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Resolving Connected Domination in Graphs

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Abstract: For an ordered subset $W = \{w_1, w_2, \cdots, w_k\}$ of vertices and a vertex v in a connected graph $G = (V, E)$, the (metric) representation of v with respect to W is the k-vector $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$. The set W is a resolving set for G if distinct vertices of G have distinct representations with respect to W. A resolving set of minimum cardinality is called a minimum resolving set and the cardinality of it is a dimension of G, denoted by $dim(G)$. In this paper, we introduce resolving connected domination number $\gamma_{rc}(G)$ of graphs. We investigate the relationship between resolving connected domination number, connected domination number, resolving domination number and dimension of a graph G. Bounds for $\gamma_{rc}(G)$ are determined. Exact values of $\gamma_{rc}(G)$ for some standard graphs are found.

Key Words: Resolving dominating set, resolving connected dominating set, resolving connected domination number, dimension of a graph.

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§1. Introduction

In this paper, we consider the connected simple graph $G = (V, E)$, that finite, have no loops, multiple and directed edges, and there is a path between any pair of its vertices. Let G be such a graph and let n and m be the number of its vertices and edges, respectively. The distance $d(u, v)$ between two vertices u and v of a graph G is the minimum length of the paths connecting them (i.e., the number of edges between them). A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a subset $S \subseteq V(G)$, the subgraph $\langle S\rangle$ of G is called the subgraph induced by S if $E(\langle S\rangle) = \{uv \in E(G)|u, v \in S\}$. We refer to [3], for graph theory notation and terminology not described here.

A set D of vertices in a graph G is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set in G. The concept of connected domination number was introduced by E. Sampathkumar and H. Walikar [7]. A dominating set D of a graph G is connected dominating set if a subgraph induced by D is connected. The connected domination number $\gamma_c(G)$ of G is the minimum cardinality of a connected dominating set in G . for more details in domination theory of graphs we refer to [5].

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Let G be a connected graph of order n and let $W = \{w_1, w_2, \cdots, w_k\}$ be an ordered set of vertices of G. For a vertex v of G, the k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k)).$$

where $d(v, u)$ represents the distance between the vertices v and u, is called the representation of v with respect to W. The set W is a resolving set for G if $r(u|W) = r(v|W)$ implies that $u = v$ for every pair u, v of vertices of G. A resolving set of minimum cardinality is called a minimum resolving set or a basis of a graph G and the cardinality of a basis of G is its dimension and denoted by $dim(G)$. The concepts of resolving set and minimum resolving set have previously appeared in the literature in [9] and later in [10],, Slater introduced these ideas and used locating set for what we have called resolving set. He referred to the cardinality of a minimum resolving set in a graph G as its location number of G. Slater described the usefulness of these ideas when working with U. S. sonar and coast guard Loran (Long range aids to navigation) stations. Independently, Harary and Melter [4], investigated these concepts as well, but used metric dimension rather than location number, the terminology that we have adopted.

C. Robert and et al. in [6], introduced the concept of resolving domination in graphs. A set D of vertices of a graph G that is both resolving and dominating is a resolving dominating set. The minimum cardinality of a resolving dominating set is called the resolving domination number $\gamma_r(G)$. Motivated by this paper, we introduce the concept of resolving connected domination number of graphs. We investigate the relationship between resolving connected domination number, connected domination number, resolving domination number and dimension of graphs. Exact values of $\gamma_{rc}(G)$ for some standard graphs are computed. Bounds for $\gamma_{rc}(G)$ of a graph are found.

Before we are starting in the main results of resolving connected domination, we consider the following useful results on dimension and resolving domination numbers of graphs.

