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The Minimum Equitable Domination Energy of a Graph

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Abstract: A subset D of V is called an equitable dominating set [8] if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$, where $deg(u)$ denotes the degree of vertex u and $deg(v)$ denotes the degree of vertex v. Recently, The minimum covering energy $E_c(G)$ of a graph is introduced by Prof. C. Adiga, and co-authors [1]. Motivated by [1], in this paper we define energy of minimum equitable domination $E_{ED}(G)$ of some graphs and we obtain bounds on $E_{ED}(G)$. We also obtain the minimum equitable domination determinant of some graph G given by $det_{ED}(G) = \mu_1\mu_2 \ldots \mu_n$ where $\mu_1,\ \mu_2, \ldots, \mu_n$ are eigenvalues of $A_{ED}(G)$.

Key Words: Minimum equitable domination set, spectrum of minimum equitable domination matrix, energy of minimum equitable domination, determinant of minimum equitable domination matrix.

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§1. Introduction

The energy of a graph and its applications to Organic Chemistry are given in detail in two important works by I. Gutman and co-authors [5, 9]. For more details with applications on the energy of a graph, one may refer [2, 4, 6, 9]. Recently, the minimum covering energy $E_c(G)$ of a graph is introduced by Prof. C. Adiga, and co-authors [1]. Motivated by [1], in this paper we define energy of minimum equitable domination $E_{ED}(G)$ of some graphs and we obtain bounds on $E_{ED}(G)$. We also obtain the minimum equitable domination determinant of some graph G given by $det_{ED}(G) = \mu_1\mu_2 \ldots \mu_n$ where $\mu_1,\ \mu_2, \ldots, \mu_n$ are eigenvalues of $A_{ED}(G)$.

Let G be a graph with set of vertices , $V = \{v_1, v_2, \cdots, v_n\}$ and set of edges, E. For a simple graph, i.e a graph without loops, multiple or directed edges, a subset D of V is called an equitable dominating set [8] if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$, where $deg(u)$ denotes the degree of vertex u and $deg(v)$ denotes the degree of vertex v. Let ED is minimum equitable domination set of a graph G.

The minimum equitable domination matrix is defined as a square matrix $A_{ED}(G) = (a_{ij})$,

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where

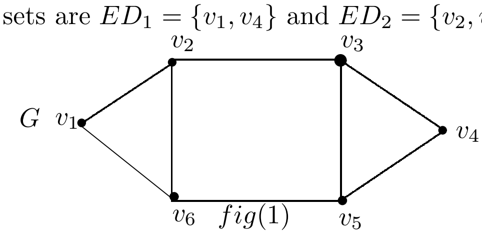
$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 1 & \text{if } i = j \text{ and } v_i \in ED \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The eigenvalues of the minimum equitable domination matrix $A_{ED}(G)$ are $\mu_1, \mu_2, \cdots, \mu_n$. Since the minimum equitable domination matrix is symmetric, its eigenvalues are real and can be written as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The energy of minimum equitable domination of a graph is defined as

$$E_{ED}(G) = \sum_{i=1}^{n} |\mu_i|. \tag{2}$$

We also obtain the minimum equitable domination determinant of some graph G given by $det_{ED}(G) = \mu_1 \mu_2 \cdots \mu_n$ where $\mu_1,\ \mu_2, \ldots, \mu_n$ are eigenvalues of $A_{ED}(G)$.

Example 1.1 The figure 1 shows the graph G with vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Then minimum equitable domination sets are $ED_1 = \{v_1, v_4\}$ and $ED_2 = \{v_2, v_5\}$,



$fig(1)$

$$A_{ED_1}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

characteristic polynomial of $A_{ED_1}(G)$ is $\Phi_6(G, \mu) = \mu^6 - 2\mu^5 - 7\mu^4 + 8\mu^3 + 12\mu^2$, the spectrum of $A_{ED_1}(G)$ is

$$Spec_{ED_1} = \begin{pmatrix} 3 & 2 & 0 & -1 & -2 \\ 1 & 1 & 2 & 1 & 1 \end{pmatrix}$$

and the energy of minimum equitable domination of ED_1 is $E_{ED_1} = 8$ and $det_{ED_1}(G) = 0$..

$$A_{ED_2}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

characteristic polynomial of $A_{ED_2}(G)$ is $\Phi_6(G,\mu) = \mu^6 - 2\mu^5 - 7\mu^4 + 6\mu^3 + 14\mu^2 - 3$, the spectrum of $A_{ED_2}(G)$ is

$$Spec_{ED_2} = \begin{pmatrix} 3.1819 & 1.8019 & 0.4450 & -0.5936 & -1.2470 & -1.5884 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and the energy of minimum equitable domination of ED_2 is $E_{ED_2} = 8.8578$ and $det_{ED_2}(G) = -3$. One can note that $det_{ED_2}(G) \neq 0$ and $E_{ED_2} > E_{ED_1}$. Also the energy of minimum equitable domination depends upon the minimum equitable domination set.

