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Extended Results on Complementary Tree Domination Number and Chromatic Number of Graphs

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Abstract: For any graph $G = (V, E)$ a subset $D \subseteq V$ is a dominating set if every vertex in $V - D$ is adjacent to at least one vertex in D. A dominating set is said to be a complementary tree dominating set if the induced subgraph $< V - D >$ is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number and is denoted by $\gamma_{ctd}(G)$. In this paper, we find an upper bound for $\gamma_{ctd}(G)+\chi(G) = 2p-5$ and $\gamma_{ctd}(G) + \chi(G) = 2p - 6$, p is the number of vertices in G.

Key Words: Domination number, complementary tree domination.

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§1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretical terms, we refer Harary [1] and for terms related to domination we refer Haynes et al. [2].

A subset D of V is said to be a dominating set in G if every vertex in $V - D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. The concept of complementary tree domination was introduced by Muthammai, Bhanumathi and Vidhya [3]. A dominating set D is called a complementary tree domination set if the induced subgraph $< V - D >$ is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G, denoted by $\gamma_{ctd}(G)$ and such a set D is called a γ_{ctd} set. The minimum number of colours required to colour all the vertices such that adjacent vertices do not receive the same colour is the chromatic number $\chi(G)$.

In this paper, we obtain sharp upper bound for $\gamma_{ctd}(G)+\chi(G) = 2p-5$ and $\gamma_{ctd}(G)+\chi(G) = 2p-6$. We use the following previous results.

Theorem 1.1([1]) *For any connected graph G, $\chi(G) \leq \Delta(G) + 1$.*

Theorem 1.2([3]) *For any connected graph G with $p \geq 2$, $\gamma_{ctd}(G) \leq p - 1$.*

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Theorem 1.3([3]) *Let G be a connected graph with $p \geq 2$. $\gamma_{ctd}(G) = p - 1$ if and only if G is a star on p vertices.*

Theorem 1.4([3]) *Let G be a connected graph containing a cycle. Then $\gamma_{ctd}(G) = p - 2$ if and only if G is isormorphic to one of the following graphs. C_p, K_p or G is the graph obtained by attaching pendant edges at at least one of the vertices of a complete graph.*

Theorem 1.5([3]) *Let T be a tree with p vertices which is not a star. Then $\gamma_{ctd}(T) = p - 2$ if and only if T is a path or T is obtained by attaching pendant edges at at least one of the end vertices.*

Theorem 1.6([4]) *For any connected graph G, $\gamma_{ctd}(G) + \chi(G) \leq 2p - 1$, $(p \geq 2)$. The equality holds if and only if $G \cong K_2$.*

Theorem 1.7([4]) *For any connected graph G, $\gamma_{ctd}(G) + \chi(G) = 2p - 2$ $(p \geq 3)$ if and only if $G \cong P_3$ or K_p, $p \geq 4$.*

Theorem 1.8([4]) *For any connected graph G, $\gamma_{ctd}(G) + \chi(G) = 2p - 3$ $(p \geq 4)$ if and only if G is a star on four vertices or G is the graph obtained by adding a pendant edge at exactly one vertex of K_{p-1}.*

Theorem 1.9([4]) *For any connected graph G, on p vertices, $\gamma_{ctd}(G) + \chi(G) = 2p - 4$ $(p \geq 5)$ if and only if G is one of the following graphs.*

(1) *G is a star on 5 vertices;*

(2) *G is a cycle on 4 (or) 5 vertices;*

(3) *G is the graph obtained by attaching exactly two pendant edges at one vertex or two vertices of K_{p-2};*

(4) *is the graph obtained by joining a new vertex to j $(2 \leq j \leq p - 2)$ vertices of K_{p-1}.*

§2. Main Results

Notation 2.1 The following notations are used in this paper:

(1) $K_n(p - n)$ is the set of graphs on n vertices obtained from K_n by attaching $(p - n)$, $(p > n)$ pendant edges at the vertices of K_n.

(2) $K_n(P_m)$ is the graph obtained from K_n by attaching a pendant edge of P_m to any one vertex of K_n.

(3) $K'_n(H)$ is the set of graphs obtained from K_n by joining each of the vertices of the graph H to the same i $(1 \leq i \leq n - 1)$ vertices of K_n.

(4) $K''_n(H)$ is the set of graphs obtained from K_n by joining each of the vertices of the graph H to distinct $(n - 1)$ vertices of K_n.

