International J.Math. Combin. Vol.4(2015), 74-83

Galilean Bobillier Formula for One-Parameter Planar Motions

Nurten (BAYRAK) GÜRSES, Mücahit AKBIYIK and Salim YÜCE

Department of Mathematics

Faculty of Arts and Sciences of Yildiz Technical University, Esenler, Istanbul, 34220, Turkey

E-mail: nbayrak@yildiz.edu.tr, makbiyik@yildiz.edu.tr, sayuce@yildiz.edu.tr

Abstract: In this present paper, Galilean Euler-Savary formula for the radius of curvature of the trajectory of a point in the moving Galilean plane (or called Isotropic plane) during one-parameter planar motion is taken into consideration. Galilean Bobillier formula is obtained by using the geometrical interpretation of the Galilean Euler-Savary formula. Moreover, a direct way is chosen to obtain Bobillier formula without using the Euler-Savary formula in the Galilean plane. As a consequence, the Galilean Euler-Savary will appear as a specific case of Bobillier formula given in the Galilean plane.

Key Words: Galilean plane, Euler-Savary formula, Bobillier formula.

AMS(2010): 53A04,53A35.

§1. Introduction

The study of kinematic analysis and synthesis to describe a motion and to design a mechanism for a desired range of motion, respectively, were examined by many researchers [1]-[9].

In 1959, H. R. Müller defined one-parameter planar motion in the Euclidean plane E^2, studied the moving coordinate system and Euler-Savary formula which gives the relationship between the curvature of trajectory curves, during one-parameter planar motions, [8]. Then A. A. Ergin, by considering the Lorentzian plane L^2, instead of the Euclidean plane E^2, introduced the one-parameter planar motion in L^2 and gave the relations between both the velocities and accelerations and also defined the moving coordinate system [10]-[11]. Euler-Savary formula is studied in Lorentzian plane for the one-parameter Lorentzian motions by using two different ways: In 2002, I. Aytun studied the this formula for the one-parameter Lorentzian motions with using the Müller's Method [12]. In 2003, T. Ikawa gave this formula on Minkowski plane by taking a new aspect without using the Müller's Method [13]. Ikawa gives relation between curvature of roulette and curvatures of these base curve and rolling curve, [13]. Euler-Savary formula is a well documented and an admitted formula in the literature and it takes place in a lot of studies of engineering and mathematics, [14]-[20].

In 1983, the kinematics in the isotropic plane is studied by O. Röschel. In [21], the fundamental properties of the point-paths are investigated, a formula analog to the well-known

[1]Received April 7, 2015, Accepted December 2, 2015.

formula of Euler-Savary is developed and special motions: an isotropic elliptic motion and an isotropic four-bar-motion are studied. Besides, in 1985, the motions $\sum / \sum_0$ in the isotropic plane is studied in [22]. Given C^2 -curve k in the moving frame $\sum$. Röschel found the enveloped curve k_0 in the fixed frame $\sum_0$ and considered the correspondence between the isotropic curvatures A and A_0 of k and k_0. Then third-order properties of the point-paths are investigated.

Moreover, M. Akar and S. Yüce, [23], introduced the one-parameter motions in the Galilean plane $\mathbb{G}^2$ (or called Isotropic) and gave same concepts analog with [8] or [10]. They analyzed the relationships between the absolute, relative and sliding velocities of one-parameter Galilean planar motion as well as the related pole lines. Also in [24], one Galilean plane moving relative to two other Galilean planes, one moving and the other fixed, was taken into consideration and the relation between the absolute, relative and sliding velocities of this motion and pole points were obtained. Also, a canonical relative system for one-parameter Galilean planar motion was defined. Furthermore, Euler-Savary formula was obtained with the aim of this canonical relative system by using Müller's method in [24]. On the other hand, Euler-Savary formula with using the Ikawa's method is examined in [25].

In 1988, M. Fayet introduced a new formula relative to the curvatures in an one planar motion Euclidean planar motion and called it *Bobillier formula* which may obtained by using Euler-Savary formula and without using Euler-Savary formula [26, 27]. In addition to this, Bobillier formula gave a new analytically aspect to graphically viewpoint of Bobillier construction which was studied by [15]-[20], [26]-[29]. Bobillier formula was established also with concerning second order properties of one-parameter planar motion in the complex plane in [30] and with regarding Lorentzian planar motion in [31].

