International J.Math. Combin. Vol.3 (2008), 56-78

The Characterization of Symmetric Primitive Matrices with exponent $n-3$

Lichao Huangfu[1] and Junliang Cai[2]

1. Beijing NO.4 High School, Beijing, P.R.China

2. School of Mathematical Sciences of Beijing Normal University, Beijing, 100875, P.R.China.

Email: caijunliang@bnu.edu.cn

Abstract: An $n \times n$ nonnegative matrix $A = (a_{ij})$ is said to be *Smarandachely primitive* if $A^k > 0$ for at least two integers $k > 0$ and *primitive* if for some integers $k > 0$. The least such integers k is called the *Smarandache exponent* or *exponent* of A and denoted by $\gamma^S(A)$ and $\gamma(A)$, respectively. The symmetric primitive matrices with exponent $\geq n-2$ has been described in articles [4]-[9]. In this paper the complete characterization of symmetric primitive matrices with exponent $n-3$ is obtained.

Key words: Smarandachely primitive matrix, Primitive matrix, Smarandache exponent, exponent, primitive graph.

AMS(2000): 05C10.

§1. Introduction

An $n \times n$ nonnegative matrix $A = (a_{ij})$ is said to be *Smarandachely primitive* if $A^k > 0$ for at least two integers $k > 0$ and *primitive* if for some integers $k > 0$. The least such integer k is called the *Smarandache exponent* or *exponent* of A and denoted by $\gamma^S(A)$ and $\gamma(A)$, respectively. The associated graph of *symmetric matrix* A, denoted by $G(A)$, is the graph with a vertex set $V(G(A)) = \{1, 2, \cdots, n\}$ such that there is an edge from i to j in $G(A)$ if and only if $a_{ij} > 0$. A graph G is called to be *primitive* if there exists an integer $k > 0$ such that for all ordered pairs of vertices $i, j \in V(G)$(not necessarily distinct), there is a walk from i to j with length k.The least such k is called the *exponent* of G, denoted by $\gamma(G)$. Clearly,a symmetric matrix A is primitive if and only if its associated graph $G(A)$ is primitive. And in this case, we have $\gamma(A) = \gamma(G(A))$. By this reason as above, we shall employ graph theory as a major tool and consider $\gamma(G(A))$ to prove our results.

Let SE_n be the exponent set of $n \times n$ symmetric primitive matrices. In 1986,Shao[4] proved $SE_n = \{1, 2, \cdots, 2n-2\} \setminus S$, where S is the set of all odd numbers among [n,2n-2] and gave the characterization of the matrix with exponent $2n-2$. In 1990,Wang[5] gave the characterization of the matrix with exponent $2n-4$. In 1991, Li[6] obtained the characterization with exponent

[1]Supported by the Priority Discipline of Beijing Normal University and NNSFC (10271017).

[2]Received July 16, 2008. Accepted September 6, 2008.

$2n-6$. In 1995, Cai and Zhang[7] derived the complete characterization of symmetric primitive matrices with exponent $2n-2r(\geq n)$. In 2003, Cai and Wang[8] got the characterization with exponent $n-1$. In 2004,Cai[9] characterized the matrix with exponent $n-2$. The purpose of this paper is to go further into the problem and give the complete characterization of symmetric primitive matrices with exponent $n-3$.

§2. Some lemmas on $\gamma(G)$

For convenience, We will narrate the lemmas with graph theory below.

Lemma 2.1[4] *G is a primitive graph iff G is connected and has odd cycles.*

The *local exponent* from vertex u to v, denoted by $\gamma(u,v)$, is the least integer k such that there exists a walk of length l from u to v for all $l \geq k$.We denote $\gamma(u,u)$ by $\gamma(u)$ for short.

Lemma 2.2[4] *If G is a primitive graph, then*

$$\gamma(G) = \max_{u,v \in V(G)} \gamma(u,v).$$

We denote by $P(u,v)$ the shortest walk from u to v in G. The length of $P(u,v)$ is called the *distance* between u and v, denoted by $d_G(u,v)$. The *diameter* of G is defined as

$$diam(G) = \max_{u,v \in V(G)} d_G(u,v).$$

Let G_1 and G_2 be two subgraphs of $G.P(G_1,G_2)$ denotes the shortest walk between G_1 and G_2.Its length

$$d_G(G_1,G_2) = \min\{d_G(u,v) \mid u \in V(G_1), v \in V(G_2)\}.$$

Lemma 2.3[9] *Let G be a primitive graph,and let $u,v \in V(G)$. If there are two walks from u to v with length k_1 and k_2, respectively,where $k_1+k_2 \equiv 1(\text{mod })$,then*

$$\gamma(u,v) \leq \max\{k_1,k_2\} - 1.$$

Let $u,v \in V(G)$,we name the walk from u to v with different parity length to $d_G(u,v)$ a *dissimilar walk*, denoted by $W(u,v)$. The shortest (u,v)-dissimilar walk is called the *primitive walk* between u and v, denoted by $W_r(u,v)$, its length is denoted by $b(u,v)$ [9].

Lemma 2.4[8] *If G is a primitive graph, then*

$$\gamma(u,v) = b(u,v) - 1.$$

Therefore,

$$\gamma(G) = \max_{u,v \in V(G)} b(u,v) - 1.$$

Lemma 2.5[8] *Let G be a primitive graph, then* *(i)* $\gamma(u,v) \geq d_G(u,v)$;

(ii) $\gamma(u,v) \equiv d_G(u,v) (\mathrm{mod}\ 2)$;

(iii) $\gamma(G) \geq diam(G)$,and $\gamma(G) \equiv diam(G) (\mathrm{mod}\ 2)$.

Lemma 2.6[8] *Suppose G is the primitive graph with order n. If there are $u, v \in V(G)$ such that $\gamma(u,v) = \gamma(G)$, then for any odd cycle C in G we have*

$$|V(P(u,v)) \cap V(C)| \leq n - \gamma(G).$$

Apparently, any (u,v)-dissimilar walk is inevitably correlative with some odd cycle. And for any odd cycle C, there is a (u,v)-dissimilar walk correlative with C, we denote it by $W(u,v,C)$. Therefore, there must be some smallest odd cycle C_0 such that $W_r(u,v) = W(u,v,C_0)$. We call C_0 a (u,v)-*primitive cycle* or the primitive cycle of $P(u,v)$. If there exists a (u,v)-shortest path which intersects with its primitive cycle C_0, then we can choose some (u,v)-shortest path, denoted by $P(u,v)$ might as well, such that their intersected vertexes can be arranged on a path.Set $p = |V(P(u,v)) \cap V(C_0)|$,then $p \leq \min\{n-\gamma(G), [\frac{n}{2}], \frac{1}{2}(|C_0|-1)\}$. Ulteriorly, we have

$$\begin{aligned}
\gamma(u,v) &= \gamma(u,v,C_0) \\
&= d_G(u,C_0) + |P(C_0)| + d_G(v,C_0) - 1 \\
&= d_G(u,v) + |C_0| - 2(p-1) - 1,
\end{aligned}$$

where $P(C_0)$ denotes the left part of C_0 which deletes the part in common with $P(u,v)$. If the (u,v)-shortest path has at most one intersected vertex with its primitive cycle C_0, there must be $w \in V(C_0)$ such that $d_G(u,C_0) = d_G(u,w)$ and $d_G(v,C_0) = d_G(v,w)$. Further we have

$$\begin{aligned}
\gamma(u,v) &= \gamma(u,v,C_0) \\
&= d_G(u,C_0) + |C_0| + d_G(v,C_0) - 1 \\
&= d_G(u,w) + |C_0| + d_G(v,w) - 1.
\end{aligned}$$

§3. Constructions of graphs

Let G be a primitive graph with order n. If there exists a vertex $w \in V(G)$ such that $\gamma(w) = \gamma(G)$, we call G a *graph of the first type*, otherwise a *graph of the second type*. Firstly, we define a class of graphs $\mathcal{N}_{n-3}$ as follows:

Denote the set $\mathcal{N}_{n-3} = \mathcal{N}_{n-3}^{(1)} \cup \mathcal{N}_{n-3}^{(3)} \cup \cdots \cup \mathcal{N}_{n-3}^{(n-2)}$,where $\mathcal{N}_{n-3}^{(d)}(1 \leq d \leq n-2, d \equiv 1(\mathrm{mod}\ 2), n \equiv 1(\mathrm{mod}\ 2))$ are defined as follows.