(5) $K'''_n(H)$ is the set of graphs obtained from K_n by joining all the vertices of H, each is adjacent to at least i $(2 \leq i \leq n - 1)$ vertices of K_n.

(6) $F_1(K_n, 2K_1)$ is the set of graphs obtained from K_n by joining one vertex of $2K_1$ to i $(2 \leq i \leq n - 1)$ vertices of K_n and the other vertex to any one vertex of K_n.

(7) $F_{21}(K_n, K_2)$ is the set of graphs obtained from K_n by joining one vertex of K_2 to i $(1 \leq i \leq n - 1)$ vertices of K_n.

(8) $F_{22}(K_n, K_2)$ is the set of graphs obtained from K_n by joining each of the vertices of K_2 to i $(1 \leq i \leq n - 1)$ distinct vertices of K_n.

(9) $F_3(K_n, 3K_1)$ is the set of graphs obtained from K_n by joining one vertex of $3K_1$ to any of the i $(1 \leq i \leq n-1)$ vertices of K_n and each of other two vertices of $3K_1$ to exactly one vertex of K_n.

(10) $F_{41}(K_n, K_2 \cup K_1)$ is the set of graphs obtained from K_n by joining one vertex of K_2 and the vertex of K_1 to distinct $(n-1)$ vertices of K_n.

(11) $F_{42}(K_n, K_2 \cup K_1)$ is the set of graphs obtained from K_n by joining one vertex of K_2 to i $(1 \leq i \leq n-1)$ vertices of K_n and the vertex of K_1 to any one vertex of K_n.

(12) $F_{43}(K_n, K_2 \cup K_1)$ is the set of graphs obtained from K_n by joining each of the vertices of $K_2 \cup K_1$ to vertices of K_n such that each vertex of $K_2 \cup K_1$ is adjacent to exactly one vertex of K_n.

(13) $F_{51}(K_n, P_3)$ is the set of graphs obtained from K_n by joining the central vertex of P_3 to i $(1 \leq i \leq n-1)$ vertices of K_n.

(14) $F_{52}(K_n, P_3)$ is the set of graphs obtained from K_n by joining a pendant vertex and the central vertex of P_3 to the same i $(1 \leq i \leq n-1)$ vertices of K_n.

(15) $F_{53}(K_n, P_3)$ is the set of graphs obtained from K_n by joining a pendant vertex and the central vertex of P_3 to distinct $(n-1)$ vertices of K_n.

In the following $\gamma_{ctd}(G) + \chi(G) = 2p - 5$ and $\gamma_{ctd}(G) + \chi(G) = 2p - 6$ are found.

Theorem 2.1 *Let G be a connected graph with p $(p \geq 6)$ vertices then $\gamma_{ctd}(G) + \chi(G) = 2p - 5$ if and only if G is one of the following graphs:*

(a) *G is a star (or) a cycle on 6 vertices;*
(b) $G \in K_{p-3}(3)$*;*
(c) $G \in K'_{p-2}(K_2)$*;*
(d) $G \in F_1(K_{p-2}, 2K_1)$*;*
(e) $G \in F_{21}(K_{p-2}, K_2)$.

Proof If G is one of the graphs given in the theorem, then $\gamma_{ctd}(G) + \chi(G) = 2p - 5$. Conversely, assume $\gamma_{ctd}(G) + \chi(G) = 2p - 5$. This is possible only if

(*i*) $\gamma_{ctd}(G) = p - 1$ and $\chi(G) = p - 4$;
(*ii*) $\gamma_{ctd}(G) = p - 2$ and $\chi(G) = p - 3$;
(*iii*) $\gamma_{ctd}(G) = p - 3$ and $\chi(G) = p - 2$;
(*iv*) $\gamma_{ctd}(G) = p - 4$ and $\chi(G) = p - 1$;
(*v*) $\gamma_{ctd}(G) = p - 5$ and $\chi(G) = p$.

Case 1. $\gamma_{ctd}(G) = p - 1$ and $\chi(G) = p - 4$.

But, $\gamma_{ctd}(G) = p - 1$ if and only if G is star $K_{1,p-1}$ on p vertices (Theorem 1.3, [3]). For a star G, $\chi(G) = 2$. Therefore, $\chi(G) = p - 4$ implies that $p = 6$ that is, G is a star on 6 vertices.

