

## Timelike-Spacelike Mannheim Pair Curves Spherical Indicators Geodesic Curvatures and Natural Lifts

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**Abstract:** In this paper, Mannheim curve is a timelike curve, by getting partner curve as a spacelike curve which has spacelike binormal, with respect to  $\mathbb{L}^3$  Lorentz Space,  $S_1^2$  Lorentz sphere, or  $H_0^2$  Hyperbolic sphere, we have calculated arc lengths of Mannheim partner curve's  $(T^*)$ ,  $(N^*)$ ,  $(B^*)$  spherical indicator curves, arc length of  $(C^*)$  fixed pol curve, and we have calculated geodesic curves of them, and also we have figured some relations among them. In addition, if the natural lifts geodesic spray of spherical indicator curvatures of Mannheim partner curve is an integral curve, we have expressed how Mannheim Curve is supposed to be.

**Key Words:** Lorentz space, Mannheim curve, geodesic curvatures, geodesic spray, natural lift.

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### §1. Introduction

There are a lot of researches to be done in 3-dimentional Euclidian Space on differential geometry of the curves. Especially, many theories were obtained by making connections two curves' mutual points and between Frenet Frames. Well known researches are Bertrand curves and Involute-Evolute curves, [6], [4], [7], [19]. Those curves were studied carefully in different spaces, therefore, so many results were gained. In Euclidian Space and Minkowski Space, Bertrand curves' Frenet frames and Involute-Evolute curves' Frenet frames create spherical indicator curves on unit sphere surface. Those spherical indicator curves' natural lifts and geodesic sprays are defined in [5], [16], [3], [8].

Mannheim curve was firstly defined by A. Mannheim in 1878. Any curve can be a Mannheim curve if and only if  $\kappa = \lambda(\kappa^2 + \tau^2)$ ,  $\lambda$  is a nonzere constant, where curvature of curve is  $\kappa$  and curvature of torsion is  $\tau$ . After a time, Manheim curve was redefined by Liu ve Wang. According to this new definition, when first curve's principal normal vector and second curve's binormal vector are linearly dependent, first curve was named as Mannheim curve, and second curve was named as Mannheim partner curve, [21], [10]. There are so many researches

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to be done by Mathematicians after Liu ve Wang's definition [12], [15], [15], [2].

## §2. Preliminaries

Let  $\alpha : I \rightarrow \mathbb{E}^3$ ,  $\alpha(t) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$  be unit speed differentiable curve. If we symbolize  $\alpha : I \rightarrow \mathbb{E}^3$  curve's Frenet as  $\{T, N, B\}$ , curvature as  $\kappa$ , and torsion as  $\tau$ , there are some equations among them;

$$\begin{cases} T'(s) = \kappa(s)N(s) \\ N'(s) = -\kappa(s)T(s) + \tau(s)B(s) \\ B'(s) = -\tau(s)N(s). \end{cases} \quad (2.1)$$

By using (2.1), we can get  $W$  Darboux vector as;

$$W = \tau T + \kappa B \quad (2.2)$$

If  $\varphi$  is the angle which is between  $W$  and  $B$ , the unit Darboux vector is that;

$$C = \sin \varphi T + \cos \varphi B \quad (2.3)$$

Let  $X$  be differentiable vector space on  $M$ . (Note that  $M$  is any vector space)

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)) \quad (2.4)$$

$\alpha$  curve is an integral curve of  $X$  if and only if  $\frac{d}{ds}(\alpha(s)) = X(\alpha(s))$ .

Suppose that  $TM = \bigcup_{P \in M} T_M(P)$ , then,

$$\bar{\alpha} : I \rightarrow TM, \bar{\alpha}(s) = (\alpha(s), \alpha'(s)) \quad (2.5)$$

$\bar{\alpha} : I \rightarrow TM$  curve is natural lift of  $\alpha : I \rightarrow M$  and for  $v \in TM$

$$X(v) = -\langle v, S(v) \rangle N|_P$$

$X$  vector space is called geodesic spray [5], [17]. Where

$$\bar{D}_X Y = D_X Y + \langle S(X), Y \rangle N \quad (2.6)$$

The equation of (2.6) is a Gauss equation on  $M$ .  $(T)$ ,  $(N)$  and  $(B)$  spherical indicator

curves' equations and  $(C)$  pol curve's equations are given respectively;

$$\begin{cases} \alpha_T(s) = T(s) \\ \alpha_N(s) = N(s) \\ \alpha_B(s) = B(s) \\ \alpha_C(s) = C(s) \end{cases}$$

With respect to  $\mathbb{E}^3$ , arc lengths and geodesic curvatures of those curves are given respectively;

$$s_T = \int_0^s \kappa ds, \quad s_N = \int_0^s \|W\| ds, \quad s_B = \int_0^s \tau ds, \quad s_C = \int_0^s \varphi' ds \quad (2.7)$$

$$\begin{cases} k_T = \frac{1}{\cos \varphi}, \\ k_N = \sqrt{1 + \left(\frac{\varphi'}{\|W\|}\right)^2}, \\ k_B = \frac{1}{\sin \varphi}, \\ k_C = \sqrt{1 + \left(\frac{\|W\|}{\varphi'}\right)^2}. \end{cases} \quad (2.8)$$

With respect to  $S^2$ , geodesic curvatures are given;

$$\gamma_T = \tan \varphi, \quad \gamma_N = \frac{\varphi'}{\|W\|}, \quad \gamma_B = \cot \varphi, \quad \gamma_C = \frac{\|W\|}{\varphi'} \quad (\text{see}[9]) \quad (2.9)$$