In this respect, we bring a new breath of Bobillier formula in the Galilean plane in this study. We introduce Bobillier formula with two ways: by using Galilean Euler-Savary equation with respect to one-parameter Galilean motion and a direct way towards to it.

§2. **Preliminaries**

The study of mechanics of rectilinear motions reduces to a geometry of two dimensional space. The geometry is invariant under transformation stated by I. M. Yaglom

$$\begin{aligned} x' &= \qquad x + a \\ y' &= y + vx + b \end{aligned}$$

which is called *Galilean transformation for rectilinear motions* [32]. This geometry is called *two dimensional Galilean geometry* is represented by $\mathbb{G}^2$. Yaglom also expressed three dimensional Galilean geometry which is obtained by plane-parallel motions, is denoted by $\mathbb{G}^3$.

Galilean geometry is a geometry of the Galilean Relativity or shortly a non-Euclidean geometry. It is a "bridge" from Euclidean geometry to Special Relativity. Two and three dimensional Galilean geometry were worked in detail in the literature and the further information about the Galilean geometry can be found in [32]-[34]. Also, many studies are conducted in the Galilean plane (or Isotropic plane) and Galilean space, [33]-[34].

The basic notation about Galilean plane geometry can be given as below:

The distance between points $A(x_1, x_2)$ and $B(y_1, y_2)$ in $\mathbb{G}^2$ is defined by as follows:

$$d(A,B) = \begin{cases} |x_{1-}y_1|, & x_1 \neq y_1 \\ |x_{2-}y_2|, & x_1 = y_1. \end{cases}$$

In this paper, we will denote the inner product of two vectors in the sense of Galilean by notation $\langle , \rangle_{\mathbb{G}}$. Moreover, we will define the Galilean cross product as below:

$$(a \times_{\mathbb{G}} b) = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} & if\ \ a_1 \neq 0 \text{ or } \mathrm{b}_1 \neq 0 \\ \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} & if\ \ a_1 = \mathrm{b}_1 = 0 \end{cases}$$

where $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$. On the other hand, Galilean circle is defined by

$$S_{\mathbb{G}}(m,r) = \{x - m \in \mathbb{G}^2 : \langle x - m, x - m \rangle_{\mathbb{G}} = r^2\}.$$

So the unit Galilean circle is $x = \pm 1$. Hence, we have

$$\cos_g \alpha = 1, \ \ \sin_g \alpha = \alpha$$

for all α. Also, by using another circle definition in Euclidean geometry, we get a set

$$ax^2 + 2b_1 x + 2b_2 y + c = 0$$

which are (Euclidean) parabolas. This set is called a *Galilean cycle* and denoted by Z.

§3. One-Parameter Planar Motion in the Galilean Plane $\mathbb{G}$

Let $\mathbb{G}'$ and $\mathbb{G}$ be fixed and moving Galilean planes with the perpendicular coordinate systems $\{O'; \mathbf{g_1'}, \mathbf{g_2'}\}$ and $\{O; \mathbf{g_1}, \mathbf{g_2}\}$, respectively. If we take M_1, M_2 and M_3 are points linked to moving Galilean plane $\mathbb{G}$ then there are the conjugate points M_1', M_2' and M_3' of these points which are the curvature centers of the trajectory drawn M_1, M_2 and M_3 in the fixed Galilean plane $\mathbb{G}'$.

The normals of this trajectory pass from an instantaneous center of rotation that is denoted by P and called as *pole point*.

Since there exist pole points in every moment t, during the one-parameter planar motion

$\mathbb{G}/\mathbb{G}'$, any pole point P is situated varied position on the planes $\mathbb{G}$ and $\mathbb{G}'$. The position of pole point P on the moving plane $\mathbb{G}$ is usually a curve called *moving pole curve* and denoted by (P). Also the position of this pole point P on the fixed plane $\mathbb{G}'$ is usually a curve called *fixed pole curve* denoted by (P') [23].

The axis $\mathbf{x}$ is the common tangent and the axis $\mathbf{y}$ is the common normal to pole curves (P) and (P') at P, see Figure 1.