Case 2. $\gamma_{ctd}(G) = p - 2$ and $\chi(G) = p - 3$.

But, $\gamma_{ctd}(G) = p - 2$ implies that G is one of the following graphs (a) C_p, cycle on p vertices (b) K_p, complete graph on p vertices (c) G is the graph obtained by attaching pendant edges at least one of the vertices of a complete graph (d) G is a path (e) G is obtained from a path of at least three vertices, by attaching pendant edges at at least one of the end vertices of the path.

G cannot be one of the graphs mentioned in (b), (d) and (e), since if $G \cong K_p$, then $\chi(G) = p$ and if G is a path (or) as in (e), then $\chi(G) = 2$ and hence $p = 5$.

If $G \cong C_p$ then $\chi(G) = p - 3$ implies $p = 5$ (or) 6. But, G has at least 6 vertices and hence $G \cong C_6$. Let G be a graph obtained by attaching pendant edge at at least one of the vertices of a complete graph.

But $\chi(G) = p - 3$ implies that, G is the graph on p vertices obtained from K_{p-3} by attaching three pendant edges.
That is, $G \in K_{p-3}(3)$.

Case 3. $\gamma_{ctd}(G) = p - 3$ and $\chi(G) = p - 2$.

$\chi(G) = p-2$ implies that either G contains or does not contain a clique K_{p-2} on $(p-2)$ vertices. Assume G contains a clique K_{p-2} on $(p-2)$ vertices. Let $V(K_{p-2}) = \{u_1, u_2, \cdots, u_{p-2}\}$ and $D = V(G) - V(K_{p-2}) = \{x, y\}$.

Since G is connected, at least one of x and y is adjacent to vertices of K_{p-2}. Also both x and y are adjacent to at most $(p-3)$ vertices of K_{p-2}.

Subcase 3.1. x and y are non adjacent.

If both x and y are adjacent to same u_i $(1 \leq i \leq p-2)$ then $V - D = V(G) - \{\text{any two vertices of } K_{p-2}\}$ forms a minimum ctd-set of G, since the pendant vertices x and y must be in any ctd-set and hence $\gamma_{ctd}(G) = p - 2$.

Similarly, if both x and y are adjacent to same i $(2 \leq i \leq p-3)$ vertices of K_{p-2}, then the set $V(G) - \{x, y, u_i, u_j\}$ where $u_i \in N(x) \cap K_{p-2}$ and $u_j \in (N(x))^c \cap K_{p-2}$ forms a minimum ctd-set and hence $\gamma_{ctd}(G) = p - 4$.

Let x be adjacent to at least i vertices of K_{p-2}, where $2 \leq i \leq p-3$. If y is adjacent to at least two vertices of K_{p-2}, then also $\gamma_{ctd}(G) = p-4$. Therefore, y is adjacent to exactly one vertex of K_{p-2}. That is, G is the graph obtained by joining two non-adjacent vertices to vertices of K_{p-2}, such that one vertex is adjacent to i $(2 \leq i \leq p-3)$ vertices and the other vertex is adjacent to exactly one vertex of K_{p-2}. That is, $G \in F_1(K_{p-2}, 2K_1)$.

Subcase 3.2. x and y are adjacent.

If $N(x) \cap K_{p-2}$ and $N(y) \cap K_{p-2}$ are distinct, then $\gamma_{ctd}(G) = p-4$, since the set $V(G) - \{x, y, u_i, u_j\}$, where $u_i \in N(x) \cap (N(y))^c \cap K_{p-2}$ and $u_j \in (N(x))^c \cap (N(y))^c \cap K_{p-2}$ forms a minimum ctd-set. Therefore, $N(x) \cap K_{p-2}$ and $N(y) \cap K_{p-2}$ are equal. Hence, G is the graph obtained from K_{p-2} by joining the two vertices of K_2 to the same i $(1 \leq i \leq p-3)$ vertices of K_{p-2} (or) G is the graph obtained from K_{p-2} by joining one vertex of K_2 to i $(1 \leq i \leq p-3)$ vertices of K_{p-2}. Therefore, $G \in K'_{p-2}(K_2)$ (or) $G \in F_{21}(K_{p-2}, K_2)$.

If G does not contain a clique on $(p-2)$ vertices then it can be seen that no new graph exists.

Case 4. $\gamma_{ctd}(G) = p - 4$ and $\chi(G) = p - 1$.

