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Enhancing the Bounds of $|\mathcal{N}_3(x) - \tau x|$ Using Special Form of ω_p-*numbers*

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ABSTRACT

Primitive weird number is weird number which are not a multiple of any smaller weird numbers. The goal of this work is to generate a square-free primitive weird number $x = c\prod_{i=1}^{n} q_i$ where $\{q_i\}_{i=1}^{n}$ be an increasing sequence of prime numbers such that q_1 is greater than $\prod_{j=1}^{r}(\bar{q}_j + 1)$ and $c = \prod_{j=1}^{r} \bar{q}_j$ is deficient number with n greater than 1, to be able to use this special form to enhancing the classic bounds of Ω-results for $(N_3 - \tau x)$.

1. Introduction

Let x be a natural number, and $\sigma(x)$ be the sum of its divisors. Then x is called abundant, deficient and perfect if $\Delta(x) = \sigma(x) - 2x > 1$, $\nabla(x) = 2x - \sigma(x) = -\Delta(x)$ and $\Delta(x) = 0$ respectively. We say that an abundant number x which can be express as a sum of distinct proper divisors is semiperfect (sp-number in this paper). We say x is weird (ω-number in this paper) if x is abundant and not sp-number. We call a ω-number x primitive (ω_p-number in this paper) if it is not a multiple of any smaller ω-number. This introduction also addresses some details from lectures about Beurling prime system and the generalized Chebyshev's function $\Psi_3(x)$ which would be the integer part of x minus 1. That is:

$$\Psi_3(x) = x + O(1)$$

Where the Beurling prime system is very closed to being discrete system and investigating how regular the corresponding generalized counting function of integers N_3 to be. Now we move our attention to address an O-results and Ω-results for $(N_3(x) - \tau x)$ which has been investigated by number of writers (for example [1]) as follows:

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On the Riemann Hypothesis the upper bound for $\left(N_{\mathfrak{z}}(x)-\tau x\right)$ can be improved to:

$N_{\mathfrak{z}}(x)-\tau x = O\left(x \exp\left\{-\frac{a \log x \log_3 x}{\log_2 x}\right\}\right)$, for every $a < 0.25$.

Furthermore, they also find lower bound for $(N_{\mathfrak{z}}(x)-\tau x)$, they have:

$N_{\mathfrak{z}}(x)-\tau x = \Omega(x e^{-\frac{a \log x \log_4 x}{\log_3 x}})$ for every $a > 1$,

As a comparison between O-results on the Riemann Hypothesis and Ω-results, they have

$N_{\mathfrak{z}}(x)-\tau x = \Omega\left(x \exp\{-\frac{b \log x \log_4 x}{\log_3 x}\}\right)$ for every $b > 1$,

$N_{\mathfrak{z}}(x)-\tau x \ll x \exp\{-\frac{c \log x \log_3 x}{\log_2 x}\}$ for every $c < \frac{1}{4}$.

This shows that there is a small gap between these results which reflects the great difficulty in determining the behaviors of $\zeta_{\mathfrak{z}}(\sigma + it)$ in the strip $0.5 < \sigma < 1$. In a previous paper [2] we have been restricted the gap between the error terms of Ω-results for $(N_{\mathfrak{z}} - \tau x)$ (where the Ω-results are generalized Ω-results for N_p as counting function of Beurling) and the error terms of O-results for $(N - \tau x)$ on R.H. and we get:

$N_{\mathfrak{z}}(x)-\tau x = \Omega(x e^{-c k_x})$ for every $c > 1$,

Where

$$k_x = \log x \sqrt{\frac{\log_4 x}{\log_3 x}}.$$

The aim of this work is to generate ω_p-number from square-free prime numbers and using this special form to enhancing the upper bound of $(N_{\mathfrak{z}} - \tau x)$ using the same approach as in [2].