If θ is the rotation angle of motion of the Galilean plane $\mathbb{G}$ with respect to $\mathbb{G}'$ at each t moment, then each point M makes a rotation motion with $\dot{\theta}$ angular velocity at the instantaneous center P.

Let $\mathbf{X_1}$, $\mathbf{X_2}$ and $\mathbf{X_3}$ be unit vectors, then these unit vectors can be given as

$$\mathbf{X_1} = \frac{\mathbf{PM_1}}{\|\mathbf{PM_1}\|_{\mathbb{G}}}, \mathbf{X_2} = \frac{\mathbf{PM_2}}{\|\mathbf{PM_2}\|_{\mathbb{G}}}, \mathbf{X_3} = \frac{\mathbf{PM_3}}{\|\mathbf{PM_3}\|_{\mathbb{G}}}. \tag{3.1}$$

If the abscissae of the points M_1 and M_1' on the axis $(P, \mathbf{X_1})$ are ρ_1 and ρ_1' respectively, then it can be written that

$$\langle \mathbf{PM_1}, \mathbf{X_1} \rangle_{\mathbb{G}} = \rho_1, \text{ and } \langle \mathbf{PM_1'}, \mathbf{X_1} \rangle_{\mathbb{G}} = \rho_1'. \tag{3.2}$$

Similarly,

$$\langle \mathbf{PM_2}, \mathbf{X_2} \rangle_{\mathbb{G}} = \rho_2, \text{ and } \langle \mathbf{PM_2'}, \mathbf{X_2} \rangle_{\mathbb{G}} = \rho_2',$$

$$\langle \mathbf{PM_3}, \mathbf{X_3} \rangle_{\mathbb{G}} = \rho_3, \text{ and } \langle \mathbf{PM_3'}, \mathbf{X_3} \rangle_{\mathbb{G}} = \rho_3'.$$

§4. Inflection Points, Inflection Cycle and Euler-Savary Formula in Galilean Plane $\mathbb{G}$

Let M be an arbitrary point on moving Galilean plane $\mathbb{G}$ and M' be its conjugate point on fixed plane $\mathbb{G}'$. Let the coordinates of points M and M' be (m_1, m_2) and (m_1', m_2') in the canonical relative system, respectively. The vectors $\mathbf{PM}$ and $\mathbf{PM'}$ have same direction which passes the pole point P. So we can write

$$m_1' = \lambda m_1, \quad m_2' = \lambda m_2,$$

where λ is an unknown ratio. From the definition of Euler-Savary equation in Galilean plane [24], we get the relation between the points M and M' such as

$$m_1' = \frac{m_1 m_2}{m_2 - m_1^2 \frac{d\theta}{ds}}, \qquad m_2' = \frac{m_2 m_2}{m_2 - m_1^2 \frac{d\theta}{ds}}.$$

From the fact that, an inflection point may be defined to be a point whose trajectory momentarily has an infinite radius of curvature [14, 15], we get the inflection cycle such that

$$m_2 = m_1^2 \frac{d\theta}{ds}.$$

Let the inflection points linked to the points M_1, M_2 and M_3, by referring to Figure 1 be

M_1^*, M_2^* and M_3^*, respectively. The locus of such points is a cycle in the moving Galilean plane $\mathbb{G}$ called as an *inflection cycle*. The abscissae of the inflection points can be written as below:

$$\langle \mathbf{PM_1^*}, \mathbf{X_1} \rangle_{\mathbb{G}} = \rho_1^*, \quad \langle \mathbf{PM_2^*}, \mathbf{X_2} \rangle_{\mathbb{G}} = \rho_2^*, \quad \langle \mathbf{PM_3^*}, \mathbf{X_3} \rangle_{\mathbb{G}} = \rho_3^*. \tag{4.1}$$

Let the diameter of the inflection cycle be h. Then there is a relationship between h and ρ_1^* as follows:

$$h \sin_g \theta_1 = \rho_1^*, \tag{4.2}$$

where θ_1 is angle of the motion $\mathbb{G}/\mathbb{G}'$.

