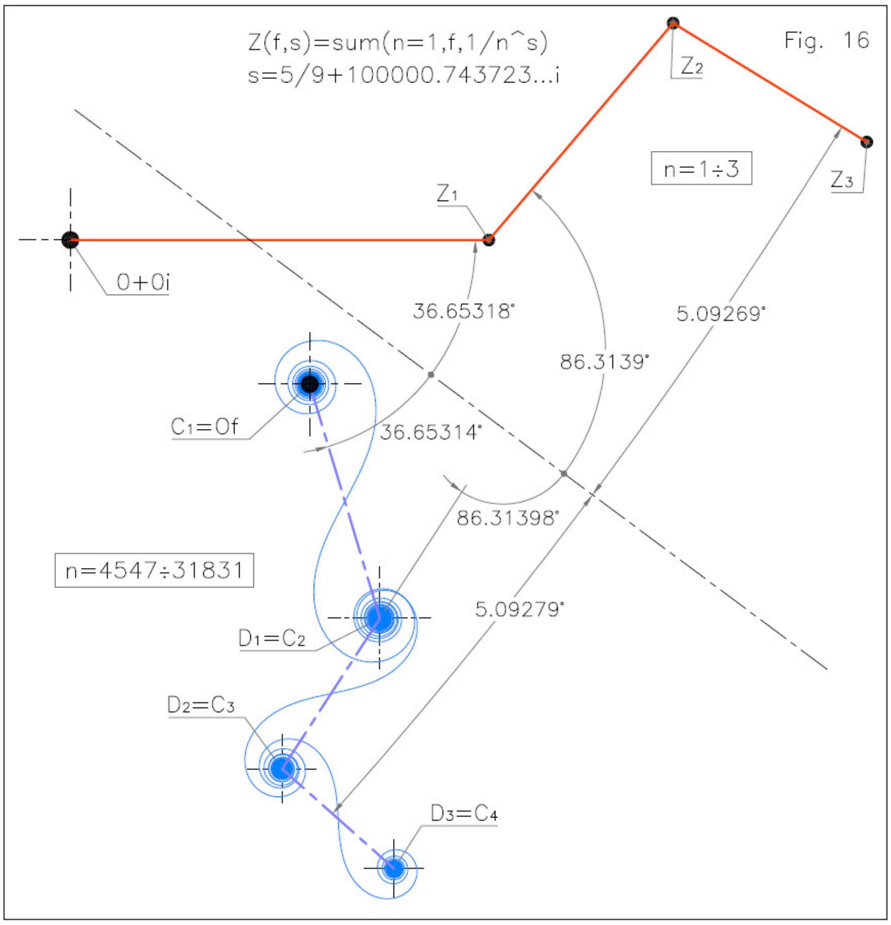
C1=Of=-0.84673021381513541714431254219361407532+0.76832263686721469640248028303464432307*I

D1=C2=-0.35502846897502637203992969161721830950-0.87126208647509133151871803201358076672*I

D2=C3=-0.98878218274601660074394047608264715379-1.8483827171981405256814121405554767117*I

D3=C4=-0.29508198044468991885208955987069630531-2.4674461728753337323752803403387896575*I
r1b=1.7117275128333209138072851381533823726
r2b=1.1646508412434703068103081075350967919
r3b=0.92975231302832797031983434662838644617
zeta(4/9+100000.743723...*I)
=-0.84673014950757096871937381022112793916 + 0.76832262463815631856022041029900861535*I

Fig. 16 concerns a trace resulting from the formulation (1) for s=5/9+ 100000.743723...i.



These are the values I have calculated for the indicated points.
Z1=1+0*I
Z2=1.4404282313967157743719222600213454175+0.51861385358704620706255711166028950774*I
Z3=1.9032535374614007212442124434898817873+0.23432153907617029996513253443733297378*I
ρ1=1/sqrtn(1^5, 9)=1
ρ2=1/sqrtn(2^5, 9)=0.68039500008718848212784018766240654944
ρ3=1/sqrtn(3^5, 9)=0.54316607407294877966782901238146633121
C1=Of=0.57203488848265085391149269945353542505 - 0.34487991058004164950854477168771736175*I
D1=C2=0.73985057675036631702924643475070948747 - 0.90446301916837973337679430090849555237*I
D2=C3=0.50623652386280126564970087804019541161 - 1.2646488609553691637231954943100021815*I
D3=C4=0.77373173602286596157379284633088441477 - 1.5033637384149378025773308650076518139*I
r1b=0.58420513341212783192918541347107579981
r2b=0.42931321757013609472032176868340751960
r3b=0.35851842330740965210028708609210130873
zeta(5/9+100000.743723...*I)
=0.57203490980482021091222463925673069972 - 0.34487991700520777400306819628490760566*I

7. For what reasons is Riemann hypothesis true for the traces resulting from formulation (3).

Formulation (3) limits itself to contrasting the sum of the results deriving from (n) even, with the sum of the results deriving from (n) odd.