Let $n = 2r+3$ and $K = (V, E)$ be a graph, where the vertex set $V = \bigcup\limits_{0 \leq i \leq r} V_i$ with $V_i \cap V_j = \phi (0 \leq i < j \leq r)$ and $V_k = \{u_{l,k} \mid l = 1, 2, \cdots, r+3\} (k = 0, 1, \cdots, r)$,the edge set $E = E_1 \cup E_2$ with $E_1 = \{uv \mid u \in V_i, v \in V_{i+1}, 0 \leq i \leq r-1\}$ and $E_2 = \{uv \mid u, v \in V_r\}$. For any odd number d such that $1 \leq d \leq n-2$, let $t = r - \frac{1}{2}(d-1)$. We put the path $P_t = u_{1,0}u_{1,1}\cdots u_{1,t}$ and the cycle $C_d = u_{1,t}u_{1,t+1}\cdots u_{1,r}u_{2,r}\cdots u_{2,t+1}u_{1,t}$,and set $K_{(d)} = P_t \cup C_d$ which we call it a *structural graph*. Let the set of induced subgraphs with order n of K which contain $K_{(d)}$ be

$K^{(d)}$. For any $N \in K^{(d)}$, we denote the spanning subgraph of N which contains subgraph $K_{(d)}$ by $N_{(d)}$, and define the set of graphs $\mathcal{N}^{(d)}$ as:

$$\mathcal{N}^{(d)} = \{N_{(d)} \mid N \in K^{(d)}, 1 \le d \le n-2, d \equiv 1 (\mathrm{mod}\ 2)\}.$$

We mark the graphs of $\mathcal{N}^{(d)}$ with $\mathcal{N}^{(d)}_{n-3}$ which satisfy the following qualifications:

(1) $diam(N_{(d)}) \le n-3$;

(2) For any odd number $d' > d$, there doesn't exist the graph $K_{(d')}$ in $N_{(d)}$;

(3) Let x be the vertex of $N_{(d)}$ such that $d_{N_{(d)}}(x, C_d) > t$, then there must exist a odd cycle C such that:

$$2d_{N_{(d)}}(x, C) + |C| \le n-2.$$

Let $u_i \in V(P(x, C_d)) \cap P_t (i \le t)$ be the vertex with the smallest subscript. If C is the odd cycle which doesn't intersect with $K_{(d)}$ and has at most one intersected vertex with $P(x, u_i)$(The shortest path from C to $P(x, u_i)$ is denoted by $P(w, z)$, where $w \in V(P(x, u_i))$ and $z \in V(C)$. And it suggests that C and $P(x, u_i)$ has only one vertex in common if $w = z$), and such that $2d_{N_{(d)}}(w, z) + |C|$ is as small as possible, then

(i) if $|C| + d = 4$ and $d_{N_{(d)}}(x, u_i) + d_{N_{(d)}}(w, z) + |C| = t+3$, then we must have

$$2d_{N_{(d)}}(w, z) + |C| \ne 2(t-i) + d.$$

(ii) if $|C| = d = 1$ and $d_{N_{(d)}}(x, u_i) + d_{N_{(d)}}(w, z) = t+1$, then we must have

$$d_{N_{(d)}}(w, z) \ne t-i.$$

(iii) if $|C| = d = 1$ and $d_{N_{(d)}}(x, u_i) + d_{N_{(d)}}(w, z) = t+2$, then we must have

$$|d_{N_{(d)}}(w, z) - (t-i)| \ge 6.$$

Another class of graphs $\mathcal{M}_{n-3}$ is defined as follows:

Let $n-3 = m+2r$, then $n-3 \equiv m (\mathrm{mod}\ 2)$. Let $T = (U, F)$ be a graph, where the vertex set $U = \bigcup_{0 \le i \le r} U_i$ with $U_i \cap U_j = \phi (0 \le i < j \le r)$ and $U_i = \{u_{i,k} \mid k = 0, 1, \cdots, n-1\} (i = 0, 1, 2, \cdots, r)$, the edge set $F = F_1 \cup F_2 \cup F_3$ with $F_1 = \{u_{i,j} u_{k,l} \mid j+l+i+k \equiv 1 (\mathrm{mod}\ 2)\}$,$F_2 = \{uv \mid u, v \in U_r\}$ and $F_3 = \{uv \mid u \in U_{r-1}, v \in U_r\}$.We defined the set of graphs $\mathcal{M}_{n-3} = \mathcal{M}^{(0)}_{n-3} \cup \mathcal{M}^{(1)}_{n-3} \cup \mathcal{M}^{(2)}_{n-3} \cup \mathcal{M}^{(3)}_{n-3}$ as follows:

(i) Construction of $\mathcal{M}^{(0)}_{n-3}$: Let d_0, d_1 be the odd numbers such that $1 \le d_0, d_1 \le 5$ and $2 \le d_0 + d_1 \le 6$,and t_0, t_1 be the positive numbers such that $2r+1 = 2t_0 + d_0 \le 2t_1 + d_1$ and $m + t_0 + t_1 + d_0 + d_1 \le n+1$. We put the path $P_0 = u_{0,j} u_{1,j} \cdots u_{t_0,j}$ and the path $P_1 = u_{0,i} u_{1,i} \cdots u_{t_1,i} (0 \le h \le i < j \le m+h \le n-1)$. Let C_{d_0} be the cycle with length d_0 which has only one intersected vertex $u_{t_0,j}$ with P_0, while C_{d_1} be the cycle with length d_1 which has only one intersected vertex $u_{t_1,i}$ with P_1 and doesn't intersect with C_{d_0}. Put $K_{d_0,d_1} = P(u_{0,h}, u_{0,m+h}) \cup P_0 \cup P_1 \cup C_0 \cup C_1$, and call it a *structural graph*. Let $V(d_0, d_1) = V_1(d_0, d_1) \cup V_2(d_0, d_1)$, where $V_1(d_0, d_1) = V(K_{d_0,d_1})$ with $|V_1(d_0, d_1)| = m + t_0 + t_1 + d_0 + d_1 - 1 \le n$,and $V_2(d_0, d_1) \subseteq U \setminus V_1(d_0, d_1)$ with $|V_2(d_0, d_1)| = t_0 + 3 - t_1 - d_1 \le 2$. Therefore, we have

$|V(d_0,d_1)| = n$. We choose the connected subgraph T_{d_0,d_1} of $T[V(d_0,d_1)]$ to form the set of graphs $\mathcal{M}^{(0)}_{n-3}$, where T_{d_0,d_1} satisfies that:

(1) $diam(T_{d_0,d_1}) \leq n-3$;

(2) $V(T_{d_0,d_1}) = V(d_0,d_1)$, and $E(K_{d_0,d_1}) \subseteq E(T_{d_0,d_1})$;

(3) there doesn't exist a path P_2 and an cycle C_{d_2} such that $2t_2+d_2 < 2t_0+d_0$ and they have only one common vertex $u_{t_2,l}$,where $P_2 = u_{0,l}u_{1,l}\cdots u_{t_2,l}$ with length $t_2 > 0$ and C_{d_2} is an odd cycle with length d_2;

(4) if there exist a $(x_{a,j}, y_{b,i})$-path with length $p = t_0+4-t_1-d_1 \leq 3$ which connects $P_0 \cup C_0$ to $P_1 \cup C_1$ in $T_{d_0,d_1} - E(K_{d_0,d_1})$, where $0 \leq a \leq t_0$ and $0 \leq b \leq t_1$, then we have

$$a+b+p > j-i, a+b+i+j \equiv p (\mathrm{mod}\ 2),$$

and

$$(2t_0+d_0)-(2t_1+d_1)-(p+i-j) \leq a-b \leq p+i-j;$$

(5) if there exists a vertex x in T_{d_0,d_1} such that $d_{T_{d_0,d_1}}(x,C_0) \geq t_0$ and $d_{T_{d_0,d_1}}(x,C_1) \geq t_0+\frac{1}{2}(d_0-d_1)$, there must exist an odd cycle C such that

$$2d_{T_{d_0,d_1}}(x,C)+|C| < m+2r+1.$$

(*ii*) Construction of $\mathcal{M}^{(1)}_{n-3}$: Let $m+2t_0+3 = n, t_0 \geq 0$. Let $C_{t_0} = u_{0,i}\cdots u_{t_0,i}u_{t_0,i+2}\cdots u_{1,i+2}u_{0,i} (0 \leq h \leq i \leq m+h \leq n-1)$, then $|C_{t_0}| = 2t_0+1$(C_{t_0} is a loop on $u_{0,i}$ if $t_0 = 0$). Put the graph $K_{m,t_0} = P(u_{0,h}, u_{0,m+h}) \cup C_{t_0}$, and call it a *structural graph*. Let $V(m,t_0) = V_1(m,t_0) \cup V_2(m,t_0)$, where $V_1(m,t_0) = V(K_{m,t_0})$ and $V_2(m,t_0) \subseteq U \setminus V_1(m,t_0)$ with $|V_2(m,t_0)| = 2$. We choose the connected subgraph T_{m,t_0} of $T[V(m,t_0)]$ to form the set of graphs $\mathcal{M}^{(1)}_{n-3}$, where T_{m,t_0} satisfies that:

(1) $diam(T_{m,t_0}) \leq n-3$;