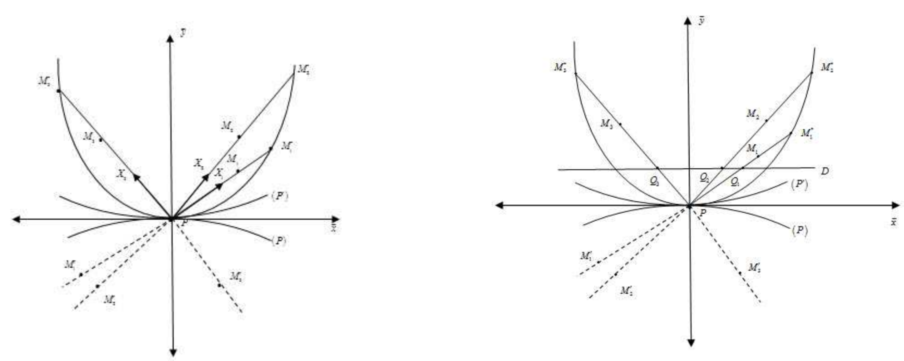
During one-parameter planar motion $\mathbb{G}/\mathbb{G}'$, the point M_1 in the moving Galilean plane $\mathbb{G}$ draws a trajectory with instantaneous curvature center M_1' in the fixed Galilean plane $\mathbb{G}'$. In reverse motion, the point M_1' in $\mathbb{G}'$ draws a trajectory in $\mathbb{G}$, being the curvature center at the point M_1, (see Figure 1). This interconnection between the points M_1 and M_1' is given by Euler-Savary formula

$$\left(\frac{1}{\rho_1'} - \frac{1}{\rho_1}\right) \sin_g \theta_1 = \frac{1}{R_1'} - \frac{1}{R_1}, \tag{4.3}$$

where R_1' and R_1 are the abscissae on $(O, \vec{y})$ of the curvature centers of pole curves (P') and (P), respectively [24]. From the equations (4.2) and (4.3) it is seen that

$$\left(\frac{1}{\rho_1'} - \frac{1}{\rho_1}\right) \sin_g \theta_1 = \frac{1}{R_1'} - \frac{1}{R_1} = \frac{ds}{d\theta_1} = \frac{1}{h}$$

in which $\frac{1}{h} = \frac{1}{R_1} - \frac{1}{R_1'}$ (first form) or $\frac{1}{h} = \pm\frac{\omega}{V}$ (second form) where ω is the angular velocity of the motion of the plane $\mathbb{G}$ with respect to $\mathbb{G}'$ and V is the common velocity of P on (P') and (P).



$\mathbf{PM_i}, \mathbf{PM_i^*}, 1 \le i \le 3$, vectors $\qquad$ $\mathbf{Q_i}, 1 \le i \le 3$, points

Figure 2

§5. Galilean Bobillier Formula Obtained by Galilean Euler-Savary Formula

Let consider the points Q_j are defined by $\mathbf{PQ_j} = \frac{1}{\rho_j^*}\mathbf{X_j}$ where $\frac{1}{\rho_j^*} = \frac{1}{\rho_j} - \frac{1}{\rho_j'}$ for $1 \leqslant j \leqslant 3$. Then the points Q_1, Q_2, and Q_3 are the images of points M_1^*, M_2^*, and M_3^* of the inflection cycle which belong to $(P,\mathbf{X_1})$, $(P,\mathbf{X_2})$ and $(P,\mathbf{X_3})$, respectively (see Figure 2). Therefore the following equations can be written as follows:

$$\langle \mathbf{PQ_1},\mathbf{X_1}\rangle_{\mathbb{G}}=\tfrac{1}{\rho_1^*},\ \ \langle \mathbf{PQ_2},\mathbf{X_2}\rangle_{\mathbb{G}}=\tfrac{1}{\rho_2^*},\ \ \langle \mathbf{PQ_3},\mathbf{X_3}\rangle_{\mathbb{G}}=\tfrac{1}{\rho_3^*},\ \ \langle \mathbf{PQ},\mathbf{x}\rangle_{\mathbb{G}}=\tfrac{1}{h}. \tag{5.1}$$