Let  $\bar{\alpha} : I \rightarrow \chi(M)$  be natural lift of  $\alpha : I \rightarrow M$ .  $X$  geodesic spray is an integral curve if and only if there is a geometric curve on  $M$ , (see [5]).

$$g : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}, \quad g(X, Y) = x_1 y_1 + x_2 y_2 - x_3 y_3$$

This inner product space is defined as Lorentz Space and symbolized as  $\mathbb{L}^3$ .  $X \in \mathbb{L}^3$  vector's norm is  $\|X\|_{\mathbb{L}} = \sqrt{|g(X, X)|}$ . For  $X = (x_1, x_2, x_3)$  and  $Y = (y_1, y_2, y_3) \in \mathbb{L}^3$

$$X \times Y = (x_3 y_2 - x_2 y_3, x_1 y_3 - x_3 y_1, x_1 y_2 - x_2 y_1)$$

$X \times Y$  is called vector product of  $X$  and  $Y$ , [1]. Let  $T$  be tangent vector of  $\alpha : I \rightarrow \mathbb{L}^3$ .  $\alpha : I \rightarrow \mathbb{L}^3$  is respectively defined as:

- (1) If  $g(T, T) > 0$ ,  $\alpha$  curve is a spacelike curve;
- (2) If  $g(T, T) < 0$ ,  $\alpha$  curve is a timelike curve;
- (3) If  $g(T, T) = 0$ ,  $\alpha$  curve is a lightlike or null curve, (see [11]).

Let  $\alpha : I \rightarrow \mathbb{L}^3$  be differentiable timelike curve. In this case,  $T$  is timelike,  $N$  and  $B$  are

spacelike, and Frenet formulas are given;

$$\begin{cases} T' = \kappa N \\ N' = \kappa T - \tau B \\ B' = \tau N. \end{cases} \quad (2.10)$$

(see [22]). Where

$$W = \tau T - \kappa B \quad (2.11)$$

(see [20]). In this situation, there are two cases for  $W$  Darboux vector; if  $W$  is spacelike, the Lorentzian timelike angle  $\varphi$  which is between  $-B$  and  $W$ , then;

$$\kappa = \|W\| \cosh \varphi, \tau = \|W\| \sinh \varphi \quad (2.12)$$

$$C = \sinh \varphi T - \cosh \varphi B \quad (2.13)$$

and unit Darboux vector is;

If  $W$  timelike,  $\kappa$  and  $\tau$  are formulized;

$$\kappa = \|W\| \sinh \varphi, \tau = \|W\| \cosh \varphi \quad (2.14)$$

and unit Darboux vector is;

$$C = \cosh \varphi T - \sinh \varphi B \quad (2.15)$$

Let  $\alpha : I \rightarrow \mathbb{M}^3$  be spacelike curve which has spacelike binormal. In this case,  $\alpha$  curve's Frenet vectors' vector product are respectively;

$T \times N = -B$ ,  $N \times B = -T$ ,  $B \times T = N$  and Frenet formulas are found as;

$$\begin{cases} T' = \kappa N \\ N' = \kappa T + \tau B \\ B' = \tau N \end{cases} \quad (2.16)$$

(see [22]). In this case, Darboux vector will be;

$$W = -\tau T + \kappa B, (see[20]). \quad (2.17)$$

Let  $\varphi$  be the angle which is between  $B$  and  $W$ . Then,

$$\kappa = \|W\| \cos \varphi, \tau = \|W\| \sin \varphi \quad (2.18)$$

and unit Darboux vector is given as;

$$C = -\sin \varphi T + \cos \varphi B \quad (2.19)$$

Let  $M$  be a Lorentz manifold, and  $\overline{M}$  be a hypersurface of  $M$ . Suppose that  $S$  is a shape operator which is obtained from  $N$  normal of  $\overline{M}$ ,  $D$  is the connection on  $M$ ,  $\bar{D}$  is the connection on  $\overline{M}$ ,

For  $X, Y \in \chi(\overline{M})$ , Gauss Equation is;

$$D_X Y = \bar{D}_X Y + \varepsilon g(S(X), Y) N \quad (2.20)$$

Where  $S(X) = -D_X N$  and  $\varepsilon = g(N, N)$ , [18].

$$S_1^2(r) = \{ X \in IR_1^3 \mid g(X, X) = r^2, r \in IR, r = fixed \}$$

is defined as Lorentz sphere,

$$H_0^2(r) = \{ X \in IR_1^3 \mid g(X, X) = -r^2, r \in IR, r = fixed \}$$

is defined as hyperbolic sphere.

Let  $\alpha : I \rightarrow \mathbb{E}^3$  and  $\alpha^* : I \rightarrow \mathbb{E}^3$  be two differentiable curves. Suppose that Frenet Frames on the points of  $\alpha(s)$  and  $\alpha^*(s)$  are respectively given as  $\{T(s), N(s), B(s)\}$  and  $\{T^*(s), N^*(s), B^*(s)\}$ . If  $\alpha$  curve's principal normal vector and  $\alpha^*$  curve's binormal vector are linearly dependent,  $\alpha$  curve is named Mannheim curve,  $\alpha^*$  curve is named Mannheim partner curve, and it is shown as  $(\alpha, \alpha^*)$ , [21]. Mannheim curve's equation is given as;

$$\alpha^*(s^*) = \alpha(s) - \lambda N(s) \text{ or } \alpha(s) = \alpha^*(s^*) + \lambda B^*(s^*) \quad [12].$$