(2) $V(T_{m,t_0}) = V(m,t_0)$, and $E(K_{m,t_0}) \subseteq E(T_{m,t_0})$;

(3) neither does there exist an odd cycle with length $2t_0+1$ that has only one intersected vertex with $P(u_{0,h}, u_{0,m+h})$, nor does there exist an odd cycle C_d with length d such that $2t+d < 2t_0+1$ in T_{m,t_0}, where $t = d_{T_{m,t_0}}(P(u_{0,h}, u_{0,m+h}), C_d) > 0$;

(4) if there exists a $(u_{b,i}, u_{a,i+2})$-path with length $p \leq 3$ which divides up C_{t_0} in $T_{m,t_0} - E(K_{m,t_0})$, where $0 \leq a, b \leq t_0$, then a, b must satisfy that:if $a+b \equiv p (\mathrm{mod}\ 2)$, then $|a-b| \leq p$; if $a+b+1 \equiv p (\mathrm{mod}\ 2)$, then $a+b+p \geq 2t_0+1$;

(5) if there exists a vertex x in T_{m,t_0} such that $d_{T_{m,t_0}}(x, C_{t_0}) \geq \frac{1}{2}m$, there must be an odd cycle C such that

$$2d_{T_{m,t_0}}(x,C)+|C| < m+2r+1;$$

(*iii*) Construction of $\mathcal{M}^{(2)}_{n-3}$: Let $m+2t_0+3 = n, t_0 \geq 0$. We put the cycle $C_{t_0} = u_{0,i}\cdots u_{t_0,i}zu_{t_0,i+1}\cdots u_{0,i+1}u_{0,i} (0 \leq h \leq i < i+1 \leq m+h \leq n-1)$, where $z = u_{t_0+1,i}$ or $u_{t_0+1,i+1}$,then $|C_{t_0}| = 2t_0+3$. Put $K_{m,t_0} = P(u_{0,h}, u_{0,m+h}) \cup C_{t_0}$, and we call it a *structural graph*. Let $V(m,t_0) = V_1(m,t_0) \cup V_2(m,t_0)$, where $V_1(m,t_0) = V(K_{m,t_0})$ and $V_2(m,t_0) \subseteq$

$U \setminus V_1(m,t_0)$ with $|V_2(m,t_0)| = 1$. We choose the connected subgraph T_{m,t_0} of $T[V(m,t_0)]$ to form the set of graphs $\mathcal{M}^{(2)}_{n-3}$,where T_{m,t_0} satisfies that:

(1) $diam(T_{m,t_0}) \le n-3$;

(2) $V(T_{m,t_0}) = V(m,t_0)$, and $E(K_{m,t_0}) \subseteq E(T_{m,t_0})$;

(3) neither does there exist an odd cycle with length less than $2(t_0+q)-1$ which have $q(1 \le q \le 2)$ intersected vertexes with $P(u_{0,h}, u_{0,m+h})$, nor does there exist an odd cycle C_d with length d such that $2t+d < 2t_0+1$ in T_{m,t_0}, where $t = d_{T_{m,t_0}}(P(u_{0,h}, u_{0,m+h}), C_d) > 0$;

(4) if there exists a $(u_{b,i}, u_{a,i+1})$-path with length $p \le 2$ that divides up C_{t_0} in $T_{m,t_0} - E(K_{m,t_0})$, where $0 \le a, b \le t_0+1$, then a, b must satisfy that: if $a+b \equiv p(\mathrm{mod}\ 2)$, then $a+b+p \ge 2t_0+2$; if $a+b+1 \equiv p(\mathrm{mod}\ 2)$,then $|a-b| \le p+1$;

(5) if there exists a vertex x in T_{m,t_0} such that $d_{T_{m,t_0}}(x, C_{t_0}) \ge \frac{1}{2}m - 1$, there must be an odd cycle C such that

$$2d_{T_{m,t_0}}(x, C) + |C| < m + 2r + 1.$$

(iv) Construction of $\mathcal{M}^{(3)}_{n-3}$: Let $m + 2t_0 + 1 = n, t_0 \ge 0$. We put the cycle $C_{t_0} = u_{0,k-1} \cdots u_{t_0,k-1} u_{t_0,k+1} \cdots u_{0,k+1} u_{0,k} u_{0,k-1} (0 \le h \le k-1 < k+1 \le m+h \le n-1)$, then $|C_{t_0}| = 2t_0 + 3$. Put $K_{m,t_0} = P(u_{0,h}, u_{0,m+h}) \cup C_{t_0}$, and call it a *structural graph.* Put $V(m,t_0) = V(K_{m,t_0})$. We choose the connected subgraph T_{m,t_0} of $T[V(m,t_0)]$ to form the set of graphs $\mathcal{M}^{(3)}_{n-3}$, where T_{m,t_0} satisfies that:

(1) $diam(T_{m,t_0}) \le n-3$;

(2) $V(T_{m,t_0}) = V(m,t_0)$,and $E(K_{m,t_0}) \subseteq E(T_{m,t_0})$;

(3) neither does there exist an odd cycle with length less than $2(t_0+q)-3$ which have $q(1 \le q \le 3)$ intersected vertexes with $P(u_{0,h}, u_{0,m+h})$, nor does there exist an odd cycle C_d with length d such that $2t+d < 2t_0+1$ in T_{m,t_0}, where $t = d_{T_{m,t_0}}(P(u_{0,h}, u_{0,m+h}), C_d) > 0$;

(4) if there exist an edge $u_{b,k-1}u_{a,k+1}$ that divides up C_{t_0} in $T_{m,t_0} - E(K_{m,t_0})$, where $0 \le a, b \le t_0$, then a, b must satisfy that:

$$a + b \equiv 1(\mathrm{mod}\ 2), |a-b| \le 3;$$

if there exists an edge $v_k x_a$(or $v_k y_b$) that divides up C_{t_0} in $T_{m,t_0} - E(K_{m,t_0})$, where $1 \le a \le t_0$(or $1 \le b \le t_0$), then a(or b) must satisfy that: $a = 2$(or $b = 2$), or $a = 1$(or $b = 1$)(iff $t_0 = 1$);

(5) if there exists a vertex x in T_{m,t_0} such that $d_{T_{m,t_0}}(x, C_{t_0}) \ge \frac{1}{2}m - 2$, there must exist an odd cycle C such that

$$2d_{T_{m,t_0}}(x, C) + |C| < m + 2r + 1.$$

§4. Main results and proofs

Theorem 4.1 *G is a graph with order n of the first type with $\gamma(G) = n-3$ iff $G \in \mathcal{N}_{n-3}$.*

Proof For the necessity, suppose G is a graph with order n of the first type with $\gamma(G) = n-3$. Then there must be a vertex u_0 and an odd cycle C in G such that

$$\gamma(u_0) = \gamma(u_0, C) = \gamma(G) = n-3.$$

We choose u_0 and C such that $d = |C|$ is as great as possible, and denote $C = C_d$. Note that

$$\gamma(G) = \gamma(u_0) \equiv d_G(u_0, u_0)(\text{mod } 2), d_G(u_0, u_0) = 0,$$

we set $\gamma(G) = 2r$. So we get $n = 2r + 3$.

Let $t = d_G(u_0, C_d)$,then

$$\gamma(u_0) = 2t + d - 1 = 2r = n - 3.$$

Thus we get

$$n = 2t + d + 2, t = r - \frac{1}{2}(d - 1), 1 \leq d \leq 2r + 1.$$

We put the path $P_t = P(u_0, C_d) = u_0u_1 \cdots u_t$, the cycle $C_d = u_tu_{t+1} \cdots u_{t+d-1}u_t$, and let

$$\begin{aligned} V_1(t,d) &= V(P_t \cup C_d), V_2(t,d) = V(G) \setminus V_1(t,d), \\ E_1(t,d) &= E(P_t \cup C_d), E_2(t,d) = E(G) \setminus E_1(t,d). \end{aligned}$$

Thus

$$n_1 = |V_1(t,d)| = t + d, n_2 = |V_2(t,d)| = t + 2.$$

It suggests above that there is a structural graph $K_{(d)} = P_t \cup C_d$ in G.To testify that $G \in \mathcal{N}_{n-3}^{(d)} \subset \mathcal{N}_{n-3}$,we shall prove that: (a) G meets the construct qualifications of $\mathcal{N}_{n-3}^{(d)}$, and (b) G is a subgraph of K.