Theorem 1.1([1, 4]) *Let G be a connected graph of order $n > 2$. Then*

(a) *$dim(G) = 1$ if and only if $G = P_n$;*

(b) *$dim(G) = n - 1$ if and only if $G = K_n$;*

(c) *For $n > 4$, $dim(G) = n - 2$ if and only if $G = K_{r,s}$, $(r, s > 1)$, $G = K_r + K_s$, $(r > 1; s > 2)$, or $G = K_r + (K_1 \cup K_s)$, $(r, s > 1)$.*

Theorem 1.1([8, 4]) (a) *For a cycle C_n , $n \geq 3$, $dim(C_n) = 2$;*

(b) *For $n \geq 3$, let $W_{1,n}$ be the wheel graph on $n + 1$ vertices. Then*

$$dim(W_{1,n}) = \begin{cases} 3, & \text{if } n = 3 \text{ or } n = 6; \\ \lfloor \frac{2n+2}{5} \rfloor, & \text{otherwise.} \end{cases}$$

The following definitions are stated in [1, 4].

Definition 1.3 *Fix a graph G. A vertex $v \in V(G)$ is called a major vertex if $d(v) \geq 3$. An* *end-vertex u is called a terminal vertex of a major vertex v if $d(u,v) < d(u,w)$ for every other major vertex w in G. The terminal degree of a major vertex v is the number of terminal vertices of v. A major vertex v is an exterior major vertex if it has positive terminal degree. Let $\sigma(G)$ denote the sum of terminal degrees of all major vertices of G, and let $ext(G)$ the number of exterior major vertices of G.*

Theorem 1.4([1]) *If T is a tree that is not a path, then $dim(T) = \sigma(T) - ext(T)$.*

Corollary 1.5([5]) *If T is a tree of order $n > 3$, then $\gamma_c(T) = n - l(T)$. Where $l(T)$ denote the number of end-vertex of T.*

Lemma 1.6([6]) *Let u and v be vertices of a connected graph G. If either*
(1) u and v are not adjacent and $N(u) = N(v)$. or
(2) u and v are adjacent and $N[u] = N[v]$, then every resolving set of G contains at least one of u and v.

Proposition 1.7([6]) *If G is a connected graph of order $n > 2$ and diameter d, then $\gamma_r \geq f(n,d)$, where*

$$f(n,d) = min\{k + \sum_{i=1}^{k} \begin{pmatrix} k \\ i \end{pmatrix} (d-1)^{k-i}\}.$$

§2. Main Results

A connected graph G ordinarily contains many dominating sets. Indeed, every superset of a dominating set is also a dominating set. The same statement is true for a connected dominating sets, also for resolving sets. In this paper we study those connected dominating sets that are resolving sets as well. Such sets will be called resolving connected dominating sets. Thus a resolving connected dominating set D of vertices of G not only dominates all the vertices of G but has the added feature that the subgraph $\langle D \rangle$ induced by it is connected, also distinct vertices of G have distinct representations with respect to D. The cardinality of a minimum resolving connected dominating set is called the resolving connected domination number of G and is denoted by $\gamma_{rc}(G)$. A resolving dominating and resolving connected dominating sets of cardinality $\gamma_r(G)$ and $\gamma_{rc}(G)$, is called a $\gamma_r(G)$-set and $\gamma_{rc}(G)$-set, for G, respectively. To illustrate these concepts, consider the following graph G in Figure 1.

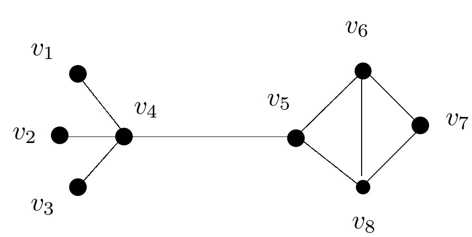

Figure 1. A graph with $\gamma = 2$, $\gamma_c = 3$, $\gamma_r = 4$, $\gamma_{rc} = 5$ and $dim = 3$.

