The Concept of Place Value

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Version 1.0, August 9, 2023

1. Motivation and aim of this paper

An insufficient development of place value understanding is one of the main causes of mathematical weakness. Even though children already need a place value understanding from Grade 1 on (e.g., interpreting the number symbol 14 as 1 bundle of tens and 4 ones), many children in upper primary and junior secondary school still struggle with place value (e.g., with decimal numbers).

Reading research publications indicates that researchers use different terminology when discussing place value, and this can cause misunderstanding for teachers. For example, Ross (1989) states four "properties of our numeration system" that concern place value: (1) the positional property; (2) the base-ten property; (3) the multiplicative property; and (4) the additive property. Houdement and Tempier (2019) indicate that the essence of place value is the juxtaposition of digits, which enables the representation of any quantity, regardless of its size, and that within this representation each individual digit also denotes a quantity. The starting point for Houdement and Tempier is the notion of unit, the foundation of all number systems, where (in the decimal system) ten ones[1] form a new unit: one ten… and so on. Bailey (2015) indicates that a fundamental notion of base ten place value is the integration of early experiences of counting by ones and grouping numbers in tens; and later linking these groupings to the corresponding written numeral and how numbers are said.

In our view, any of the existing accounts of what constitutes place value understanding is incomplete in some way and has led us to the point where we feel that there is an important need to clarify the concept of place value. In this regard, the aims of this paper are:

- to clarify the concept of place value;
- to distinguish the overarching concept of place value from prerequisite knowledge and from several sub-concepts of the concept of place value; and
- to propose a way to develop a connected understanding of place value.

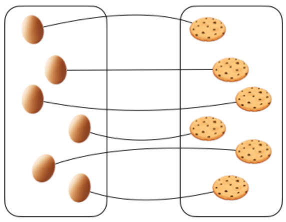
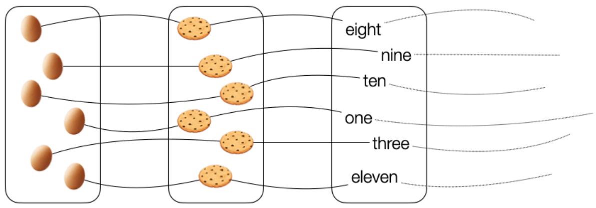
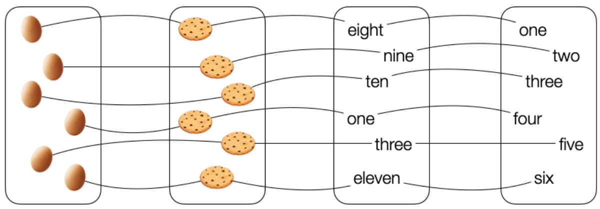
[1] We only use capitals when labelling columns in charts or when we use the abbreviation of the bundle units (e.g., H).

2. Prerequisite knowledge for the concept of place value

2.1. Cardinal numbers and counting

> The **concept of cardinal numbers** is the understanding that
> - we can build quantities,
> - number words (e.g., six) are designations for quantities with the same cardinality, and
> - number symbols (e.g., 6) are symbols for quantities with the same cardinality.

Essential for an understanding of the place value concept is the acquisition of the concept of cardinal numbers, and hence the requirement to understand the properties of quantities. There are several steps of abstraction needed to establish the concept of cardinal numbers (Table 1).

1st step	We compare different quantities (e.g., of eggs, biscuits) and explore whether they have the same cardinality. This is done by a **one-to-one-correspondence**.	
2nd step	We can also explore the quantity represented by number words by a one-to-one-correspondence.	
3rd step	To introduce designations for quantities, we need to take **an initial sequence of the infinite number word sequence** (one, two, three, four, five, six, …) and complete a one-to-one-correspondence. This is what we do when we **count an amount of objects**. Consequently, the number word sequence has to be reproduced in the correct order (also known as **principle of stable order**).	

4th step	All quantities, that have equal cardinality, are designated with the same name, that is the last word in the number word sequence (i.e., six in the example here). Already when counting an amount of objects, we need the **part-whole concept**, because we need to understand that the sum of all the single ones (parts) added together equal the amount of the whole (e.g., 1 + 1 + 1 + 1 + 1 + 1 = 6) (also known as **cardinal principle**).	"six"
5th step	We introduce the number symbol for this last word in the number sequence (e.g., 6).	6

Table 1: Steps of abstraction to establish the concept of cardinal numbers

If we count the following objects (Fig. 1)

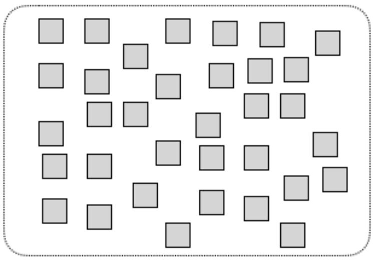


Fig. 1: An amount of objects

"one, two, …ten, …thirty, thirty-one, thirty-two, thirty-three, thirty-four",
it is not necessary that the *"ten"* has a special meaning. It can be a number word like any other number word (e.g., ten things is one more than nine things and one less than eleven things). And likewise the symbol "34" represents the number of objects, without the necessity of understanding that the symbol also represents three bundles of ten and four single ones. Thus, one may be able to say the number word (e.g., "thirty-four") and/or write the number symbol "34", without any concept of place value. Counting is, nonetheless, an important prerequisite knowledge for place value as, without being able to count, we are not able to **group** (as explained in section 2.3).

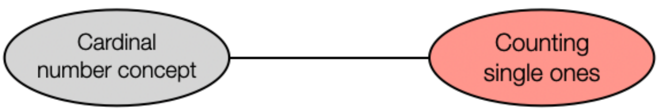


Fig. 2: Knowledge package of place value - part 1

2.2. The concept of part-whole

The **concept of part-whole** (Fig. 3) is the understanding that
- a whole can be partitioned into parts, and
- the amount of the whole is the sum of the amounts of the parts.

Fig. 3: Part-whole

The number of parts may be two (as in Fig. 3, left), but more than two parts are also possible (Fig. 3, right). For example, an amount of six eggs can be partitioned into, for example, two eggs and four eggs (Fig. 4 left), two eggs and three eggs and one egg (Fig. 4 middle), or two eggs and one egg and two eggs and one egg (Fig. 4 right). The total number of eggs is always six eggs (i.e., the sum of the amounts in each of the parts).

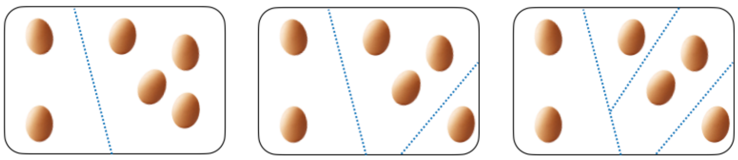


Fig. 4: Examples of part-whole with eggs

If a whole is partitioned into two parts, we call this a **number triple**, consisting of the whole and the two parts.

Written with number symbols, the part-whole-connection for the example above may be represented as in Fig. 5.

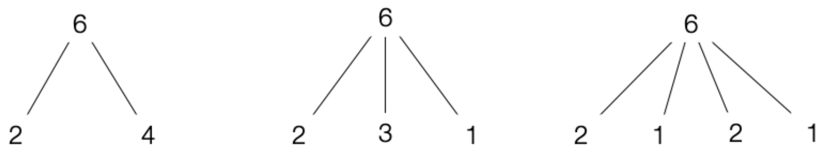


Fig. 5: Part-whole with number symbols

The acquisition of the part-whole concept is essential for several mathematical aspects (see Chapter 5). As well as its importance in understanding the concept of place value, the part-whole concept is also a foundational understanding for **addition** and **subtraction** (see Section 5.1):

$W = P_1 + P_2$	(e.g., $7 = 3 + 4$)	or	$P_1 + P_2 = W$ (e.g., $3 + 4 = 7$)
$W = P_2 + P_1$	(e.g., $7 = 4 + 3$)	or	$P_2 + P_1 = W$ (e.g., $4 + 3 = 7$)
$P_1 = W - P_2$	(e.g., $3 = 7 - 4$)	or	$W - P_2 = P_1$ (e.g., $7 - 4 = 3$)
$P_2 = W - P_1$	(e.g., $4 = 7 - 3$)	or	$W - P_1 = P_2$ (e.g., $7 - 3 = 4$)

Attention:
In order to conceptualise the equal sign as a relation, it is helpful to have the addition or subtraction sometimes on the left and sometimes on the right side of the equal sign.

If all of the parts are equal in size, the arithmetic operations of **multiplication** and **division** can be deduced (Fig. 6) (also see Section 5.2):

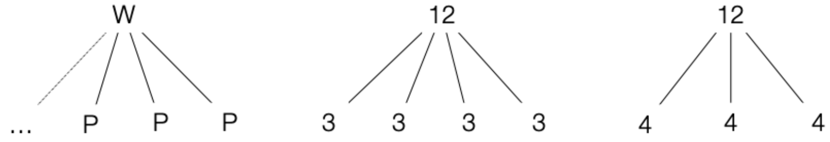


Fig. 6: From part-whole with equal parts to multiplication

$W = P + P + \ldots + P = n \times P$ (e.g., $12 = 4 + 4 + 4 = 3 \times 4$)
$W - P - P - \ldots - P = 0$; $W/P = n$ or $W/n = P$ (e.g., $12 - 4 - 4 - 4 = 0$; $12/4 = 3$ or $12/3 = 4$)

Equal-sized parts are also the basis for the **decimal part-whole concept** (see below, first step), as in the decimal part-whole concept the parts have the same size, namely, a power of ten (e.g., 10^0, 10^1, 10^2).

The **subconcept of decimal part-whole** is the understanding that
- a whole (multiples of the same power of tens, for example, $W_1 = n \times 10^0$, $W_2 = m \times 10^1$, $W_3 = k \times 10^2$) can be partitioned into parts that are all of the same power of tens (1st step, Fig. 7), and
- the amount of the whole (2nd step, Fig. 9) is the sum of the amounts of parts (multiples of powers of tens (for example, $P_1 = n \times 10^0$, $P_2 = m \times 10^1$, $P_3 = k \times 10^2$).

In this regard, we have two steps in understanding the decimal part-whole subconcept, in the direction from parts to whole:
- 1st step: The whole is the multiple of the same power of tens
  $W = n \times P = P_1 + P_2 + \ldots + P_n$ (Fig. 7).

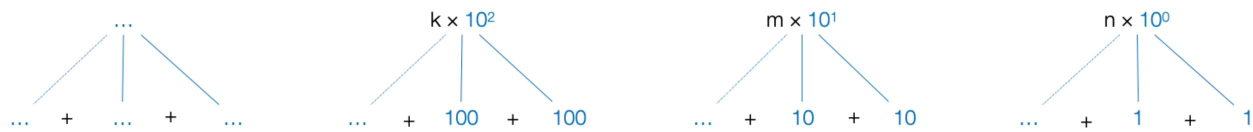


Fig. 7: There are n times ones ($n \times 10^0$), m times tens ($m \times 10^1$), k times hundreds ($k \times 10^2$), and so on.

We have this first step for each power of ten, for example,1 + 1 + 1 + 1 is the same as 4 × 1; 10 + 10 + 10 is the same as 3 × 10; and 100 + 100 + 100 + 100 + 100 is the same as 5 × 100 (Fig. 8).

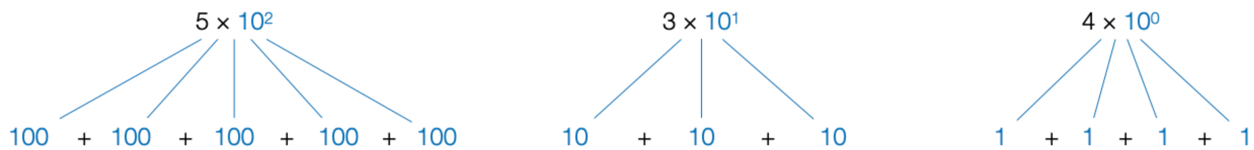


Fig. 8: Example for wholes that are multiples of powers of ten

This step demonstrates the **multiplicative property** of our decimal number system and is important in determining the **value of a digit** (see Section 3.2.2).

- 2^{nd} step: The amount of the whole is the sum of the amounts of the parts (see 1^{st} step) (Fig. 9).