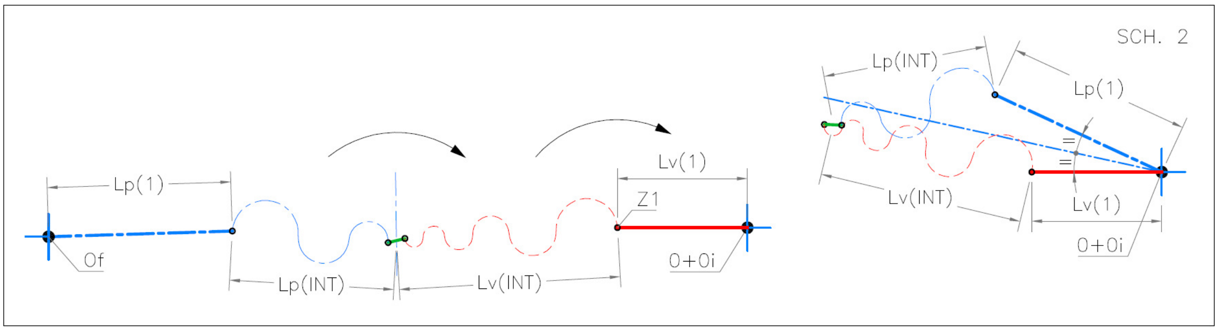
This is why only the last origins of the traces resulting from this formulation, which coincide with the origin of the complex plane, coincide with the last origins of the traces resulting from the other two formulations.

The Riemann hypothesis concerns non-trivial zeros, consequently what is valid for the formulation (3) is valid for the Riemann hypothesis.

In the traces resulting from formulation (3) there is never a symmetry between the two parts of the trace; only for a=1/2is there a compensation between the two parties.

The lack of symmetry in the traces resulting from formulation (3) for a=1/2 could lead one to think that there exist reasons other than those I have highlighted for formulation (1); actually the reasons are the same.

SCH. 2 wants to schematically show that there can be no symmetry in the traces resulting from formulation (3); but, if a=1/2, there can be compensation.



In SCH2 it should be evident that the distance between the two origins of the last pseudo-clothoid is greater than the length of the first vector; only if a=1/2 is this ratio between distances and lengths constant.

In SCH. 2 I used the same labels present and described in SCH. 1.

Since there cannot be symmetry in the traces resulting from formulation (3), the second and third reasons for which Riemann hypothesis is true must be described differently; but the substance is the same as already described for formulation (1).

If a=1/2 the value of (b) is neutral, with respect to the distances between the two origins of the pseudo-clothoids.

On the contrary if a≠1/2, (b) has an influence proportional to its value, on the distance between the two origins of the pseudo-clothoids. In section 9 I explain why (b) results neutral if a=1/2.

Later I present the functions valid for the traces resulting from formulation (3); for now I limit myself to providing the information needed to justify, the descriptions of the second and third reasons for which Riemann hypothesis is true, also for formulation (3).

In the traces resulting from formulation (3), the correspondences between pseudo-clothoids and vectors present in the first part of the trace concern only the vectors resulting from (n) odd.

This means that the last pseudo-clothoid has a correspondence with the first vector, the penultimate with the third, the antepenultimate with the fifth, ...

For formulation (3) it is with these vectors that the symmetry exists between orientations and angles.

According to formulation (3) it is with these vectors that there is a precise relationship, concerning the distance between the two origins of the pseudo-clothoid and the length of the vector.

Here are now the reasons for the validity of Riemann hypothesis for formulation (3).

- In the traces resulting from any of the three analyzed formulations, for any value of (a) and for b≥2π, the orientations of the pseudo-clothoids are always symmetric to the angles of the vectors, corresponding in the first part of the trace; *this is the first reason why Riemann hypothesis can only be true.*
- In the traces resulting from formulation (3), if a=1/2 the distances between the two origins of the pseudo-clothoids have a constant ratio, with the lengths of the corresponding vectors in the first part of the trace; *this is the second reason why Riemann hypothesis can only be true.*

The third reason why Riemann hypothesis can only be true is the following.

In the traces resulting from formulation (3), if a≠1/2 the distances between the two origins of the pseudo-clothoids are influenced by the value of (b), more and more evidently as the value of (b) increases; this influence is evident in two ways.

- If a<1/2 the distance between the two origins of the pseudo-clothoids, it is always greater than it should be (to make the compensation), depending on its position in the trace.
- If a>1/2 the distance between the two origins of the pseudo-clothoids is always smaller than it should be (to make the compensation), depending on its position in the trace.

At this point I have explained the reasons why Riemann hypothesis can only be true for the traces resulting from formulation (3).

I have not forgotten the vectors which in the first part of the trace correspond to an even value of (n); evidently they are compensated by the greater distance between the two origins of the pseudo-clothoids.

8. Analysis of the traces resulting from the formulation (3).

Fig. 17 uses a trace resulting from formulation (3) which presents a situation similar to the traces inserted in Fig. 3 and Fig. 4.

In this case I use the trace to highlight the correspondence of the pseudo-clothoids, with the vectors of the first part of the trace resulting from (n) odd.

Being a=1/2; I also highlight the relationship between the distances between two origins of the pseudo-clothoids and the lengths of the corresponding vectors in the first part of the trace, resulting from (n) odd.

The following descriptions concern the traces indicated by the five references shown in the image.

(1) This is the complete trace, I used two colors to distinguish the two parts.
(2) The vectors of the first part of the trace corresponding to (n) odd are highlighted in green.
(3) Group the highlighted vectors and indicate the length of the first vector, for which n=1.
(4) The trace obtained in (3) is mirrored with respect to the axis of symmetry; magnified by the indicated value, knowing that p=1.
(5) Superposition of the trace obtained in (4) on the trace (1).

1/sqrt(0.5)=sqrt(2) is the constant ratio existing between the distances of the two origins of the pseudo-clothoids and the lengths of the corresponding vectors in the first part of the trace; for any value of (b) provided that a=1/2.

Note: for the traces represented in Fig. 17÷20 the points resulting from n=1÷5000 were used.