By Lemma 1.6, every resolving set of G contains at least two vertices from set $W = \{v_1, v_2, v_3\}$. Since no 2-element subset of W is a resolving set, it follows that $dim(G) \geq 3$. On the other hand, the set $\{v_1, v_2, v_6\}$ is a resolving set for G, implying that $dim(G) = 3$. The set $\{v_4, v_6\}$ is a γ-set of G and so $\gamma(G) = 2$, the set $\{v_4, v_5, v_6\}$ is a γ_c-set of G so $\gamma_c(G) = 3$, the set $\{v_1, v_2, v_4, v_6\}$ is a γ_r-set of G so $\gamma_r(G) = 4$ and the set $\{v_1, v_2, v_4, v_5, v_6\}$ is a γ_{rc}-set of G and so $\gamma_{rc}(G) = 5$.

2.1 Exact Values of Resolving Connected Domination of Some Standard Graphs

In this section, we present The exact values of resolving connected domination numbers of some well-known classes of graphs as following:

Proposition 2.1

(1) $\gamma_{rc}(K_n) = \gamma_{rc}(K_{1,n}) = n - 1$, *for* $n \geq 2$*;*
(2) $\gamma_{rc}(P_n) = n - 2$, *for* $n \geq 4$*;*
(3) $\gamma_{rc}(C_n) = n - 2$, *for* $n \geq 3$*;*
(4) $\gamma_{rc}(K_{r,s}) = r + s - 2$, *for* $r, s \geq 2$*;*
(5) *For integers* $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ *with* $n_1 + n_2 + \cdots + n_k = n$ *and* $k \geq 2$, $\gamma_{rc}(K_{n_1, n2, \cdots, nk}) = n - k$.

Theorem 2.2 *For a wheel graph* $W_{1,n}$ *of order* $n \geq 7$

$$\gamma_{rc}(W_{1,n}) = \left\lfloor \frac{2n+2}{5} \right\rfloor + 1.$$

Proof In $W_{1,n} = K_1 + C_n,\ \ n \geq 7$, let $V(W_{1,n}) = \{v_0, v_1, v_2, \cdots, v_n\}$, where v_0 is a central vertex and $v_1, v_2, \cdots, v_n$ are vertices of C_n. Let R be a minimum resolving set of $W_{1,n}$. Since $d(v_0, v_i) = 1$ for all i with $1 \leq i \leq n$ it follows that v_0 does not belong to any minimum resolving set of $W_{1,n}$. Hence, $v_0 \notin R$. In other hand, the set $\{v_0\}$ is a γ-set of $W_{1,n}$ and it is a connected set so the set $\{v_0\}$ is also a γ_c-set of $W_{1,n}$. Thus, the set $D = \{v_0\} \cup R$ is a γ_r-set of $W_{1,n}$. Since the subgraph $\langle D \rangle$ is connected it follows that the set D is a γ_{rc}-set of $W_{1,n}$. Therefore, by this and Theorem 1.2 we get

$$|D| = |R \cup \{v_0\}| = |R| + |\{v_0\}| = \left\lfloor \frac{2n+2}{5} \right\rfloor + 1.$$

And this completes the proof. □

Theorem 2.3 *Let* T *be a tree of order* $n \geq 4$*, that is not a path. If every major vertex of* T *adjacent to its terminal vertex, then*

$$\gamma_{rc}(T) = n - ext(T).$$

Proof Let W, S and D be a resolving set, a connected dominating set and a resolving connected dominating set of a tree T, respectively, with minimum cardinality. Since every superset of a resolving set is a resolving set and every superset of a connected dominating set is a dominating set it follows that $S \cup W$ is a resolving connected dominating set of T. Thus, $D \subseteq (S \cup W)$. it follows that

$$|D| \leq |S \cup W| \leq |S| + |W| \tag{1}$$

Conversely, Since a resolving connected dominating set is both a resolving set and a connected dominating set it follows that $W \subseteq D$ and $S \subseteq D$. Hence, $(W \cup S) \subseteq D$. Therefore, $|S| + |W| - |S \cap W| \leq |D|$. Now, let $L(T)$ be the set of all end-vertex of T. Then from Definition 1.3 we get $|L(T)| = l(T) = \sigma(T)$. From Theorem 1.4 and lemma 1.6 and since every major vertex of T adjacent to it terminal vertex, we conclude that a connected domination set S dose not containing any resolving set. Then $W \subset L(T)$. Since $L(T) \subseteq V(T) - S$ it follows that $S \cap W = \phi$. Hence,