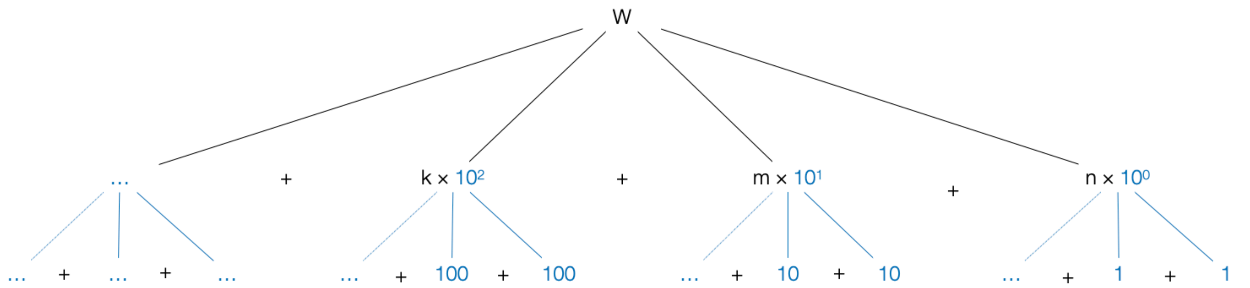


Fig. 9: The whole is the sum of n times ones ($n \times 10^0$), m times tens ($m \times 10^1$), k times hundreds ($k \times 10^2$), and so on: $W = n \times 10^0 + m \times 10^1 + k \times 10^2 + \ldots$

This step demonstrates the **additive property** of our decimal number system and is important in determining the **value of a number** (see Section 3.2.3).

In the example above, the whole would be $W = 5 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$ (Fig. 10).

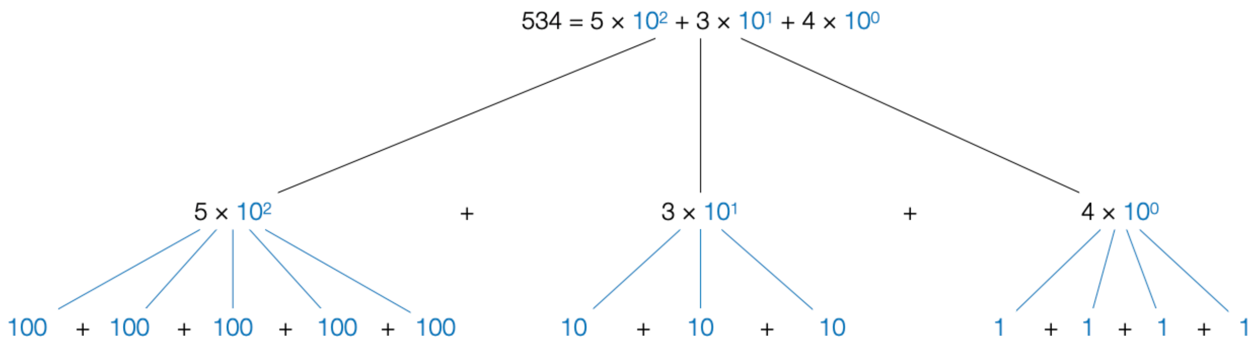


Fig. 10: Example for the whole $534 = 4 \times 10^0 + 3 \times 10^1 + 5 \times 10^2$

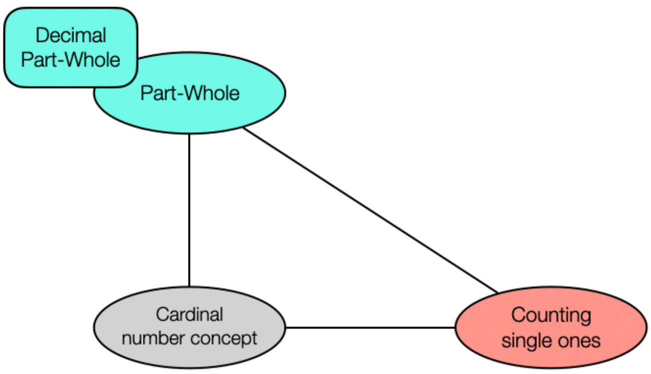


Fig. 11: Knowledge package of place value - part 2

2.3. The concept of grouping

The **concept of grouping an amount of objects** is the understanding that
- there are different possibilities to group for a given group size (in the sense of repeating the grouping),
- there are different possibilities to group due to the group size,
- the grouping can be repeated (in the sense of grouping groups), and
- the different possibilities to group with respect to a base are all equivalent to each other in terms of the total amount (application of the part-whole concept).

2.3.1. Different possibilities to group for a given group size (in the sense of repeating the grouping)

When counting larger amounts, it is very helpful to group objects (to ensure that no objects are uncounted, or are counted twice). For example, if there are 34 objects, we can make one group of 10 and we have 24 single ones left. However, it is also possible to form two or three groups of ten (Fig. 12). If we form groups of ten, repeating the grouping of ten is possible if there are **more than 19** objects.

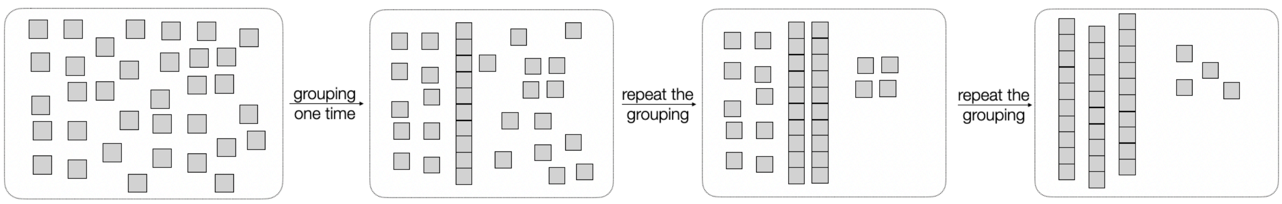


Fig. 12: Different possibilities to group (repeating the grouping)

In this regard, we develop the understanding that we can repeat the grouping (as seen in Fig. 12 in the 3rd and 4th example from left), and form more than one group of ten (or of other group sizes as shown in Section 2.3.2) - but we do not necessarily have to. Usually however, we try to form as many groups of the same size as possible. This corresponds to partitioning the amount into equal parts by division, which will often leave a remainder. For example, having 34 objects, we can form three groups of 10 (there are no more groups of ten possible), and four single ones remain (Fig. 13).

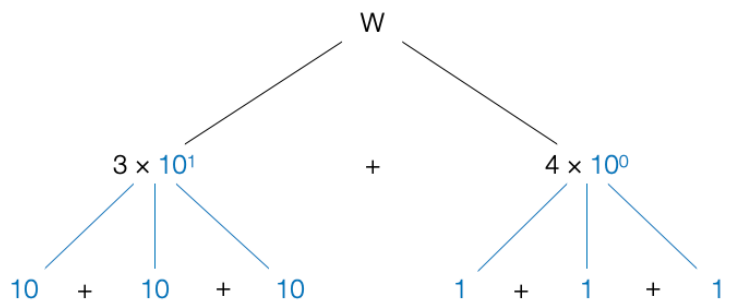


Fig. 13: Partitioning the amount into power of ten parts

When forming as many groups of a given size as possible, the number of groups is uniquely determined. This is important for **positional notation** (see Section 3.2).

2.3.2. Different possibilities to group due to the group size

There are always different possibilities to group an amount of objects (Fig. 14), depending on the group size.

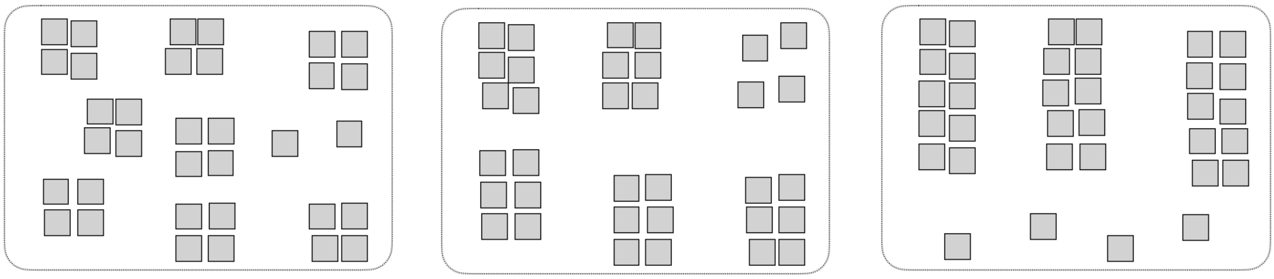


Fig. 14: 8 groups of four and 2 single ones (left), 5 groups of six and 4 single ones (middle), 3 groups of ten and 4 single ones (right)

When we use grouping, we can count to 34 in a range of ways. Depending on the chosen group size, the way we count changes, for example,:

- in groups of four and then ones: "*four, eight, twelve, sixteen, twenty, twenty-four, twenty-eight, thirty-two, thirty-three, thirty-four*" or
- in groups of six and then ones: "*six, twelve, eighteen, twenty-four, thirty, thirty-one, thirty-two, thirty-three, thirty-four*" or
- in groups of ten and then ones: *“ten, twenty, thirty, thirty-one, thirty-two, thirty-three, thirty-four”*.

Counting groups and single ones is referred to as **abbreviated counting**, and can be an efficient way to count an amount of objects.

Although the group size used for counting is arbitrary, it is often suggested by the context (e.g., when counting eggs we often use the group size 6, 10, or 12). Most often we use the group size 10, which refers to the **decimal property** of our number system.

2.3.3. Repeated grouping (in the sense of grouping groups)

Grouping ones into larger groups is a grouping of first **order**. Grouping can also be repeated in the same way by grouping the groups, for example, grouping 10 groups of tens into one group of a hundred, 10 groups of hundreds into one group of a thousand, and so on (Fig. 15). That leads to groupings of the 2nd, 3rd, 4th, … nth order. When grouping in tens, 2nd order grouping is only possible with **more than 99** objects.

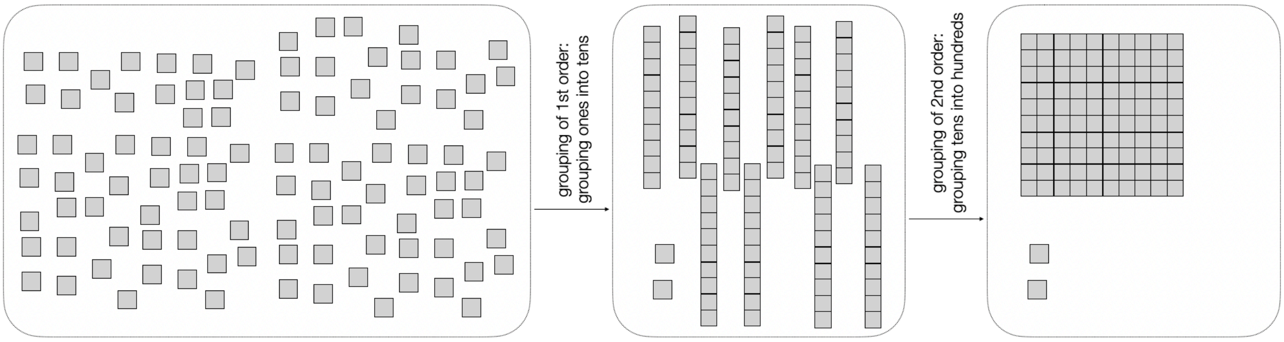


Fig. 15: Repeated grouping

Grouping up the 4th order leads to the 4th level, grouping up to the nth order leads to the nth **level**.

It is not self-evident to use the same group size on every level. For example, in the Mayan number system, when writing dates, groups of 20 were made in the 1st level, groups of 18 in the 2nd level, and groups of 20 in the 3rd and subsequent levels. Another, more common example would be time measurements, where 60 seconds are grouped in one minute, 60 minutes are grouped in one hour, 24 hours are grouped in one day, and seven days are grouped in one week.

If we use the same group size for every order, then the group size is called the **base**. In our decimal number system we always group in powers of ten, so the base of the system is 10. In this regard the 1st order grouping is to group 10 ones (10^0) into one group of 10 ($10 \times 10^0 = 1 \times 10^1$), the 2nd order grouping is to group 10 groups of tens into one group of 100 ($10 \times 10^1 = 1 \times 10^2$), the 3rd order grouping is to group 10 groups of hundreds into one group of 1000 ($10 \times 10^2 = 1 \times 10^3$) and so on. The level of grouping can also be seen in the **exponent** of the group size (e.g., 10^2 is a group in the 2nd order).

A special type of grouping, that is common if we have a base, is achieved by **maximal grouping**. This is the case if we group, and keep on grouping the groups, until it is no longer possible to group at all. This maximal grouping is necessary for positional notation (see Section 3.2).

2.3.4. Different groupings generate the same amount

It is a consequence of the part-whole concept that different groupings are all equivalent to each other in terms of the total amount. This means that we need to apply the part-whole concept to the grouping. In this sense, we see that the addition of the different partitions (e.g., $1 + 1 + 1 + \ldots + 1 = 30 + 4 = 20 + 14 = 10 + 24 = 34$) always equals the total of 34 (Fig. 16).