2. Generating ω_p-number

In this section we provide a way for generate ω_p-number from square-free prime numbers, so we start with:

2.1. Lemma

Let $c = \prod_{j=1}^{r} \bar{q}_j$ with $\Delta(c) \leq -1$ (c to be deficient number) and there exists a prime $q > \overline{q_r}$ such that cq is abundant, then for $n > 0$ there are an increasing sequence of prime numbers $\{q_i\}_{i=1}^{n}$ such that $q_1 > \overline{q_r}$, $c\prod_{i=1}^{n} q_i$ is abundant and $c\prod_{i=1}^{s} q_i$ is deficient for all $s < n$. Hence $x = c\prod_{i=1}^{n} q_i$ is PA-number.

Proof:

To prove that x is PA-number (i.e. x is abundant and $c\prod_{i=1}^{s} q_i$ is deficient for all $1 \leq s < n$). If $s = 1$ then $x = cq_1$ is PA-number, by [3, lemma 2.1] we get $x = cq_1$ is abundant and since for all $d|x$ is deficient (either $d = q_1$ or $d|c$ which are deficient by [3, lemma 2.2]).

Now, for $s = 2$, let $x = c_1 q_2$ where $c_1 = cq_1$. By [3, lemma 4.1] we have x is abundant. Now, If there is divisor d of x then either d divide c_1 or $d = q_2$. So when $d|c_1$ either $d = q_1$ or $d|c$. Therefore, d is deficient by [3, lemma 2.2]. Continue in this manner applying corollary 2.11 in [4] for n-times we get that x is PA-number.

2.2. Proposition

Let $c = \prod_{j=1}^{r} \bar{q}_j$ such that $\Delta(c) \leq -1$ (c to be deficient number) and n greater than 1. Let $\{q_i\}_{i=1}^{n}$ be an increasing sequence of prime numbers such that q_1 is greater than $\prod_{j=1}^{r}(\bar{q}_j + 1)$. Let

$$\ell = flow(\frac{q_1 - \prod_{j=1}^{r}(\bar{q}_j + 1)}{q_n - q_1}).$$

Let $x = c\prod_{i=1}^{n} q_i$ and

$$\mathcal{K} = \cup_{k=0}^{\ell} \left\{ a \in \mathbb{N} | kq_n + \prod_{j=1}^{r} (\bar{q}_j + 1) < a < (k+1)q_1 \right\}$$

Then for all $a \in \mathcal{K}$, a cannot be expressible as a sum of distinct divisors of x.

Proof:

Since $\prod_{j=1}^{r}(\bar{q}_j + 1) + 1 \leq q_1$, then $\mathcal{K}$ is not empty set. If

$$\bar{\ell} < \frac{q_1 - \prod_{j=1}^{r}(\bar{q}_j + 1)}{q_n - q_1}$$

So,

$$\prod_{j=1}^{r}(\bar{q}_j + 1) + \bar{\ell} q_n < (\bar{\ell} + 1) q_1.$$

Let $k \leq \ell$ and let a belongs to $\mathcal{K}$, (i.e. $\prod_{j=1}^{r}(\bar{q}_j + 1) + kq_n < a < (k+1)q_1$). We went to prove that $a \neq \sum_{b|x} b$ (sum of distinct divisors of x).

Suppose that $a = \sum_{b|x} b$ (sum of distinct divisors of x). As

$$a < (k+1)q_1 < (\bar{\ell} + 1)q_1 < (\tfrac{q_1}{2} + 1) < {q_1}^2.$$

So these divisors take one of the forms $\bar{q}q_1$ or $\bar{q}$ with $\bar{q}|c$ and $\bar{q}$ belongs to the set $\{\overline{q_1}, \overline{q_2}, \dots, \overline{q_r}\}$.