$\chi(G) = p-1$ implies that either G contains or does not contain a clique K_{p-1} on $(p-1)$ vertices. Assume G contains a clique K_{p-1} on $(p-1)$ vertices. Let $V(G) - V(K_{p-1}) = \{x\}$. Since G is connected, x is adjacent to at least one of the vertices of K_{p-1}. Also, x is not adjacent to all the vertices of K_{p-2}, since otherwise $G \cong K_p$. Then either $V(G) - \{u_i, u_j\}$ (or) $V(G) - \{x, u_i, u_j\}$, where $u_i \in N(x) \cap K_{p-1}$ and $u_j \in (N(x))^c \cap K_{p-1}$ forms a minimum ctd-set. Hence in this case, no graph exists. If G does not contain a clique K_{p-1} on $(p-1)$ vertices.

Case 5. $\gamma_{ctd}(G) = p - 5$ and $\chi(G) = p$.

$\chi(G) = p$ implies $G \cong K_p$. But, $\gamma_{ctd}(K_p) = p - 2$. Here also, no graph exists. From cases 1 - 5, G can be one of the following graphs:

(a) G is a star (or) a cycle on 6 vertices;

(b) $G \in K_{p-3}(3)$;

(c) $G \in K'_{p-2}(K_2)$;
(d) $G \in F_1(K_{p-2}, 2K_1)$;
(e) $G \in F_{21}(K_{p-2}, K_2)$. □

Remark 2.1 For any connected graph with p $(3 \leq p \leq 5)$ vertices, $\gamma_{ctd}(G) + \chi(G) = 2p - 5$ if and only if G is one of the following graphs.

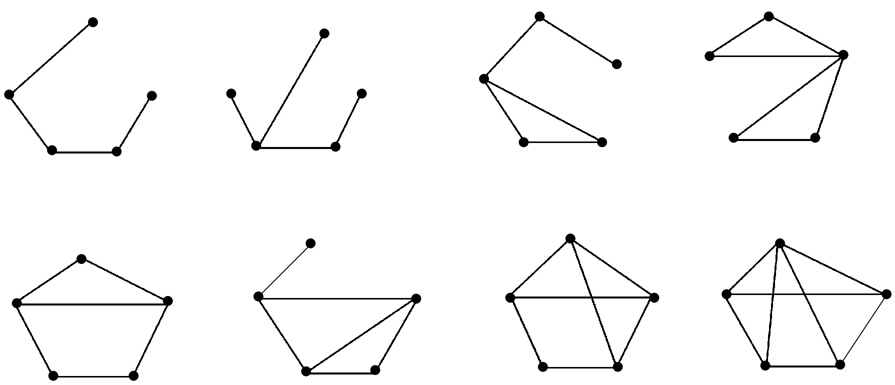

Fig.1

Theorem 2.2 *For any connected graph G with p $(p \geq 7)$ vertices, $\gamma_{ctd}(G) + \chi(G) = 2p - 6$ if and only if G is one of the following graphs:*

(a) *G is a star (or) a cycle on 7 vertices;*
(b) $G \in K_{p-4}(4)$*;*
(c) $G \in F_3(K_{p-3}, 3K_1)$*;*
(d) $G \in K'_{p-3}(K_3)$*;*
(e) $G \in K''_{p-3}(K_3)$*;*
(f) $G \in F_{41}(K_{p-3}, K_2 \cup K_1)$*;*
(g) $G \in F_{42}(K_{p-3}, K_2 \cup K_1)$*;*
(h) $G \in F_{43}(K_{p-3}, K_2 \cup K_1)$*;*
(i) $G \in K_{p-3}(P_4)$*;*
(j) $G \in F_{51}(K_{p-3}, P_3)$*;*
(k) $G \in F_{52}(K_{p-3}, P_3)$*;*
(l) $G \in F_{53}(K_{p-3}, P_3)$*;*
(m) $G \in K''_{p-3}(P_3)$*;*
(n) $G \in K'''_{p-2}(2K_1)$*;*
(o) $G \in F_{22}(K_{p-2}, K_2)$.