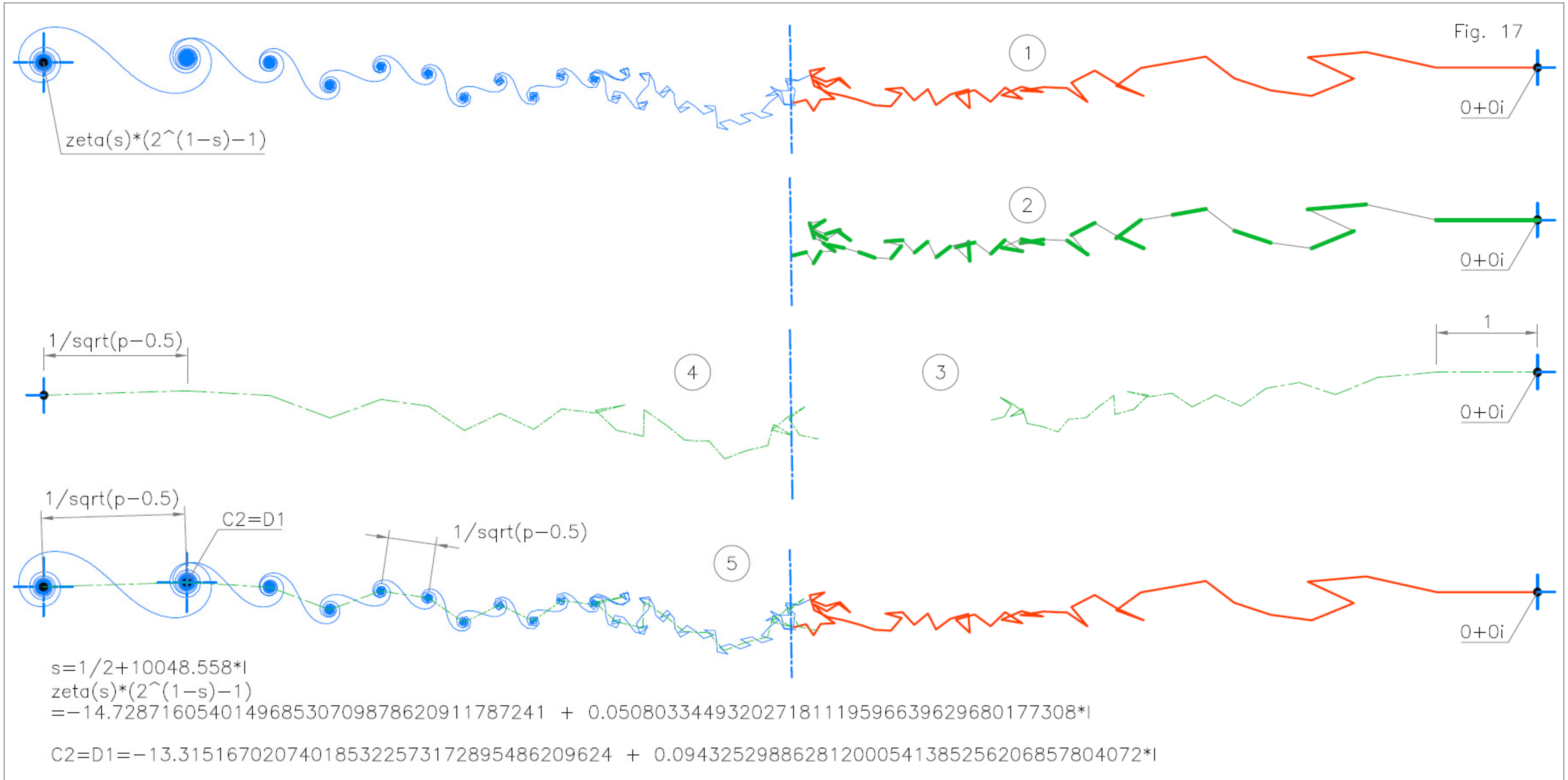


Fig. 17
1
2
3
4
5
0+0i
0+0i
0+0i
0+0i
zeta(s)*(2^(1-s)-1)
1
1/sqrt(p-0.5)
1/sqrt(p-0.5)
C2=D1
1/sqrt(p-0.5)
s=1/2+10048.558*I
zeta(s)*(2^(1-s)-1)
=-14.72871605401496853070987862091178724 1 + 0.050803344932027181119596639629680177308*I
C2=D1=-13.31516702074018532257317289548620962 4 + 0.0943252988628120005413852562068578040 72*I

- Three functions that allow to calculate, for a given pseudo-clothoid, the values of (n) corresponding to the first, last and intermediate vector; valid for formulation (3).

The value of (n) corresponds to the end point of the vector and (p) is the position of the pseudo-clothoid; p=1 corresponds to the last pseudo-clothoid.

Last vector (rounding to the previous integer) $$n(p) = \frac{b}{\pi \cdot (2 \cdot p - 2)} \quad (10)$$

Point (M_p) (rounding to the nearest integer) $$n(p) = \frac{b}{\pi \cdot (2 \cdot p - 1)} \quad (11)$$

First vector (rounding to the next integer) $$n(p) = \frac{b}{2 \cdot \pi \cdot p} \quad (12)$$

I note that for the function (10) the minimum value of (p) can only be 2.

From functions (10), (11) and (12) it is clear that the divisor of (b), for p=1 is: none, π, 2π, for p=2 is 2π, 3π, 4π, and so on.