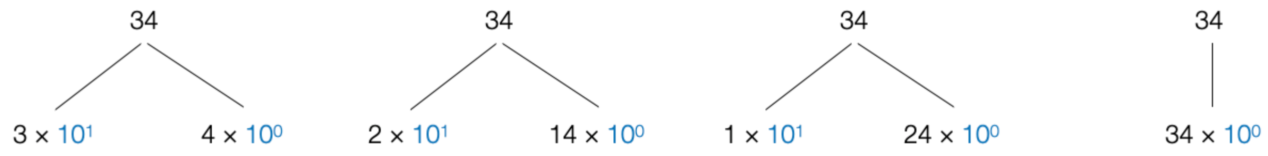


Fig. 16: Applying the part-whole concept to different groupings

This equivalence can also be deduced from the principle of **invariance**, that is, rearranging objects does not change the number of objects.

Note that, as was the case with counting, grouping can be done without necessarily understanding place value.

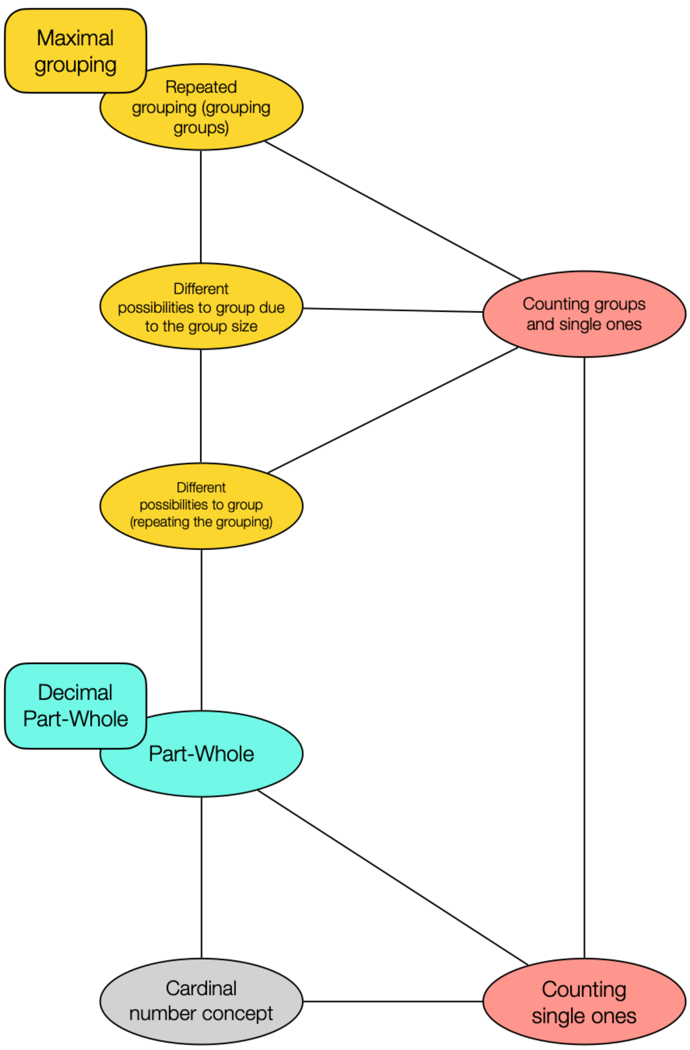


Fig. 17: Knowledge package of place value - part 3

2.4. The concept of bundling and unbundling

The next prerequisite knowledge for place value is bundling, a concept that may at first glance look similar to, but in fact is distinct from, grouping.

The **concept of bundling objects** is the understanding that
- an amount of objects is grouped and **labelled**, that means, the group is recognised as a new object with its own identifier (name).

The **concept of unbundling objects** is the inverse operation, which means that
- a (named) bundle of objects can be reinterpreted as individual objects (Fig. 18).

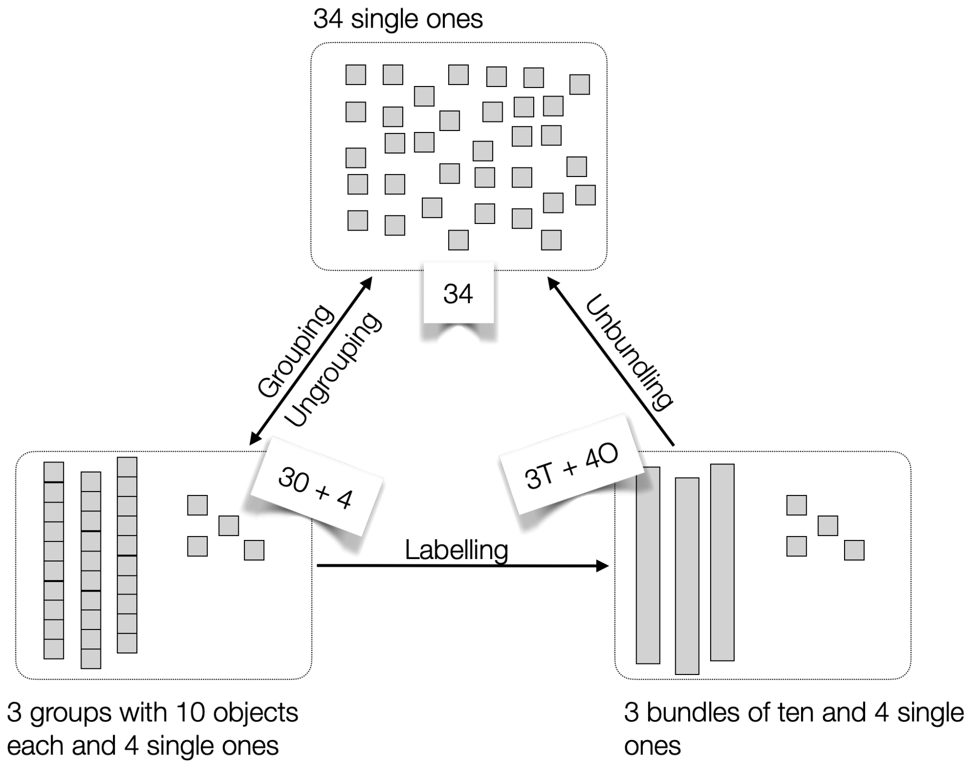


Fig. 18: Single ones, groups and single ones, bundles and single ones

We can group an amount of objects with the result being a number of groups consisting of 10 (or 5 or 7 or …) objects. In contrast, if we bundle the 10 (or 5 or 7 or …) objects, each group of 10 (or 5 or 7 or …) objects is labelled as a new object, a bundle of 10 (or 5 or 7 or … objects). Bundling is grouping, together with the additional step of labelling *10 ones* in a group to *one ten* (at the 1st level). For example, if we use grouping (in tens), the amount 34 is the same as three groups of 10 and four single ones (34 = 30 + 4). If we label the groups and thus bundle, the amount 34 is represented as three tens and four ones (34 = 3T + 4O, Fig. 18). For place value understanding, the step from having 10 ones in a group to having this amount represented as one “bundle of ten” (or short: “ten”) –and vice-versa– is critical. Grouping and labelling is the operation of bundling and its **inverse** is unbundling.

Although a group and a bundle may represent the same cardinality, they describe the amount differently (e.g., one group of ten objects vs. one ten). Also, as was the case with grouping,

- there are different bundling possibilities for a given base (in the sense of repeating the bundling),
- there are different bundling possibilities due to the base,
- the bundling can be repeated (in the sense of bundling bundles),
- the different bundling possibilities with respect to a base are all equivalent to each other in terms of the total amount (application of the part-whole concept).

2.4.1. Different possibilities to bundle due to the bundling size

It is important to note at this juncture that there are a range of contexts where we don't use ten as the bundling size. For example, when calculating with time we can bundle in sixties (60 seconds = 1 minute, 60 minutes = 1 hour) or sevens (7 days = 1 week) or fourteens (14 days = 1 fortnight). Calculating in the digital world, we bundle in twos and hence only need the digits 1 and 0. The definition of colours in computer-based designs uses the bundle size 16. In this regard, bundling and unbundling do not necessarily require a bundling size of 10 (Fig. 20).

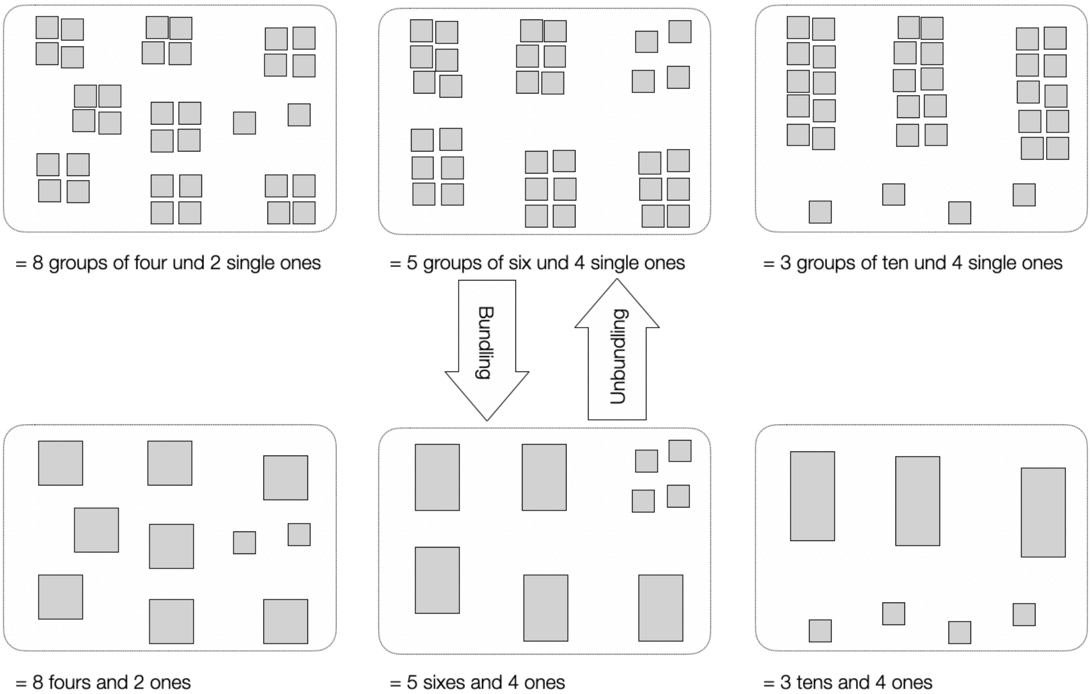


Fig. 20: From grouping to bundling and visa verse

The way of counting also changes when bundling. We can count 34 objects, for example,

- in bundles of four and then ones: "*one four, two fours, three fours, four fours, five fours, six fours, seven fours, eight fours, and one, two ones*" or
- in bundles of six and then ones: "*one six, two sixes, three sixes, four sixes, five sixes, and one, two, three, four ones*" or
- in bundles of ten and then ones: "*one ten, two tens, three tens, and one, two, three, four ones*".

2.4.2. Repeated bundling (in the sense of bundling bundles)

From this point on we will only discuss bundling and unbundling in base 10.

When working with amounts less than 99 objects in base 10, only 1st order bundling is required. This means, that it is possible to bundle 10 ones to one ten, but there is no repeated bundling of bundles. For example, 99 objects can be bundled maximally in the 1st order to nine tens and nine ones. If there are **100 objects or more**, there is a need to group and to label again. In this regard, bundling can be done not only with the single ones bundled into ten (bundling of 1st order), it can also be repeated on higher orders. That means, that it is also possible to bundle the bundles, which leads to bundles of 2nd (hundreds = 10^2), 3rd (thousands = 10^3), 4th (ten thousands = 10^4), … or nth order.

As bundling is composed of grouping and labelling, it is not possible to label without having grouped previously. But it is possible to alternate between grouping and labelling to group maximally and then label. Fox example, when bundling 100 ones, we have the options:

- *Alternate grouping and labelling:* we can form groups of 1st order (100 = 10 × (10 O)), bundle those groups in 1st order (10 × (10 O) = 10 × 1T), and repeat the grouping and bundling in 1st order (10 × 1T = 1 × (10T) = 1H), as shown in Fig. 21.