There are some following equations among those curves;

$$\begin{cases} T = \cos \theta T^* + \sin \theta N^* \\ N = B^* \\ B = -\sin \theta T^* + \cos \theta N^*. \end{cases} \quad (2.21)$$

$$\cos \theta = \frac{ds^*}{ds}, \quad \sin \theta = \lambda \tau^* \frac{ds^*}{ds} \quad (2.22)$$

$$\begin{cases} T^* = \cos \theta T - \sin \theta B \\ N^* = \sin \theta T + \cos \theta B \\ B^* = N. \end{cases} \quad (2.23)$$

Where  $\angle (T, T^*) = \theta$ , [3]. Let  $\kappa$  be curvature of  $\alpha$ ,  $\tau$  be torsion of  $\alpha$ , and let  $\kappa^*$  be curvature of  $\alpha^*$ ,  $\tau^*$  be torsion of  $\alpha^*$ . Then, there are the following equations;

$$\begin{cases} \kappa = \tau^* \sin \theta \cdot \frac{ds^*}{ds} \\ \tau = -\tau^* \cos \theta \cdot \frac{ds^*}{ds}. \end{cases} \quad (2.24)$$

$$\begin{cases} \kappa^* = \frac{d\theta}{ds^*} \\ \tau^* = (\kappa \sin \theta - \tau \cos \theta) \cdot \frac{ds^*}{ds}. \end{cases} \quad (2.25)$$

$$\tau^* = \frac{\kappa}{\lambda\tau}, [3] \quad (2.26)$$

Let  $\alpha : I \rightarrow \mathbb{E}^3$  be Mannheim curve,  $\alpha^* : I \rightarrow \mathbb{E}^3$  be Mannheim partner curve. Suppose that Frenet frames are respectively given as  $\{T(s), N(s), B(s)\}$  and  $\{T^*(s), N^*(s), B^*(s)\}$ . Let  $\theta$  be the angle which is between  $T$  and  $T^*$ , and let  $\varphi$  be the angle which is between  $B$  and  $W$ . In this case, the following equations hold.

$$C = T^* \quad (2.27)$$

$$\begin{cases} \sin \varphi = \cos \theta \\ \cos \varphi = -\sin \theta \end{cases} \quad (2.28)$$

(see [15]). If we consider (2.28), (2.8), (2.9), (2.22) and (2.23) will respectively turn the following equations;

$$\cos \theta = \frac{\tau}{\|W\|}, \quad -\sin \theta = \frac{\kappa}{\|W\|} \quad (2.29)$$

$$\begin{cases} k_T = -\frac{1}{\sin \theta}, \\ k_N = \sqrt{1 + \left(\frac{\theta'}{\|W\|}\right)^2}, \\ k_B = \frac{1}{\cos \theta}, \\ k_C = \sqrt{1 + \left(\frac{\|W\|}{\theta'}\right)^2}. \end{cases} \quad (2.30)$$

$$\gamma_T = -\cot \theta, \quad \gamma_N = \frac{\theta'}{\|W\|}, \quad \gamma_B = -\tan \theta, \quad \gamma_C = \frac{\|W\|}{\theta'}$$

$$\sin \varphi = \frac{ds^*}{ds}, \quad \cos \varphi = \lambda\tau^* \frac{ds^*}{ds} \quad (2.31)$$

(see [15]).

### §3. Timelike-Spacelike Mannheim Curve Pairs

**Definition 3.1** Let  $\alpha : I \rightarrow \mathbb{L}^3$  be timelike curve and let  $\alpha^* : I \rightarrow \mathbb{L}^3$  be spacelike curve which has spacelike binormal. Suppose that  $\alpha$  curve's Frenet frames on the point of  $\alpha(s)$  is

$\{T(s), N(s), B(s)\}$  and  $\alpha^*$  curve's Frenet frames on the point of  $\alpha^*(s)$  is  $\{T^*(s), N^*(s), B^*(s)\}$ . If  $\alpha$  curve's principal normal vector and  $\alpha^*$  curve's binormal vektore are linearly dependent,  $\alpha$  curve is called Mannheim curve and  $\alpha^*$  curve is called Mannheim partner curve. This pair curve is briefly symbolized as  $(\alpha, \alpha^*)$  and it is named timelike-spacelike Mannheim curve pairs.

**Theorem 3.1** *The distance which is between  $(\alpha, \alpha^*)$  timelike-spacelike Mannheim curve pairs is constant.*

*Proof* It can be written that;

$$\alpha(s) = \alpha^*(s^*) + \lambda(s^*)B^*(s^*).$$

If this equation is derived with respect to  $s^*$  parameter, we can write that;

$$T \frac{ds}{ds^*} = T^* + \lambda \tau^* N^* + \lambda' B^*$$

If we get inner product of the last equation and  $B^*$ , then;

$$\lambda' = 0.$$

From the definition of Euclidean distance, we can write that;

$$\begin{aligned} d(\alpha^*(s^*), \alpha(s)) &= \|\alpha(s) - \alpha^*(s^*)\| \\ &= |\lambda| = \text{constant} \end{aligned} \quad \square$$

