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Study Map of Orthotomic of a Circle

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Abstract: In this paper, we calculate and discuss the Study map of an spherical orthotomic of a circle which lies on the dual unit sphere in D-module. In order to do this we use matrix equation of Study mapping. Finally we give some special cases each of which is a geometric result.

Key Words: Study mapping, D-module, orthotomic, spherical orthotomic.

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§1. Introduction

In linear algebra, dual numbers are defined by W. K. Clifford (1873) by using real numbers. Dual numbers are extended the real numbers by adjoining one new element ε with the property $\varepsilon^2 = 0$. Dual numbers have the form $x + \varepsilon y$, where $x, y \in R$. The dual numbers set is two-dimensional commutative unital associative algebra over the real numbers. Its first application was made by E. Study. He used dual numbers and dual vectors in his research on the geometry of lines and kinematics [15]. He devoted special attention to the representation of directed line by dual unit vectors and defined the mapping which is called with his name. There exists one to one correspondence between dual unit points of dual unit sphere and the directed lines of the Euclidean line space E^3.

Let α be a regular curve and $\overrightarrow{T}$ be its tangent, and u be a source. Orthotomic of α with respect to the source (u) is the locus of reflection of u about the tangents $\overrightarrow{T}$ [7]. Bruce and Giblin studied the unfolding theory to the evolutes and orthotomics of plane and space curves [3], [4] and [5]. Georgiou, Hasanis and Koutroufiotis investigated the orthotomics in Euclidean (n + 1)-space E^{n+1} [6]. Alamo and Criado studied the antiorthotomics in Euclidean (n+1)-space E^{n+1} [1]. Xiong defined the spherical orthotomic and the spherical antiorthotomic [16]. In this paper we examine the Study Map of the spherical orthotomic of a circle which lies on the dual unit sphere in D-Module.

§2. Preliminaries

If a and a^* are real numbers and $\varepsilon^2 = 0$ but $\varepsilon \notin R$, a dual number can be written as $A = a + \varepsilon a^*$,

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where $\varepsilon = (0,1)$ is the dual unit.

The set $D = \{A = a + \varepsilon a^* \,|a, a^* \in R\}$ of dual numbers is a commutative ring over the real number field and is denoted by D. Then the set

$$D^3 = \left\{ \overrightarrow{A} = (A_1, A_2, A_3) \,|A_i \in D, 1 \leq i \leq 3 \right\}$$

is a module over the ring D which is called a D-Module, under the addition and the scalar multiplication on the set D ([12]). The elements of D^3 are called dual vectors. Thus a dual vector has the form $\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*$, where $\overrightarrow{a}$ and $\overrightarrow{a}^*$ are real vectors at R^3. Then, for any vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ in D^3, the inner product and the vector product of these vectors are defined as

$$\left\langle \overrightarrow{A}, \overrightarrow{B} \right\rangle = \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle + \varepsilon \left(\left\langle \overrightarrow{a}, \overrightarrow{b}^* \right\rangle + \left\langle \overrightarrow{a}^*, \overrightarrow{b} \right\rangle \right)$$

and

$$\overrightarrow{A} \wedge \overrightarrow{B} = \overrightarrow{a} \wedge \overrightarrow{b} + \varepsilon \left(\overrightarrow{a} \wedge \overrightarrow{b}^* + \overrightarrow{a}^* \wedge \overrightarrow{b} \right),$$

respectively. The norm $\left\| \overrightarrow{A} \right\|$ of $\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*$ is defined as

$$\left\| \overrightarrow{A} \right\| = \| \overrightarrow{a} \| + \varepsilon \frac{\langle \overrightarrow{a}, \overrightarrow{a}^* \rangle}{\| \overrightarrow{a} \|}, \quad \overrightarrow{a} \neq 0.$$

The dual vector $\overrightarrow{A}$ with norm 1 is called a dual unit vector. The set of dual unit vectors

$$S^2 = \left\{ \overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^* \in D^3 \,\middle|\, \left\| \overrightarrow{A} \right\| = 1; \in D, \overrightarrow{a}, \overrightarrow{a}^* \in R^3 \right\}$$

is called the dual unit sphere.