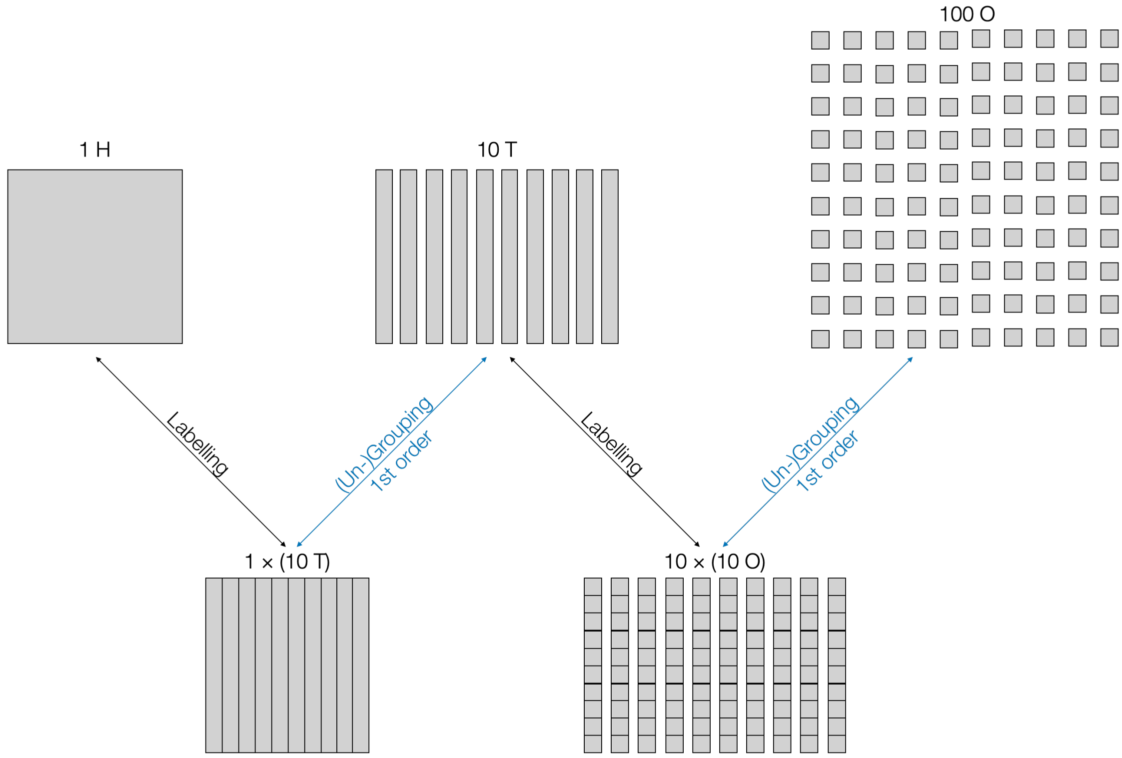


Fig. 21: Grouping and labelling in alternation

- *Group only, and label at the end:* we can form groups (100 O = 10 × (10 O)), and repeat the grouping (100 O = 10 × (10 O) = 1 × (10 × (10 O)) = 1 H) until it is not possible any more, and then label in the last step (Fig. 22).

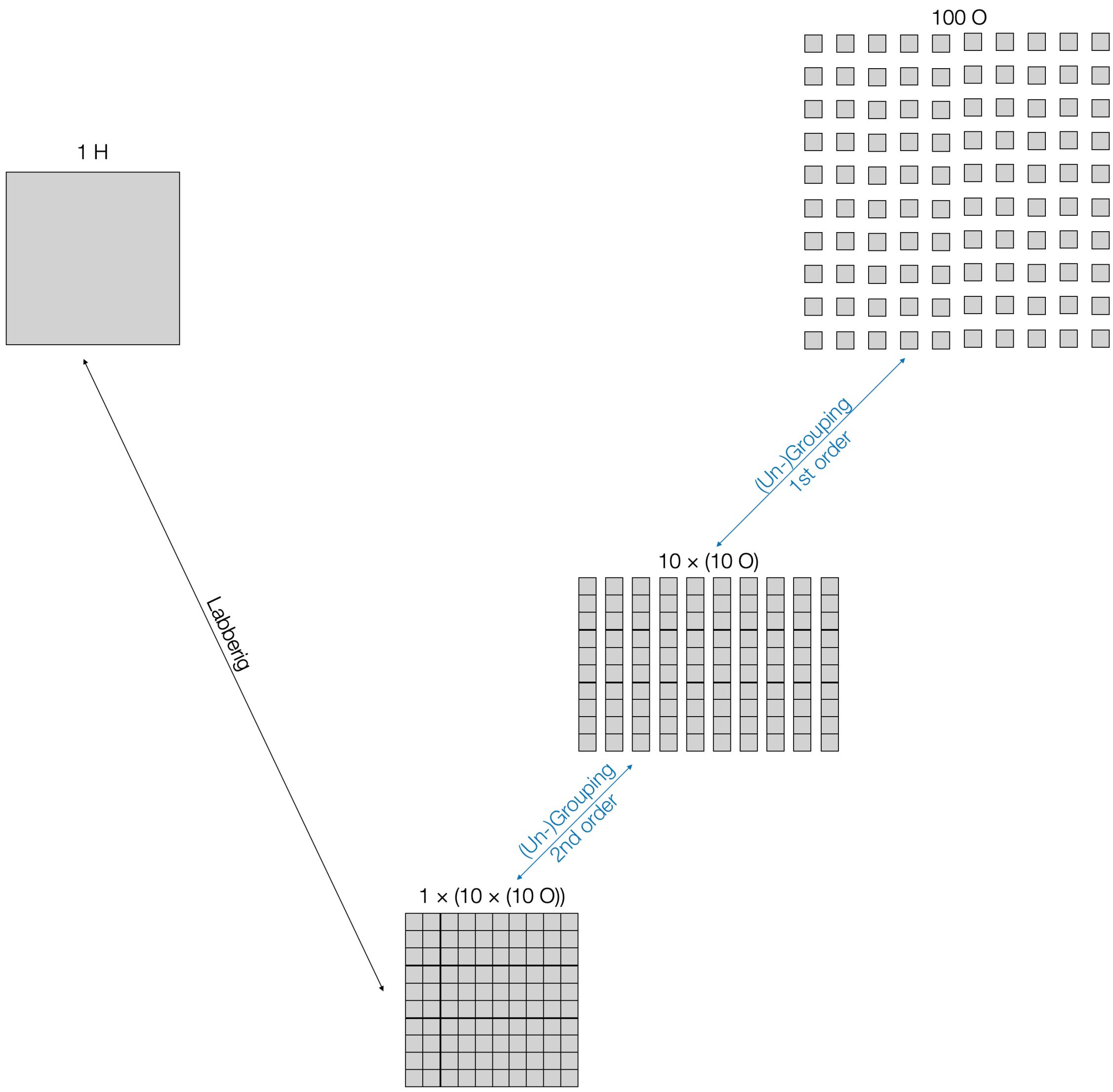


Fig. 22: Way of only grouping 100 ones and bundling at the end

Usually, we do not group in the 2nd order (Fig. 22), but alternate between grouping 1st order and labelling (Fig. 21).
Fig. 23 provides an overview regarding different ways of grouping and labelling when there are more than 99 / 999 / 9999 / … objects.

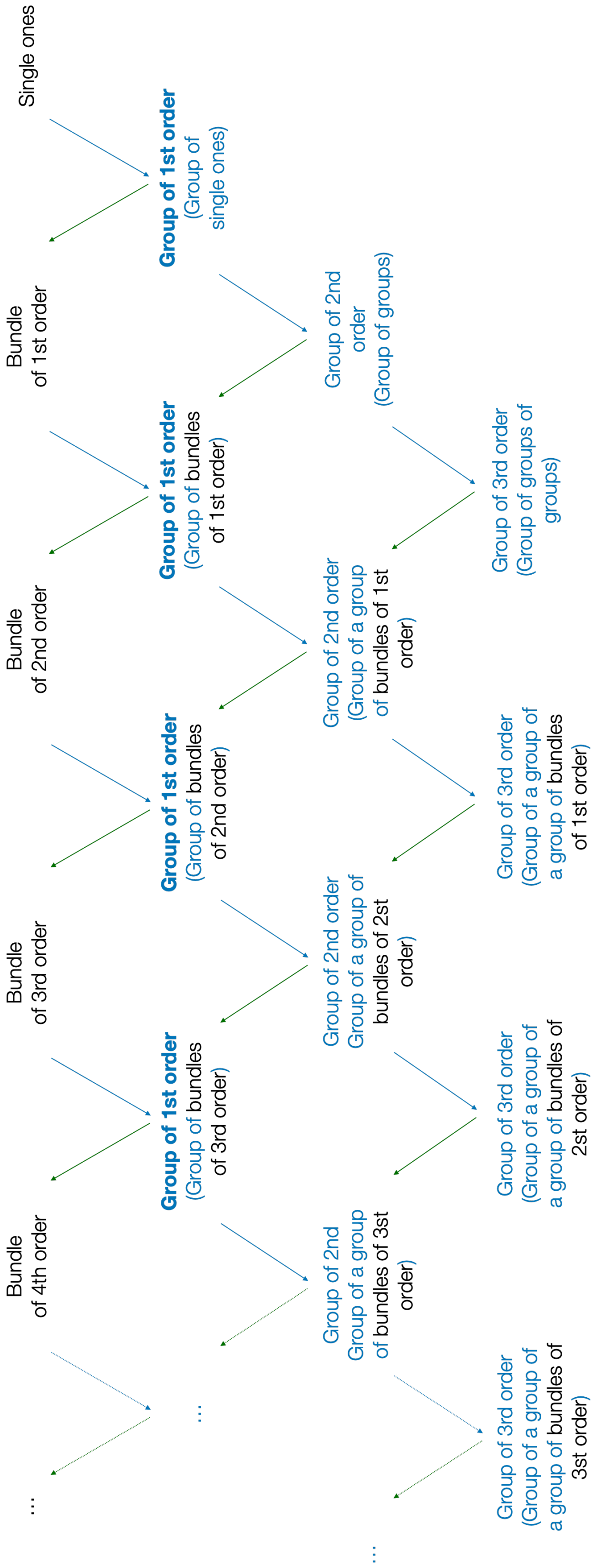


Fig. 23: Ways to repeat the bundling

The capability to alternate flexibly between (un-)grouping and labelling is also important, for example, when using written algorithms or using different calculating strategies (see Chapter 5). We describe this capability as having a **flexible understanding of place value** (see Section 3.1.3).

A special kind of bundling is **maximal bundling**. This occurs if we label groups of 1^{st} order, and continue grouping and labelling, (2^{nd} order, 3^{rd} order…) until it is no longer possible to create any further groups.

2.4.3. Different bundlings generate the same amount

As was the case with grouping, we need to develop the understanding that different bundlings are equivalent to each other in terms of the total amount. Applying the part-whole concept to bundling, we need to understand that, for example, 34 = 3T 4O = 2T 14O = 1T 24O = 0T 34O.

Note that, as was the case with counting and grouping, bundling can be done without place value understanding.

Fig. 24: Knowledge Package of Place Value - part 4

2.4.4. Levels of abstraction when bundling

Building upon counting and grouping, the action of bundling an amount of objects can be completed on two levels of abstraction:

- 1st level of abstraction: the value of the bundle is **visible in the volume of the bundle**. When we bundle material (e.g., eggs, sticks, cubes of the base ten material), the volume of the bundle is visible in the material. For example, in the Japanese textbook みんなと学ぶ 小学校 算数 (Minna to manabu Shōgakkō Sansū, “Learn with everybody –Primary School –

Arithmetic") the step from having 10 tens in a group to having one hundred is visualised with base ten blocks as follows (Fig. 25). The new object –the hundred– is visualised with a blue frame taping together the tens to a new object.

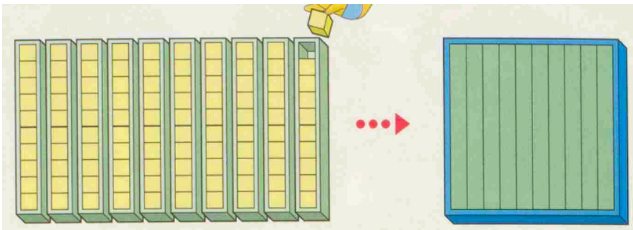


Fig. 25: Bundling 10 tens to 1 hundred with base ten blocks
(みんなとまなぶ しょうがっこう さんすう 1ねん, p. 132)

Also 10 hundreds can be grouped and labelled to 1 thousand, and be shown as in Fig. 26 below with the yellow ribbon.

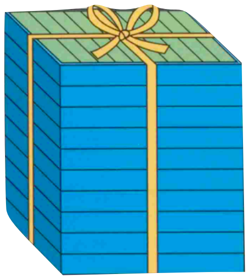


Fig. 26: Bundling 10 hundreds to 1 thousand with base ten blocks
(みんなと学ぶ 小学校 こくご 二年上, p. 15)

- 2nd level of abstraction: the value of the bundle is **visible in the symbol for the bundle**. For example, the Egyptian number system used bundling with base 10: 10 strokes were bundled into 1 cattle, 10 cattle were bundled into 1 rope, and so on (Fig. 27).