**Theorem 3.2** *Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs. Suppose that  $\alpha$  curve's and  $\alpha^*$  curve's Frenet frames are respectively  $\{T, N, B\}$  and  $\{T^*, N^*, B^*\}$ . In this case, there are the following equations;*

$$\begin{cases} T^* = -\sinh \theta T + \cosh \theta B \\ N^* = -\cosh \theta T + \sinh \theta B \\ B^* = N. \end{cases} \quad (3.1)$$

$$\sinh \theta = \frac{ds^*}{ds}, \quad \cosh \theta = -\lambda \tau^* \frac{ds^*}{ds} \quad (3.2)$$

$$\begin{cases} T = \sinh \theta T^* - \cosh \theta N^* \\ N = B^* \\ B = \cosh \theta T^* - \sinh \theta N^*. \end{cases} \quad (3.3)$$

*Proof* If we derive  $\alpha^*(s^*) = \alpha(s) - \lambda N(s)$  with respect to  $s$  parameter, we can write that;

$$T^* \frac{ds^*}{ds} = (1 - \lambda \kappa) T(s) - \lambda \tau B \quad (3.4)$$

If we get inner product of (3.4) and  $T$ , then;

$$-\sinh \theta \frac{ds^*}{ds} = 1 - \lambda \kappa \quad (3.5)$$

If we get inner product of (3.4) and  $B$ , then;

$$\cosh \theta \frac{ds^*}{ds} = \lambda \tau \quad (3.6)$$

If (3.5) and (3.6) are plugged into (3.4), we can write that;

$$T^* = -\sinh \theta T + \cosh \theta B$$

From Frenet formulas, the following equations can be found.

$$N^* = -\cosh \theta T + \sinh \theta B,$$

$$B^* = N$$

Obviously, we have shown that the equation of (3.1). If we arrange this equation with respect to  $T$  and  $B$ , we can find the equation of (3.3). If  $\alpha(s) = \alpha^*(s^*) + \lambda B^*(s^*)$  is derived with respect to sparameter, it can be found that;

$$T = T^* \frac{ds^*}{ds} + \lambda \tau^* \frac{ds^*}{ds} N^*$$

If we consider the corresponding value of  $T$  from (3.3), the equation of (3.2) is proven.

**Theorem 3.3** *Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs. Let  $\kappa$  be curvature of  $\alpha$ , and let  $\tau$  be torsion of  $\alpha$ . In this case, there is the following equation*

$$\lambda \kappa - \mu \tau = 1$$

*Proof* If we divide (3.5) to (3.6), we can write that;

$$\tanh \theta = \frac{\lambda \kappa - 1}{\lambda \tau}$$

If we get  $\mu = \lambda \tanh \theta$ , the result is proven.  $\square$

**Theorem 3.4** *Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs. In this case, there are the following equations among curvatures.*

$$\kappa = -\tau^* \cosh \theta \frac{ds^*}{ds}, \quad \tau = -\tau^* \sinh \theta \frac{ds^*}{ds} \quad (3.7)$$

$$\kappa^* = \frac{d\theta}{ds^*} = \theta' \frac{ds}{ds^*}, \quad \tau^* = -\kappa \cosh \theta \frac{ds}{ds^*} + \tau \sinh \theta \frac{ds}{ds^*}. \quad (3.8)$$



*Proof* If  $\langle T, B^* \rangle = 0$  is derived, then;  $\kappa = -\tau^* \cosh \theta \frac{ds^*}{ds}$ ,

If  $\langle B, B^* \rangle = 0$  is derived, then;  $\tau = -\tau^* \sinh \theta \frac{ds^*}{ds}$ ,

If  $\langle T, T^* \rangle = \sinh \theta$  is derived, then;  $\kappa^* = \frac{d\theta}{ds^*} = \theta' \frac{ds}{ds^*}$ ,

If  $\langle N, N^* \rangle = 0$  is derived, then;  $\tau^* = -\kappa \cosh \theta \frac{ds}{ds^*} + \tau \sinh \theta \frac{ds}{ds^*}$ . Therefore, the result is proven.  $\square$

**Theorem 3.5** *Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs. Let  $\kappa^*$  be curvature of  $\alpha^*$ , let  $\tau^*$  be torsion of  $\alpha^*$ , and let  $\tau$  be torsion of  $\alpha$ . The following equation holds.*

$$\tau^* = -\frac{\kappa}{\lambda\tau}. \quad (3.9)$$

*Proof* If we get equations from (3.2) and if we multiply side by side (3.5) and (3.6), the result is proven.  $\square$

**Theorem 3.6** *Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs. There is the following equation between  $\alpha$  curve's  $W$  Darboux vector and  $\alpha^*$  curve's  $T^*$  tangent vector.*

$$W = \tau^* \frac{ds^*}{ds} T^* \quad (3.10)$$

*Proof* We know that  $W = \tau T - \kappa B$ . If we get the corresponding values of  $T$  and  $B$  from (3.3), and then if we plug into those values in (3.10), Then, if we get the corresponding values of  $\kappa$  and  $\tau$  from (3.7), and then if we plug into those values in (3.10), the result is proven.  $\square$

**Result 3.1** *Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs. Let  $\varphi$  be the angle which is between  $\alpha$  curve's Darboux vector and  $\alpha$  curve's binormal vector. There are the following equations between  $\theta$  and  $\varphi$ .*

If  $W$  is a spacelike vector;

$$\begin{cases} \sinh \varphi = -\sinh \theta \\ -\cosh \varphi = \cosh \theta \end{cases} \quad (3.11)$$