- Command lines for PARI-GP useful for identifying the origins of pseudo-clothoids, in the traces resulting from formulation (3).

```
a=
b=
p=
s=a+b*I
x1=b/(Pi*(2*p-2))
x2=truncate(b/(Pi*(2*p-2)))
Z(f,s)=sum(n=1,f,(-1)^n*n^(-s))
if((x1-x2) < 0.5, f1=truncate(b/(Pi*(2*p-2)))-1, f1=truncate(b/(Pi*(2*p-2))))
f2=f1+1
z1=Z(f1,s)
z2=Z(f2,s)
Cp=(real(z1)+real(z2))/2+((imag(z1)+imag(z2))/2)*I
```

The requested data are (a), (b) and (p).

The difference between (x1) and (x2) determines which of the two vectors has its midpoint closest to the origin.

(z1) and (z2) are the extremes of the vector used and (C_p) is the midpoint which we assume as the origin $C_p=D_{p-1}$.

Note: These command lines use function (10); for this function the minimum value of (p) can only be 2.

The custom function Z(f,s) has the drawback of starting over from 1 at each increase in the value of (n), for values of (b) which involve long times, use the following command line variant.

I somehow solved the problem by calculating a value (Base) just smaller than (z1).

```
a=
b=
p=
s=a+b*I
x1=b/(Pi*(2*p-2))
x2=truncate(b/(Pi*(2*p-2)))
Z(f,s)=sum(n=1,f,(-1)^n*n^(-s))
if((x1-x2) < 0.5, f1=truncate(b/(Pi*(2*p-2)))-1, f1=truncate(b/(Pi*(2*p-2))))
f2=f1+1
fb1=f1-2
fb2=f1-1
Base=Z(fb1,s)
Z(f,s)=Base+sum(n=fb2,f,(-1)^n*n^(-s))
z1=Z(f1,s)
z2=Z(f2,s)
Cp=(real(z1)+real(z2))/2+((imag(z1)+imag(z2))/2)*I
```

- Distance between the two origins of a pseudo-clothoid; function valid for formulation (3).

This function, for the traces resulting from formulation (3) and only if a=1/2, allows to calculate the distance between the two origins of a pseudo-clothoid.

$$\mathrm{r(p)} = \frac{1}{\sqrt{p-0.5}} \qquad (13)$$

It is clear that it is derived from function (9) which in turn is derived from function (4).

Also for this function I point out that only the square root is required; in function (4) the root is square only if a=1/2.

In another way it is possible to calculate r(p) when a≠1/2; in this case I use the following command lines for PARI-GP.

With these command lines starting from the resulting value of r(p) for a=1/2, the distance between the two origins of the pseudo-clothoid is obtained, given the values of (b), (p) and a≠1/2.

```
b=
p=
nr=
dr=
medio=b/(Pi*(2*p-1))    \\ In this case, rounding would be counterproductive.
rBase=1/sqrt(p-0.5)
rho1=1/sqrt(medio)
rho2=1/sqrtn(medio^nr,dr)
rpb=(rho2/rho1)*rBase
```

As for function (4), (nr) and (dr) used in the penultimate line, are the numerator and denominator of (a).

- Inclination of the axis of symmetry.

When the value of (b) is large enough, I use the following command lines for PARI-GP with which I calculate (in decimal degrees) the inclination of the symmetry axis which I call Gamma; as for the vectors the zero is at 3.00 and for (b) positive the growth is hourly.

```
b=
p=1
n(p)=round(b/(Pi*(2*p-1)))
if(type(n(p)/2) == "t_INT", ThetaM(p)=b*log(n(p)), ThetaM(p)=b*log(n(p))+Pi)
ThetaMp=ThetaM(p)*180/Pi-(truncate((ThetaM(p)*180/Pi)/360)*360)
Beta=ThetaMp-45
Gamma=Beta/2 \\ It can even be Gamma=180+Beta/2
```

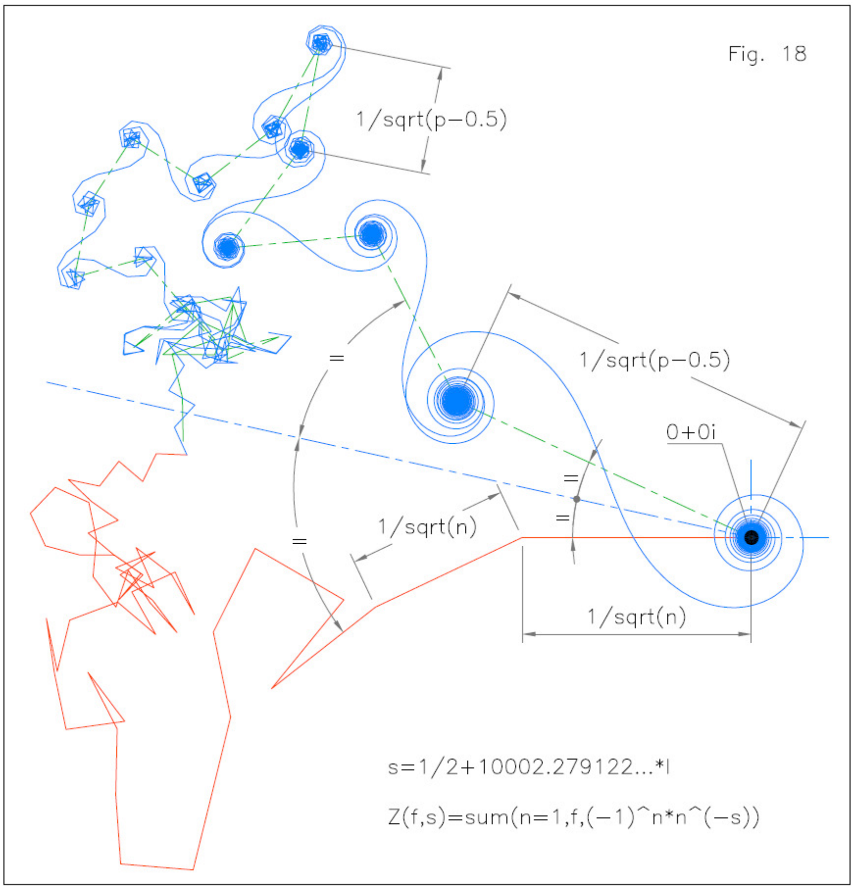
Once n(p) has been calculated, it is checked whether it is even or odd, if it is odd, π is added to ThetaM(p).