From the definition of Euler-Savary formula in Galilean plane [24] and the equation (5.1)

$$\mathbf{PQ_1}\sin_g\theta_1=\frac{1}{\rho_1^*}\mathbf{X_1}\sin_g\theta_1\ =\frac{1}{h}\mathbf{X_1},$$

$$\mathbf{PQ_2}\sin_g\theta_2=\frac{1}{\rho_2^*}\mathbf{X_2}\sin_g\theta_2\ =\frac{1}{h}\mathbf{X_2},$$

and

$$\mathbf{PQ_3}\sin_g\theta_3=\frac{1}{\rho_3^*}\mathbf{X_3}\sin_g\theta_3\ =\frac{1}{h}\mathbf{X_3},$$

are obtained. By taking into account the last three equations, we have

$$\langle \mathbf{PQ}_1,\mathbf{X_1}\rangle_{\mathbb{G}}\sin_g\theta_1=\langle \mathbf{PQ_2},\mathbf{X_2}\rangle_{\mathbb{G}}\ \sin_g\theta_2=\langle \mathbf{PQ_3},\mathbf{X_3}\rangle_{\mathbb{G}}\ \sin_g\theta_3=\frac{1}{h}.$$

Thus, the set of the points Q is a straight line which is denoted by D parallel to real axis $\mathbf{x}$. Thus the line $\mathbf{x}$ is an image of the inflection cycle by this inversion, see Figure 1. From the fact that the vectors $\mathbf{PQ_1}-\mathbf{PQ_2}$ and $\mathbf{PQ_2}-\mathbf{PQ_3}$ are linearly dependent, the following equation can be written:

$$(\mathbf{PQ_1}\times\mathbf{PQ_2})-(\mathbf{PQ_2}\times\mathbf{PQ_2})-(\mathbf{PQ_1}\times\mathbf{PQ_3})+(\mathbf{PQ_2}\times\mathbf{PQ_3})=\mathbf{0}.$$

Since $\mathbf{PQ_1}=\frac{1}{\rho_1^*}\mathbf{X}_1$, $\mathbf{PQ_2}=\frac{1}{\rho_2^*}\mathbf{X}_2$, $\mathbf{PQ_3}=\frac{1}{\rho_3^*}\mathbf{X}_3$ and $\rho_1^*\rho_2^*\rho_3^*$ never vanishes, we get

$$\rho_1^*\left(\mathbf{X}_2\times\mathbf{X}_3\right)+\rho_2^*\left(\mathbf{X}_3\times\mathbf{X}_1\right)+\rho_3^*\left(\mathbf{X}_1\times\mathbf{X}_2\right)=\mathbf{0}$$

and for the sake of brevity, if we take

$$\theta_3-\theta_2=\theta_{23},\theta_1-\theta_3=\theta_{31},\theta_2-\theta_1=\theta_{12}, \tag{5.2}$$

then we find

$$\rho_1^*\sin_g\theta_{23}+\rho_2^*\sin_g\theta_{31}+\rho_3^*\sin_g\theta_{12}=0\ \ , \tag{5.3}$$

where $\frac{1}{\rho_j^*}=\frac{1}{\rho_j}-\frac{1}{\rho_j'}$ for $1\leqslant j\leqslant 3$.

This is Bobillier formula for one-parameter planar motion in Galilean plane $\mathbb{G}$ analog with Bobillier formula given in Euclidean plane [27], complex plane [30] and Lorentzian plane [31]. With using the Galilean trigonometric properties we can write

$$\rho_1^*\theta_{23}+\rho_2^*\theta_{31}+\rho_3^*\theta_{12}=0\ \ . \tag{5.4}$$

The equation (5.4) is called the *Galilean Bobillier formula* during the one-parameter planar motions $\mathbb{G}/\mathbb{G}'$.

§6. Galilean Bobillier Formula Deduced from a Direct Way in the Galilean Plane $\mathbb{G}$

In this section, we will introduce Galilean Bobillier formula from a direct way. Let us examine the trajectory velocity and trajectory acceleration of the points in moving Galilean plane $\mathbb{G}$. Suppose that $\mathbf{V}'(M_1)$ and $\mathbf{J}'(M_1)$ are absolute velocity and absolute acceleration vector of the point M_1, respectively. Let denote the angular velocity of planar motion $\mathbb{G}/\mathbb{G}'$ by ω, then $\omega = \frac{\Delta\theta}{\Delta t}$ where θ is the rotation angle. By taking an orthogonal vector to the Galilean planes $\mathbb{G}$ and $\mathbb{G}'$ as $\mathbf{z}$, the angular velocity vector can be defined by $\omega = \omega\mathbf{z}$. Moreover, the sliding velocity vector of the point M_1 is

$$\mathbf{V}(M_1) = \omega \times \mathbf{PM_1} = \omega \left\|\mathbf{PM_1}\right\|_{\mathbb{G}} \sin_g \theta. \tag{6.1}$$