For $a = \sum_{j=1}^{N} \bar{q}_j q_1 + \sum_{i=1}^{M} \tilde{q}_i$ where $\{\tilde{q}_k\}_{i=1}^{M}$ is the set of distinct divisors of c. As $a < (k+1)q_1$, then $\sum_{j=1}^{N} \bar{q}_j$ must be less than or equal k. So

$$\sum_{i=1}^{M} \tilde{q}_i = a - \sum_{j=1}^{N} \bar{q}_j q_1 > a - kq_n > \prod_{j=1}^{r} (\bar{q}_j + 1)$$

This is a contradiction with our assumption. Hence $a \neq \sum_{b|x} b$ (b distinct divisors of x). Therefore, there is no elements in $\mathcal{K}$ can be expressed as a sum of distinct divisors of x.

2.3. Theorem

Let $c = \prod_{j=1}^{r} \bar{q}_j$ such that $\Delta(c) \leq -1$ (deficient number) and there exists a prime $q > \overline{q_r}$ such that cq is abundant. And for n greater than 1, let $\{q_i\}_{i=1}^{n}$ be an increasing sequence of prime numbers such that q_1 is greater than $\prod_{j=1}^{r}(\bar{q}_j + 1)$. Let

$$x = c \prod_{i=1}^{n} q_i.$$

If x is abundant and $\Delta(x)$ belongs to $\mathcal{K}$, then x is ω_p-number.

Proof:

To prove that x is ω_p-number (i.e. x is ω-numbers and PA-numbers).

First we prove that x is ω-number by using lemma 2 in [5] and as $\Delta(x)$ belongs to $\mathcal{K}$, and since there is no elements in $\mathcal{K}$ can be expressed as a sum of distinct divisors of x. We get that x is ω-number.

The second part of the prove is to show that x is PA-number it is enough to show that $\Delta(c \prod_{i=1}^{s} q_i) < 0$ for all $s < n$.

If $n = 2$, then

$$\Delta(cq_1) = \sigma\left(\prod_{j=1}^{r} \bar{q}_j\right)(q_1 + 1) - 2 \prod_{j=1}^{r} \bar{q}_j \, q_1$$

$$= \left(\prod_{j=1}^{r} (\bar{q}_j + 1) - 2 \prod_{j=1}^{r} \bar{q}_j \right) q_1 + \prod_{j=1}^{r} (\bar{q}_j + 1)$$

$$= \Delta\left(\prod_{j=1}^{r} \bar{q}_j \right) q_1 + \prod_{j=1}^{r} (\bar{q}_j + 1)$$

So $\Delta(cq_1)$ less than zero (deficient), since $\Delta(c) \leq -1$ and q_1 greater than $\prod_{j=1}^{r}(\bar{q}_j + 1)$.

Now, we went to show that $\Delta(cq_1 \dots q_{n-1}) < 0$ for $n \geq 3$. We have

$$\Delta\left(c \prod_{i=1}^{n-1} q_i \right) = \Delta\left(\frac{c \prod_{i=1}^{n} q_i}{q_n} \right) = \sigma\left(\frac{c \prod_{i=1}^{n} q_i}{q_n} \right) - 2 \frac{c \prod_{i=1}^{n} q_i}{q_n}$$

$$= \frac{\sigma(c\prod_{i=1}^{n} q_i)}{q_n+1} - 2\frac{c\prod_{i=1}^{n} q_i}{q_n}$$

$$= \frac{\Delta(c\prod_{i=1}^{n} q_i) - 2\frac{c\prod_{i=1}^{n} q_i}{q_n}}{q_n+1}$$

Therefore, $\Delta\left(c\prod_{i=1}^{n-1} q_i\right) < 0$ since for $\Delta(x) \in \mathcal{K}$, so $\Delta(x) < (k+1)q_1 \leq (\ell+1)q_1 < (\frac{q_1}{n}+1)q_1 < q_1^2$.