§2. Bounds for the Minimum Equitable Domination Energy of a Graph

We first need the following Lemma.

Lemma 2.1 *Let G be a graph with vertices $\{v_1, v_2, \ldots, v_n\}$ and let $A_{ED}(G)$ be the minimum equitable domination matrix of G. Let $\Phi_n(A_{ED}(G)) = det(\mu I_n - A_{ED}(G)) = c_0\mu^n + c_1\mu^{n-1} + c_2\mu^{n-2} + \cdots + c_n$ be the characteristic polynomial of $A_{ED}(G)$. Then*

(1) $c_0 = 1$;
(2) $c_1 = -|ED|$;
(3) $c_2 = \binom{|ED|}{2} - m$.

Proof (1) $c_0 = 1$ follows directly from the definition $\Phi_n(G,\mu) = det(\mu I_n - A_{ED}(G))$, i.e $c_0 = 1$.

(2) c_1 = sum of determinants of all 1×1 principal submatrices of $A_{ED}(G)$,

$$\text{i.e } c_1 = (-1)^1 \text{ trace of } A_{ED}(G) = -|ED|.$$

(3) c_2 = sum of determinants of all 2×2 principal submatrices of $A_{ED}(G)$,

$$\begin{aligned} i.e\ c_2 &= (-1)^2 \sum_{i<j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = \sum_{i<j}(a_{ii}a_{jj} - a_{ij}a_{ji}) = \sum_{i<j} a_{ii}a_{jj} - \sum_{i<j} a_{ij}^2 \\ c_2 &= \binom{|ED|}{2} - m. \qquad \square \end{aligned}$$

Lemma 2.2 *Let G be a connected graph and let $\mu_1, \mu_2, \cdots, \mu_n$ be the eigenvalues of minimum equitable dominating matrix $A_{ED}(G)$. Then*

$$\sum_{i=1}^{n} \mu_i = |ED|$$

and

$$\sum_{i=1}^{n} \mu_i^2 = 2m + |ED|.$$

Proof The sum of diagonal elements of $A_{ED}(G)$ is $\sum_{i=1}^{n} \mu_i = \text{trace}[A_{ED}(G)] = \sum_{i=1}^{n} a_{ii}$ $= |ED|$.

Similarly, the sum of squares of the eigenvalues of $A_{ED}(G)$ is trace of $[A_{ED}(G)]^2$,

$$\begin{aligned} \sum_{i=1}^{n} \mu_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\ a_{ji} \\ &= \sum_{i=1}^{n}(a_{ii})^2 + \sum_{i\neq j} a_{ij}\ a_{ji} \\ &= \sum_{i=1}^{n}(a_{ii})^2 + 2\sum_{i<j}(a_{ij})^2 \end{aligned}$$

$$\sum_{i=1}^{n} \mu_i^2 = 2m + |ED|.$$ □

Theorem 2.3 *Let G_1 and G_2 be two graphs with n vertices and m_1, m_2 are number of edges in G_1 and G_2 respectively. Let $\mu_1, \mu_2, \cdots, \mu_n$ are eigenvalues of $A_{ED_1}(G_1)$ and $\mu_1', \mu_2', \cdots, \mu_n'$ are eigenvalues of $A_{ED_2}(G_2)$. Then*

$$\sum_{i=1}^{n} \mu_i\ \mu_i' \leq \sqrt{(2m_1 + |ED_1|)(2m_2 + |ED_2|)},$$

where $A_{ED_i}(G_i)$ is minimum equitable domination matrix of G_i $(i = 1, 2)$ and ED_1, ED_2 be minimum equitable domination sets of G_1 and G_2 respectively.