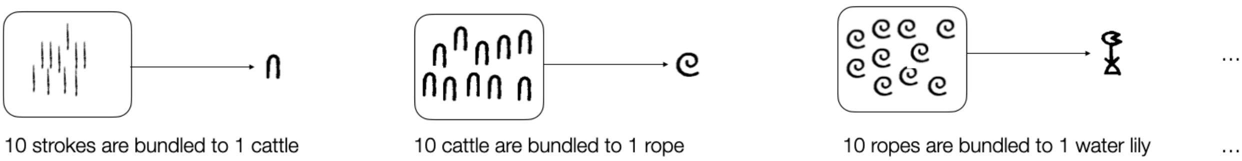


Fig. 27: Bundling by the use of different symbols

In this way, the symbol itself gives information about its value (e.g., 1 rope has the value of a hundred) irrespective of its position in the written representation of a number (e.g., 134 can be represented as 4 strokes, 1 rope, and 3 cattle or 3 cattle, 4 strokes, and 1 rope). One problem with these hieroglyphs is that the Egyptian number system required a new symbol for each new bundle size. Regardless which representation you choose (different materials for different bundle sizes or different symbols for different bundle sizes), to distinguish the different bundle sizes, there is always the need of an infinite number of materials/symbols. The concept of place value is superior to this approach, as it requires only ten symbols in base ten, as can be seen in Chapter 3.

3. The concept of place value
3.1. The subconcept of the value of the place

The **subconcept of the value of the place** is the understanding that
- there are designated places in the notation of numbers, and that it is only the place (position) of a number, that gives information about its bundle size.

3.1.1. The decimal place value system

In principle, there are different possibilities to arrange the places for different bundle sizes. For example, the Mayan positioned the places for their different bundle sizes from bottom to top, increasing vertically (Table 2).

Mayan number symbols	Values of the places	Values of the Mayan number symbols according to their place
	$20^2 \bullet 18$	$4 \bullet (20^2 \bullet 18) = 28\,800$
	$20^1 \bullet 18$	$6 \bullet (20^1 \bullet 18) = 2160$
	20^1	$17 \bullet 20^1 = 340$
	20^0	$8 \bullet 20^0 = 8$

Table 2: The value of the places in the Mayan number system in writing dates

In the decimal place value system, we have a unique place in the notation of numbers for the amount of ones (10^0), another for the amount of tens (10^1), another for the amount of hundreds (10^2), and so on. The places for the bundle sizes are positioned horizontally, increasing from right to left (Table 3).

...	Th	H	T	O
...	10^3	10^2	10^1	10^0

Table 3: The value of the places

3.1.2. From bundling to the value of the place

There are several steps to progress from counting bundles to an understanding of the values of the places (Table 4):

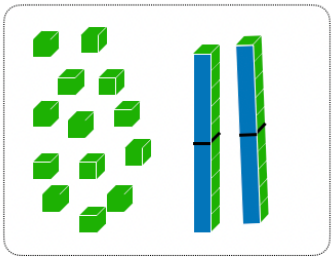
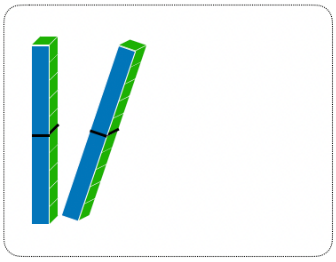
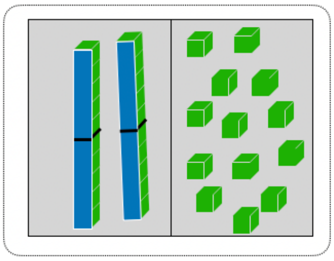
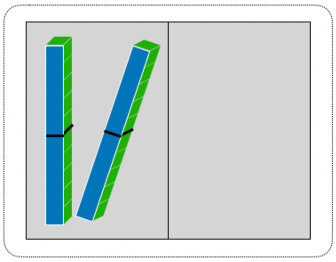
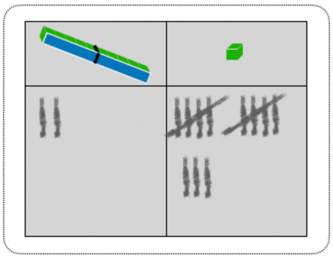
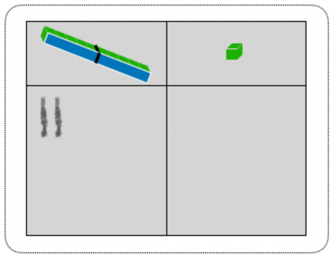
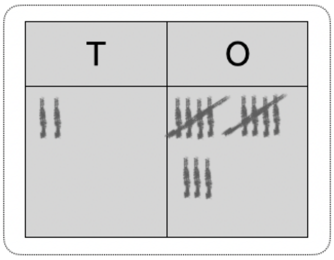
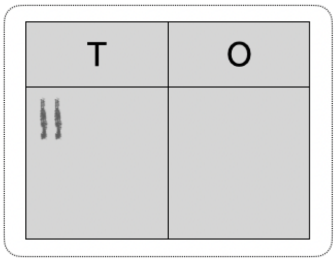
Step	Description of the step	Example 1	Example 2
1st step	Sorting the bundles according to their bundle size		
2nd step	Sorting the bundles according to the positional notation (see Section 3.2). This is done via the use of a **sorting chart**		
3rd step	Introducing the **place value chart** Noting the quantity of bundles in the relevant column with tokens		
4th step	Naming the places with the corresponding names of bundle units		

Table 4: Steps from counting bundles to the value of the place

Attention:
Note that bundles should not be placed in the place value chart (Fig. 28 left), as the sorting chart would then be confused with the place value chart, and 2T in the tens column would have the value of 20T (Fig. 28 right), that is 2H. You can see in Table 4 that the two tens of the second step (representing twenty) are replaced with two tokens (representing two) in the third step.

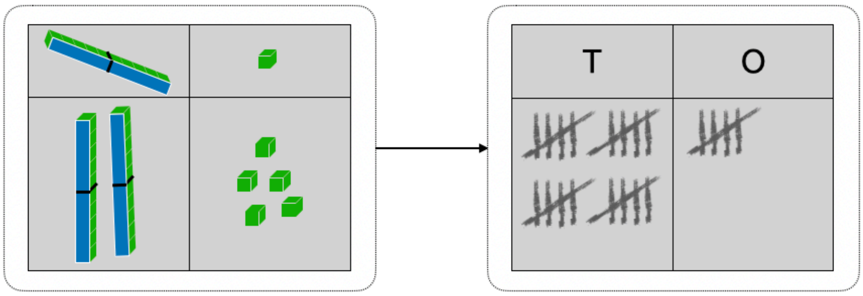
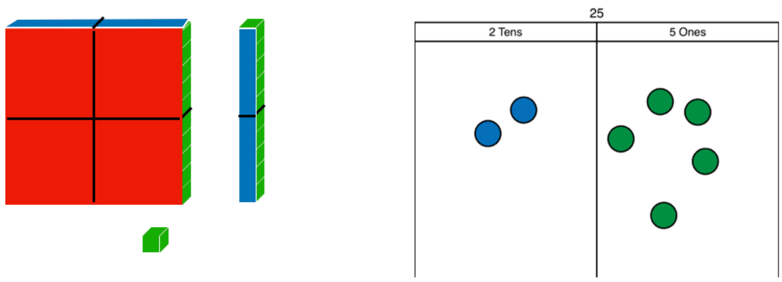


Fig. 28: A non-recommended mix of sorting chart and place value chart

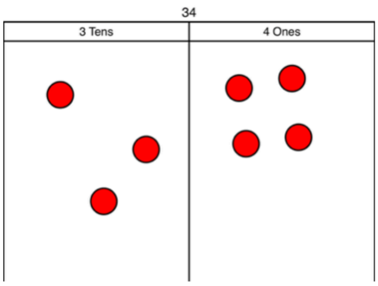
An additional step is often made by the use of **colours** (as also seen in the bundling material in Table 4); however, there is no need for different colours, as in a place value system it is only the place that gives information about the value of the number. Colours may be used to support children in understanding the different bundle sizes (Fig. 29), but this should only be an intermediate step.



...	Th	H	T	O
...	7	5	2	3

Fig. 29: Different colours to support the distinction of different bundle sizes

The step from bundling materials to tokens or numbers that are all of the same type, is a further essential step of abstraction leading to positional notation. The place of a token or a number determines its value (Fig. 30), and not its volume or color.



...	Th	H	T	O
...	7	5	2	3

Fig. 30: Tokens or numbers of the same type

3.1.3. Flexible understanding of place value

One suitable tool to support the concept of place value is the app "Place Value" (Kortenkamp, 2012). In the app, there is a place value chart, where the amount of bundles is not noted with numbers, but by the use of the corresponding quantity of tokens (Fig. 31).

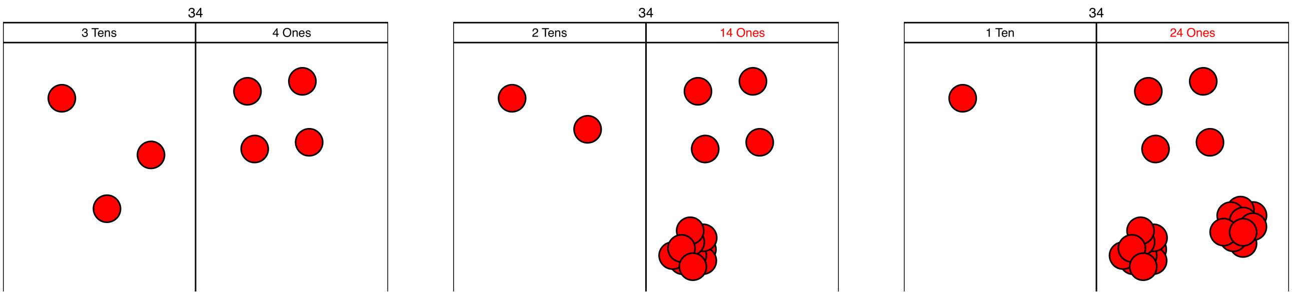


Fig. 31: The app „Place Value"

In that way, the app enables the user to experience the action of bundling and unbundling in an enactive way, with an automatic linking of the amount of tokens of a special bundle size to the corresponding number. The aim is to support the flexible understanding of place value, which is important e.g., when using algorithms or calculating strategies – see Chapter 5. As a specific example, 16 832 can easily be divided by 4, if we use the strong nonstandard partitioning (explained below) of 16T 8H 32O: (16T 8H 32O) / 4 = 4T 2H 8O = 428

The **flexible understanding of place value** (Fig. 33) is the ability to change the partitioning of a number between
- standard partitioning;
- strong nonstandard partitioning; and
- weak nonstandard partitioning.

The partitioning that arises out of maximal bundling is called **standard partitioning**: the parts are multiples of powers of ten that are maximally bundled (e.g., 16 832 as 1TTh 6Th 8H 3T 2O).

As well as standard partitioning, it is important for number fluency to also understand two forms of **nonstandard partitioning: strong** and **weak**[2]:

- *strong nonstandard partitioning:* Starting from standard partition, some bundle sizes are unbundled completely. The parts are still bundled multiples of powers of ten (e.g., 16 832 as 16Th 8H 32O or 168H 3T 2O). This means that the nonstandard partitioning you can always "see" the original digits in the representation. Another way of creating the strong nonstandard partitions is to "cut" the standard representation into segments and using these segments for creating the representation: 16|83|2 would lead to 16Th 83T 2O, 1|6832 would lead to 1Th 6832 O. This also shows that there are at most $2^{n-1} - 1$ strong nonstandard partitions for an n-digit number, as there are $n - 1$ possible places for cuts, and having no cuts is the standard partitioning.

[2] In earlier publications, Ladel and Kortenkamp used the term "not strong" instead of "weak" (Ladel and Kortenkamp, 2016).

- *weak nonstandard partitioning:* All other partitions in multiples of powers of tens. Usually, it is not possible to “see” all of the original digits in the representation (e.g., 16 832 as 15T 18H 2T 12O or 14T 26H 23T 2O).