Now, we give the definition of spherical normal, spherical tangent and spherical orthotomic of a spherical curve α. $\left\{ \overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B} \right\}$ be the Frenet frame of α. The spherical normal of α is the great circle which passing through $\alpha(s)$ and normal to α at $\alpha(s)$ and is given by

$$\begin{cases} \langle \overrightarrow{x}, \overrightarrow{x} \rangle = 1 \\ \left\langle \overrightarrow{x}, \overrightarrow{T} \right\rangle = 0 \end{cases}$$

where x is an arbitrary point of spherical normal. The spherical tangent of α is the great circle which tangent to α at $\alpha(s)$ and is given by

$$\begin{cases} \langle \overrightarrow{y}, \overrightarrow{y} \rangle = 1 \\ \left\langle \overrightarrow{y}, (\overrightarrow{\alpha} \wedge \overrightarrow{T}) \right\rangle = 0 \end{cases}$$

where y is an arbitrary point of spherical tangent.

Let $u \in S^2$ be a source. Xiong defined the spherical orthotomic of α relative to u to be ([17]) the set of reflections of u about the planes whose lie on the above great circles for all $s \in I$ and given by

$$\overrightarrow{\widetilde{u}} = 2 \langle (\overrightarrow{\alpha} - \overrightarrow{u}), \overrightarrow{v} \rangle \overrightarrow{v} + \overrightarrow{u}$$

where $\overrightarrow{v} = \dfrac{\overrightarrow{B} - \left\langle \overrightarrow{B}, \overrightarrow{\alpha} \right\rangle \overrightarrow{\alpha}}{\left\| \overrightarrow{B} - \left\langle \overrightarrow{B}, \overrightarrow{\alpha} \right\rangle \overrightarrow{\alpha} \right\|}$.

§3. Study Mapping

Definition 3.1 *The Study mapping is an one to one mapping between the dual points of a dual unit sphere, in D-Module, and the oriented lines in the Euclidean line space E^3.*

Let K, O and $\left\{O; \overrightarrow{E}_1, \overrightarrow{E}_2, \overrightarrow{E}_3\right\}$ denote the unit dual sphere, the center of K and dual orthonormal system at O, respectively where

$$\overrightarrow{E}_i = \overrightarrow{e}_i + \varepsilon \overrightarrow{e}_i^*; 1 \leq i \leq 3. \tag{3.1}$$

Let S_3 be the group of all the permutations of the set $\{1, 2, 3\}$, then it can be written as

$$\begin{cases} \overrightarrow{E}_{\sigma(1)} = sgn(\sigma) \overrightarrow{E}_{\sigma(2)} \wedge \overrightarrow{E}_{\sigma(3)}, sgn(\sigma) = \pm 1, \\ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ \sigma(1) & \sigma(2) & \sigma(3) \end{pmatrix} \end{cases} \tag{3.2}$$

In the case that the orthonormal system

$$\{O; \overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3\}$$

is the system of the line space E^3. We can write the moment vectors $\overrightarrow{e}_i^*$ as

$$\overrightarrow{e}_i^* = \overrightarrow{MO} \Lambda \overrightarrow{e}_i, 1 \leq i \leq 3. \tag{3.3}$$

Since these moment vectors are the vectors of R^3, we may write that

$$\overrightarrow{e}_i^* = \sum^3 \lambda_{ij} \overrightarrow{e}_i, \lambda_{ij} \in R, 1 \leq i \leq 3. \tag{3.4}$$