As data only (b) is required, in these command lines n(p) has the same role that in function (5) it is played by (n); n(p) is calculated using function (11).

The ThetaMp angle is obtained by transforming ThetaM(p) into decimal degrees and removing the multiples of 360°.

8.1 Verification of the method of calculating the origins of pseudo-clothoids.

Fig.18 represents the trace resulting from formulation (3) by s=1/2+ 10002.279122…i; also in this case I highlighted the two parts of the trace with two different colors.



As for Fig. 17, I reconstructed the trace of the pseudo-clothoid axes and inserted it into the image.

I wanted to highlight some of the symmetrical orientations with respect to the axis and also the different functions that determine the length of the vectors in the first part of the trace and the distances between the two origins of the pseudo-clothoids.

With reference to the trace inserted in Fig. 18, I calculated the values (Zn) of the final points of the first five vectors and using function (10) I calculated the three origins preceding the last one.

I also computed the length of the first three vectors resulting from (n) odd and the distance between the two origins of the last three pseudo-clothoids.

These are the results I got.

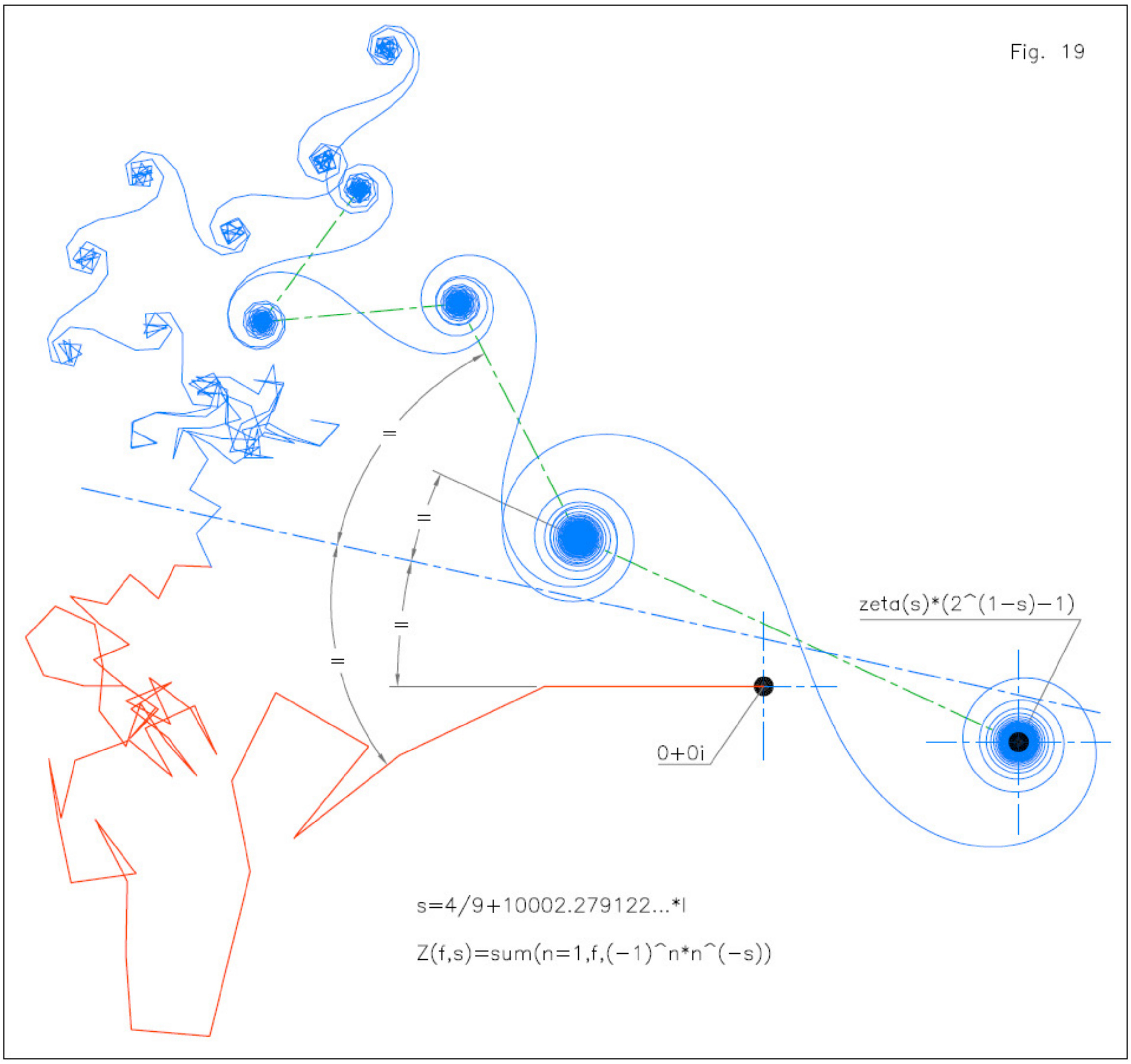
Z1=-1+0*I
Z2=-1.6387241578770326061673355258846009802-0.30336685735965873000749491272729956951*I
Z3=-2.0932844409861232943998892520421958166-0.65932792595159991350193454510822278802*I
Z4=-1.7773473412746743467185393286536847319-0.27179244496190012616860730575843503345*I
Z5=-2.1640013494466602003413002550950671151-0.047073335929005692335089485278568278095*I
D1=C2=-1.2831624785034068437811529167550530593 + 0.59455696862755512716652011872739571095*I
D2=C3=-1.6547858488086230615806952349901977723 + 1.3215781665232812810050067371200290218*I
D3=C4=-2.2844336314388701834957268448875660138 + 1.2630252925624625700562009250442914373*I
ρ1=1
ρ3=0.57735026918962576450914878050195745565
ρ5=0.44721359549995793928183473374625524709
r1=1.4142135623730950488016887242096980786
r2=0.81649658092772603273242802490196379732
r3=0.63245553203367586639977870888654370674