And

$$2\frac{c\prod_{i=1}^{n} q_i}{q_n} = 2c\prod_{i=1}^{n-1} q_i > 2cq_1^2 > q_1^2.$$

So $\Delta(c\prod_{i=1}^{n} q_i) < 2c\prod_{i=1}^{n-1} q_i$. Hence $c\prod_{i=1}^{s} q_i$ is deficient for all $s < n$.

As $x = c\prod_{i=1}^{n} q_i$ is abundant and by using lemma 4.4, so x is PA-number. Hence x is ω_p-number.

3. Enhancing the bounds of $(N_3(x) - \tau x)$

We will need to recall a fundamental theorem, which will be used in the proof of the main theorem.

3.1. Theorem

For k greater than or equal 1 the k –prime q_k satisfies the inequalities

$$\frac{1}{6}klogk < q_k < 12(k\, log\, k + klog12e^{-1}).$$

Proof: for a proof see [6, Theorem 4.7].

After preparing the necessary concepts, now we are able to prove theorem.

3.2. Theorem

Let $\{x_n\}_{n=1}^{\infty}$ to be an infinite sequence of a square-free ω_p-numbers, then

$$N_3(x_n) - \tau x_n = \Omega(x_n e^{-AG(x_n)}) \qquad \text{for all } A > 1,$$

Where

$$G(x_n) = \log x_n (\log_3 x_n)^{-\frac{1}{2}}.$$

Proof:

In previous paper [2], we proved that:

$$N_3(x) - \tau x = \Omega(xe^{-ck_x}) \text{ for every } c > 1,$$

where

$$k_x = \log x \left(\frac{\log_4 x}{\log_3 x}\right)^{\frac{1}{2}}.$$

This is valid for any x goes to ∞. As a special case, let $\{x_n\}$ to be an infinite sequence of square-free ω_p-number $x_n = \prod_{i=1}^{k} q_i$, then one can get:

$$N_3(x_n) - \tau x_n \geq B\prod_{i=1}^{k} q_i \; e^{-c\log\prod_{i=1}^{k} q_i\left(\frac{\log_4\prod_{i=1}^{k} q_i}{\log_3\prod_{i=1}^{k} q_i}\right)^{\frac{1}{2}}}$$

$$\geq B\prod_{i=1}^{k} q_i \; e^{-c\log\prod_{i=1}^{k} q_i\left(\frac{\log_3\left(\sum_{i=1}^{k}\log q_i\right)}{\log_3\prod_{i=1}^{k} q_i}\right)^{\frac{1}{2}}}$$

$$> B\prod_{i=1}^{k} q_i \; e^{-c\log\prod_{i=1}^{k} q_i\left(\frac{\log_3(k\log q_k)}{\log_3\prod_{i=1}^{k} q_i}\right)^{\frac{1}{2}}}$$

$$> B\prod_{i=1}^{k} q_i \; e^{-c\log\prod_{i=1}^{k} q_i\left(\frac{\log_3(kq_k)}{\log_3\prod_{i=1}^{k} q_i}\right)^{\frac{1}{2}}}$$

$$\geq B\prod_{i=1}^{k} q_i \; e^{-A\log\prod_{i=1}^{k} q_i\left(\frac{1}{\log_3\prod_{i=1}^{k} q_i}\right)^{\frac{1}{2}}}$$

Since $kq_k < 12k(k\, log\, k + klog12e^{-1})$ by (theorem 2.1.) Therefore,

$N_3(x_n) - \tau x_n = \Omega\left(x_n e^{-A\log x_n(\log_3 x_n)^{-\frac{1}{2}}}\right)$ for every $A > 1$.

4. Conclusion

This work explains how could the form of ω_p-numbers effect on the bound of the generalized counting function of integers N_3 and thus restrict the gap between the error terms of Ω-results for $(N_3 - \tau x)$ and the error terms of O-results for $(N - \tau x)$ on Riemann Hypothesis. One can use this form to show the effect on O-results for $\left(N_{\mathfrak{z}}(x) - \tau x\right)$.

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