Hence (3.3) and (3.4) give us

$$\lambda_{ii} = 0, \lambda_{ij} = -\lambda_{ji}, 1 \leq i, j \leq 3$$

and so the scalars λ_{ij} are denoted by λ_i, that is,

$$\lambda_{ij} = \lambda_i.$$

Then (3.4) reduces to

$$\begin{bmatrix} \overrightarrow{e}_1^* \\ \overrightarrow{e}_2^* \\ \overrightarrow{e}_3^* \end{bmatrix} = \begin{bmatrix} 0 & \lambda_1 & -\lambda_3 \\ -\lambda_1 & 0 & \lambda_2 \\ \lambda_3 & -\lambda_2 & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{e}_1 \\ \overrightarrow{e}_2 \\ \overrightarrow{e}_3 \end{bmatrix}. \tag{3.5}$$

Hence the Study mapping

$$K \to E^3$$

can be given as a mapping from the dual orthonormal system to the real orthonormal system. By using the relations (3.1) and (3.5), we can express Study mapping in the matrix form as follows:

$$\begin{bmatrix} \overrightarrow{E}_1 \\ \overrightarrow{E}_2 \\ \overrightarrow{E}_3 \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1\varepsilon & -\lambda_3\varepsilon \\ -\lambda_1\varepsilon & 1 & \lambda_2\varepsilon \\ \lambda_3\varepsilon & -\lambda_{21}\varepsilon & 1 \end{bmatrix} \begin{bmatrix} \overrightarrow{e}_1 \\ \overrightarrow{e}_2 \\ \overrightarrow{e}_3 \end{bmatrix} \tag{3.6}$$

which says that Study mapping corresponds with a dual orthogonal matrix. Since we know [14] that the linear mappings are in one to one correspondence with the matrices we may give the following theorem.

Theorem 3.2 *A Study mapping is a linear isomorphism.*

Since the Euclidean motions in E^3 leave do not change the angle and the distance between two lines, the corresponding mapping in D-Module leave the inner product invariant.

This is the action of an orthogonal matrix with dual coefficients. Since the center of the dual unit sphere K must remain fixed the transformation group in D-module (the image of the Euclidean motions) does not contain any translations. Hence, in order to represents the Euclidean motions in D-Module we can apply the following theorem [8].

Theorem 3.3 *The Euclidean motions in E^3 are in one to one correspondence with the dual orthogonal matrices.*

Definition 3.4 *A ruled surface is a surface that can be swept out by moving a line in space. This line is the generator of surface.*

A differentiable curve

$$t \in R \to \overrightarrow{X}(t) \in K$$

on the dual unit sphere K,depending on a real parameter t,represents differentiable family of straight lines of which is ruled surface ([2], [8]). The lines $\overrightarrow{X}(t)$ are the generators of the surface.

Let X, Y be two different points of K and Φ be the dual angle $\left(\overrightarrow{OX}, \overrightarrow{OY}\right)$. The dual angle Φ has a value $\varphi + \varepsilon\varphi^*$ which is a dual number, where φ and φ^* are the angle and the minimal distance between the two lines $\overrightarrow{X}$ and $\overrightarrow{Y}$, respectively. Then we have the following theorem ([8]).

Theorem 3.5 *Let* $\overrightarrow{X}$, $\overrightarrow{Y} \in K$. *Then we have*

$$\left\langle \overrightarrow{X} , \overrightarrow{Y} \right\rangle = \cos \Phi,$$

where

$$\cos \Phi = \cos \varphi - \varepsilon \varphi^* \sin \varphi. \tag{3.7}$$

The following special cases of Theorem 3.5 are important ([9]):

$$\left\langle \overrightarrow{X} , \overrightarrow{Y} \right\rangle = 0 \implies \varphi = \frac{\pi}{2} \text{ and } \varphi^* = 0; \tag{3.8}$$

meaning that the lines $\overrightarrow{X}$ and $\overrightarrow{Y}$ meet at a right angle.