Proof If G is one of the graphs given in the theorem, then $\gamma_{ctd}(G) + \chi(G) = 2p - 6$. Conversely, assume $\gamma_{ctd}(G) + \chi(G) = 2p - 6$. This possible, only if

(i) $\gamma_{ctd}(G) = p - 1$ and $\chi(G) = p - 5$;
(ii) $\gamma_{ctd}(G) = p - 2$ and $\chi(G) = p - 4$;
(iii) $\gamma_{ctd}(G) = p - 3$ and $\chi(G) = p - 3$;
(iv) $\gamma_{ctd}(G) = p - 4$ and $\chi(G) = p - 2$;
(v) $\gamma_{ctd}(G) = p - 5$ and $\chi(G) = p - 1$;

(vi) $\gamma_{ctd}(G) = p - 6$ and $\chi(G) = p$.

Case 1. $\gamma_{ctd}(G) = p - 1$ and $\chi(G) = p - 5$

But, $\gamma_{ctd}(G) = p - 1$ if and only if G is a star $K_{1,p-1}$ on p vertices. But, for a star $\chi(G) = 2$. Hence, $p = 7$. That is, G is a star on 7 vertices.

Case 2. $\gamma_{ctd}(G) = p - 2$ and $\chi(G) = p - 4$

But, $\gamma_{ctd}(G) = p - 2$ if and only if

(a) $G \cong C_p$;

(b) $G \cong K_p$;

(c) G is the graph obtained by attaching pendant edges at at least one of the vertices of a complete graph;

(d) G is a path;

(e) G is obtained from of path of at least three vertices by attaching pendant edges at at least one of the end vertices of the path.

As in case 2 of Theorem 2.1.

G is a cycle on 7 vertices (or) G is the graph on p vertices obtained from K_{p-4} by attaching four pendant edges. That is, $G \cong C_7$ (or) $K_{p-4}(4)$.

Case 3. $\gamma_{ctd}(G) = \chi(G) = p - 3$

$\chi(G) = p - 3$ implies that either G contains or does not contains a clique K_{p-3} on $(p - 3)$ vertices. Assume G contains a clique K_{p-3} on $(p - 3)$ vertices. Let $V(K_{p-3}) = \{u_1, u_2, \cdots, u_{p-3}\}$ and $D = V(G) - V(K_{p-3}) = \{x, y, z\}$. Each of x, y, z is not adjacent to all the vertices of K_{p-3}. $< D >= \overline{K_3}, K_3, P_3$ (or) $K_2 \cup K_1$.

Subcase 3.1. $< D > \cong \overline{K_3}$.

Since G is connected, every vertex of D is adjacent to at least one vertex of K_{p-3}. Let x be adjacent to i $(1 \leq i \leq p - 4)$ vertices of K_{p-3}.

If y (or) z is adjacent to at least two vertices of K_{p-3}, then $\gamma_{ctd}(G) \leq p - 4$. Therefore, both y and z are adjacent to exactly one vertex of K_{p-3}. That is, G is the graph obtained from K_{p-3} by joining vertices of $3K_1$ to the vertices of K_{p-3} such that one is adjacent to any of the i $(1 \leq i \leq p-4)$ vertices of K_{p-3} and each of the remaining two is adjacent to exactly one vertex of K_{p-3} and hence $G \in F_3(K_{p-3}, 3K_1)$.

Subcase 3.2. $< D > \cong K_3$.

Since G is connected, at least one vertex of K_3 is adjacent to vertices of K_{p-3}. If there exist vertices $u_i, u_j \in K_{p-3}$ such that $u_i \in N(x) \cap (N(y))^c \cap K_{p-3}$ and $u_j \in (N(x))^c \cap (N(y))^c \cap K_{p-3}$, then the set $V(G) - \{x, y, u_i, u_j\}$ is a γ_{ctd}-set of G and hence $\gamma_{ctd}(G) = p - 4$.

Similarly in the case, when $u_i \in N(y) \cap (N(z))^c$ and $u_j \in (N(x))^c \cap (N(y))^c$ (or) $u_i \in N(z) \cap (N(x))^c$ and $u_j \in (N(z))^c \cap (N(x))^c$ in K_{p-3}.

Therefore, either (i) $N(x) \cap K_{p-3} = N(y) \cap K_{p-3} = N(z) \cap K_{p-3}$ (or) (ii) $N(x) \cap K_{p-3}$, $N(y) \cap K_{p-3}$, $N(z) \cap K_{p-3}$ are mutually distinct and each has $(p-4)$ vertices. That is, G is the graph obtained from K_{p-3} by joining each of the vertices of K_3 either to the same i $(1 \leq i \leq p-4)$ vertices of K_{p-3} (or) to distinct $(p-4)$ vertices of K_{p-3}. Therefore, $G \in K'_{p-3}(K_3)$ (or) $G \in K''_{p-3}(K_3)$.