Proof Let $\mu_1, \mu_2, \cdots, \mu_n$ are eigenvalues of $A_{ED_1}(G_1)$ and $\mu_1', \mu_2', \cdots, \mu_n'$ are eigenvalues of $A_{ED_2}(G_2)$. Then by the Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

If $a_i = \mu_i$, $b_i = \mu_i'$ then

$$\begin{aligned} \left(\sum_{i=1}^{n} \mu_i\mu_i'\right)^2 &\leq \left(\sum_{i=1}^{n} \mu_i^2\right)\left(\sum_{i=1}^{n} (\mu_i')^2\right) \\ \left(\sum_{i=1}^{n} \mu_i\mu_i'\right)^2 &\leq (2m_1 + |ED_1|)\ (2m_2 + |ED_2|) \\ &\Rightarrow \sum_{i=1}^{n} \mu_i\mu_i' \leq \sqrt{(2m_1 + |ED_1|)\ (2m_2 + |ED_2|)}. \end{aligned}$$

Hence the theorem. □

Theorem 2.4 *Let G be a graph with n vertices, m edges. Let ED is the minimum equitable* *domination set. Then*

$$\sqrt{(2m+|ED|)+n(n-1)\ |detA_{ED}(G)|^{2/n}}\ \leq\ E_{ED}(G)\ \leq\ \sqrt{n(2m+|ED|)}.$$

Proof This proof follows the ideas of McClelland's bounds [6] for graphs $E(G)$. For the upper bound, let $\mu_1, \mu_2, \cdots, \mu_n$ be the eigenvalues of minimum equitable dominating matrix $A_{ED}(G)$. Apply the Cauchy-Schwartz inequality to $(1, 1, \cdots, 1)$ and $(\mu_1, \mu_2, \cdots, \mu_n)$ is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

If $a_i = 1$, $b_i = |\mu_i|$ then

$$\begin{aligned}\left(\sum_{i=1}^{n} |\mu_i|\right)^2 &\leq \left(\sum_{i=1}^{n} 1^2\right)\left(\sum_{i=1}^{n} |\mu_i|^2\right)\\ &\Rightarrow [E_{ED}(G)]^2 \leq n(2m+|ED|)\end{aligned}$$

from the above $\left(\sum_{i=1}^{n} \mu_i^2 = 2m+|ED|\right)$,

$$E_{ED}(G) \leq \sqrt{n(2m+|ED|)},$$

which is upper bound.

For the lower bound, by using arithmetic mean and geometric mean inequality, we have

$$\begin{aligned}\frac{1}{n(n-1)}\sum_{i\neq j} |\mu_i|\ |\mu_j| &\geq \left(\prod_{i\neq j} |\mu_i|\ |\mu_j|\right)^{\frac{1}{n(n-1)}}\\ \sum_{i\neq j} |\mu_i|\ |\mu_j| &\geq n(n-1)\left(\prod_{i=1}^{n} |\mu_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}\end{aligned}$$

$$\sum_{i\neq j} |\mu_i|\ |\mu_j| \geq n(n-1)\left(\prod_{i=1}^{n} |\mu_i|\right)^{2/n}. \tag{3}$$

Consider

$$E_{ED}(G)]^2 = \left[\sum_{i=1}^{n} |\mu_i|\right]^2 = \sum_{i=1}^{n} |\mu_i|^2 + \sum_{i\neq j} |\mu_i|\ |\mu_j|.$$

From (3) we have

$$[E_{ED}(G)]^2 \geq \sum_{i=1}^{n} |\mu_i|^2 + n(n-1)\left(\prod_{i=1}^{n} |\mu_i|\right)^{2/n},$$

$$[E_{ED}(G)]^2 \geq\ (2m+|ED|)+n(n-1)\ |detA_{ED}(G)|^{2/n}$$

$$\Rightarrow [E_{ED}(G)] \geq \sqrt{(2m + |ED|) + n(n-1)\ |detA_{ED}(G)|^{2/n}},$$

which is lower bound. □

Theorem 2.5 *If the minimum equitable domination energy $E_{ED}(G)$ is a rational number, then $E_{ED}(G) \equiv |ED|(mod2)$, where ED is minimum equitable domination set.*

Proof Let $\mu_1, \mu_2, \cdots, \mu_n$ be the eigenvalues of minimum equitable domination matrix $A_{ED}(G)$. Then the trace of

$$A_{ED}(G) = \sum_{i=1}^{n} a_{ii} = |ED|.$$

Let $\mu_1, \mu_2, \cdots, \mu_r$ be positive and remaining eigenvalues are non-positive then,

$$\begin{aligned} E_{ED}(G) &= \sum_{i=1}^{n} |\mu_i| = (\mu_1 + \mu_2 + \cdots + \mu_r) - (\mu_{r+1} + \mu_{r+2} + \cdots + \mu_n) \\ &= 2(\mu_1 + \mu_2 + \cdots + \mu_r) - (\mu_1 + \mu_2 + \cdots + \mu_n) \\ &= 2(\mu_1 + \mu_2 + \ldots + \mu_r) - |ED| \\ \Rightarrow E_{ED}(G) &\equiv |ED|(mod2). \end{aligned}$$