If  $W$  is timelike vector;

$$\begin{cases} \cosh \varphi = -\sinh \theta \\ -\sinh \varphi = \cosh \theta \end{cases} \quad (3.12)$$

$$\varphi' = -\theta' \quad (3.13)$$

*Proof* Suppose that  $W$  is a spacelike vector, and the following equations hold because of the equations of (2.13) and (3.1),

$$C = \sinh \varphi T - \cosh \varphi B,$$

$$T^* = -\sinh \theta T + \cosh \theta B$$

If we consider that  $C = T^*$ , we can write that;

$$\begin{cases} \sinh \varphi = -\sinh \theta \\ -\cosh \varphi = \cosh \theta \end{cases},$$

Similarly, suppose that  $W$  is a timelike vector, and from the equation of (2.15), we can write that;

$$C = \cosh \varphi T - \sinh \varphi B$$

If we consider that  $C = T^*$ , we can write that;

$$\begin{cases} \cosh \varphi = -\sinh \theta \\ -\sinh \varphi = \cosh \theta \end{cases}$$

If we divide two equations to each other in the equation of (3.12), we can easily write that;

$$\cot h\varphi = \tanh \theta$$

If we derive last equation, the following equation holds.

$$\varphi' = -\theta'.$$

**Theorem 3.7** *Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs. There is the following equation which is between  $\alpha$  curve's  $W$  Darboux vector and  $\alpha^*$  curve's  $W^*$  Darboux vector;*

$$W^* = \frac{-1}{\sinh \theta} W - \frac{\theta' \kappa}{\lambda \kappa \|W\|} N \quad (3.14)$$

*Proof* It can be written  $\tau^* T^* = -W^* + \kappa^* B^*$  because  $\alpha^*$  Mannheim partner curve's  $W^*$  vector is spacelike. If this equation is plugged into (3.10), and if we consider that  $B^* = N$ , the following equation holds.

$$W = \frac{ds^*}{ds} (-W^* + \kappa^* N)$$

If the corresponding value of  $\frac{ds}{ds^*}$  from (3.2) is written in this equation, then;

$$W^* = \frac{-1}{\sinh \theta} W + \kappa^* N \quad (3.15)$$

On the other hand, from the equation of (3.7), we can write that  $\frac{ds}{ds^*} = \frac{\tau^*}{\|W\|}$ . If the corresponding value of  $\tau^*$  in (3.9) is written in this equation, then;

$$\frac{ds}{ds^*} = \frac{-\kappa}{\lambda \tau \|W\|}.$$

If we plug this equation into  $\kappa^* = \theta' \frac{ds}{ds^*}$ , the following equation holds.

$$\kappa^* = -\theta' \frac{\kappa}{\lambda_T \|W\|} \quad (3.16)$$

If we write this value of  $\kappa^*$  in (3.15), the result is proven.

Let  $s_T$  be  $\alpha : I \rightarrow \mathbb{L}^3$  timelike Mannheim curve's  $(T)$  tangent indicator's arc length, then;

$$s_T = \int_0^s \kappa ds \quad (3.17)$$

Similarly,  $(N)$  principal normal,  $(B)$  binormal, and  $(C)$  fixed pol curve's arc lengths are respectively;

$$s_N = \int_0^s \|W\| ds, \quad (3.18)$$

$$s_B = \int_0^s |\tau| ds, \quad (3.19)$$

$$s_C = \int_0^s |\phi'| ds \quad (3.20)$$

Let  $k_T$  be  $(T)$  tangent indicator's geodesic curvature on  $\mathbb{L}^3$ . Suppose that  $T_T$  is unit tangent vector of  $(T)$ , then;

$$k_T = \|D_{T_T} T_T\|$$

If  $\alpha_T(s) = T(s)$  tangent indicator is derived with respect to  $s_T$  parameter, we can write that;

$$T_T = N \quad (3.21)$$

If we derive one more time and simplify the equation, the following equation holds.

$$D_{T_T} T_T = T - \frac{\tau}{\kappa} B \quad (3.22)$$

From the definition of geodesic curvature;

$$k_T = \sqrt{\left| -1 + \frac{\tau^2}{\kappa^2} \right|} \quad (3.23)$$

If we consider the equations of (2.12) and (3.11), we can write that;

$$k_T = \frac{1}{\cosh \theta} \quad (3.24)$$

Similarly, if  $\alpha_N(s) = N(s)$  principal normal indicator is derived with respect to  $s_N$  parameter, and the equation of (2.12) is plugged into (3.25), and then the equation of (3.11) is considered, we can write that;

$$T_N = -\cosh \theta T + \sinh \theta B \quad (3.25)$$

If we derive one more time, the following equation holds.

$$D_{T_N} T_N = (\theta' \sinh \theta T + \|W\| N - \theta' \cosh \theta B) \frac{1}{\|W\|} \quad (3.26)$$

From the definition of geodesic curvature;

$$k_N = \sqrt{\left| 1 + \left( \frac{\theta'}{\|W\|} \right)^2 \right|} \quad (3.27)$$

If  $\alpha_B(s) = B(s)$  binormal indicator is derived with respect to  $s_B$  parameter, and if we chose the positive routing.

$$D_{T_B} T_B = \frac{\kappa}{\tau} T - B \quad (3.28)$$

From the equation of (2.12);

$$k_B = \frac{1}{\sinh \theta} \quad (3.29)$$

If  $\alpha_C(s) = C(s)$  fixed pol curve is derived with respect to  $s_C$  parameter, we can write that;

$$T_C = \cosh \varphi T - \sinh \varphi B$$

If we derive one more time, and if we consider the equations of (3.11) and (3.13), we can write that;

$$D_{T_C} T_C = -\sinh \theta T + \cosh \theta B \pm \frac{\|W\|}{\theta'} N \quad (3.30)$$