Subcase 3.3. $< D > \cong K_2 \cup K_1$.

Let $x, y \in V(K_2)$ and $z \in V(K_1)$ since G is connected, at least one of the vertices of K_2 and z is adjacent to vertices of K_{p-3}. Denote $G \cap K_{p-3}$ by G_1.

(i) Let one of x and y, say x be adjacent to vertices of K_{p-3}. That is, $deg_{G_1}(y) = 1$.

Let x be adjacent to at least two vertices of K_{p-3}. That is, $deg_{G_1}(x) \geq 2$. Assume $deg_{G_1}(z) \geq 2$. If there exist $u_i, u_j \in K_{p-3}$ such that $u_i \in N(x) \cap N(z)$ and $u_j \in (N(x))^c \cap (N(z))^c$ or if $N(x) \cap K_{p-3} = N(z) \cap K_{p-3}$ and if each set has $(p-4)$ vertices, then $\gamma_{ctd}(G) = p-4$. Therefore, we have the following cases:

(a) $N(x) \cap K_{p-3}$ and $N(z) \cap K_{p-3}$ are distinct, and each set has $(p-4)$ vertices or

(b) $deg_{G_1}(z) = 1$. That is, G is the graph obtained from K_{p-3} by joining exactly one of the vertices of K_2 and a new vertex to distinct $(p-4)$ vertices of K_{p-3} or G is the graph obtained from K_{p-3} by attaching a pendant edge and joining exactly one vertex of K_2 to i $(1 \leq i \leq p-4)$ vertices of K_{p-3}. That is, $G \in F_{41}(K_{p-3}, K_2 \cup K_1)$ or $G \in F_{42}(K_{p-3}, K_2 \cup K_1)$.

(ii) If each of x, y, z is adjacent to at least two vertices of K_{p-3}, then either $V(G) - \{x, y, z, u_i, u_j\}$, where $u_i \in N(x) \cap (N(y))^c \cap (N(z))^c \cap K_{p-3}$ and $u_j \in N(z) \cap (N(x))^c \cap (N(y))^c \cap K_{p-3}$ (or) $V(G) - \{x, y, u_i, u_j\}$, where $u_i \in N(x) \cap (N(y))^c \cap K_{p-3}$ and $u_j \in (N(x))^c \cap (N(y))^c \cap K_{p-3}$ is a γ_{ctd}-set of G.

Similarly, if either $N(x) \cap K_{p-3} = N(y) \cap K_{p-3} = N(z) \cap K_{p-3}$ and $2 \leq |N(x) \cap K_{p-3}| \leq p-4$. (or) $N(x) \cap K_{p-3}$, $N(y) \cap K_{p-3}$, and $N(z) \cap K_{p-3}$ are distinct and each set has the same number i $(2 \leq i \leq p-4)$ of elements, then also $\gamma_{ctd}(G) = p-4$.

Hence, each of x, y and z is adjacent to exactly one vertex of K_{p-3}. That is, G is the graph obtained from K_{p-3} by attaching a pendant edge and joining two vertices of K_2 to vertices of K_{p-3} such that each is adjacent to exactly one vertex of K_{p-3}. Hence, $G \in F_{43}(K_{p-2}, K_2 \cup K_1)$.

Subcase 3.4. $< D > \cong P_3$.

Since G is connected, at least one of the vertices of P_3 is adjacent to vertices of K_{p-3}. Let x and z be the pendant vertices and y be the central vertex of P_3.

(i) Assume exactly one of x, y, z is adjacent to vertices of K_{p-3}. If $deg_{G_1}(x) \geq 2$, then $\gamma_{ctd}(G) = p-4$. Hence, $deg_{G_1}(x) = 1$. That is, G is the graph obtained from K_{p-3} by attaching a path of length 3 at a vertex of K_{p-3} (or) that is, $G \in K_{p-3}(P_4)$ (or) G is the graph obtained from K_{p-3} by joining the central vertex of P_3 to i $(1 \leq i \leq p-4)$ vertices of K_{p-3}, that is, $G \in F_{51}(K_{p-3}, P_3)$.

(ii) Assume any two of x, y, z are adjacent to vertices of K_{p-3}.

(a) If x and z are adjacent to vertices of K_{p-3}, then $\gamma_{ctd}(G) = p-4$.