(a) Note that $diam(G) \leq \gamma(G) = n - 3$, then the first construct qualification meets.By the choose of C_d, there doesn't exist the structural graph $K_{(d')}$(d' is an odd number with $d' > d$) in G, thus the second qualification meets. Suppose that there exists a vertex x such that $d_G(x, C_d) > t$, then

$$\gamma(x, C_d) = 2d_G(x, C_d) + d - 1 > 2t + d - 1 = n - 3.$$

If $2d_G(x, C) + |C| > n - 2$ for any odd cycle C which is different from C_d in G, we can get

$$\gamma(x, C) = 2d_G(x, C) + |C| - 1 > n - 3.$$

Thus we get a contradiction

$$\gamma(G) \geq \gamma(x) > n - 3 = \gamma(G).$$

Let $u_i \in V(P(x, C_d)) \cap P_t (i \leq t)$ be the vertex with the smallest subscript. Then $P(x, u_i)$ is a shortest path from C to P_t. Let C be the odd cycle which doesn't intersect with $K_{(d)}$ and has at most one intersected vertex with $P(x, u_i)$(The shortest path from C to $P(x, u_i)$ is denoted by $P(w, z)$, where $w \in V(P(x, u_i))$ and $z \in V(C)$. It suggests that C and $P(x, u_i)$ have only one vertex in common if $w = z$), and such that $2d_{N_{(d)}}(w, z) + |C|$ is as small as possible. Note that

$$\begin{aligned} \gamma(x, u_0, C) &\leq d_G(u, u_0) + 2d_G(w, z) + |C| - 1, \\ \gamma(x, u_0, C_d) &\leq d_G(u, u_0) + 2d_G(u_i, u_t) + d - 1, \end{aligned}$$

we then have

$$\begin{aligned}&\gamma(x,u_0,C)+\gamma(x,u_0,C_d)\\&=2(d_G(u,u_0)+d_G(u_i,u_t)+d_G(w,z)+|C|+d-1)-(d+|C|).\end{aligned}$$

(1) Suppose that $|C|+d=4$. If $d_G(x,u_i)+d_G(w,z)+|C|=t+3$ and $2d_G(w,z)+|C|=2(t-i)+d$, then we have

$$\gamma(x,u_0,C)=\gamma(x,u_0,C_d)$$

and

$$d_G(x,u_i)+d_G(w,z)+|C|-1=t+2=|V_2(d)|.$$

Therefore,

$$d_G(x,u_0)+d_G(w,z)+d_G(u_i,u_t)+|C|+d-1=n.$$

Thus we get

$$\gamma(x,u_0,C)+\gamma(x,u_0,C_d)=2n-4=2(n-2)$$

and

$$\gamma(x,u_0,C)=\gamma(x,u_0,C_d)=n-2.$$

(2) Suppose that $|C|=d=1$. If $d_G(x,u_i)+d_G(w,z)=t+1$ and $d_G(w,z)=t-i$. Then we have

$$\gamma(x,u_0,C)=\gamma(x,u_0,C_d)$$

and

$$d_G(x,u_i)+d_G(w,z)+|C|-1=t+1=|V_2(d)|-1.$$

Therefore,

$$d_G(x,u_0)+d_G(w,z)+d_G(u_i,u_t)+|C|+d-1=n-1.$$

Thus we get

$$\gamma(x,u_0,C)+\gamma(x,u_0,C_d)=2(n-1)-2=2(n-2)$$

and

$$\gamma(x,u_0,C)=\gamma(x,u_0,C_d)=n-2.$$

(3) Suppose that $|C|=d=1$.If $d_G(x,u_i)+d_G(w,z)=t+2$ and $|d_G(w,z)-t-i|<6$. Then we have

$$|\gamma(x,u_0,C)-\gamma(x,u_0,C_d)|<6,$$

and

$$d_G(x,u_i)+d_G(w,z)+|C|-1=t+2=|V_2(d)|.$$

Therefore,

$$d_G(x,u_0)+d_G(w,z)+d_G(u_i,u_t)+|C|+d-1=n.$$

Thus we get

$$\gamma(x,u_0,C)+\gamma(x,u_0,C_d)=2n-2=2(n-1).$$

Note that

$$\gamma(x, u_0, C) \equiv \gamma(x, u_0, C_d)(\text{mod } 2).$$

Hence we get

$$\min\{\gamma(x, u_0, C), \gamma(x, u_0, C_d)\} \geq n-2.$$

The three cases lead to a common contradiction

$$\gamma(x, u_0) = \min\{\gamma(x, u_0, C), \gamma(x, u_0, C_d)\} \geq n-2.$$

So the third qualification meets.

(b) Let

$$V(G) = U_0 \cup U_1 \cup \cdots \cup U_{r-1} \cup U_r,$$

where

$$\begin{aligned} U_i &= \{u \mid d_G(u_0, u) = i, u \in V(G)\}, \\ U_r &= \{u \mid d_G(u_0, u) \geq r, u \in V(G)\}, \end{aligned} \qquad (i = 0, 1, \cdots, r-1).$$

Then $G[U_i](i = 0, 1, \cdots, r-1)$ must be a null graph. Otherwise, there must be some odd cycle in $G' = G[U_0 \cup U_1 \cup \cdots \cup U_{r-1}]$. Let C be the odd cycle such that $d_G(u_0, C) + \frac{1}{2}(|C|-1)$ is as small as possible in G'. Then we have

$$d_G(u_0, C) + \frac{1}{2}(|C|-1) < r.$$

This implies a contradiction

$$\gamma(u_0) \leq \gamma(u_0, C) = 2d_G(u_0, C) + |C| - 1 < 2r = n-3 = \gamma(u_0).$$

Note that $|U_i| \geq 1 (i = 0, 1, \cdots, r)$. Then we have

$$|U_i| \leq 2r + 3 - r = r + 3.$$

So we can assert that G is a subgraph of K. Therefore, $G \in \mathcal{N}_{n-3}^{(d)} \subset \mathcal{N}_{n-3}$.

For the sufficiency, without loss of generality, we let $G \in \mathcal{N}_{n-3}^{(d)}$ with $1 \leq d \leq n-2$ and $d \equiv 1(\text{mod } 2)$. It is obvious that G is connected and has $K_{(d)} = P_t \cup C_d$ as its structural graph.

In the following argument, we shall prove two results:

(1) $\gamma(u_0) = n-3$

Clearly, we have

$$\gamma(u_0, C_d) = 2d_G(u_0, C_d) + |C_d| - 1 = 2t + d - 1 = n-3.$$

Hence we have $n = 2t + d + 2$. Put

$$n_1 = |V_1(d)| = |V(P_t \cup C_d)| = t + d,$$

and

$$n_2 = |V_2(d)| = |V(G) \setminus V_1| = t + 2.$$

If there is an odd cycle C in G such that $\gamma(u_0, C) < n-3 = 2r$, then $2d_G(u_0, C) + |C| - 1 < 2r$, i.e. $d_G(u_0, C) + \frac{1}{2}(|C|-1) < r$. This implies that $G[U']$ contains the odd cycle C,where $U' = \{u \| \ d_G(u_0, u) < r, u \in V(G)\}$. Because the induced subgraph $K[V']$ of K about $V' = \{u \| \ d_K(u_0, u) < r, u \in V(K)\}$ is bipartite, its subgraph $G[U']$ doesn't contain any odd cycles,a contradiction. So we have $\gamma(u_0) = n-3$.

(2) $\forall \ u, v \in V(G), \gamma(u, v) \leq n-3$

It is obvious that $\gamma(u) \leq n-3$ for any vertex in G. In what follows, it suffices to prove $\gamma(u, v) \leq n-3$ for any two distinct vertexes u and v in $V(G)$.

If $d_G(u, C_d) + d_G(v, C_d) \leq 2t$, We can easily get $\gamma(u, v) \leq n-3$. So we put $d_G(u, C_d) + d_G(v, C_d) > 2t$, and without loss of generality we let $d_G(u, C_d) > t$,then there must be an odd cycle C in G such that $2d_G(u, C) + |C| \leq n-2$. Suppose that $V(P(u, C)) \cap V(P_t) \neq \phi$, let $w \in V(P(u, C)) \cap V(P_t)$ be the first vertex along $P(u, C)$ from u to C, then $d_G(u, w) > d_G(u_0, w)$. We then have

$$\begin{aligned} \gamma(u_0) &\leq \gamma(u_0, C) \leq 2(d_G(u_0, w) + d_G(w, C)) + |C| - 1 \\ &< 2(d_G(u, w) + d_G(w, C)) + |C| - 1 \\ &= 2d_G(u, C) + |C| - 1 \leq n - 3 = \gamma(u_0), \end{aligned}$$

a contradiction. Therefore $P(u, C)$ doesn't intersect with P_t.

Let M be the component with u of $G[V_2(d)]$ in G, we shall complete our arguments in the following three cases:

(I) $V(C) \cap V(C_d) \neq \phi$

By the connectivity of G and $|V_2| = t+2$, we have $d_G(u, C_d) = t+1$ or $t+2$ which correspond to the following six cases.