From the definition of geodesic curvature;

$$k_C = \sqrt{\left| 1 + \left( \frac{\|W\|}{\theta'} \right)^2 \right|} \quad (3.31)$$

**Result 3.2** Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs.  $\alpha$  Mannheim curve's spherical indicators are  $(T), (N), (B)$  and also  $\alpha$  Mannheim curve's fixed pol curve which is  $(C)$ , with respect to  $\mathbb{I}\mathbb{L}^3$ , geodesic curvatures of  $(T), (N), (B)$  and  $(C)$  are respectively;

$$\left\{ \begin{array}{l} k_T = \frac{1}{\cosh \theta} \\ k_N = \sqrt{\left| 1 + \left( \frac{\theta'}{\|W\|} \right)^2 \right|} \end{array} \right.,$$

$$\left\{ \begin{array}{l} k_B = \frac{1}{\sinh \theta} \\ k_C = \sqrt{\left| 1 + \left( \frac{\|W\|}{\theta'} \right)^2 \right|}. \end{array} \right.$$

Let  $D$  be  $\alpha : I \rightarrow \mathbb{M}^3$  Mannheim curve's connection on  $\mathbb{M}^3$ , let  $\bar{D}$  be  $\alpha : I \rightarrow \mathbb{M}^3$  Mannheim curve's connection on  $S_1^2$ , and let  $\bar{\bar{D}}$  be  $\alpha : I \rightarrow \mathbb{M}^3$  Mannheim curve's connection on  $H_0^2$ .

Suppose that  $\xi$  is unit normal vector space of  $S_1^2$  and  $H_0^2$ , then;

$$D_X Y = \bar{D}_X Y + \varepsilon g(S(X), Y)\xi, \quad \varepsilon = g(\xi, \xi)$$

$$D_X Y = \bar{\bar{D}}_X Y + \varepsilon g(S(X), Y)\xi, \quad \varepsilon = g(\xi, \xi).$$

Where  $S$  is shape operator of  $S_1^2$  and  $H_0^2$ , and corresponding matrix is;

$$S = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

(see[?]). Let  $\gamma_T$  be  $(T)$  tangent indicator's geodesic curvature on  $H_0^2$ , then;

$$\gamma_T = \left\| \bar{\bar{D}}_{T_T} T_T \right\|$$

From Gauss Equation, we can write that;

$$D_{T_T} T_T = \bar{\bar{D}}_{T_T} T_T + \varepsilon g(S(T_T), T_T) T$$

where

$$\varepsilon = g(T, T) = -1, \quad S(T_T) = -T_T, \quad \text{and } g(S(T_T), T_T) = -1$$

If we write those values in Gauss equation, and if we consider (3.22), we can write that;

$$\bar{\bar{D}}_{T_T} T_T = -\frac{\tau}{\kappa} B. \quad (3.32)$$

And also from the equation of (2.12) and (3.11);

$$\gamma_T = \tanh \theta. \quad (3.33)$$

Similarly, Let  $\gamma_N$  be geodesic curvature of  $(N)$  principal normal indicator on  $S_1^2$ , then;

$$\gamma_N = \left\| \bar{\bar{D}}_{T_N} T_N \right\|$$

From Gauss Equation, we can write that;

$$D_{T_N} T_N = \bar{\bar{D}}_{T_N} T_N + \varepsilon g(S(T_N), T_N) N$$

where

$$\varepsilon = g(N, N) = +1, \quad S(T_N) = -T_N, \quad \text{and } g(S(T_N), T_N) = +1.$$

If those values are written in Gauss equation, and if we consider (3.26), we can write that;

$$\bar{\bar{D}}_{T_N} T_N = \frac{\theta'}{\|W\|} (\sinh \theta T - \cosh \theta B) \quad (3.34)$$

If we get norm of the equation;

$$\gamma_N = \frac{\theta'}{\|W\|} \quad (3.35)$$

Let  $\gamma_B$  be geodesic curvature of  $(B)$  binormal indicator on  $S_1^2$ , then;

$$\gamma_B = \|\bar{D}_{T_B} T_B\|.$$

From Gauss equation, we can write that;

$$D_{T_B} T_B = \bar{D}_{T_B} T_B + \varepsilon g(S(T_B), T_B) B$$

where

$$\varepsilon = g(B, B) = +1, \quad S(T_B) = -T_B \quad \text{and} \quad g(S(T_B), T_B) = -1.$$

If we write those values in Gauss equation, and if we consider (3.28), we can write that;

$$\bar{D}_{T_B} T_B = \frac{\kappa}{\tau} T \quad (3.36)$$

From the equations of (2.12) and (3.11), it can be written that;

$$\gamma_B = \coth \theta \quad (3.37)$$

Let  $\gamma_C$  be geodesic curvature of  $(C)$  fixed pol curve on  $S_1^2$ , then;

$$\gamma_C = \|\bar{D}_{T_C} T_C\|.$$

From Gauss equation

$$D_{T_C} T_C = \bar{D}_{T_C} T_C + \varepsilon g(S(T_C), T_C) C$$

where

$$\varepsilon = g(C, C) = +1, \quad S(T_C) = -T_C \quad \text{and} \quad g(S(T_C), T_C) = -1.$$