The relation between velocities during one-parameter planar motion in Galilean plane is

$$\mathbf{V}'(M_1) = \mathbf{V}'(P) + \mathbf{V}(M_1),$$

where $\mathbf{V}'(M_1), \mathbf{V}'(P)$ and $\mathbf{V}(M_1)$ denote the absolute, sliding and relative velocity vectors of $\mathbb{G}/\mathbb{G}'$, respectively [23]. With using the equation (6.1), we have

$$\mathbf{V}'(M_1) = \mathbf{V}'(P) + (\omega \times \mathbf{PM_1}) \ . \tag{6.2}$$

By differentiating the equation (6.2) with respect to time t , we obtain

$$\mathbf{J}'(M_1) = \mathbf{J}'(P) + \left(\dot{\omega}\mathbf{z} \times \mathbf{PM_1}\right) + \omega^2 \mathbf{PM}_1, \tag{6.3}$$

where $\mathbf{J}'(P)$ is acceleration vector of the point on $\mathbb{G}'$ that coincides instantaneously with P. Here the first term is the trajectorywise invariant acceleration component, the second term is tangential acceleration component and the third term is centripental acceleration component. With considering this explanation for the inflection points whose acceleration normal is zero, then the absolute velocity vector and acceleration vectors of the point M_1^* on the inflection cycle becomes linearly dependent, so

$$\mathbf{V}'(M_1^*) \times \mathbf{J}'(M_1^*) = \mathbf{0}$$

can be written. If we substitute the method of the (6.2) and (6.3) into the last equation, the equation rewritten as follows:

$$(\mathbf{V}'(P) + (\omega\mathbf{z} \times \mathbf{PM_1^*})) \times \left(\mathbf{J}'(P) + \left(\dot{\omega}\mathbf{z} \times \mathbf{PM_1^*}\right) + \omega^2 \mathbf{PM}_1^*\right) = \mathbf{0}.$$

From $\mathbf{V}'(P) = \mathbf{0}$ and the equation (4.1) $\left\|\mathbf{PM_1^*}\right\|_{\mathbb{G}} = \rho_1^*$, we obtain

$$\left\langle \mathbf{PM_1}^*, \mathbf{J}'(P)\right\rangle_{\mathbb{G}} \mathbf{z} - \omega^2 \left\|\mathbf{PM}_1^*\right\|_{\mathbb{G}}^2 \mathbf{z} = \mathbf{0}.$$

With simplifying calculations and using $\mathbf{PM_j}^* = \left\|\mathbf{PM_j}^*\right\|_{\mathbb{G}} \mathbf{X}_j$ for $1 \leq j \leq 3$, we obtain the equations as follows,

$$\rho_1^* = \frac{\left\langle \mathbf{X}_1, \mathbf{J}'(P) \right\rangle_{\mathbb{G}}}{\omega^2} \tag{6.4}$$

$$\rho_2^* = \frac{\left\langle \mathbf{X}_2, \mathbf{J}'(P) \right\rangle_{\mathbb{G}}}{\omega^2}, \tag{6.5}$$

$$\rho_3^* = \frac{\left\langle \mathbf{X}_3, \mathbf{J}'(P) \right\rangle_{\mathbb{G}}}{\omega^2}, \tag{6.6}$$

for the points M_1^*, M_2^* and M_3^*, respectively. It is easily seen from the equations (6.4), (6.5) and (6.6) that ρ_1^*, ρ_2^*, and ρ_3^* are the orthogonal projections of the same vector $\frac{\mathbf{J}'(P)}{\omega^2}$ onto the vectors $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$, respectively. The relationship between these unit vectors is indicated with the equation

$$\lambda_1 \mathbf{X}_1 + \lambda_2 \mathbf{X}_2 + \lambda_3 \mathbf{X}_3 = \mathbf{0},$$

where λ_1, λ_2 and λ_3 are quantities. By successive Galilean cross products with $\mathbf{X}_1$ and $\mathbf{X}_2$ the quantities λ_1, λ_2 and λ_3 are obtained and using the specification (5.2), then the previous linear combination becomes

$$\sin_g \theta_{12} \mathbf{X}_3 + \sin_g \theta_{23} \mathbf{X}_1 + \sin_g \theta_{31} \mathbf{X}_2 = \mathbf{0}.$$