$$|S \cup W| \leq |D|. \tag{2}$$

From equations (1) and (2) we have $|D| = |S| + |W|$. Therefore, by Theorem 1.4 and Corollary 1.5 we get

$$\begin{aligned}\gamma_{rc}(T) = |D| = |S| + |W| &= \gamma_c(T) + dim(T)\\ &= n - \sigma(T) + \sigma(T) - ext(T) = n - ext(T).\end{aligned}$$

□

Corollary 2.4 *Let T be a tree of order $n \geq 4$, that is not a path. If every major vertex of T adjacent to its terminal vertex, then*

$$\gamma_{rc}(T) = \gamma_c(T) + dim(T).$$

2.2 Bounds on Resolving Connected Domination Number

In this section we investigate with some bounds on resolving connected domination number of graphs.

Theorem 2.5 *For any connected graph of order $n \geq 2$, $\gamma_{rc}(G) \leq n - 1$. The bound is sharp, K_n and $K_{1,n}$ attainting this bound.*

Corollary 2.6 *For any connected graph G of order n and size m, $\gamma_{rc}(G) \leq m$.*

Theorem 2.7 *For any tree T of order $n \geq 4$, that is not a star,*

$$\gamma_{rc}(T) \leq n - 2.$$

Proof Let T be a tree of order $n \geq 4$, that is not a star, on contrary we suppose that $\gamma_{rc}(T) \geq n - 1$. if $T = P_n$ then by proposition 1.1 $\gamma_{rc}(T) = n - 2$, contradiction. Now, if $T \neq P_n$, then T has at least one vertex (say v) with $d(v) \geq 3$. Then v is a major vertex of T which is an exterior vertex. Consider the following cases.

Case 1. T has only one a vertex v as a major vertex. Since T not a star, it follows that there exists a vertex $u \in V(T)$ such that $d(v,u) \geq 2$. Without loss the generality, and for simplicity we consider T is a broom graph (see Figure 2).

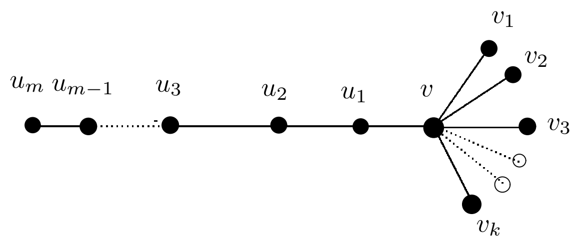

Figure 2. A broom graph

The set $S = V(T) - \{v_1, v_2, \cdots, v_k, u_m\}$ is a γ_c-set of T and a set $W = \{v_1, v_2, \cdots, v_{k-1}, u_1\}$ is a resolving set of T. Hence, a set $D = S \cup W$ is a γ_{rc}- set of T with minimum cardinality. Therefore, $\gamma_{rc}(T) = n-2 \leq n-1$, contradiction to hypothesis.

Case 2. T has at least two an exterior vertices, then $\gamma_{rc}(T) \leq n - ext(T) \leq n-1$, contradiction. Therefore, the theorem is true. $\square$

Proposition 2.8 *For every connected graph, necessarily,*

$$dim(G) \leq \gamma_r(G) \leq \gamma_{rc}(G),$$
$$\gamma(G) \leq \gamma_c(G) \leq \gamma_{rc}(G)$$

and

$$\gamma(G) \leq \gamma_r(G) \leq \gamma_{rc}(G).$$

Theorem 2.9 *Let G be a connected graph of order and size n, m, respectively. Then $\gamma_{rc}(G) = m$ if and only if $G = K_{1,n}$.*

Proof If $G = K_{1,n}$, then $\gamma_{rc}(G) = n-1 = m$.