If we write those values in Gauss equation, and if we consider (2.13), (3.11) and (3.30), we can write that;

$$\gamma_C = \frac{\|W\|}{\theta'}. \quad (3.38)$$

**Result 3.3** Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs.  $\alpha$  Mannheim curve's spherical indicators are  $(T), (N), (B)$  and also  $\alpha$  Mannheim curve's fixed pol curve is  $(C)$ , with respect to  $S_1^2$  Lorentz sphere or  $H_0^2$  hyperbolic sphere, geodesic curvatures of  $(T), (N), (B)$  and  $(C)$  are respectively;

$$\gamma_T = \tanh \theta, \gamma_N = \frac{\theta'}{\|W\|}, \gamma_B = \coth \theta, \gamma_C = \frac{\|W\|}{\theta'}.$$

Let  $\alpha^* : I \rightarrow \mathbb{L}^3$  be spacelike Mannheim curve which has spacelike binormal, and let  $s_{T^*}$

be  $\alpha^* : I \rightarrow \mathbb{L}^3$  curve's  $(T^*)$  tangent indicator's arc length, then,

$$s_{T^*} = \int_0^s \theta' ds \quad (3.39)$$

Similarly, arc lengths of  $(N^*)$ ,  $(B^*)$  and  $(C^*)$  are found;

$$s_{N^*} = \int_0^s \sqrt{(\theta')^2 + \|W\|^2} ds \quad (3.40)$$

$$s_{B^*} = \int_0^s \|W\| ds \quad (3.41)$$

$$s_{C^*} = \int_0^s (\varphi^*)' ds. \quad (3.42)$$

On the other hand, let  $\varphi^*$  be the angle which is between  $W^*$  and  $B^*$ . unit Darboux vector can be written as;

$$C^* = -\sin \varphi^* T^* + \cos \varphi^* B^*$$

where

$$\sin \varphi^* = \frac{\tau^*}{\|W^*\|} \quad \text{and} \quad \cos \varphi^* = \frac{\kappa^*}{\|W^*\|} \Rightarrow \tan \varphi^* = \frac{\tau^*}{\kappa^*}.$$

$C^*$  is derived and then if we simplify the equation, we can write that;

$$(\varphi^*)' = \frac{\left(\frac{\tau^*}{\kappa^*}\right)'}{1 + \left(\frac{\tau^*}{\kappa^*}\right)^2}$$

If the values of  $\kappa^*$  and  $\tau^*$  are written in the equation, we can write that;

( Note that the values of  $\kappa^*$  and  $\tau^*$  are corresponding values of (3.8) and (3.12)

$$(\varphi^*)' = \frac{\left(\frac{\|W\|}{\theta'}\right)'}{1 + \left(\frac{\|W\|}{\theta'}\right)^2} \quad (3.43)$$

If (3.43) is written in the equation of (3.42), we can easily find that;

$$s_{C^*} = \int_0^s \frac{\left(\frac{\|W\|}{\theta'}\right)'}{1 + \left(\frac{\|W\|}{\theta'}\right)^2} ds \quad (3.44)$$

If (3.23) is considered, we can obtain that;

$$(\varphi^*)' = \frac{\left(\sqrt{k_C^2 - 1}\right)'}{k_C^2} \quad (3.45)$$

If (3.45) is written in the equation of (3.42), we can find that;

$$s_{C^*} = \int_0^s \frac{\left(\sqrt{k_C^2 - 1}\right)'}{k_C^2} ds. \quad (3.46)$$

**Result 3.4** Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs. On the point of  $\alpha^*(s)$ ,  $\alpha^*$  curve's Frenet vectors' spherical indicator curves and  $(C^*)$  fixed pol curve which is drawn by the unit Darboux vector. In terms of  $\mathbb{I}\mathbb{L}^3$ ,  $(C^*)$  fixed pol curve's arc lengths are respectively;

$$\left\{ \begin{array}{l} s_{T^*} = \int_0^s \theta' ds, \\ s_{N^*} = \int_0^s \sqrt{(\theta')^2 + \|W\|^2} ds, \\ s_{B^*} = \int_0^s \|W\| ds, \\ s_{C^*} = \int_0^s \frac{\left(\frac{\|W\|}{\theta'}\right)'}{1 + \left(\frac{\|W\|}{\theta'}\right)^2} ds. \end{array} \right.$$

**Result 3.5** Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs, and let  $k_C$  be  $\alpha$  timelike Mannheim curve's geodesic curvature. In this case, the arc length of  $(C^*)$  fixed pol curve is;

$$s_{C^*} = \int_0^s \frac{\left(\sqrt{k_C^2 - 1}\right)'}{k_C^2} ds.$$

In terms of  $\mathbb{I}\mathbb{L}^3$ , Let  $k_{T^*}$  be  $\alpha_{T^*}^*(s) = T^*(s)$  tangent indicator's geodesic curvature, let  $s_{T^*}$  be arc parameter, and let  $T_{T^*}$  be unit tangent vector. Then, we can say that;

$$T_{T^*} = -\cosh \theta T + \sinh \theta B \quad (3.47)$$

If we derive one more time the equation of (3.47), it can be written that;

$$D_{T_{T^*}} T_{T^*} = -\sinh \theta T + \cosh \theta B + \frac{\|W\|}{\theta'} N \quad (3.48)$$

or if we get norm of (3.48), we can write that;

$$k_{T^*} = \sqrt{\left|1 + \left(\frac{\|W\|}{\theta'}\right)^2\right|} \quad (3.49)$$