The product of the previous equation with the vector $\frac{\mathbf{J}'(I)}{\omega^2}$ is given as below:

$$\sin_g \theta_{12} \frac{\langle \mathbf{X}_3, \mathbf{J}'(P) \rangle_{\mathbb{G}}}{\omega^2} + \sin_g \theta_{23} \frac{\langle \mathbf{X}_1, \mathbf{J}'(P) \rangle_{\mathbb{G}}}{\omega^2} + \sin_g \theta_{31} \frac{\langle \mathbf{X}_2, \mathbf{J}'(P) \rangle_{\mathbb{G}}}{\omega^2} = \mathbf{0}.$$

Finally, if we substitute the equations (6.4), (6.5) and (6.6) into the last equation, we obtain the equation (5.4) which was called *Bobillier formula*. It can be noticed that, the direct way gives us the Bobillier formula without using the Euler-Savary formula. Therefore the following theorem can be given.

Theorem 6.1 *During the one-parameter planar motion $\mathbb{G}/\mathbb{G}'$, the relation between the distances of inflection points of points in the moving plane $\mathbb{G}$ and the pole point is given by the equation (5.4) which is called Bobillier formula.*

Let us analyze a particular case of Theorem 6.1. If a point K linked to moving plane $\mathbb{G}$ is coincident with instantaneous pole center P, then $\mathbf{V}'(K) = \mathbf{0}$ and similarly $\mathbf{J}'(K) = \mathbf{0}$. From this place, the vector $\mathbf{X}_2$ is equal to $\mathbf{x}$ which is the normal to the path of K at P. Hence, in the equation (6.5) ρ_2^* is equal to zero. Thus we can express the following corollary.

Corollary 6.2 *Let a point K linked to moving plane $\mathbb{G}$ be coincident with instantaneous pole center P. In that case Bobillier formula in the Galilean plane becomes*

$$\rho_1^* + \rho_3^* \theta_{12} = 0.$$

In conclusion, the corollary simply a particular case of Bobillier formula in the Galilean plane $\mathbb{G}$.

References

[1] Beyer R., *The Kinematic Synthesis of Mechanisms*, Chapman and Hall, Ltd., London, England, (1963).

[2] Erdman A. G., *Modern Kinematics, Developments in the Last Forty Years*, John Wiley and Sons, Inc., Wiley Series in Design Engineering, New York, (1993).

[3] Erdman A. G., Sandor, G. N., and Kota, S., *Mechanism Design*, Prentice-Hall,Inc., Upper Saddle River, New Jersey, Volume 1, Fourth Edition, (2001).

[4] Hain K., *Applied Kinematics*, McGraw-Hill Book Co., Inc., New York,Second Edition, (1967).

[5] Hall A. S., Jr., *Kinematics and Linkage Design*, Waveland Press, Inc., Prospect Heights, Illinois, (1986), (Originally published by Prentice-Hall,Inc.,1961).

[6] Hunt K. H., *Kinematic Geometry of Mechanisms*, Clarendon Press, Ltd., Oxford, England, (1978).

[7] Waldron K. J., and Kinzel, G. L., *Kinematics, Dynamics, and Design of Machinery*, John Wiley and Sons, Inc., New York, (1999).

[8] Blaschke W. and Müller, H. R., *Ebene Kinematik*,Velag von R. Oldenbourgh, München, (1956).

[9] Uicker J. J. Jr., Pennock, G. R., and Shigley, J. E., *Theory of Machines and Mechanisms*, Oxford University Press, Inc., New York,Fourth Edition, (2011).

[10] Ergin A. A., On the one-parameter Lorentzian motion, *Comm. Fac. Sci. Univ. Ankara*, Series A 40, 59–66, (1991).