Hence the theorem. □

§3. Minimum Equitable Domination Energy and Determinant of Certain Standard Graphs

Theorem 3.1 *For $n \geq 4$, the minimum equitable domination energy of star graph $S_{1,n-1}$ is $(n-2) + 2\sqrt{n-1}$.*

Proof Consider the star graph $S_{1,n-1}$ with vertex set $V = \{v_0, v_1, \cdots, v_{n-1}\}$. The minimum equitable domination set is $ED = \{v_0, v_1, \cdots, v_{n-1}\}$. Then minimum equitable domination matrix is

$$A_{ED}(S_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \ldots & 1 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}_{n \times n}.$$

The characteristic polynomial of $A_{ED}(S_{1,n-1})$ is

$$\Phi_n(S_{1,n-1},\mu)=\begin{vmatrix} \mu-1 & -1 & -1 & \dots & -1 & -1 \\ -1 & \mu-1 & 0 & \dots & 0 & 0 \\ -1 & 0 & \mu-1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \dots & \mu-1 & 0 \\ -1 & 0 & 0 & \dots & 0 & \mu-1 \end{vmatrix}_{n\times n}$$

$$=-(-1)^{n+1}\begin{vmatrix} -1 & -1 & \dots & -1 & -1 \\ \mu-1 & 0 & \dots & 0 & 0 \\ 0 & \mu-1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots\mu-1 & 0 & 0 \\ 0 & 0 & \dots & \mu-1 & 0 \end{vmatrix}+(\mu-1)\begin{vmatrix} \mu-1 & -1 & \dots & -1 & -1 \\ -1 & \mu-1 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & \dots & \mu-1 & 0 \\ -1 & 0 & \dots & 0 & \mu-1 \end{vmatrix}$$

$$\Phi_n(S_{1,n-1},\mu)=-(\mu-1)^{n-2}+(\mu-1)\Phi_{n-1}(S_{1,n-2},\mu). \tag{4}$$

Now change n to $n-1$ in (1), we get

$$\Phi_{n-1}(S_{1,n-2},\mu)=-(\mu-1)^{n-3}+(\mu-1)\Phi_{n-2}(S_{1,n-3},\mu). \tag{5}$$

Substitute (5) in (4),

$$\Phi_n(S_{1,n-1},\mu)=-2(\mu-1)^{n-2}+(\mu-1)^2\Phi_{n-2}(S_{1,n-3},\mu) \tag{6}$$

Continuing this process, we get

$$\begin{aligned} \Phi_n(S_{1,n-1},\mu) &= -(n-4)(\mu-1)^{n-2}+(\mu-1)^{n-4}\Phi_4(S_{1,3},\mu) \\ &= -(n-4)(\mu-1)^{n-2}+(\mu-1)^{n-4}[(\mu-1)^2(\mu^2-2\mu-2)] \\ \Phi_n(S_{1,n-1},\mu) &= (\mu-1)^{n-2}[\mu^2-2\mu-(n-2)]. \end{aligned}$$

The spectrum of minimum domination energy of a graph is

$$Spec_{ED}(S_{1,\ n-1})=\begin{pmatrix} 1+\sqrt{n-1} & 1 & 1-\sqrt{n-1} \\ 1 & n-2 & 1 \end{pmatrix}.$$

The energy of minimum equitable domination of a graph is

$$E_{ED}(S_{1,\ n-1})=(n-2)+2\sqrt{n-1},\ n\geq 4.$$ □

Theorem 3.2 *Let $K_{s,t}$ be complete bipartite graph with $s+t$ vertices and energy of minimum equitable complete bipartite graph $E_{ED}(K_{s,t})$ is*

$$E_{ED}(K_{s,t})=(s+t-2)+2\sqrt{st}$$

if $|s-t|\geq 2,\ \ s,t\geq 2.$

Proof Let complete bipartite graph $K_{s,t}$ with $|s-t| \geq 2$ where $s, t \geq 2$ with vertex set $V = \{v_1, v_2, \cdots, v_s, u_1, u_2, \cdots, u_t\}$. The minimum equitable domination set is $ED = \{v_1, v_2, \cdots, v_s, u_1, u_2, \cdots, u_t\}$. Then