- What changes if a≠1/2.

Fig.19 represents the trace resulting from formulation (3) by s=4/9+ 10002.279122…i; also in this case I highlighted the two parts of the trace with two different colors.

To the calculations I added the one relating to the second origin of the last pseudo-clothoid (point of convergence), to obtain it I multiplied the result provided by the zeta(s) function by s=4/9+10002.279122…i by (2^(1-s)-1).

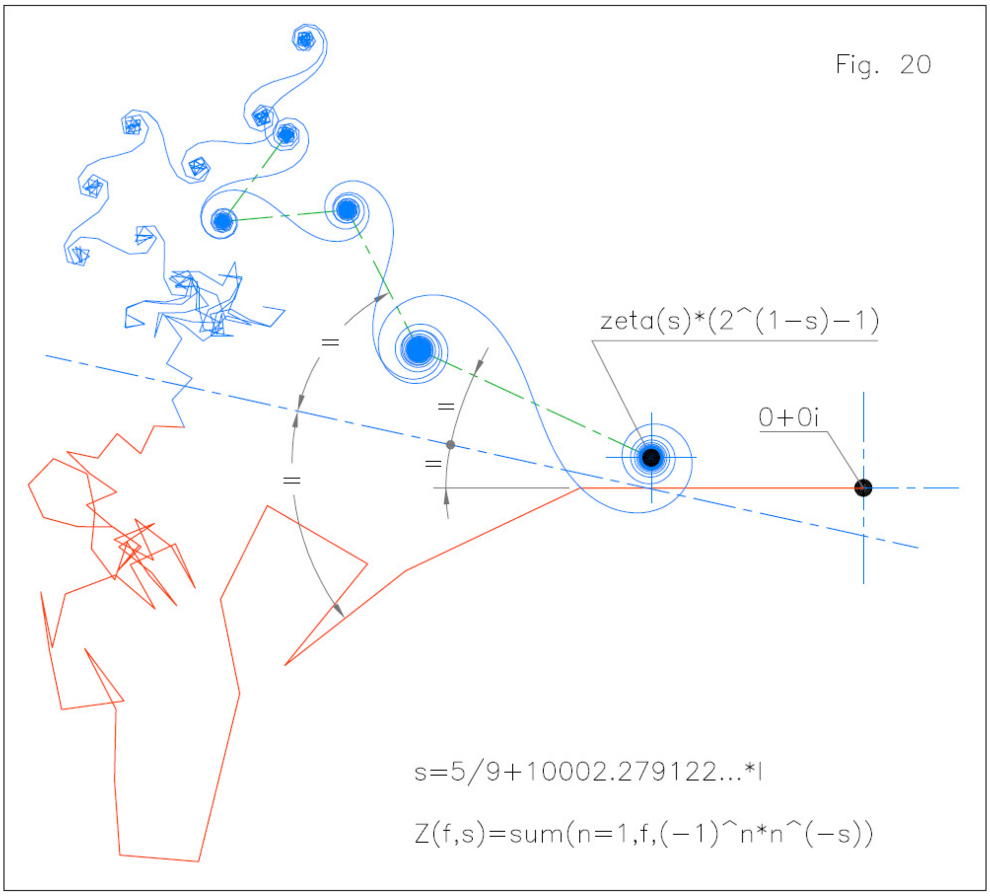
Note: In Fig. 19 and Fig. 20 I chose to position the axis of symmetry so that it crosses the same cross vector in Fig. 18.



These are the results I got.

Z1=-1+0*I
Z2=-1.6637999739631258581260225266815013522-0.31527680538331156746944176829038536607*I
Z3=-2.1469680404957559604407276180672159031-0.69364030889068071040716242489060579422*I
Z4=-1.8057370990750159110433478612952503042-0.27507883848144127666949027282052990549*I
Z5=-2.2285557960306174736074180172738230654-0.029341205212038361999818404457252685689*I
D1=C2=-0.84408384921468031150614212044173099486 + 0.67613419015044884350374536941051742547*I
D2=C3=-1.3913582997186369968273064227151994515 + 1.7467857402545804454153332853808567235*I
D3=C4=-2.2926708547621702190108217180840080868 + 1.6629708655278800097820144530821619705*I
ρ1=1
ρ3=0.61368584903291603508168767420564100200
ρ5=0.48904256961953771033394713081192870859
r1b=2.2137230189003707100535479809979535230
r2b=1.2024192412610305484757424099534070630
r3b=0.90532937475058220152542575646836098340
s=4/9+10002.2791222728886154882113785932532*I
zeta(s)*(2^(1-s)-1)
=1.1644998291705930772762092306984473297 - 0.25454887989348652459591764438178983764*I

Fig.20 represents the trace resulting from formulation (3) by s=5/9+ 10002.279122…i; also in this case I highlighted the two parts of the trace with two different colors.