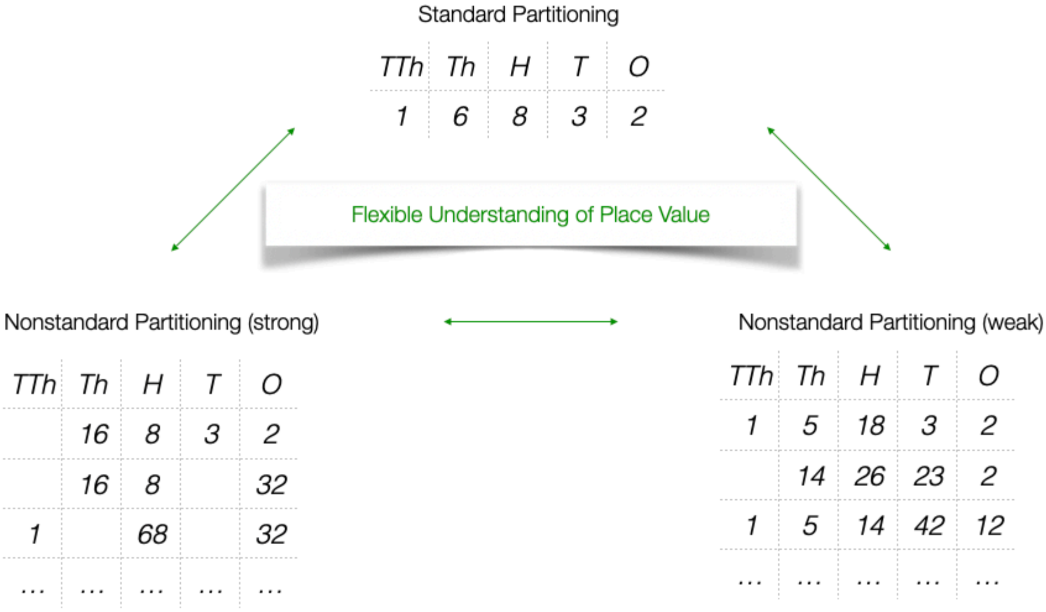


Fig. 32: Flexible understanding of place value

The flexible understanding of place value corresponds to:
- different bundling possibilities for a given base (in the sense of repeating the bundling) (see Section 2.4.1)
- repeated bundling (in the sense of bundling bundles) (see Section 2.4.2) and
- alternating between bundling and grouping.

3.1.4. Expanding the place value chart from right to left

The places in the place value chart correspond to the bundling sizes. As we have repeated bundling (in the sense of bundling bundles, see Section 2.4.2), with the possibility to bundle as long as it is possible, we can expand the places in the place value chart from right to left as long as it is necessary to make new bundles (e.g., if there are 134 ones, we can bundle them into 13 tens and four single ones remain, Fig. 34 left). Repeating the bundling, we now add an additional column/place to the left and bundle 10 of the 13 tens into 1 hundred and 3 tens (Fig. 33 right).

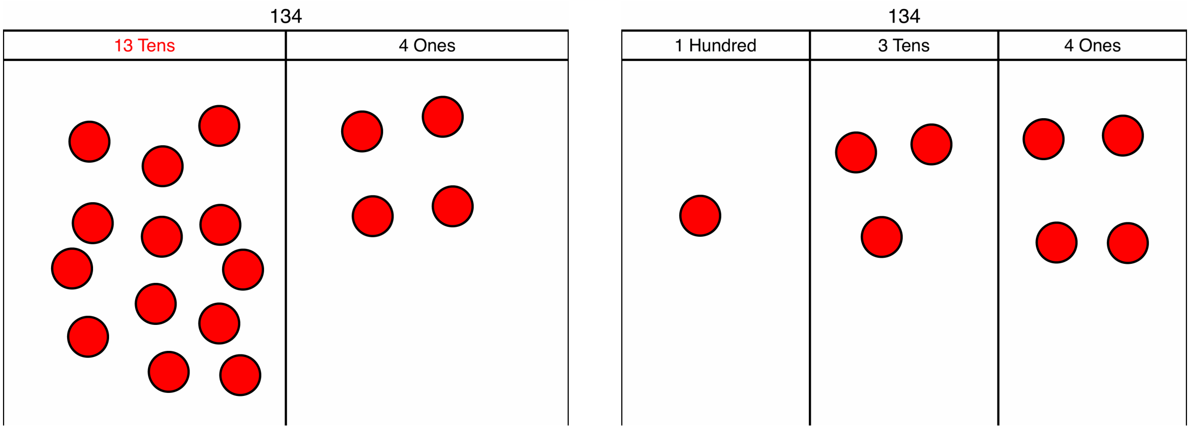


Fig. 33: Expanding the places from right to left via repeating the bundling in higher orders

3.1.5. Expanding the place value chart from left to right

The inverse operation of bundling is unbundling. Just as we have repeated bundling, it is also possible to perform repeated unbundling, as the inverse operation (Fig. 34).

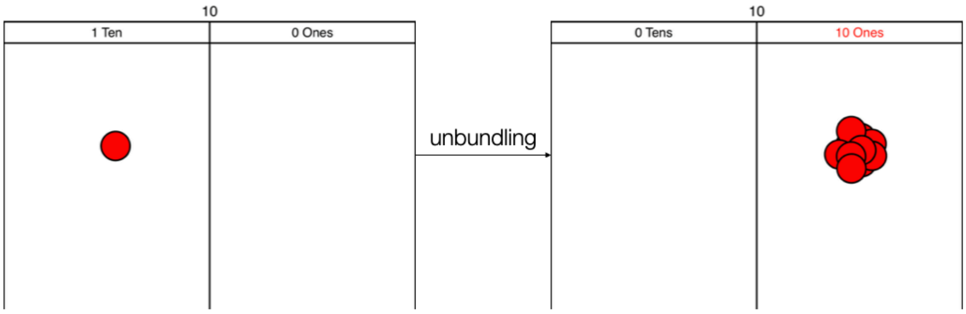


Fig. 34: Unbundling as the inverse operation of bundling

As as example, 1H can be unbundled into 10T or 1T can be unbundled into 10O. Although it is *often* possible to bundle amounts, it is *always* possible to unbundle, and hence it is possible to unbundle 1O (Fig. 35) as 10 tenths. At this point, **decimals** are required to represent parts **less than 1 whole**.

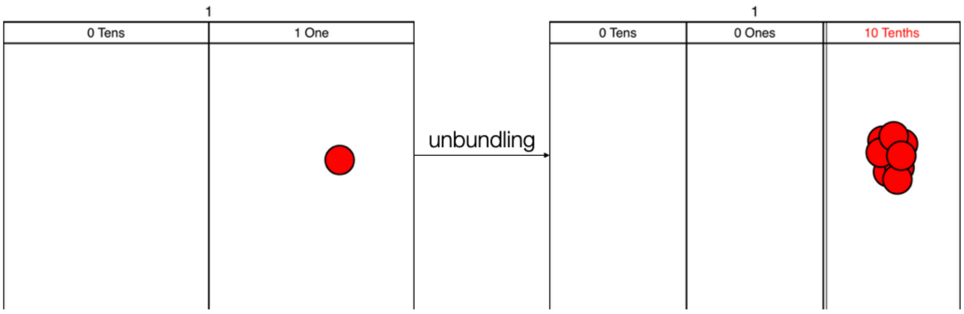


Fig. 35: Unbundling 1 ones into 10 tenths

When working with natural numbers the smallest unit are the ones. However, already when measuring (e.g., length or mass) we are often required to use rational numbers. When representing measurements, we extend the places to the right (Fig. 36) to include decimal fractions. In terms of measurement, unbundling allows for more accurate measurement.

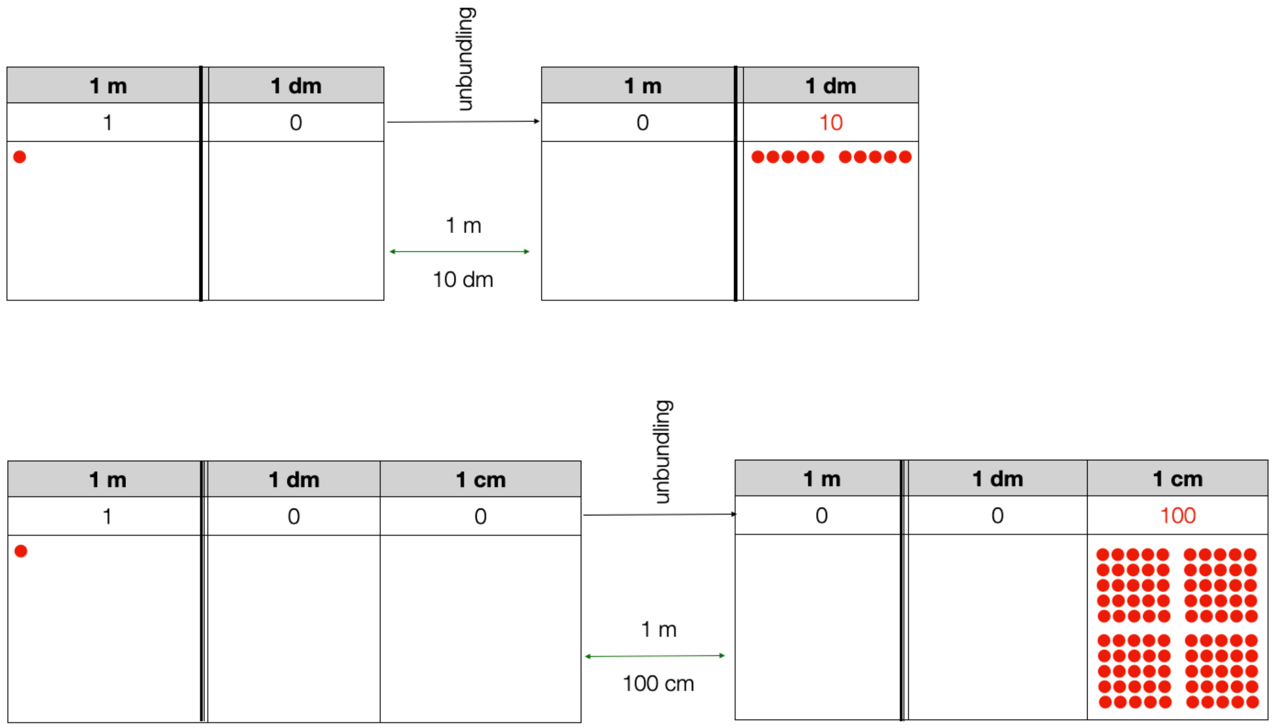


Fig. 36: Unbundling 1m to 10 dm (above) or 100 cm (below)

In the process of representing amounts smaller than one whole, we extend the places from the left to the right, but continue the same conceptual process of unbundling a unit for several ones of a smaller unit (e.g., 1 m = 10 dm = 100 cm).

As noted above, while repeated bundling is limited up to maximal bundling, there is no limit when unbundling. It is always possible to unbundle to a smaller unit (for example, 3.45 m = 34.5 dm = 345 cm = 3450 mm = …)

In this way we build upon the **understanding of the connection of neighbouring places** (Fig. 37).

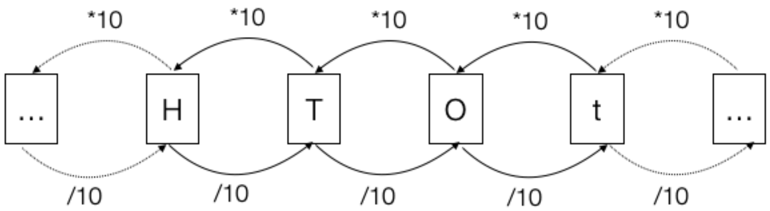


Fig. 37: The connection of neighbouring places

3.2. The subconcept of positional notation

The **subconcept of positional notation** is the understanding that
- there is a requirement for maximal bundling in positional notation

3.2.1. Requirement for maximal bundling in positional notation

Using a place value chart, the value of the place is determined by the named columns. If there are no columns, and hence only the position of a digit gives information regarding its value, the assignment of the position to its place value has to be clearly defined. This is the case if there is only one digit per place and position. We reach this by **maximal bundling** (Fig. 38).

TTh	Th	H	T	O
	16	8		32

TTh	Th	H	T	O
1	6	8	3	2

16 832

Fig. 38: Non-maximal and maximal bundling in the place value chart

Maximal bundling is achieved when no further bundling is possible (i.e., when bundling in the decimal system, it is not possible to bundle any amounts of ones, tens, hundreds, thousands etc. into a bundle of ten of those amounts). For example, the amount 16 832 is maximally bundled when it is represented as 1 ten-thousand, 6 thousands, 8 hundreds, 3 tens, and 2 ones, but not maximally bundled if the amount is represented as, for example, 16 thousands 8 hundreds and 32 ones, as it is possible to bundle 10 of the thousands into 1 ten-thousand and 30 of the ones into 3 tens[3] .

Maximal bundling is why we only require 10 digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) in our number system, as more than nine objects are always bundled to a new bundle.

[3] The cardinality does not change in these examples, only how the cardinality is represented changes.

3.2.2. The value of a digit at a place

The **value of a digit at a place**
- is determined by the product of the digit with the value of its place. The same value can be found by unbundling all bundles of the according place into single ones.

If an amount of objects is maximally bundled then there is only one digit per place, as each digit in a certain number (e.g., 341) stands for the number of bundles according to the places of the digits (e.g., 3H 4T 1O). The value of a digit at a place corresponds to unbundling all bundles indicated by the digit into ones. In this regard, we get a digit's value at a place by multiplying the digit with the value of the place, that is, the bundle size (e.g., $3 \cdot 10^2$, $4 \cdot 10^1$, $1 \cdot 10^0$). According to the subconcept of decimal part-whole (see Section 2.2), this is the first step in direction from whole to parts (Fig. 39).