$$A_{ED}(K_{s,t}) = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 1 & \ldots & 0 & 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & 1\ldots & 0 & 0 & 1 & 1 & \ldots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & \ldots & 0 & 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & \ldots & 1 & 1 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 1 & \ldots & 1 & 1 & 0 & 1 & \ldots & 0 & 0 \\ 1 & 1 & \ldots & 1 & 1 & 0 & 0 & 1\ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 1 & 1 & 0 & 0 & \ldots & 1 & 0 \\ 1 & 1 & \ldots & 1 & 1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}_{(s+t)\times(s+t)}.$$

The characteristic polynomial of $A_{ED}(K_{s,t})$ is

$$\Phi_{s+t}(K_{s,t},\mu) = \begin{vmatrix} \mu-1 & 0 & \ldots & 0 & 0 & -1 & -1 & \ldots & -1 & -1 \\ 0 & \mu-1 & \ldots & 0 & 0 & -1 & -1 & \ldots & -1 & -1 \\ 0 & 0 & \ldots & 0 & 0 & -1 & -1 & \ldots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \mu-1 & 0 & -1 & -1 & \ldots & -1 & -1 \\ 0 & 0 & \ldots & 0 & \mu-1 & -1 & -1 & \ldots & -1 & -1 \\ -1 & -1 & \ldots & -1 & -1 & \mu-1 & 0 & \ldots & 0 & 0 \\ -1 & -1 & \ldots & -1 & -1 & 0 & \mu-1 & \ldots & 0 & 0 \\ -1 & -1 & \ldots & -1 & -1 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \ldots & -1 & -1 & 0 & 0 & \ldots & \mu-1 & 0 \\ -1 & -1 & \ldots & -1 & -1 & 0 & 0 & \ldots & 0 & \mu-1 \end{vmatrix}$$

$$= \begin{vmatrix} (\mu-1)I_s & -J_{s\times t}^T \\ -J_{t\times s} & (\mu-1)I_t \end{vmatrix},$$

where $J_{t\times s}$ is a matrix with all entries equal to one,

$$\begin{aligned}\Phi_{s+t}(K_{s,t},\mu) &= |(\mu-1)I_s|\ |(\mu-1)I_t-(-J)\frac{I_s}{\mu-1}(-J^T)| \\ &= (\mu-1)^{s-t}\ |(\mu-1)^2I_t-JJ^T| \\ &= (\mu-1)^{s-t}\ P_{JJ^T}[(\mu-1)^2] \\ &= (\mu-1)^{s-t}\ P_{sJ_t}[(\mu-1)^2],\end{aligned}$$

where P_{sJ_t} is the characteristic polynomial of the matrix sJ_t

$$\begin{aligned}\Phi_{s+t}(K_{s,t},\mu) &= (\mu-1)^{s-t}[(\mu-1)^2-st]\ [(\mu-1)^2]^{t-1} \\ &= (\mu-1)^{s-t}(\mu-1)^{2t-2}\ [\mu^2+1-2\mu-st] \\ \Phi_{s+t}(K_{s,t},\mu) &= (\mu-1)^{s+t-2}\ [\mu^2-2\mu-(st-1)]\end{aligned}$$

is the characteristic polynomial of minimum equitable domination matrix of $K_{s,t}$. The spectrum of minimum equitable domination matrix of $K_{s,t}$ is

$$Spec_{ED}(K_{s,\ t}) = \begin{pmatrix} 1+\sqrt{st} & 1 & 1-\sqrt{st} \\ 1 & s+t-2 & 1 \end{pmatrix}.$$

The minimum equitable domination energy of a graph is

$$E_{ED}(K_{s,\ t}) = (s+t-2)+2\sqrt{st}. \qquad \square$$

A *crown graph* S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{v_1, v_2, \ldots, v_n, u_1, u_2, \cdots, u_n\}$ and edge set $\{v_iu_i : 1 \leq i, j \leq n,\ i \neq j\}$. Therefore S_n^0 coincides with the complete bipartite graph $K_{n,n}$ with horizontal edges removed [1].

Theorem 3.3 *For $n \geq 3$, the minimum equitable domination energy of the crown graph S_n^0 is equal to $2(n-2)+\sqrt{n^2-2n+5}+\sqrt{n^2+2n-3}$.*

Proof Consider crown graph S_n^0 with vertex set $V = \{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n\}$. The minimum equitable domination set is $ED = \{v_1, u_1\}$.