[11] Ergin A. A., Three Lorentzian planes moving with respect to one another and pole points, *Comm. Fac. Sci.Univ. Ankara*, Series A41, 79-84(1992).

[12] Aytun I., *Euler-Savary Formula for One-parameter Lorentzian Plane Motion and its Lorentzian Geometrical Interpretation*, M.Sc. Thesis, Celal Bayar University, (2002).

[13] Ikawa T., Euler-Savary's formula on Minkowski geometry, *Balkan Journal of Geometry and Its Applications*, 8 (2), 31-36(2003).

[14] Sandor G. N., Xu, Y., and Weng T-C., A graphical method for solving the Euler-Savary equation, *Mechanism and Machine Theory*, 25 (2), 141–147(1990).

[15] Sandor G. N., Arthur G. E. and Raghavacharyulu E., Double Valued Solutions of the Euler-Savary Equation and Its Counterpart in Bobillier's Construction, *Mechanism and Machine Theory*, 20 (2), 145–178, (1985).

[16] Alexander J. C. and Maddocks J. H., On the Maneuvering of Vehicles, *SIAM J. Appl. Math.*, 48(1): 38–52(1988).

[17] Buckley R. and Whitfield E. V., The Euler-Savary formula, *The Mathematical Gazette*, 33(306): 297–299(1949).

[18] Dooner D. B. and Griffis M. W., On the Spatial Euler-Savary Equations for Envelopes, *J. Mech. Design*, 129(8): 865–875 (2007).

[19] Ito N. and Takahashi K., Extension of the Euler-Savary Equation to Hypoid Gears, *JSME Int.Journal. Ser C. Mech Systems*, 42(1): 218-224 (1999).

[20] Pennock G. R. and Raje N. N., Curvature Theory for the Double Flier Eight-Bar Linkage, *Mech. Theory*, 39: 665-679 (2004).

[21] Röschel O., Zur Kinematik der isotropen Ebene, *Journal of Geometry*, 21, 146–156(1983).

[22] Röschel O., Zur Kinematik der isotropen Ebene II, *Journal of Geometry*, 24, 112–1221985).

[23] Akar M., Yüce, S. and Kuruoğlu, N., One-parameter Planar Motion in the Galilean plane, *International Electronic Journal of Geometry* (IEJG), Vol. 6, Issue 1, pp. 79–88 (2013).

[24] Akbıyık M. and Yüce S., The Moving Coordinate System And Euler-Savary's Formula For The One Parameter Motions On Galilean (Isotropic) Plane, *International Journal of Mathematical Combinatorics*, Vol.2(2015), 88–105.

[25] Akbıyık M. and Yüce S., *Euler Savary's Formula on Galilean Plane*, Submitted.

[26] Fayet M., Une Nouvelle Formule Relative Aux Courbures Dans un Mouvement Plan, *Mech. Mach. Theory*, 23 (2), 135–139(1988).

[27] Fayet M., Bobillier Formula as a Fundamental Law in Planar Motion, *Z. Angew. Math. Mech.*, 82 (3), 207–210(2002).

[28] Garnier R., *Cours de cinematique*, Gauthier-Villar, Paris, 38, (1956).

[29] Dijskman E. A., *Motion Geometry of Mechanism*, Cambridge University Press, Cambridge, (1976).

[30] Ersoy S. and Bayrak N., Bobillier Formula for One Parameter Motions in the Complex Plane, *J. Mech. and Robotics*, Vol. 4, pp. 245011-245014, DOI: 10.1115/1.4006195, (2012).

[31] Ersoy S. and Bayrak N., Lorentzian Bobillier Formula, *Applied Mathematics E-Notes*, Issue 13, 25-35(2013).

[32] Yaglom I. M., *A simple non-Euclidean Geometry and its Physical Basis*, Springer-Verlag, New York, (1979).

[33] Helzer G., Special relativity with acceleration, *The American Mathematical Monthly*, Vol. 107, No. 3, 219–237, (2000).

[34] Röschel O., *Die Geometrie des Galileischen Raumes*, Habilitationsschrift, Institut fr Math. und Angew. Geometrie, Leoben, (1984).

[35] Pavković B. J., Kamenarović I., The Equiform Differential Geometry of Curves in the Galilean Space, *Glasnik Matematićki*, 22(42), 449–457, (1987).