(b) Let x and y be adjacent to vertices of K_{p-3}. If there exist vertices $u_i, u_j \in K_{p-3}$ such that $u_i \in N(x) \cap (N(y))^c$ and $u_j \in (N(x))^c \cap (N(y))^c$, then also $\gamma_{ctd}(G) = p-4$. Therefore, either

(a) $N(x) \cap K_{p-3} = N(y) \cap K_{p-3}$ or

(b) $N(x) \cap K_{p-3}$ and $N(y) \cap K_{p-3}$ are distinct and each set has $(p-4)$ vertices. That is, G is the graph obtained from K_{p-3} by joining one pendant vertex and the central vertex of P_3 to the same i $(1 \leq i \leq p-4)$ vertices of K_{p-3} (or) G is the graph obtained from K_{p-3} by joining one pendant vertex and the central vertex of P_3 to the distinct $(p-4)$ vertices of K_{p-3}. i.e., $G \in F_{52}(K_{p-3}, P_3)$ or $G \in F_{53}(K_{p-3}, P_3)$.

(iii) Assume x, y and z are adjacent to vertices of K_{p-3}. As in Subcase 3.3, if $N(x) \in K_{p-3} = N(y) \cap K_{p-3} = N(z) \cap K_{p-3}$ and $1 \leq |N(x) \cap K_{p-3}| \leq (p-4)$ or $N(x) \cap K_{p-3}$, $N(y) \cap K_{p-3}$ and $N(z) \cap K_{p-3}$ are distinct and each of these sets are distinct and has $(p-4)$ vertices. Hence, G is the graph obtained from K_{p-3} by joining each of the vertices of P_3 to distinct $(p-4)$ vertices of K_{p-3}.

That is, $G \in K''_{p-3}(P_3)$.

If G does not contain a clique K_{p-3} on $(p-3)$ vertices, then it can be verified that no new graph exists.

Case 4. $\gamma_{ctd}(G) = p - 4$ and $\chi(G) = p - 2$.

$\chi(G) = p-2$ implies that G either contains or does not contains a clique K_{p-2} on $(p-2)$ vertices. Assume G contains a clique K_{p-2} on $p-2$ vertices. Let $V(G) - V(K_{p-2}) = \{x, y\}$. If x and y are non-adjacent then as in Subcase 3.1 of Theorem 2.1, G is the graph obtained from K_{p-2} by joining two non-adjacent vertices to vertices of K_{p-2} such that each is adjacent to at least i $(2 \leq i \leq p-3)$ vertices of K_{p-2}. That is, $G \in K'''_{p-2}(2K_1)$.

If x and y are adjacent, then as in subcase 3.2 of Theorem 2.1, G is the graph obtained from K_{p-2} by joining each of the vertices of K_2 to i $(1 \leq i \leq p-3)$ distinct vertices of K_{p-2}. That is, $G \in F_{22}(K_{p-2}, K_2)$. If G does not contains a clique on $p-2$ vertices, then no new graph exists. For the cases $\gamma_{ctd}(G) = p-5$ and $\chi(G) = p-1$ and $\gamma_{ctd}(G) = p-6$ and $\chi(G) = p$, no graph exists.

From cases 1 - 4, we can conclude that G can be one of the graphs given in the theorem. □

Remark 2.2 For any connected graph with p $(4 \leq p \leq 6)$ vertices, $\gamma_{ctd}(G) + \chi(G) = 2p - 6$ if and only if G is one of the following graphs.