These are the results I got.

Z1=-1+0*I

Z2=-1.6145956099094200319381130789882233040-0.2919068214751296558738933678779221 7845*I

Z3=-2.0422419410567259636378950514539191087-0.6267918756057504763318816149008064 5091*I

Z4=-1.7497237697605071811229347104916359766-0.2679825736432954761025988027457113 2608*I

Z5=-2.1033063409947225331040234653989171393-0.0624842244784524888645693262439846 38741*I

D1=C2=-1.5684199192309171330315246912014026172 + 0.48641429598161383204261044527791018195*I

D2=C3=-1.8207686410648106948575620882944358470 + 0.98009475150499125549737005217846987595*I

D3=C4=-2.2606341411364050313951605235175590662 + 0.93918973024251001719419233893056975096*I

ρ1=1

ρ3=0.54316607407294877966782901238146633121

ρ5=0.40896235302295821244831773986522580160

r1b=0.90345539298474025548422164203563630405

r2b=0.55443778990720713714153709901370795626

r3b=0.44182814692188663789834246991783309753

s=5/9+10002.279122272888615488211378593 2532*I

zeta(s)*(2^(1-s)-1)

=-0.74868499564114998567516748826150696652 + 0.10658817322468463888193154585657886072*I

9. For what reasons, if a=1/2, the value of (b) is neutral as regards the distance between the two origins of the pseudo-clothoids.

The distance between the two origins of the pseudo-clothoids is the result of the average length of the vectors, present in the central part of the pseudo-clothoids, multiplied by the number of these vectors; this explanation may be crude but it is very close to reality and I don't know of a better one at the moment.

The length of the vectors can be precisely calculated using function (4), the number of vectors present in the central part of a pseudo-clothoid, depends on the value of (b) and on the position of the pseudo-clothoid in the trace, therefore on (p) .

As results from function (4) the length of the vectors always starts from 1 and varies in a different way for each value of (a), as a function of (n).

The value of (a) determines how much the length of the vector changes with each increment of (n).

It is evident that there is a synchrony, between the influence of the value of (b) on the number of vectors of a pseudo-clothoid and the variation of the length of the vectors as (n) increases, for a=1/2.

That the value of (b) determines the number of vectors present in a pseudo-clothoid (and therefore also in its central part) is demonstrated by the link between (b), the multiples of π and the pseudo-clothoids.

From the above relation I obtained the functions (6), (7) and (8) for the formulation (1) and the functions (10), (11) and (12) for the formulation (3).

Function (9), for a=1/2, allows to calculate the distance between the two origins of a pseudo-clothoid, present in a trace resulting from formulation (1), given its position (p).

Function (13), for a=1/2, allows to calculate the distance between the two origins of a pseudo-clothoid, present in a trace resulting from formulation (3), given its position (p).

Following both the functions I have presented command lines for PARI-GP, with which it is possible to calculate the distance between the two origins of a pseudo-clothoid, when a≠1/2.

How command lines work.

- The rho2/rho1 comparison concerns the lengths of two vectors ending in the point (M), deriving from a≠1/2 and a=1/2.
- The lengths in question therefore result from the value of (medio) and from the different values of (a).
- The value of (medio) depends on (b) and increases as its value increases.
- Increasing the value of (medio) inevitably increases the gap between (rho2) and (rho1); except for the case where also (rho2) results from a=1/2.
- The value of (rBase) comes from (p); even if in an opaque way (rBase) contains the number of vectors of the central part, of the pseudo-clotoid in question.

From the analysis of the results I deduced two new formulas, one for the formulation (1) and one for formulation (3). These two formulas are more explicitly based on the number of vectors present in the part central of the pseudo-clothoids and a “useful” length; furthermore work for any value of (a). I will present these two formulas in the same article, in which I will present the formula that allows you to calculate the value of (n), corresponding to the hinge vector.

Thanks to the functions and command lines for PARI-GP that I have provided in this article, it is possible to verify what I have just stated; it will also be evident that, when a≠1/2, the influence of (b) on the distance between the two origins of the pseudo-clothoids, increases with the increase in its value.

Starting from function (4), the formulas presented in this article all derive exclusively from mine understanding of the results, which I have obtained from time to time.

I highlight that from the information I have provided in this article, it is possible to deduce the formula that allows to calculate the value of (n) relating to the hinge vector; the importance of which I have no doubt.

I wish that for the traces resulting from the Riemann zeta function, the concept that a zero can be overcome correspond only and only to a point of convergence, as normally understood; I would like the latest origins of the traces resulting from formulation (1) were considered points of convergence.

The following two sections, as well as provide further help in understanding what is set out in this article, want to highlight that what has just been stated is far from absurd and insurmountable.

10. Consequences of part $(-1)^n$ on the traces of the Riemann zeta function.

Part $(-1)^n$ reverses the direction of the vectors corresponding to (n) odd, which means one vector in each of each pair of vectors that follow each other.

The first effect is found with the first vector of the traces resulting from formulation (3), which terminates for any value of (a) or of (b) in -1+0i; remember that the first vector in the traces resulting from the formulation (1), ends for any value of (a) or of (b) in 1+0i.