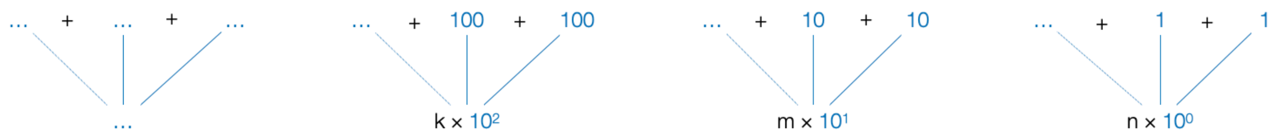


Fig. 39: The parts are of the same power of tens

In the "Place Value" App, the value of the digit "3" in the number "300" can be seen when the 3 hundreds are unbundled into 300 ones (Fig. 40).

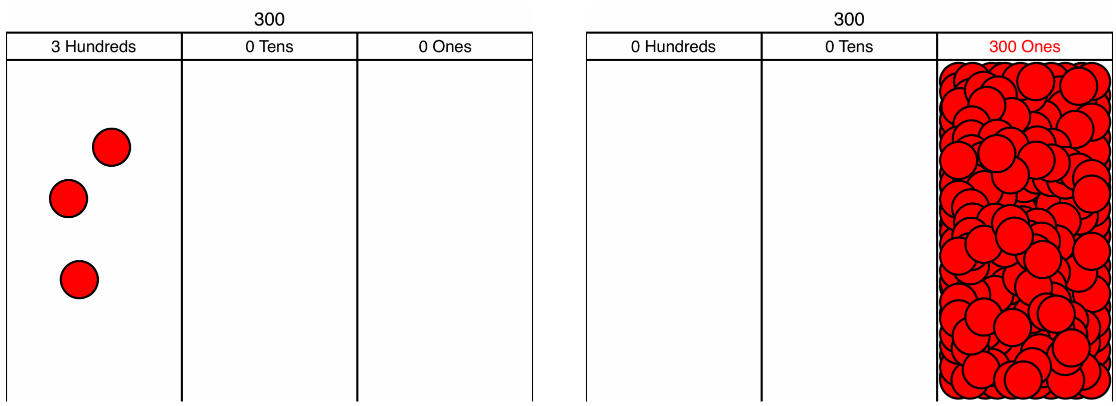


Fig. 40: Value of the digit "3" in the number "300"

It is possible that there are amounts of objects that, when maximally bundled, do not have bundles of each place value size (e.g., the amount of 503 objects can be bundled in 5 hundreds and 3 ones with no bundle of the size of a ten). This is unproblematic for several representations:
- *Noting bundles* - we do not note the bundle unit(s) that do not exist (Fig. 41 left).
- *Number words* - there is no need to say that there are 0 tens (Fig. 41 2nd from left).
- *Place Value Chart* - There is no need to fill in a "0" in the tens-column (Fig. 41 3rd from left).

However, the positional notation leads to the need for a symbol that can be used to represent any position that is "empty" (see Fig. 41 right). Therefore, we need the digit "0".

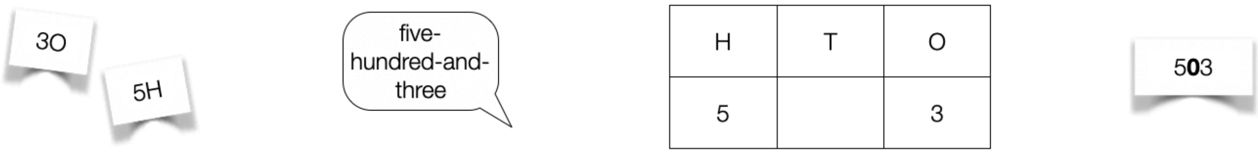


Fig. 41: Different possibilities to represent numbers without “0” and with “0” in the positional notation

In this regard, coming from noting the amount of bundles in the place value chart, we need to ensure maximal building and to supplement the digit “0” if necessary, for the aim to write a number without place value chart (Table 5).

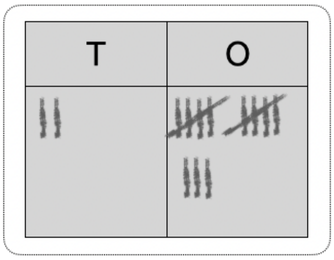
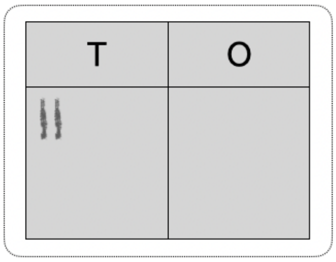
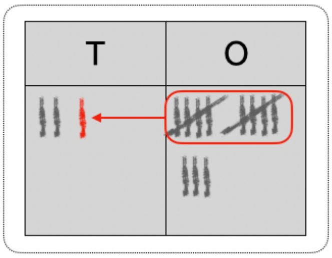
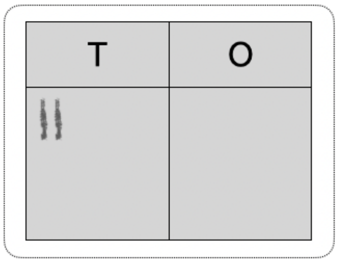
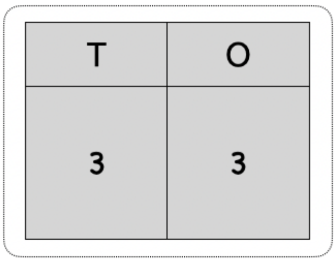
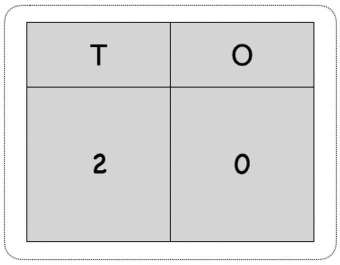
Last step of bundling to place value			
4th step	Naming the places, that correspond to the bundle units		
From place value to positional notation			
1st step	Ensure maximal bundling		
2nd step	Notation of the quantity of bundles with digits, supplement the digit „0“ if necessary		
3rd step	Writing the number without place value chart	33	20

Table 5: Steps from the place value chart to positional notation

3.2.3. The value of a number

The **value of a number**
- is the sum of the values of the digits (application of the part-whole concept).

The value of a number corresponds to unbundling all bundles at all places into ones. As the value of each digit corresponds to unbundling the respective bundles into ones, the overall value of the number is determined when we add all the values of the digits of a number (e.g., 134 = 100 + 30 + 4, Fig. 42).

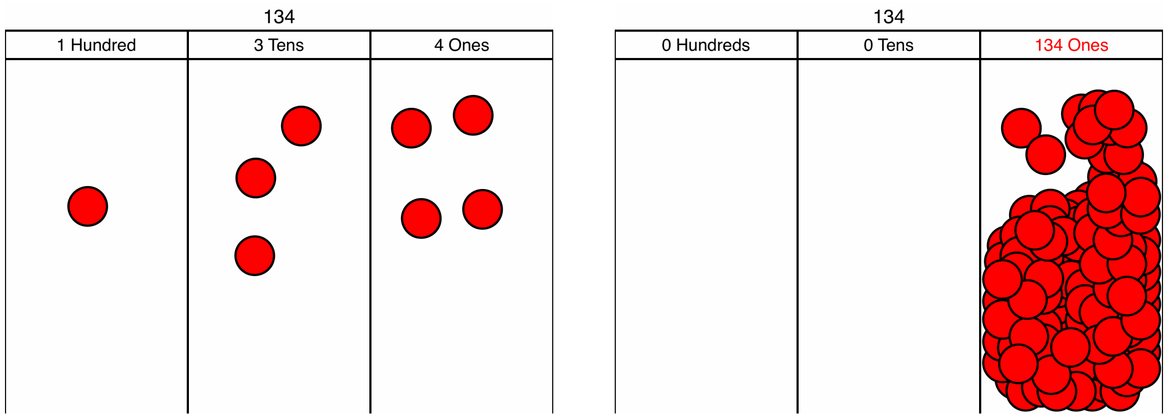


Fig. 42: Value of the number 134

This also requires the understanding of the **decimal part-whole subconcept**, and hence the understanding that the amount of the whole is the sum of the amount of the parts (values of digits). It is the 2nd step of the decimal part-whole subconcept (see Section 2.2) in direction from the whole to the parts (Fig. 43).

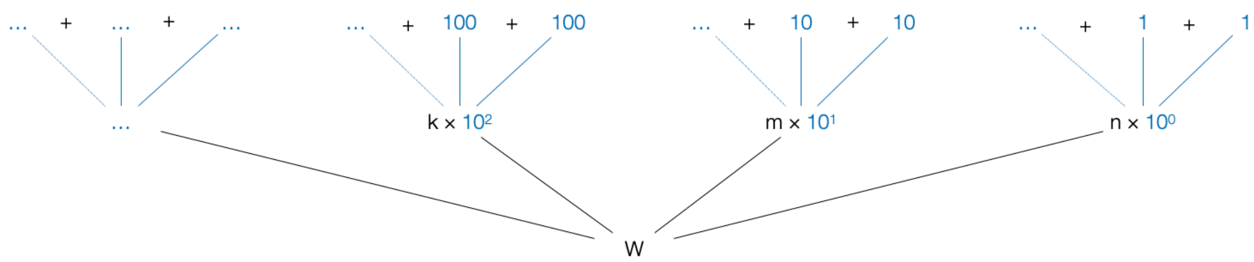


Fig. 43: The parts are the parts of the same power of tens

The value of a digit and the value of a number are not new concepts, but facts that result from the concept of place value.

Fig. 44: Knowledge package of place value - part 5

4. Linguistic influences on the concept of place value

4.1. The concept of place value und language

It is important not to confuse the ability to say the number word for an amount, or to write the number symbol (Fig. 46), with an understanding of place value (Fig. 45).

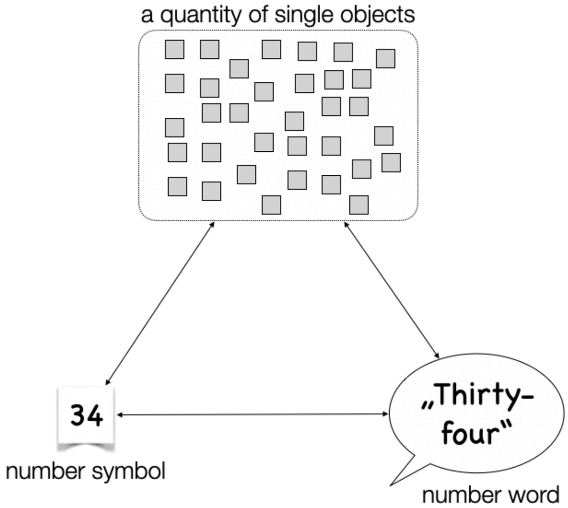


Fig. 45: Knowledge about the number symbol and the number word

If language is considered as a conceptual component of place value, it may be the case that a person could have an understanding of the concept of place value in one language, but not in another one. This important point accepted, language clearly plays an important role, as teaching and learning mathematics is influenced by language and this is clearly the case in learning place value.

To count an amount of objects correctly, and hence say the corresponding number word, it is necessary to have a concept of cardinal numbers (see Section 2.1). This includes the ability,
- to be able to apply a one-to-one-correspondence between a number word and an object,
- to reproduce the number word sequence in the correct order (principle of stable order), and
- to apply the cardinal principle, that means to understand, that the last said number word not only designates the last counted object, but also designates the whole amount of counted objects.

In this regard, saying a number word, for example “thirty four”, is not automatically connected to the understanding of “thirty-four” as 3T and 4O. The same applies for the number symbol, as being able to write the number symbol 34 is also not automatically connected to the understanding of 34 as 3T and 4O. This is why we need to connect the number words and symbolic representations to the subconcept of positional notation (see Section 3.2).

4.2. Formation rules for number words

The rules to form the number words in German and in English language are as follows (Table 6). The hyphens are inserted to highlight the application of rules, they are not necessarily part of the number word.