$$\left\langle \overrightarrow{X} , \overrightarrow{Y} \right\rangle = \text{pure dual} \implies \varphi = \frac{\pi}{2} \text{ and } \varphi^* \neq 0; \tag{3.9}$$

meaning that the lines $\overrightarrow{X}$ and $\overrightarrow{Y}$ are orthogonal skew lines.

$$\left\langle \overrightarrow{X} , \overrightarrow{Y} \right\rangle = \text{pure real} \implies \varphi \neq \frac{\pi}{2} \text{ and } \varphi^* = 0; \tag{3.10}$$

this means that the lines $\overrightarrow{X}$ and $\overrightarrow{Y}$ intersect each other.

$$\left\langle \overrightarrow{X} , \overrightarrow{Y} \right\rangle = \pm 1 \implies \varphi^* = 0 \text{ and } \varphi = 0 (\text{or } \varphi = \pi); \tag{3.11}$$

meaning that the lines $\overrightarrow{X}$ and $\overrightarrow{Y}$ are coincide (their senses are either same or opposite).

§4. The Study Map of a Circle

Let g be the straight line corresponding to the unit dual vector $\overrightarrow{E}_3$. If we choose the point M on g then we have

$$\lambda_2 = \lambda_3 = 0$$

and so the matrix, from (3.6) reduces to

$$\begin{bmatrix} \overrightarrow{E}_1 \\ \overrightarrow{E}_2 \\ \overrightarrow{E}_3 \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 \varepsilon & 0 \\ -\lambda_1 \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overrightarrow{e}_1 \\ \overrightarrow{e}_2 \\ \overrightarrow{e}_3 \end{bmatrix} \tag{4.1}$$

The inverse of this mapping is

$$\begin{bmatrix} \overrightarrow{e}_1 \\ \overrightarrow{e}_2 \\ \overrightarrow{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & -\lambda_1 \varepsilon & 0 \\ \lambda_1 \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overrightarrow{E}_1 \\ \overrightarrow{E}_2 \\ \overrightarrow{E}_3 \end{bmatrix} \tag{4.2}$$

Let

$$S = \{\overrightarrow{X} \ \mid \left\langle \overrightarrow{X} \ , \overrightarrow{E}_3 \right\rangle = \cos\Phi = \text{constant}, \ \overrightarrow{X} \in K\}$$

be the circle on the unit dual sphere K. The spherical orthotomic of the great circle which lies on the plane spanned by $\left\{\overrightarrow{E}_1, \overrightarrow{E}_2\right\}$ relative to S is a reflection of S about this plane and for $\overrightarrow{X} = (X_1, X_2, X_3) \in S$, it is given by $\overrightarrow{\widetilde{X}} = (X_1, X_2, -X_3)$. Thus the dual vector $\overrightarrow{\widetilde{X}}$ can be expressed as

$$\overrightarrow{\widetilde{X}} = \sin\Phi\cos\Psi\,\overrightarrow{E}_1 + \sin\Phi\sin\Psi\,\overrightarrow{E}_2 - \cos\Phi\,\overrightarrow{E}_3 \tag{4.3}$$

where $\Phi = \varphi + \varepsilon\varphi^*$ and $\Psi = \psi + \varepsilon\psi^*$ are the dual angles. Since we have the relations

$$\begin{cases} \overrightarrow{\widetilde{X}} = \overrightarrow{\widetilde{x}} + \varepsilon\,\overrightarrow{\widetilde{x}}^* \\ \sin\Phi = \sin\varphi + \varepsilon\varphi^*\cos\varphi, \ \ \sin\Psi = \sin\psi + \varepsilon\psi^*\cos\psi \\ \cos\Phi = \cos\varphi - \varepsilon\varphi^*\sin\varphi, \ \ \cos\Psi = \cos\psi - \varepsilon\psi^*\sin\psi \end{cases}$$