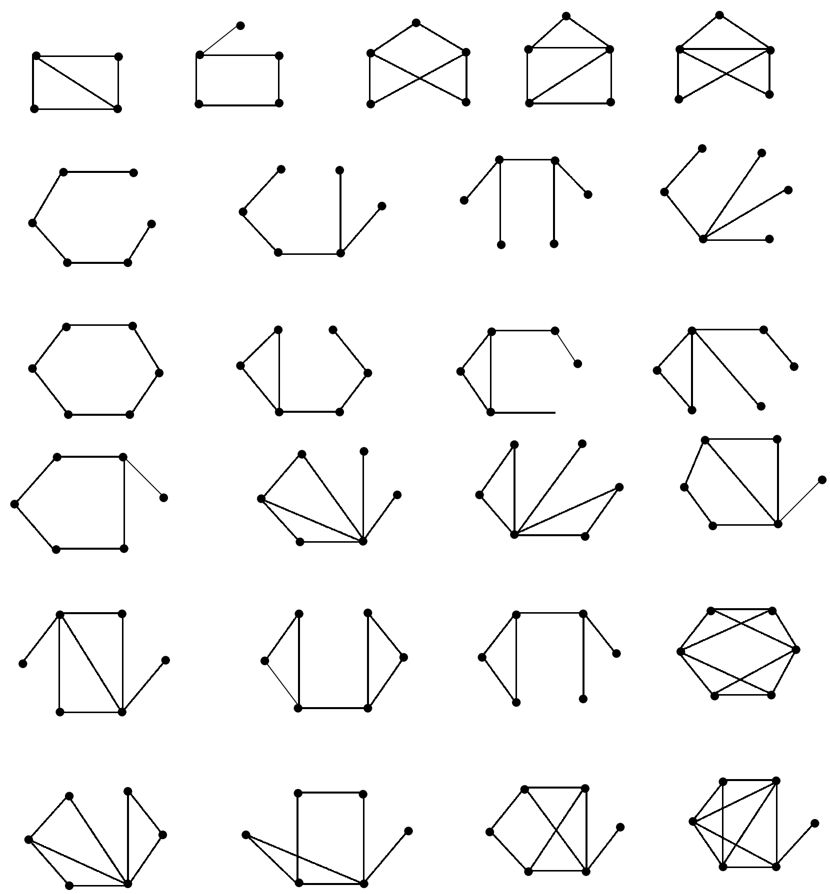

Fig.2

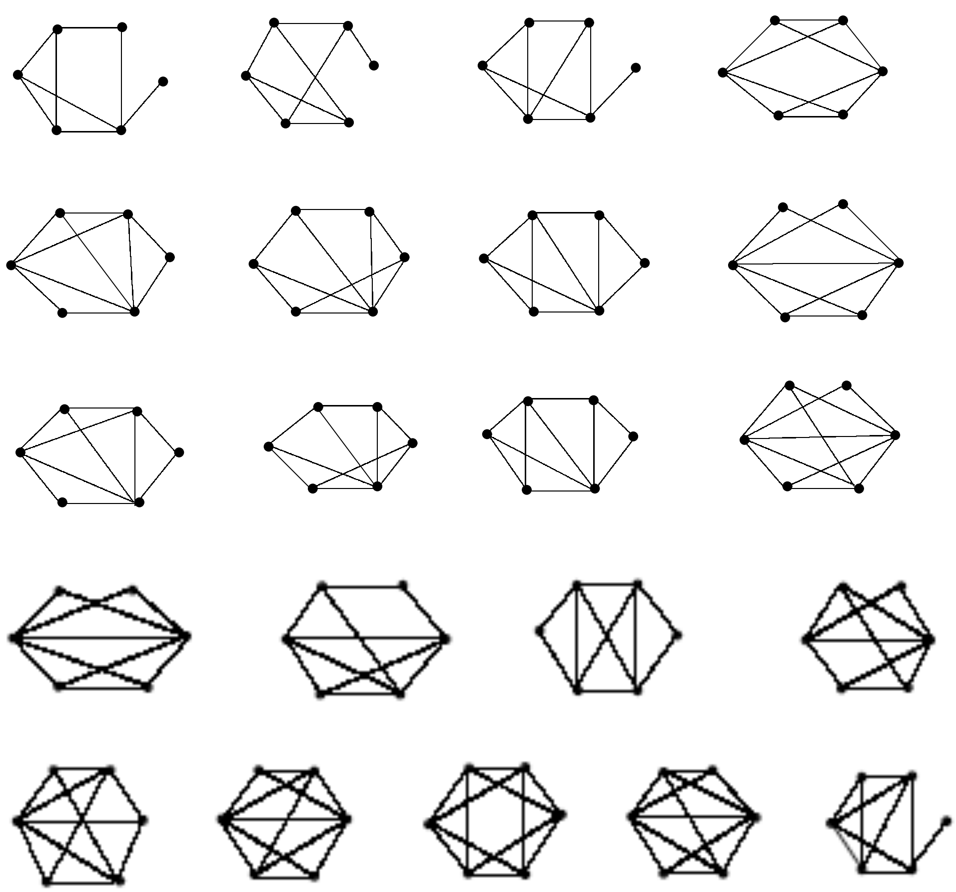

Fig.3

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