Bundle size	Number symbol	German language		English language	
		Rule	**Exceptions**	**Rule**	**Exceptions**
ones	0	null		zero	
	1	eins		one	
	2	zwei		two	
	3	drei		three	
	4	vier		four	
	5	fünf		five	
	6	sechs		six	
	7	sieben		seven	
	8	acht		eight	
	9	neun		nine	
tens		append the suffix „-zig“ to the number word of the ones:		append the suffix „-ty“ to the number word of the ones:	
	10		zehn		ten
	20		zwan-zig		twen-ty
	30		drei-ßig		thir-ty
	40	vier-zig			for-ty
	50	fünf-zig			fif-ty
	60		sech-zig	six-ty	
	70		sieb-zig	seven-ty	
	80	acht-zig		eigh-ty	
	90	neun-zig		nine-ty	
hundreds		Add the word “-hundert” after the number word of the ones:		Add the word “-hundred” after the number word of the ones:	
	100	ein-hundert		one-hundred	
	200	zwei-hundert		two-hundred	
	300	drei-hundert		three-hundred	
	400	vier-hundert		four-hundred	
	500	fünf-hundert		five-hundred	
	600	sechs-hundert		six-hundred	
	700	sieben-hundert		seven-hundred	
	800	acht-hundert		eight-hundred	
	900	neun-hundert		nine-hundred	

thousands		Add the word “-tausend” after the number word of the ones:	Add the word “-thousand” after the number word of the ones:
	1000	ein-tausend	one-thousand
	2000	zwei-tausend	two-thousand
	3000	drei-tausend	three-thousand
	4000	vier-tausend	four-thousand
	5000	fünf-tausend	five-thousand
	6000	sechs-tausend	six-thousand
	7000	sieben-tausend	seven-thousand
	8000	acht-tausend	eight-thousand
	9000	neun-tausend	nine-thousand
...			

Table 6: Number word formation rules for the powers of ten

	Number symbol	**German language**		**English language**	
		Rule	**Exceptions**	**Rule**	**Exceptions**
numbers from 11 to 19:		append the suffix “-zehn” to the number word of the ones:		append the suffix “-teen” to the number word of the ones:	
	11		elf		eleven
	12		zwölf		twelve
	13	drei-zehn			thir-teen
	14	vier-zehn		four-teen	
	15	fünf-zehn			fif-teen
	16		sech-zehn	six-teen	
	17		sieb-zehn	seven-teen	
	18	acht-zehn		eight-teen	
	19	neun-zehn		nine-teen	
2-digit numbers > 20	**35**	(**quantity of ones**) - „und“ - (**quantity of tens**): e.g. **fünf**-und-**dreißig**		(**quantity of tens**) - (**quantity of ones**): e.g. **thirty**-**five**	

Table 7: Number word formation rules for 2-digit numbers.

A difficulty in the German language (as well as in some other languages) is “**number inversion**”, which means that the sequence of the values of places from the written one (left to right) is changed in the verbal and written number word sequence. Whereas the value of places in the positional notation (see Section 3.2) decreases from left to right, the rule of the number word formation of 2-digit numbers is the other way round (Fig. 46)

Values of places using number symbols:	
T	O

Values of places using number words in German language:	
O	T

Fig. 46: Number inversion in 2-digit numbers

This number inversion of ones and tens persists with larger numbers (Table 8).

	Number symbol	German language	English language
3-digit numbers:		(**quantity of hundreds**) - (**quantity of ones**) - „und“ - (**quantity of tens**): e.g.	(**quantity of hundreds**) “and" (**quantity of tens**) - (**quantity of ones**): e.g.
	235	**zweihundert-fünf**-und-**dreißig**	**two hundred and thirty-five**

Table 8: Number word formation rules for 3-digit numbers.

The formulation of number words in German and English continue in steps by three places, whereby the inversion of the ones and the tens (ones and tens thousands, ones and tens millions, ones and tens milliards, …) persists (Table 9).

HMio	TMio	Mio	HTh	TTh	Th	H	T	O

	Number symbol	German language	English language
more-digit numbers:		(**quantity of hundreds**) - (**quantity of ones**) - und - (**quantity of tens**): e.g.	(**quantity of hundreds**) - (**quantity of tens**) - (**quantity of ones**): e.g.
	123 456 789	**einhundert-drei**-und-**zwanzig**-Millionen-**vierhundert-sechs**-und-**fünfzig**-Tausend-**siebenhundert-neun**-und-**achtzig**	**one hundred** and **twenty-three**-million **four hundred** and **fifty-six**-thousand **seven hundred** and **eighty-nine**

Table 9: Number word formation rules for more-digit numbers.

In this regard, the formation of number words is a linguistic problem, and the number words have to be learned like the number symbols. But as learning processes are almost always language-based, the formation of number words also influences the mathematics. That is the reason why we have to take care regarding the semantic meaning of number words and also the constructed meaning that we associate with the number words. It is important to separate the values of the different parts of a number word (e.g., “thirty”/ “dreißig”, “four”/

“vier”) and to connect those parts to their corresponding bundle size (for example “tens”, “ones”). Also, it can be helpful to discuss artificial number words like “twenty-fourteen” („Vierzehn-und-zwanzig“) that require further calculation (2T+1T = 3T) to find the correct amount they refer to (3 tens and 4 ones). Therefore, the concepts of grouping (see Section 2.3) and labelling (see Section 2.4) are indispensable (Fig. 47).

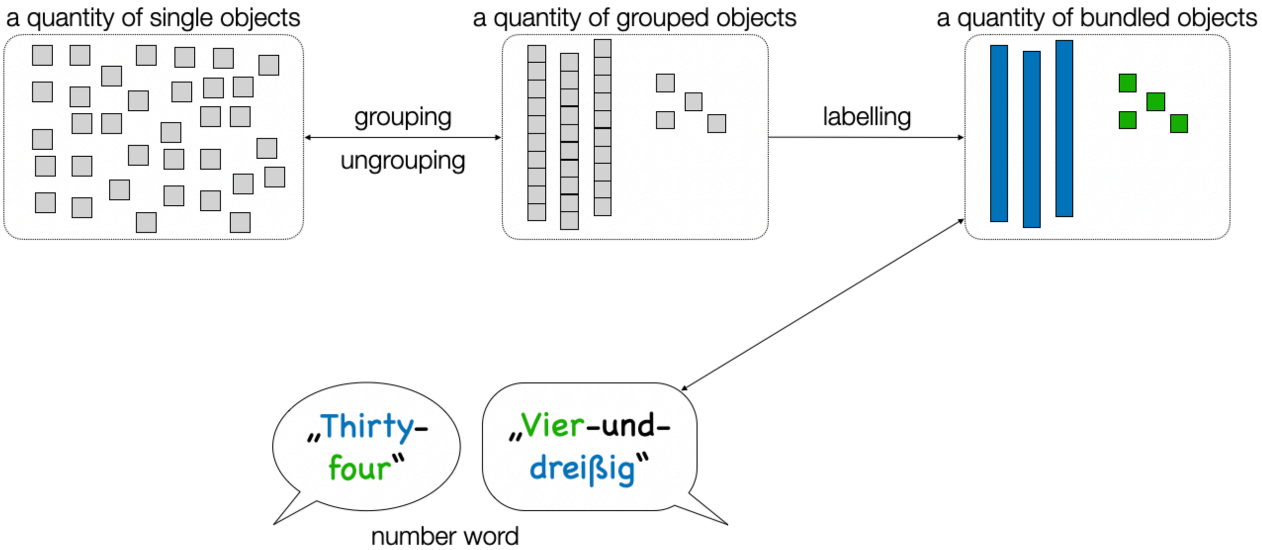


Fig. 47: Understanding of number words according to the concept of bundling

As can be seen in Fig. 48, the understanding of number words is not connected to positional notation, as it is sufficient to connect the parts of the number words only to bundled objects.

The understanding of the different parts of the number symbol corresponds to the concept of place value (see Chapter 3) and this understanding has to be connected to the understanding of number words according to the concept of bundling (Fig. 48).

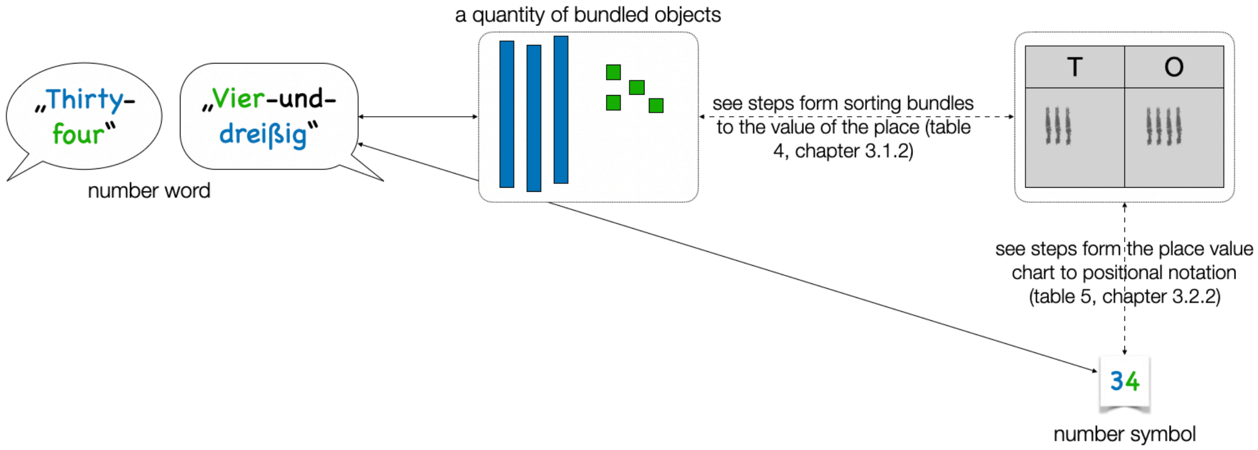


Fig. 48: Connecting the understanding of number words according to the concept of bundling to number symbols

4.3. Designations for bundles

Another difficulty regarding language is that the designations for bundles are not always the same as the designations for a group of objects. As already shown in Section 2.4, when bundling, a group of objects is not only recognised as a new object, but is also labelled with a new name:

For example, the designation for a **group** of 10 objects is
- in German language: “zehn” (“(ein-)hundert”, “(ein-)tausend”, …)
- in English language: “ten” (“(one-)hundred”, “(one-)thousand”, …)

The designation for the **bundle** is (hyphens added for clarity)
- in German language: “Zehn-er” (“Hundert-er”, “Tausend-er”, …)
- in English language: “ten-s” (“hundred-s”, “thousand-s”, …)

Although the difference in the designation is minor, with only the addition of an “-er” or “-s, it may still lead to difficulties. That is why it is important to develop the concept of grouping (see Section 2.3) and especially its connection to the concept of bundling (see Section 2.4, Fig. 49).

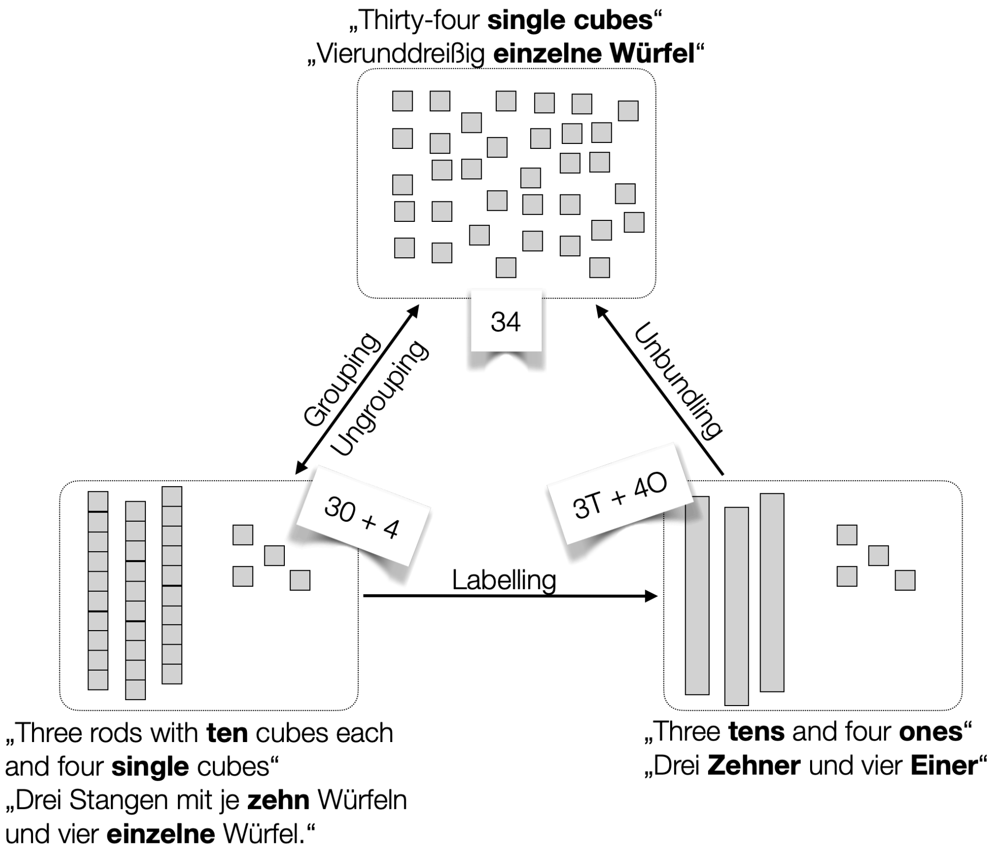


Fig. 49: Different designations for single ones, groups and single ones, bundles

5. References

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