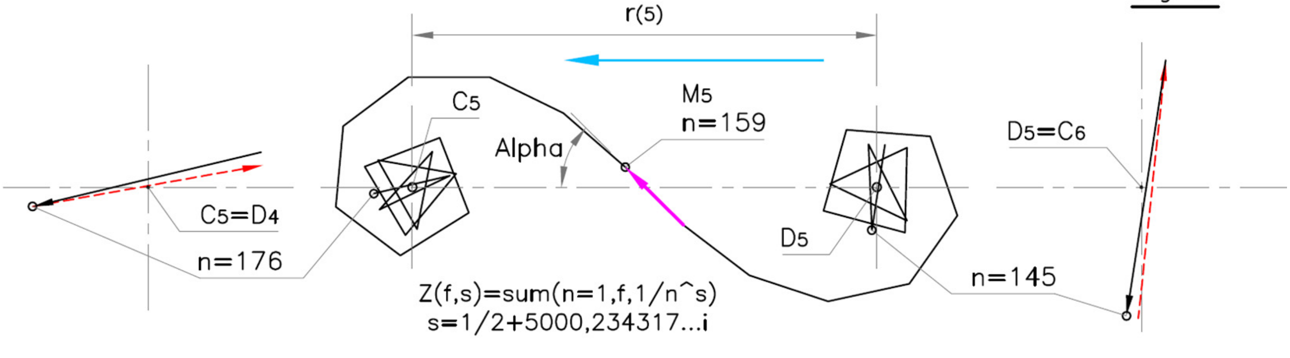
It is necessary to analyze the last pseudo-clothoids, to understand the effect that the $(-1)^n$ part has on the pseudo-clothoids and in which way convergence is possible.

I repeat Fig. 5 remembering that it is extracted from a trace resulting from formulation (1), in which it is not present the $(-1)^n$ part.

It can be observed that in correspondence of the two origins, the following vector tends to reverse the direction of previous carrier; as the comparison with Figs. 6 and 7 also shows, this is happening more and more evident towards the last pseudo-clothoid and with increasing value of (b).

In the same figure it can be seen that in the central part of the pseudo-clothoids, the next vector tends to keep the same direction as the previous vector; this too is happening more and more clearly towards the last pseudo-clothoid and with increasing value of (b).

Fig. 5



It appears evident that in the traces resulting from formulation (3), the inversion of one vector every two transforms the origins of the pseudo-clothoids in the central parts and the central parts in the origins.

Basically, the $(-1)^n$ part causes (for the same value of (b)) a backward offset of half a pseudo-clothoid, in the traces resulting from formulation (3) with respect to the traces resulting from formulation (1).

The same conclusion is reached by comparing functions (10), (11) and (12) with functions (6), (7) and (8).

The offset described can escape, if the analysis of the pseudo-clothoids preceding the last one is not careful enough.

As regards the last pseudo-clothoid, section (11) will be of help; now I want to mention that it is the trend of the θ(n) curve for a given value of (b), to put an end to the succession of pseudo-clothoids.

In the traces resulting from formulation (1) a spiral residue remains after the last pseudo-clothoid divergent polygonal spiral; however, this residue tends to disappear as the value of (b) increases.

In the traces resulting from formulation (3), the $(-1)^n$ part prevents the formation of the above residue diverging polygonal spiral; the problem, as I describe below, is the collateral effect.

11. Considerations on the last origins, for (b) tending to infinity.

As results from the function (4), for a>0 the length of the segments tends to 0 for (n) which tends to infinity; for the first values of (n) the decrease is different according to the value of (a), but the final trend is obviously the same for each value of a>0.

For a>0, therefore, the following results.

- Formulation (1).

For values of (b) tending to infinite, for p=1 from the function (6) results that n(p) tends to infinity.

As (n) tends to infinity the length (ρ) of the vectors tends to 0; remember that the function (6) provides the value of (n) corresponding to the last vector, of the pseudo-clothoid indicated by (p).

The consequence of what has been highlighted is the tendency to disappear, as the value of (b) increases, of the part of the divergent polygonal spiral following the last origin (Of).

- Formulation (2) and (3).

I remind you that the function (11) supplies the value of (n) corresponding to the vector ending in the point (M) and that it is identical to the function (6).
I also remind you that the point (M) is located halfway between the two origins of the pseudo-clothoids, as far as already highlighted for the formulation (1), the following follows.

As the value of (b) increases, the last pseudo-clothoid tends to end at point (M); this means that the last origin, commonly called the point of convergence, tends not to be reached.

For finite values of (b) it is easily verifiable that the value of n(p) resulting from function (6), if used for formulations (2) or (3) with (b) being equal, the resulting traces stop at point (M).

- Consequences and my conclusion.

I have no doubts about the validity of the (6) and (11) functions, consequently I believe these last considerations an important point in favor of formulation (1) and an important reason for meditation with regard to the other two formulations.

It cannot be escaped that the second part of the traces resulting from the Riemann zeta function, tends towards perfection with increasing value of (b); of the three formulations the only one that completely agrees with this one trend is formulation (1).

My conclusion is therefore that formulation (1) is and will remain unsurpassable.

This is the link that corresponds to this preprint.	http://doi.org/10.5281/zenodo.8026759
This is the link to the Italian version of this preprint.	http://doi.org/10.5281/zenodo.8026728

The following links correspond to some of my preprints published on zenodo.org

Primality test. My second contribution.	http://doi.org/10.5281/zenodo.6397327
Twin primes. Where they can be found.	http://doi.org/10.5281/zenodo.5902559
News on the mechanism of prime numbers.	http://doi.org/10.5281/zenodo.5844231
Goldbach's conjecture. Because I think it's true.	http://doi.org/10.5281/zenodo.5707187
How and Why to Use my Basic Scheme to make Polygonal Spirals	http://doi.org/10.5281/zenodo.5575215