These equations (4.1) and (4.3) give us the vectors $\overrightarrow{\widetilde{x}}$ and $\overrightarrow{\widetilde{x}}^*$ in the matrix form:

$$\begin{cases} \overrightarrow{\widetilde{x}} = \begin{bmatrix} \overrightarrow{e}_1 & \overrightarrow{e}_2 & \overrightarrow{e}_3 \end{bmatrix} \begin{bmatrix} \sin\varphi\cos\psi \\ \sin\varphi\sin\psi \\ -\cos\varphi \end{bmatrix} \\ \overrightarrow{\widetilde{x}}^* = \begin{bmatrix} \overrightarrow{e}_1 & \overrightarrow{e}_2 & \overrightarrow{e}_3 \end{bmatrix} \begin{bmatrix} \varphi^*\cos\varphi\cos\psi - (\psi^* + \lambda_1)\sin\varphi\sin\psi \\ \varphi^*\cos\varphi\sin\psi + (\psi^* + \lambda_1)\sin\varphi\cos\psi \\ \varphi^*\sin\varphi \end{bmatrix} \end{cases} \tag{4.4}$$

On the other hand, the point $\overrightarrow{X}$ is on the circle with center on the axis $\overrightarrow{E}_3$. As $\overrightarrow{\widetilde{X}}$ is spherical orthotomic of $\overrightarrow{X}$, $\overrightarrow{\widetilde{X}}$ is on the circle which is reflected about the plane spanned by $\left\{\overrightarrow{E}_1, \overrightarrow{E}_2\right\}$.Thus we may write

$$\left\langle \overrightarrow{\widetilde{X}} \ , \overrightarrow{E}_3 \right\rangle = \cos\Phi = \cos\varphi - \varepsilon\varphi^*\sin\varphi = \text{constant} \tag{4.5}$$

which means that $\varphi = c_1(\text{constant})$ and $\varphi^* = c_2(\text{constant})$.

The equation (4.4) and (4.5) let us to write the following relations:

$$\begin{cases} \left\langle \overrightarrow{\widetilde{x}}, \overrightarrow{\widetilde{x}} \right\rangle = 1 \\ \left\langle \overrightarrow{\widetilde{x}}, \overrightarrow{\widetilde{x}}^* \right\rangle = 0 \\ \left\langle \overrightarrow{\widetilde{x}}, \overrightarrow{e}_3 \right\rangle - \cos\varphi = 0 \\ \left\langle \overrightarrow{\widetilde{x}}, \overrightarrow{e}_3^* \right\rangle + \left\langle \overrightarrow{\widetilde{x}}^*, \overrightarrow{e}_3 \right\rangle + \varphi^*\sin\varphi = 0 \end{cases} \tag{4.6}$$

The equations (4.6) have only two parameter ψ and ψ^* so (4.6) represents a line congruence in R^3. This congruence is called spherical orthotomic congruence.

Now we may calculate the equations of this spherical orthotomic congruence in Plucker coordinates. Let $\overrightarrow{y}$ be a point of the spherical orthotomic congruence then we have ([12]).

$$\overrightarrow{y} = \overrightarrow{\widetilde{x}}(\psi, \psi^*) \wedge \overrightarrow{\widetilde{x}}^*(\psi, \psi^*) + v\,\overrightarrow{\widetilde{x}}(\psi, \psi^*). \tag{4.7}$$

If the coordinates of $\overrightarrow{y}$ are (y_1, y_2, y_3) then (4.7) give us

$$\begin{cases} y_1 = \varphi^* \sin\psi + (\psi^* + \lambda_1)\sin\varphi\cos\varphi\cos\psi + v\sin\varphi\cos\psi \\ y_2 = -\varphi^* \cos\psi + (\psi^* + \lambda_1)\sin\varphi\cos\varphi\sin\psi + v\sin\varphi\sin\psi \\ y_3 = (\psi^* + \lambda_1)\sin^2\varphi - v\cos\varphi \end{cases} \tag{4.8}$$

In this case that $\varphi \neq \dfrac{\pi}{2}$ (4.8) give us

$$\frac{y_1^2}{c_2^2} + \frac{y_2^2}{c_2^2} - \frac{[y_3 - (\psi^* + \lambda_1)]^2}{[c_2 \cot c_1]^2} = 1 \tag{4.9}$$

which has two parameters ψ^* and λ_1 so it represents a line congruence with degree two. The lines of this congruence are located so that

a) The shortest distance of these lines and the line g is $\varphi^* = c_2$;

b) The angle of these lines and the line g is $\varphi = c_1$.