Conversely, suppose that $\gamma_{rc}(G) = m$. Then by Theorem 2.5, $m \leq n-1$. Since G is a connected it follows that $m = n-1$. Hence G must be a tree. If $n \leq 3$, it is clear that G is a star and the theorem is holding. Otherwise if $n \geq 4$, by Theorem 2.7 $\gamma_{rc}(T) \leq n-2 < n-1 = m$, contradiction. Therefore, must G be a star. $\square$

Theorem 2.10 *Let G be a connected graph of order $n \geq 2$ such that the complement $\bar{G}$ of its is a connected. Then*

$$\gamma_{rc}(G) + \gamma_{rc}(\bar{G}) \leq \frac{n^2-n}{2}.$$

The equality is holding if and only if $G = k_{1,2}$.

Proof Let m and $m^{'}$ be the size of G and $\bar{G}$, respectively. By Corollary 2.6, we have

$$\gamma_{rc}(G) + \gamma_{rc}(\bar{G}) \leq m + m^{'} = \frac{n^2-n}{2}.$$

To prove the second part of theorem, let $G = K_{1,2}$. Then $\gamma_{rc}(G) = 2$ and $\gamma_{rc}(\bar{G}) = \gamma_{rc}(K_2) = 1$. Hence, $\gamma_{rc}(G) + \gamma_{rc}(\bar{G}) = 3 = \frac{9-3}{2}$.
Conversely, if $\gamma_{rc}(G) + \gamma_{rc}(\bar{G}) = \frac{n^2-n}{2}$, we should have $\gamma_{rc}(G) = m$ and $\gamma_{rc}(\bar{G}) = m'$. but this imply by Theorem 2.8, that G is a star. which requires that $n = 3$. So that $G \cong k_{1,2}$. This completes the proof. □

There are only finitely many connected graphs having a fixed resolving connected domination number. To verify this, we first, motivation by the lower bound of a resolving domination number in Proposition 1.7, establish a lower bound for a resolving connected domination number of a graph.

Theorem 2.11 *Let G be a connected graph of order $n \geq 2$ and diameter d. Then*

$$\gamma_{rc}(G) \geq f(n, d).$$

From Theorem 2.11 we have the following result.

Corollary 2.12 *Let G be a connected graph of order $n \geq 2$, diameter d and resolving connected domination number k. Then*

$$n \leq k + \sum_{i=1}^{k} \binom{k}{i} (d-1)^{k-i}.$$

Theorem 2.13 *For every positive integer k, there are only finitely many connected graphs G with resolving connected domination number k.*

Proof Let G be a connected graph of order $n \geq 2$ with $\gamma_{rc}(G) = k$. Since $\gamma_c(G) \leq \gamma_{rc}(G) = k$ it follows that the diameter of G is at most $k+1$. By Corollary 2.12 we get

$$n \leq k + \sum_{i=1}^{k} \binom{k}{i} k^{k-i}.$$

Hence n is finite, and the result is follows. □

It is an immediate observation that the only nontrivial graph having resolving connected domination number 1 is K_2. It is clear form the previous theorem , the order of any connected graph G with resolving connected domination number 2 is at most 5. By Theorem 2.13, the order of any connected graph G with resolving connected domination number 3 is at most 40. In fact, we can improve upon this statement.

Theorem 2.14 *The order of every connected graph of order n with resolving connected domination number 3 is at most 12.*

Proof Let G be a connected graph with $\gamma_{rc}(G) = 3$ and let $D = \{v_1, v_2, v_3\}$ be a γ_{rc}-set for G. Since every vertex in $V(G) - D$ is adjacent to at least one vertex of D and has distance at most 3 from the other, the representations $(v|D)$ of a vertex v in $V(G) - D$ with respect to D is 3-vector, every coordinate of which is a positive integer not exceeding 3, at least one coordinate of which is 1. The only possible representations $(v|D)$ for every $v \in V(G) - D$ are $(1,1,1), (1,1,2), (1,2,1), (1,2,2), (1,2,3), (2,1,1), (2,1,2), (2,2,1)$ and $(3,2,1)$. Then the order of G at most 12. □

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