Then the minimum equitable domination matrix of S_n^0 is same as the minimum domination matrix of S_n^0 by [7]. Therefore

$$E_{ED}(S_n^0) = 2(n-2)+\sqrt{n^2-2n+5}+\sqrt{n^2+2n-3}. \qquad \square$$

Theorem 3.4 *For $n \geq 2$, the minimum equitable domination energy of complete graph K_n is $(n-2)+\sqrt{n^2-2n+5}$*

Proof Consider the complete graph K_n with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. The minimum equitable domination set is $ED = \{v_1\}$. Then the minimum equitable domination matrix of K_n is same as the minimum domination matrix of K_n by [7]. Therefore,

$$E_{ED}(K_n) = (n-2) + \sqrt{n^2 - 2n + 5}. \qquad \square$$

Let us obtain the minimum equitable domination determinant of some graph G given by $det_{ED}(G) = \mu_1\mu_2\cdots\mu_n$, where $\mu_1,\ \mu_2, \cdots, \mu_n$ are eigenvalues of $A_{ED}(G)$.

Proposition 3.5 *Let $S_{1,n-1}$ $(n \geq 4)$, K_n $(n \geq 2)$ be the star and complete graphs with n vertices, respectively, S_n^0 $(n \geq 3)$, is crown graph with $2n$ vertices and $K_{s,t}$ $(|s-t| \geq 2)$ be the complete bipartite graph with $s+t$ vertices. Then*

(1) $det_{ED}(S_{1,n-1}) = -(n-2)$;
(2) $det_{ED}(K_n) = (-1)^{(n-1)}$;
(3) $det_{ED}(K_{s,t}) = (1-st)$;
(4) $det_{ED}(S_n^0) = (-1)^{n-1}(3-2n)$.

Proof w.k.t $det_{ED}(G) = \mu_1\ \mu_2\cdots\mu_n$, where $\mu_1, \mu_2, \cdots, \mu_n$ are eigenvalues of G.

Case 1. $det_{ED}(S_{1,n-1}) = (1+\sqrt{n-1})^1\ 1^{n-2}\ (1-\sqrt{n-1})^1 = -(n-2)$, where $n \geq 4$.

Case 2.

$$\begin{aligned} det_{ED}(K_n) &= \left(\frac{(n-1)+\sqrt{n^2-2n+5}}{2}\right)^1 (-1)^{n-2} \left(\frac{(n-1)-\sqrt{n^2-2n+5}}{2}\right)^1 \\ &= (-1)^{n-1},\ where\ \ n \geq 2. \end{aligned}$$

Case 3. $det_{ED}(K_{s,t}) = (1+\sqrt{st})^1\ (1)^{s+t-2}\ (1-\sqrt{st})^1 = (1-st)$, where $|s-t| \geq 2$.

Case 4.

$$\begin{aligned} det_{ED}(S_n^0) &= \left(\frac{(n-1)\pm\sqrt{n^2-2n+5}}{2}\right)^1 \left(\frac{(3-n)\pm\sqrt{n^2+2n-3}}{2}\right)^1 (1)^{n-2}(-1)^{n-2} \\ &= (-1)^{n-1}(3-2n), \end{aligned}$$

where $n \geq 3$. $\square$

Theorem 3.6 *If the graph G is non-singular (i.e no eigenvalues of $A_{ED}(G)$ is equal to zero) then $E_{ED}(G) \geq n$, (non-hypoenergetic).*

Proof. Let $\mu_1, \mu_2, \cdots, \mu_n$ are non-zero eigenvalues of $A_{ED}(G)$. Then inequality between the arithmetic and geometric mean, we have

$$\begin{aligned} \frac{dv}{dx}\frac{|\mu_1|+|\mu_2|+\cdots+|\mu_n|}{n} &\geq (|\mu_1||\mu_2|\cdots|\mu_n|)^{1/n} \\ \frac{1}{n}E_{ED}(G) &\geq (detA_{ED}(G))^{1/n}. \end{aligned}$$

The determinant of the $A_{ED}(G)$ matrix is necessary an integer. Because no eigenvalues is zero, $|detA_{ED}(G)| \geq 1$ then $|detA_{ED}(G)|^{1/n} \geq 1$. Therefore $E_{ED}(G) \geq n$. $\square$

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