(a) $d_G(u, C_d) = t+2, d_G(v, C_d) = t-1$

If $v \in V(P_t)$, we have

$$\begin{aligned} \gamma(u, v) &\leq \gamma(u, v, C_d) \leq d_G(u, C_d) + d_G(v, C_d) + |C_d| - 2 \\ &= (t+2) + (t-1) + d - 2 = 2t + d - 1 = n - 3. \end{aligned}$$

If $v \in V(P(u, C))$, we have

$$\begin{aligned} \gamma(u, v) &\leq \gamma(u, v, C) = d_G(u, C) + d_G(v, C) + |C| - 1 \\ &< 2d_G(u, C) + |C| - 1 \leq n - 3. \end{aligned}$$

(b) $d_G(u, C_d) = t+2, d_G(v, C_d) = t$

If $v \in V(P_t)$,note that $P(u, C)$ has no intersected vertex with P_t, we then have

$$|V(P(u, v) \cup V(C_d)| = 2t + d + 2 = n.$$

Hence the odd cycle C such that $2d_G(u, C) + |C| \leq n-2$ must be a loop on $P(u, v)$, this means $|C| = 1$. So we get

$$\begin{aligned} \gamma(u, v) &\leq \gamma(u, v, C) = d_G(u, v) + |C| - 1 \\ &= d_G(u, v) \leq diam(G) \leq \gamma(G). \end{aligned}$$

If $v \in V(P(u,C))$, we have

$$\begin{aligned}\gamma(u,v) &\leq \gamma(u,v,C) = d_G(u,C) + d_G(v,C) + |C| - 1 \\ &< 2d_G(u,C) + |C| - 1 \leq n-3.\end{aligned}$$

(c) $d_G(u,C_d) = t+2, d_G(v,C_d) = t+1$

This suggests that $v \in V(P(u,C))$, i.e. $uv \in E(P(u,C))$, hence we have

$$\begin{aligned}\gamma(u,v) &\leq \gamma(u,v,C) = d_G(u,C) + d_G(v,C) + |C| - 1 \\ &< 2d_G(u,C) + |C| - 1 \leq n-3.\end{aligned}$$

(d) $d_G(u,C_d) = t+1, d_G(v,C_d) = t$

The argument is similar to (a).

(e) $d_G(u,C_d) = t+1, d_G(v,C_d) = t+1$

Let $uw \in E(P(u,C))$,there must be $vw \in E(G) \setminus (E(K_d) \cup E(P(u,C)))$. Hence we have

$$\begin{aligned}\gamma(u,v) &\leq \gamma(u,v,C) \leq d_G(u,C) + d_G(v,C) + |C| - 1 \\ &= 2d_G(u,C) + |C| - 1 \leq n-3.\end{aligned}$$

(f) $d_G(u,C_d) = t+1, d_G(v,C_d) = t+2$

The argument is similar to (c).

(II) $V(C) \cap V(C_d) = \phi, V(C) \cap V(P_t) \neq \phi$

Let $u_i, u_j \in V(C) \cap V(P_t)$ be the vertexes with the smallest and biggest subscripts respectively, where $i \leq j \leq t-1$. By the construct qualification (2), we have

$$2d(u_0,u_i) + |C| > 2d(u_0,u_t) + d,$$

i.e.

$$\frac{1}{2}(|C|-1) \geq t - i + \frac{1}{2}(d+1).$$

By $d(u,C_d) \geq t+1$, we have

$$d(u,C) + \frac{1}{2}(|C|-1) + (t-j) \geq t+1,$$

i.e.

$$d(u,C) + \frac{1}{2}(|C|-1) \geq j+1.$$

Hence,

$$d(u,c) + |C| - (j-i+1) \geq t+1+\frac{1}{2}(d+1).$$

In addition, notice that $|V_2(d)| = t+2$. We have

$$d(u,C) + |C| - (j-i+1) \leq t+2.$$

So we have

$$t+1+\frac{1}{2}(d+1) \leq d(u,c) + |C| - (j-i+1) \leq t+2.$$

This means

$$d = 1, |C| = 2t - 2i + 3,$$

and

$$d(u, C) = i + j - t(i + j \geq t).$$

If $v \in V(M)$, it is obvious that

$$\gamma(u, v) < \gamma(u, C) \leq \gamma(G).$$

If $v \notin V(M)$, clearly we have

$$\begin{aligned} \gamma(u, v) &\leq \gamma(u, u_0) \leq \gamma(u, u_0, C) \leq d(u, C) + d(u_0, C) + \frac{1}{2}(|C| - 1) \\ &= (i + j - t) + i + (t - i + 1) = i + j + 1 < \gamma(G). \end{aligned}$$

(III) $V(C) \cap V(C_d) = \phi, V(C) \cap V(P_t) = \phi$

Let $u_i \in V(P(u, C_d)) \cap V(P_t)(i \leq t)$ be the vertex with the smallest subscript, then $P(u, u_i)$ is the shortest path from u to P_t. We shall discuss in the two following cases.

(a) Suppose C and $P(u, u_i)$ have at least two intersected vertexes. Then $|C| \geq 3$.

Let $v \in V(M)$.If $P(u, C)$ intersects with $P(v, C)$, then we have

$$\begin{aligned} \gamma(u, v) &\leq \gamma(u, v, C) \\ &\leq 2\max\{d(u, C), d(v, C)\} + |C| - 1 \\ &\leq 2(|V_2(d)| - |C|) + |C| - 1 \\ &= 2t - |C| + 3 \leq 2t \leq \gamma(G). \end{aligned}$$

If $P(u, C)$ doesn't intersect with $P(v, C)$, then we have

$$\gamma(u, v) \leq \gamma(u, v, C) \leq |V_2(d)| - 1 = t + 1 \leq \gamma(G).$$

Let $v \notin V(M)$ and $|V_1^{'}| = |V_1(d) \setminus V(P(u_0, u_i))| \geq 2$. Then we have

$$\gamma(u, v) \leq \gamma(u, v, C) \leq n - |V_1^{'}| - 1 \leq n - 3 = \gamma(G).$$

If $|V_1^{'}| = 1$, it means that $i = t - 1$ and $d = 1$. Note that $d(u, C_d) \geq t + 1$, we have $d(u, u_i) \geq i + 1 = t$. Note that $|V_2(d)| = t + 2, |C| \geq 3$,we have $|C| \leq 5$: if $|C| = 3$, there must be only two intersected vertexes of C and $P(u, u_i)$; if $|C| = 5$, there must be just three intersected vertexes of C and $P(u, u_i)$. Thus we can easily have

$$\gamma(u, v) \leq \gamma(u, u_0, C) \leq 2t \leq \gamma(G).$$

(b) Suppose that there is at most one intersected vertex of C and $P(u, u_i)$. Let $P(w, z)$ be the shortest path from C to $P(u, u_i)$, where $w \in V(P(u, u_i))$ and $z \in V(C)(w = z$ suggests that there is only one intersected vertex of C and $P(u, u_i))$.

Let $v \in V(M)$. If $P(u, C)$ doesn't intersect with $P(v, C)$, we have

$$\gamma(u, v) \leq \gamma(u, v, C) \leq |V_2(d)| - 1 = t + 1 \leq \gamma(G).$$

If $P(u,C)$ intersects with $P(v,C)$, note that $2d(u,C)+|C|\le 2t+d$, we then have

$$d(u,C)\le t+\frac{1}{2}(d-|C|).$$

If $d(v,C)<t+2-|C|$, i.e. $d(v,C)+|C|-1\le t$, we have

$$\begin{aligned}\gamma(u,v) &\le \gamma(u,v,C)\le d(u,C)+d(v,C)+|C|-1\\ &\le (t+\frac{1}{2}(d-|C|))+t\le 2t+d-1=\gamma(G).\end{aligned}$$

If $d(v,C)\ge t+2-|C|$, note that $d(v,C)+|C|\le |V_2(d)|=t+2$, we then have

$$d(v,C)=t+2-|C|.$$

Now it is clear that u is just on $P(v,C)$ and $d(v,C_d)\ge t+1$. So there must be an odd cycle C' such that

$$2d(v,C')+|C'|\le 2t+d.$$

If C' is a loop on $P(u,v)$, we then have

$$\gamma(u,v)\le d(u,v)\le diam(G)\le \gamma(G).$$

Otherwise, C' doesn't intersect with $P(u,v)$. This suggests that $d(u,C')\le d(v,C')$. Hence we have

$$\gamma(u,v)\le \gamma(v,C')\le \gamma(G).$$

If $|C'|\ge 3$,then C' must intersects with C. Similarly, $d(u,C')\le d(v,C')$. So we have

$$\gamma(u,v)\le \gamma(v,C')\le \gamma(G).$$

Let $v\notin V(M)$. Note that

$$\gamma(u,u_0,C)=d(u,u_0)+2d(w,z)+|C|-1,$$
$$\gamma(u,u_0,C_d)=d(u,u_0)+2d(u_i,u_t)+d-1,$$

we have

$$\begin{aligned}\gamma(u,u_0,C) &+ \gamma(u,u_0,C_d)\\ &= 2(d(u,u_0)+d(u_i,u_t)+d(w,z)+|C|+d-1)-(d+|C|).\end{aligned}$$

If $d+|C|\ge 6$, we have

$$\gamma(u,u_0)=\min\{\gamma(u,u_0,C),\gamma(u,u_0,C_d)\}\le n-3.$$

Therefore, we get

$$\gamma(u,v)\le \gamma(u,u_0)\le \gamma(G).$$

In what follows, it suffices to discuss the case such that $|C|+d\le 4$.