Thus, it can be seen that the lines of spherical orthotomic congruence intersect the generators of a cylinder whose radius is φ^* =constant, and the axis is g, under the angle φ^* =constant.

Definition 4.1 *If all the lines of a line congruence have a constant angle with a definite line then the congruence is called an inclined congruence.*

According to this definition, (4.9) represents an inclined congruence. Then, we have the following theorem.

Theorem 4.2 *Let S be a circle with two parameter on the unit dual sphere K The Study map of orthotomic of S is an inclined congruence with degree two.*

In other respect, we know that the shortest distance between the axis g of the cylinder and the lines of the spherical orthotomic congruence is c_2. Therefore, this cylinder is the envelope of the lines of the spherical orthotomic congruence. So, we have the following theorem.

Theorem 4.3 *Let K be a unit dual sphere and*

$$S = \{\overrightarrow{X} \mid \left\langle \overrightarrow{X}, \overrightarrow{G} \right\rangle = \cos(\varphi + \varepsilon\varphi^*) = \textit{constant}, \overrightarrow{X} \in K, \overrightarrow{G} \in K\}$$

be a circle on K. Let ζ and g be the Study maps of spherical orthotomic of$S$ and G, respectively. Then the lines of ζ has an envelope which is a circular cylinder whose axis is g and radius is c_2.

In the case that $\psi^* = -\lambda_1, \varphi \neq 0$ and $\varphi^* \neq 0$ (4.9) reduces to

$$\frac{y_1^2}{c_1^2} + \frac{y_2^2}{c_1^2} - \frac{y_3^2}{k^2} = 1, k = c_2 \cot c_1 = \text{constant}, c_1 = \varphi, c_2 = \varphi^* \tag{4.10}$$

which represents an hyperboloid of one sheet.

Since ψ^* and λ_1 are two independent parameters, it can be said that the Study map of spherical orthotomic of S is, in general, a family of hyperboloids of one sheet with two parameters. Therefore we can give the following theorem.

Theorem 4.4 *Let S be a circle on the unit dual sphere K. Then the Study map of spherical orthotomic of S is a family of hyperboloid of one sheet with two parameters.*

4.1 The Case that $\varphi^* \neq 0$ and $\varphi = \frac{\pi}{2}$

In this case the lines of the spherical orthotomic congruence (4.9) orthogonally intersect the generators of the cylinder whose axis is g and the radius is φ^*. Since (4.8) reduces to

$$\begin{cases} y_1 = \varphi^* \sin\psi + v\cos\psi \\ y_2 = -\varphi^* \cos\psi + v\sin\psi \\ y_3 = \psi^* + \lambda_1 \end{cases} \tag{4.11}$$

Then (4.9) becomes

$$\begin{cases} y_1^2 + y_2^2 = c_2^2 + v^2 \\ y_3 = \psi^* + \lambda_1 \end{cases} \tag{4.12}$$

4.2 The Case that $\varphi^* \neq 0$ and $\varphi = 0$ (or $\varphi = \pi$)

In this case the lines of the spherical orthotomic congruence ζ coincide with the generators of the cylinder which is the envelope of the lines of φ. This means noting but the Study map of spherical orthotomic of S reduces to cylinder whose equations, from (4.8), are

$$\begin{cases} y_1^2 + y_2^2 = c_2^2 \\ y_3 = -v \end{cases}$$

4.3 The Case $\varphi^* = 0$ and $\varphi = 0$ (or $\varphi = \pi$)

In this case all of the lines of the spherical orthotomic congruence ζ are coincided with the line g. Indeed, in this case, (4.8) reduces to

$$\begin{cases} y_1^2 + y_2^2 = 0 \\ y_3 = -v \end{cases}$$

which represents the line g.