Suppose that $|C|+d=4$ and $d(u,u_i)+d(w,z)+|C|\le t+2$, we have

$$d(u,u_i)+d(w,z)+|C|-1\le t+1=|V_2(d)|-1,$$

i.e.

$$d(u,u_0)+d(u_i,u_t)+d(w,z)+|C|+d-1\le n-1.$$

Hence we have

$$\gamma(u,u_0,C)+\gamma(u,u_0,C_d)\le 2(n-1)-4=2(n-3).$$

This suggests that

$$\min\{\gamma(u,u_0,C),\gamma(u,u_0,C_d)\}\le n-3.$$

Suppose that $d(u,u_i)+d(w,z)+|C|\ge t+3$, note that

$$d(u,u_i)+d(w,z)+|C|-1\le |V_2(d)|=t+2,$$

we then have

$$d(u,u_i)+d(w,z)+|C|-1=|V_2(d)|,$$

i.e.

$$d(u,u_0)+d(u_i,u_t)+d(w,z)+|C|+d-1=n.$$

Hence

$$\gamma(u,u_0,C)+\gamma(u,u_0,C_d)\le 2n-4=2(n-2).$$

By the construction of the G, we have

$$2d(w,z)+|C|\ne 2(t-i)+d,$$

i.e.

$$\gamma(u,u_0,C)\ne\gamma(u,u_0,C_d).$$

This suggests that

$$\min\{\gamma(u,u_0,C),\gamma(u,u_0,C_d)\}\le n-3.$$

Suppose that $|C|=d=1$ and $d(u,u_i)+d(w,z)\le t$, we then have

$$d(u,u_i)+d(w,z)+|C|-1=t=|V_2(d)|-2,$$

i.e.

$$d(u,u_0)+d(u_i,u_t)+d(w,z)+|C|+d-1\le n-2.$$

We then have

$$\gamma(u,u_0,C)+\gamma(u,u_0,C_d)\le 2(n-2)-2=2(n-3).$$

Thus we have

$$\min\{\gamma(u,u_0,C),\gamma(u,u_0,C_d)\}\le n-3.$$

Suppose that $d(u,u_i)+d(w,z)\ge t+1$. Note that

$$d(u,u_i)+d(w,z)+|C|-1\le |V_2(d)|=t+2,$$

we then have

$$t+1 \le d(u,u_i)+d(w,z) \le t+2.$$

If $d(u,u_i)+d(w,z)=t+1$, we thus get

$$d(u,u_0)+d(u_i,u_t)+d(w,z)+|C|+d-1=n-1.$$

It means that

$$\gamma(u,u_0,C)+\gamma(u,u_0,C_d)=2(n-1)-2=2(n-2).$$

Note that $d(w,z)\neq t-i$, we have

$$\gamma(u,u_0,C)\neq\gamma(u,u_0,C_d).$$

We therefore get

$$\min\{\gamma(u,u_0,C),\gamma(u,u_0,C_d)\}\le n-3.$$

Suppose that $d(u,u_i)+d(w,z)=t+2$, then we have

$$d(u,u_0)+d(u_i,u_t)+d(w,z)+|C|+d-1=n.$$

Hence

$$\gamma(u,u_0,C)+\gamma(u,u_0,C_d)=2n.$$

If $|d(w,z)-t-i|>6$, we then get

$$|\gamma(u,u_0,C)-\gamma(u,u_0,C_d)|>6.$$

This suggests that

$$\min\{\gamma(u,u_0,C),\gamma(u,u_0,C_d)\}\le n-3.$$

From those as above, we can easily get

$$\gamma(u,v)\le\gamma(u,u_0)\le\gamma(G).$$

Hence, $\forall\ u,v\in V(G)$, we have $\gamma(u,v)\le n-3$. □

Theorem 4.2 *G is a graph with order n of the second type with $\gamma(G)=n-3$ iff $G\in\mathcal{M}_{n-3}$.*

Proof For the sufficiency, $\forall\ G\in\mathcal{M}_{n-3}$, we have $\gamma(G)=n-3$ and $\gamma(w)<\gamma(G)$ for all $w\in V(G)$ by a direct verification.

Now for the necessity, suppose G is a graph of order n of the second type with $\gamma(G)=n-3$. Then there must be two distinct vertexes u and v and an odd cycle C_0 such that

$$\gamma(u,v)=\gamma(u,v,C_0)=\gamma(G)=n-3.$$

We put the path $P(u,v)=v_0v_1\cdots v_m$, where $v_0=u$ and $v_m=v$ with $n-3\equiv m(\mathrm{mod}\ 2)$. Without loss of generality,we set

$$n-3=m+2r, d_0=|C_0|\equiv 1(\mathrm{mod}\ 2).$$

Suppose that C is an odd cycle in G, then we have

$$|V(P(u,v)) \cap V(C)| \leq n - \gamma(G) = 3.$$

In the following, we shall complete our arguments in four cases.

(I) Suppose that any odd cycle doesn't intersect with any (u,v)-shortest path in G, then we have

$$t_0 = d_G(P(u,v), C_0) > 0.$$

By the equation $\gamma(u,v) = \gamma(u,v,C_0)$, we can easily get

$$n = m + 2t_0 + d_0 + 2.$$

We put the path $P_0 = P(P(u,v), C_0) = x_0x_1 \cdots x_{t_0}$, where $x_0 = v_j$ and $x_{t_0} \in V(C_0)$. Set

$$V_1 = V(P(u,v)) \cup V(C_0) \cup V(P_0), V_2 = V(G) \setminus V_1,$$

then we have

$$|V_1| = m + t_0 + d_0, |V_2| = t_0 + 2.$$

Suppose that the odd cycles C_1 and C_2 satisfy the following qualifications respectively.

$$\gamma(u) = \gamma(u, C_1), \gamma(v) = \gamma(v, C_2).$$

If $V(P(u,C_1)) \cap V(P_0) \neq \phi$ and $V(P(v,C_2)) \cap V(P_0) \neq \phi$, it is clear that

$$\gamma(u,C_1) = \gamma(u,C_0), \gamma(v,C_2) = \gamma(v,C_0).$$

Hence

$$\begin{aligned}
\gamma(u) &= \gamma(u,C_0) = 2d_G(u,C_0) + d_0 - 1 = 2d_G(u,x_{t_0}) + d_0 - 1 < n-3, \\
\gamma(v) &= \gamma(v,C_0) = 2d_G(v,C_0) + d_0 - 1 = 2d_G(v,x_{t_0}) + d_0 - 1 < n-3.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\gamma(G) = \gamma(u,v) &= \gamma(u,v,C_0) = d_G(u,x_{t_0}) + d_G(v,x_{t_0}) + d_0 - 1 \\
&< n - 3 = \gamma(G),
\end{aligned}$$

a contradiction. So we assume $V(P(u,C_1)) \cap V(P_0) = \phi$ without loss of generality. Suppose $v_i \in V(P(u,C_1)) \cap V(P(u,v))$ is the intersected vertex with the biggest subscript, put the path $P_1 = P(P(u,v), C_1) = y_0y_1 \cdots y_{t_1}$ with $d_1 = |C_1|$ and $t_1 = d_G(v_i, C_1)$, where $y_0 = v_i$ and $y_{t_1} \in V(C_1)$. Then we have $V(P_0) \cap V(P_1) = \phi (i < j)$ and

$$t_1 \leq t_1 + d_1 - 1 \leq |V_2| \leq t_0 + 2.$$

By the choose of $P(u,v)$ and C_0, we have

$$2t_0 + d_0 \leq 2t_1 + d_1.$$

Hence

$$2t_1 + 2d_1 - 6 + d_0 \leq 2t_0 + d_0 \leq 2t_1 + d_1 \leq 2t_0 + 5.$$

So we get

$$2 \leq d_0 + d_1 \leq 6, |t_0 - t_1| \leq 2.$$

Set $K_{d_0,d_1} = P(u,v) \cup P_0 \cup P_1 \cup C_0 \cup C_1$, then we have

$$|V(K_{d_0,d_1})| = m + t_0 + t_1 + d_0 + d_1 - 1 \leq n,$$

and

$$|V(G) \setminus V(K_{d_0,d_1})| = t_0 + 3 - t_1 - d_1 \leq 2.$$

It is easy to verify that G meets the first three construct qualifications (1),(2) and (3) of T_{d_0,d_1}. We shall prove that G meets the other qualifications:

Suppose there exists a (x_a, y_b)-path with length p which connects $P_0 \cup C_0$ to $P_1 \cup C_1$ in $G - E(K_{d_0,d_1})$, where $0 \leq a \leq t_0$ and $0 \leq b \leq t_1$. Clearly,$p = t_0 + 4 - t_1 - d_1 \leq 3$. Because any odd cycle doesn't intersect with the (u,v)-shortest path $P(u,v)$, then we have

$$a + b + p > j - i, a + b + i + j \equiv p (\text{mod } 2),$$

and

$$\begin{aligned} d_G(v_0,v_i) + d_G(v_i,y_b) + p \quad &+ \quad d_G(x_a,v_j) + d_G(v_j,v_m) + 2d_G(x_a,x_{t_0}) + |C_0| - 1 \\ &\geq \quad \gamma(u,v), \end{aligned}$$

and

$$\begin{aligned} d_G(v_0,v_i) + d_G(v_i,y_b) + p \quad &+ \quad d_G(x_a,v_j) + d_G(v_j,v_m) + 2d_G(y_b,y_{t_1}) + |C_1| - 1 \\ &\geq \quad \gamma(u,v). \end{aligned}$$

Hence we have

$$i + b + p + a + m - j + 2(t_0 - a) + d_0 - 1 \geq m + 2t_0 + d_0 - 1,$$

and

$$i + b + p + a + m - j + 2(t_1 - b) + d_1 - 1 \geq m + 2t_0 + d_0 - 1.$$

Therefore,we get

$$(2t_0 + d_0) - (2t_1 + d_1) - (p + i - j) \leq a - b \leq p + i - j.$$

The qualification (4) thus meets.

Suppose that there exists a vertex x in G such that $d_G(x, C_0) \geq t_0$ and $d_G(x, C_1) \geq t_0 + \frac{1}{2}(d_0 - d_1)$, then we can conclude that

$$\gamma(x, C_0) \leq \gamma(G), \gamma(x, C_1) \leq \gamma(G).$$

Hence there must be an odd cycle C in G such that

$$\gamma(x) = \gamma(x, C) < \gamma(G),$$

i.e.

$$2d_G(x,C)+|C|-1<n-3=m+2r.$$

The fifth qualification (5) meets.

Clearly, $G\subseteq T$, then we have $G\in \mathcal{M}^{(0)}_{n-3}$.

(II) Suppose that there exists an odd cycle C and a path $P(u,v)$ in G such that $|V(P(u,v))\cap V(C)|=1$, but for any odd cycle C' and any (u,v)-shortest path $P'(u,v)$, $|V(P'(u,v))\cap V(C')|\geq 2$ doesn't come into existence. Without loss of generality, we assume that C is the smallest odd cycle which meets the above qualifications, and $V(P(u,v))\cap V(C)=\{v_i\}$. Hence

$$n-3=\gamma(u,v)\leq\gamma(u,v,C)=m+|C|-1=|V(P(u,v)\cup C)|-1\leq n-1.$$

Note that $n-3\equiv m(\mathrm{mod}\ 2)$, we have $m+|C|=n-2$ or $m+|C|=n$. Suppose that $m+|C|=n$, then we can assert that C isn't a primitive cycle of $P(u,v)$, and $V(G)=V(P(u,v))\cup V(C)$. Note that

$$m=d_G(u,v)\leq\gamma(u,v)=n-3,$$

then we have $|C|\geq 3$. Suppose that the odd cycles C_1 and C_2 satisfy the following equations respectively:

$$\gamma(u)=\gamma(u,C_1),\gamma(v)=\gamma(v,C_2).$$

If both C_1 and C_2 intersect with C, clearly we have

$$\gamma(u,C_1)=\gamma(u,C),\gamma(v,C_2)=\gamma(u,C).$$

It is easy to verify

$$\max\{\gamma(u,C),\gamma(v,C)\}\geq\gamma(G),$$

a contradiction. Hence, we assume that C_1 doesn't intersect with C without loss of generality. Therefore, C_1 must intersect with $P(u,v)$, and C_1 is a loop on v_i of $P(u,v)$(without loss of generality,we set $i<j$). This contradicts the choose of C.Hence,$m+|C|\neq n$. Therefore,we set $m+|C|=n-2$. We then have

$$n-3=\gamma(u,v)=\gamma(u,v,C).$$

We might as well put the cycle $C=C_0=y_0\cdots y_{t_0}x_{t_0}\cdots x_1y_0$, where $y_0=v_i$. Thus, we have

$$|C|=2t_0+1,n=m+2t_0+3.$$

We put the graph $K_{m,t_0}=P(u,v)\cup C_0$. It is obvious that its order is $n-2$. It is easy to verify that G meets the first three construct qualifications (1), (2) and (3) of T_{m,t_0}. We shall prove that G meets the other qualifications:

Suppose that there is a (x_a,y_b)-path with length p which divides up C_0 in $G-E(K_{m,t_0})$, where $0\leq a,b\leq t_0$.Clearly,$p\leq 3$. Note that any odd cycle in G has only one intersected vertex with the (u,v)-shortest path, and C_0 is the smallest odd cycle which has only one intersected vertex with $P(u,v)$:

If $a+b+1 \equiv p(\text{mod } 2)$, then we have

$$d_G(v_i, x_a) + d_G(v_i, y_b) + p \geq |C_0|,$$

i.e.

$$a + b + p \geq 2t_0 + 1.$$

If $a+b \equiv p(\text{mod } 2)$, then we have

$$d_G(v_0, v_m) + 2d_G(x_a, v_i) + |C_0| - d_G(x_a, v_i) - d_G(v_i, y_b) + p - 1 \geq \gamma(u, v)$$

and

$$d_G(v_0, v_m) + 2d_G(v_i, y_b) + |C_0| - d_G(x_a, v_i) - d_G(v_i, y_b) + p - 1 \geq \gamma(u, v),$$

i.e.

$$m + a + 2t_0 - b + p \geq m + 2t_0$$

and

$$m + b + 2t_0 - a + p \geq m + 2t_0.$$

Hence, $|a-b| \leq p$, and the fourth qualification (4) comes into existence.

Suppose there exists a vertex x in G such that $d_G(x, C_0) \geq \frac{1}{2}m$, then we have $\gamma(x, C_0) \geq \gamma(G)$. Therefore, there must be an odd cycle C such that

$$\gamma(x) = \gamma(x, C) < \gamma(G),$$

i.e.

$$2d_G(x, C) + |C| - 1 < n - 3 = m + 2r.$$

The fifth qualification (5) comes into existence.

Clearly, $G \subseteq T$, then we have $G \in \mathcal{M}^{(1)}_{n-3}$.

(III) Suppose that there exists an odd cycle C and a path $P(u,v)$ in G such that $|V(P(u,v)) \cap V(C)| = 2$, but for any odd cycle C' and any (u,v)-shortest path $P'(u,v)$, $|V(P'(u,v)) \cap V(C')| \geq 3$ doesn't come into existence. Without loss of generality, we assume that C is the smallest odd cycle satisfying the above qualifications, and $V(P(u,v)) \cap V(C) = \{v_i, v_j (i < j)\}$. Clearly, $j = i+1$. Hence, we have

$$\begin{aligned} n - 3 &= \gamma(u,v) \leq \gamma(u,v,C) = i + (m - j) + |C| - 2 \\ &= m + |C| - 3 = |V(P(u,v) \cup C)| - 2 \leq n - 2. \end{aligned}$$

Note that $n - 3 \equiv m(\text{mod } 2)$. We have $\gamma(u,v) = \gamma(u,v,C)$. Hence, C is a (u,v)-primitive cycle, where $n = m + |C|$. We might as well put the cycle $C = C_0 = y_0 \cdots y_{t_0} z x_{t_0} \cdots x_0 y_0$,where $y_0 = v_i$ and $x_0 = v_{i+1}$. Hence, we have

$$|C_0| = 2t_0 + 3, n = m + 2t_0 + 3.$$

We put the graph $K_{m,t_0} = P(u,v) \cup C_0$. It is obvious that its order is $n-1$. It is easy to verify that G meets the first three construct qualifications (1),(2) and (3) of T_{m,t_0}. We shall prove that G meets the other qualifications:

Suppose that there exists a (x_a, y_b)-path with length p which divides up C_0 in $G-E(K_{m,t_0})$, where $0 \le a, b \le t_0$.Clearly,$p \le 2$. Note that any odd cycle has at most two intersected vertexes with any (u,v)-shortest path in G, and C_0 is the smallest odd cycle which has just two intersected vertexes with $P(u,v)$.

(a) If $a+b+1 \equiv p(\text{mod } 2)$, then we have

$$\begin{aligned} d_G(v_0,v_m)+2d_G(v_j,x_a)+|C_0| &- d_G(v_j,x_a)-d_G(v_i,y_b)-d_G(v_i,v_j)+p-1 \\ &\ge \gamma(u,v) \end{aligned}$$

and

$$\begin{aligned} d_G(v_0,v_m)+2d_G(v_i,y_b)+|C_0| &- d_G(v_j,x_a)-d_G(v_i,y_b)-d_G(v_i,v_j)+p-1 \\ &\ge \gamma(u,v). \end{aligned}$$

Hence, we have

$$m+a+2t_0-b+p+1 \ge m+2t_0$$

and

$$m+b+2t_0-a+p+1 \ge m+2t_0.$$

Therefore, we have

$$|a-b| \le p+1.$$

(b) If $a+b \equiv p(\text{mod } 2)$, because C_0 is the the smallest odd cycle which has just two intersected vertexes with $P(u,v)$, we have

$$d_G(v_j,x_a)+d_G(v_i,y_b)+d_G(v_i,v_j)+p \ge |C_0|.$$

We thus get

$$a+b+p \ge 2t_0+2.$$

Suppose that there exists a (z,x_a)-path(or (z,y_b)-path) with length p in $G-E(K_{m,t_0})$ which divides up C_0, where $0 \le a,b \le t_0$. Clearly, $p \le 2$. If $a+t_0+1 \equiv p(\text{mod } 2)$, note that C_0 is the smallest odd cycle which has just two intersected vertexes with $P(u,v)$, then we have

$$d_G(v_j,x_a)+p+d_G(z,y_{t_0})+d_G(y_{t_0},v_i)+d_G(v_i,v_j) \ge |C_0|.$$

We thus get

$$a+p \ge t_0+1.$$

If $a+t_0 \equiv p(\text{mod } 2)$, then we have

$$d_G(v_0,v_m)+2d_G(x_a,v_j)+(p+d_G(x_a,x_{t_0})+d_G(z,x_{t_0}))-1 \ge \gamma(u,v),$$

i.e.

$$m+2a+(p+t_0-a+1)-1 \ge m+2t_0.$$

We then get

$$a+p \ge t_0.$$

Using analogous argument, we can get the corresponding restrained qualifications for b. Hence the fourth construct qualification (4) comes into existence.

Suppose that there exists a vertex x in G such that $d_G(x, C_0) \geq \frac{1}{2}m - 1$, then we have $\gamma(x, C_0) \geq \gamma(G)$. Hence, there must exist some odd cycle C such that

$$\gamma(x) = \gamma(x, C) < \gamma(G),$$

i.e.

$$2d_G(x, C) + |C| - 1 < n - 3 = m + 2r.$$

The fifth qualification (5) thus comes into existence.

Clearly, $G \subseteq T$, then we have $G \in \mathcal{M}^{(2)}_{n-3}$.

(IV) Suppose that there exists an odd cycle C and a path $P(u, v)$ in G such that $|V(P(u, v)) \cap V(C)| = 3$. Without loss of generality,we assume that C is the smallest odd cycle which meets the above qualifications, where $V(P(u, v)) \cap V(C) = \{v_i, v_k, v_j(i < k < j)\}$. Clearly, $j = k + 1, i = k - 1$. Hence,we have

$$\begin{aligned} n - 3 &= \gamma(u, v) \leq \gamma(u, v, C) = i + (m - j) + |C| - 3 \\ &= m + |C| - 5 = |V(P(u, v) \cup C)| - 3 \leq n - 3. \end{aligned}$$

Therefore, we have

$$\gamma(u, v) = \gamma(u, v, C), |V(P(u, v)) \cup V(C)| = |V(G)| = n.$$

We might as well put the cycle $C = C_0 = y_0y_1 \cdots y_{t_0}x_{t_0} \cdots x_1x_0wy_0$, where $y_0 = v_{k-1}, w = v_k$ and $x_0 = v_{k+1}$. Hence, we have

$$|C_0| = 2t_0 + 3, n = m + 2t_0 + 1.$$

We put the graph $K_{m,t_0} = P(u, v) \cup C_0$. It is easy to verify that G meets the first three construct qualifications (1), (2) and (3) of T_{m,t_0}. We shall prove that G meets the other qualifications:

Suppose that there exists an edge x_ay_b in $G - E(K_{m,t_0})$ which divides C_0, where $0 \leq a, b \leq t_0$. Note that C_0 is the smallest odd cycle which has just three intersected vertexes with the (u, v)-shortest path $P(u, v)$, then we have $a + b \equiv 1(\text{mod } 2)$. In addition, we have

$$\begin{aligned} d_G(v_0, v_m) + 2d_G(x_a, v_j) &+ |C_0| - d_G(x_a, v_j) - d_G(v_i, v_j) - d_G(v_i, y_b) \\ &\geq \gamma(u, v) \end{aligned}$$

and

$$\begin{aligned} d_G(v_0, v_m) + 2d_G(v_i, y_b) &+ |C_0| - d_G(x_a, v_j) - d_G(v_i, v_j) - d_G(v_i, y_b) \\ &\geq \gamma(u, v). \end{aligned}$$

Hence, we have

$$m + a + 2t_0 - b + 1 \geq m + 2t_0 - 2$$

and

$$m + b + 2t_0 - a + 1 \geq m + 2t_0 - 2.$$

We thus get $|a-b| \leq 3$.

Suppose that there exists an edge $v_k x_a$ in $G - E(K_{m,t_0})$ which divides up C_0, where $1 \leq a \leq t_0$.

If a is an odd number, then we have

$$d_G(v_0, v_m) - d_G(v_k, v_j) + 1 + d_G(v_j, x_a) - 1 \geq \gamma(u, v),$$

i.e.

$$m - 1 + 1 + a - 1 \geq m + 2t_0 - 2.$$

Thus, we have $a \geq 2t_0 - 1$. Therefore, we have $t_0 = 1, a = 1$.

If a is an even number, then we have

$$\begin{aligned} d_G(v_0, v_m) - d_G(v_i, v_k) \quad &+ \quad |C_0| - d_G(v_j, x_a) - d_G(v_i, v_j) + d_G(v_k, x_a) - 1 \\ &\geq \quad \gamma(u, v), \end{aligned}$$

i.e.

$$m - 1 + (2t_0 + 3) - a - 2 + 1 - 1 \geq m + 2t_0 - 2,$$

We thus get $a \leq 2$. Hence, we have $a = 2$.

Suppose that there is an edge $v_k y_b$ in $G - E(K_{m,t_0})$ which divides C_0, where $1 \leq b \leq t_0$. Using an analogous argument, we have $b = 2$, or $b = 1$ (iff $t_0 = 1$). Hence, the fourth qualification (4) comes into existence.

Suppose that there is a vertex x in G such that $d_G(x, C) \geq \frac{1}{2}m - 2$, then we have $\gamma(x, C) \geq \gamma(G)$. Therefore, there must be an odd cycle C such that

$$\gamma(x) = \gamma(x, C) < \gamma(G),$$

i.e.

$$2d_G(x, C) + |C| - 1 < n - 3 = m + 2r.$$

Hence, the fifth qualification (5) comes into existence.

Clearly,$G \subseteq T$, then we have $G \in \mathcal{M}_{n-3}^{(3)}$. □

Using the connection between the exponent of a matrix and the exponent of a graph stated above, we have get the following result by combining Theorems 4.1 with 4.2.

Theorem 4.3 *Let A be a symmetric primitive matrix with order n, then $\gamma(A) = n - 3$ iff $G(A) \in \mathcal{N}_{n-3} \cup \mathcal{M}_{n-3}$.*

References

[1] J.A.Bondy and U.S.R.Murty, *Graph Theory with Applications*, Macmillan Press, London(1976).

[2] R.A.Brualdi and H.J.Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, New York(1991).

[3] B.L.Liu, *Combinatorial Matrix Theory(second edition,in chinese)*, Sience Press, Beijing(2005).

[4] J.Y.Shao, The exponent set of symmetric primitive matrices, *Scientia Sinica*, A, Vol.9 (1986), pp.931-939.

[5] J.Z.Wang, The character of symmetric primitive matrices with certain exponents, *J.Taiyuan Machinery College.*, (1991)No.2.

[6] G.R.Li, The characterization of symmetric primitive matrices with exponent $2n-6$, *J.Nanjing Univ.*(Graph Theory), Vol.27(1991),pp.87-92.

[7] J.L.Cai and K.M.Zhang, The characterization of symmetric primitive matrices with exponent $2n-2r(\geq n)$, *Linear Multilin.Alg.*, 39(1995), pp.391-396.

[8] J.L.Cai and B.Y.Wang, The characterization of symmetric primitive matrices with exponent $n-1$, *Linear Alg.Appl.*, 364(2003), pp.135-145.

[9] B.L.Liu, B.D.McKay, N.C.Wormald and K.M.Zhang, The exponent set of symmetric primitive (0,1) matrices with zero trace, *Linear Alg.Appl.*, 133(1990), pp.121-131.