4.4 The Case $\varphi^* = 0$ and $\varphi \neq 0$

In this case, all of the lines of ζ intersect the axis g under the constant angle φ. So, we can say that the lines of the spherical orthotomic congruence ζ are the common lines of two linear line complexes [13]. From (4.8), the equations of ζ give us that

$$y_1^2 + y_2^2 - \frac{[y_3 - (\psi^* + \lambda_1)]^2}{[\cot c_1]^2} = 0.$$

4.5 The Case that $\varphi^* = 0$ and $\varphi = \frac{\pi}{2}$

In this case S is a great circle on K. Then all of the lines of ζ orthogonaly intersect the axis g. This means that the spherical orthotomic inclined congruence reduces to a linear line complex whose axis is g. Then (4.8) gives us that the equation of ζ as

$$\begin{cases} y_1 = v\cos\psi \\ y_2 = v\sin\psi \\ y_3 = \lambda_1 + \psi^* \end{cases}$$

or

$$\begin{cases} y_1^2 + y_2^2 = v^2 \\ y_3 = \lambda_1 + \psi^* \end{cases}$$

Definition 4.5 *If all the lines of a line congruence orthogonally intersect a constant line then the congruence is called a recticongruence.*

Therefore we can give the following theorem.

Theorem 4.6 *Let S be a great circle on K, that is,*

$$S = \{\overrightarrow{X} \mid \left\langle \overrightarrow{X}, \overrightarrow{G} \right\rangle = 0, \overrightarrow{X}, \overrightarrow{G} \in K\}.$$

Then the Study map ζ of orthotomic of S is a recticongruence.

In the case that $\lambda_1 = c_3\psi$ and (4.11) reduces to

$$\begin{cases} y_1 = v\cos\psi \\ y_2 = v\sin\psi \\ y_3 = c_3\psi \end{cases}$$

or

$$y_3 = c_3 \arctan\frac{y_2}{y_1}$$

which represents a right helicoid

Since λ_1 is a parameter, we can choose it as $\lambda_1 = c_3\psi$ and so under the corresponding

mapping the image of spherical orthotomic congruence reduce to a right helicoid. Hence we can give the following theorem.

Theorem 4.8 *It is possible to choose the Study mapping such that the Study maps of spherical orthotomic of dual circles are right helicoids.*

For the spherical orthotomic of great circle which lies on $Sp\left\{\overrightarrow{E}_1, \overrightarrow{E}_3\right\}$, we have the following theorem.

Theorem 4.9 *The Study map of spherical orthotomic of S is given by*

$$\frac{y_1^2}{c_2^2} + \frac{y_3^2}{c_2^2} - \frac{\left[y_2 - (\psi^* + \lambda_3)\right]^2}{\left[c_2 \cot c_1\right]^2} = 1$$

which has two parameters, so it represents a line congruence with degree two.

For a plane spanned by $\left\{\overrightarrow{E}_2, \overrightarrow{E}_3\right\}$, we obtain the following theorem.

Theorem 4.10 *The Study map of spherical orthotomic of S is given by*

$$\frac{y_2^2}{c_2^2} + \frac{y_3^2}{c_2^2} - \frac{\left[y_1 - (\psi^* + \lambda_2)\right]^2}{\left[c_2 \cot c_1\right]^2} = 1$$

which has two parameters, so it represents a line congruence with degree two.

By using the above two theorems, one way modify the study of this paper with choosing the plane spanned by $\left\{\overrightarrow{E}_1, \overrightarrow{E}_3\right\}$ or $\left\{\overrightarrow{E}_2, \overrightarrow{E}_3\right\}$.

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