Similarly, in terms of  $\mathbb{I}\mathbb{L}^3$ , Let  $k_{N^*}$  be  $\alpha_{N^*}^*(s) = N^*(s)$  principal normal indicator's geodesic curvature, let  $s_{N^*}$  be arc parameter, and let  $T_{N^*}$  be unit tangent vector. Then, we can say



that;

$$T_{N^*} = \frac{1}{\sqrt{\left|1 + \left(\frac{\|W\|}{\theta'}\right)^2\right|}}(-\sinh \theta T + \cosh \theta B) + \frac{1}{\sqrt{\left|1 + \left(\frac{\theta'}{\|W\|}\right)^2\right|}}N \quad (3.50)$$

If we consider the equations of (3.27) and (3.31), it can be written that;

$$T_{N^*} = \frac{1}{k_C}(-\sinh \theta T + \cosh \theta B) + \frac{1}{k_N}N \quad (3.51)$$

If we derive one more time the equation of (3.51), after simplifying, it can be written that;

$$\begin{aligned} D_{T_{N^*}} T_{N^*} = & \left( \left[ \frac{\left(\frac{-\sinh \theta}{k_C}\right)' + \left(\frac{\kappa}{k_N}\right)}{\sqrt{(\theta')^2 + \|W\|^2}} \right] T + \left[ \frac{\left(\frac{1}{k_N}\right)'}{\sqrt{(\theta')^2 + \|W\|^2}} \right] N \right. \\ & \left. + \left[ \frac{\left(\frac{\cosh \theta}{k_C}\right)' - \frac{\tau}{k_N}}{\sqrt{(\theta')^2 + \|W\|^2}} \right] B \right) \end{aligned} \quad (3.52)$$

If we get norm of (3.52), we can write that;

$$k_{N^*} = \sqrt{\frac{\left[ \left(\frac{-\sinh \theta}{k_C}\right)' + \left(\frac{\kappa}{k_N}\right) \right]^2 + \left[ \left(\frac{1}{k_N}\right)' \right]^2 + \left[ \left(\frac{\cosh \theta}{k_C}\right)' - \frac{\tau}{k_N} \right]^2}{(\theta')^2 + \|W\|^2}}. \quad (3.53)$$

In terms of  $\mathbb{L}^3$ , Let  $k_{B^*}$  be  $\alpha_{B^*}^*(s) = B^*(s)$  binormal indicator's geodesic curvature, let  $s_{B^*}$  be arc parameter, and let  $T_{B^*}$  be unit tangent vector. Then, we can say that;

$$T_{B^*} = \frac{\kappa}{\|W\|}T - \frac{\tau}{\|W\|}B \quad (3.54)$$

If we consider the equations of (2.12) and (3.11), it can be written that;

$$T_{B^*} = -\cosh \theta T + \sinh \theta B$$

If we derive,  $T_{B^*}$  after simplifying, it can be written that;

$$D_{T_{B^*}} T_{B^*} = \frac{\theta'}{\|W\|}(-\sinh \theta T + \cosh \theta B) + N \quad (3.55)$$

or if we get norm of  $T_{B^*}$ , we can write that;

$$k_{B^*} = \sqrt{1 + \left(\frac{\theta'}{\|W\|}\right)^2} \quad (3.56)$$

In terms of  $\mathbb{L}^3$ , Let  $k_{C^*}$  be  $\alpha_{C^*}^*(s) = C^*(s)$  fixed pol curve's geodesic curvature, let  $s_{C^*}$  be arc parameter, and let  $T_{C^*}$  be unit tangent vector. Then, we can say that;

$$T_{C^*} = -\cos \varphi^* T^* + \sin \varphi^* B^* \quad (3.57)$$

If we derive one more time the equation of (3.57), it can be written that;

$$D_{T_{C^*}} T_{C^*} = -\sin \varphi^* T^* + \cos \varphi^* B^* - \frac{\|W^*\|}{(\varphi^*)'} N^* \quad (3.58)$$

If we get norm of (3.57), we can write that;

$$k_{C^*} = \sqrt{\left| 1 + \left( \frac{\|W^*\|}{(\varphi^*)'} \right)^2 \right|} \quad (3.59)$$

If the values of  $\kappa^*$  and  $\tau^*$  are written in  $\|W^*\| = \sqrt{[(\tau^*)^2 + (\kappa^*)^2]}$ , we can find that ( Note that the value of  $\kappa^*$  and  $\tau^*$  are corresponding values of (3.8) and (3.9).)

$$\|W^*\| = \frac{\kappa}{\lambda\tau} \sqrt{\left| 1 + \left( \frac{\|W\|}{(\varphi)'} \right)^2 \right|}$$

From the equation of (3.31) and (3.45), it can be written that;

$$\frac{\|W^*\|}{(\varphi^*)'} = \frac{\kappa(k_C)^3}{\lambda\tau \left( \sqrt{k_C^2 - 1} \right)'} \quad (3.60)$$

If the value of (3.60) is written in, we can say that;

$$k_{C^*} = \sqrt{\left| 1 + \left( \frac{\kappa(k_C)^3}{\lambda\tau \left( \sqrt{k_C^2 - 1} \right)'} \right)^2 \right|}$$

**Result 3.6** Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs. In terms of  $\mathbb{L}^3$ ,  $\alpha^*$  curve's  $(T^*), (N^*), (B^*)$  spherical indicator curves' and  $(C^*)$  fixed pol curve's geodesic curvatures are

respectively;

$$\left\{ \begin{array}{l} k_{T^*} = \sqrt{\left| 1 + \left( \frac{\|W\|}{\theta'} \right)^2 \right|}, \\ k_{N^*} = \sqrt{\frac{\left[ \left( \frac{-\sinh \theta}{k_C} \right)' + \left( \frac{\kappa}{k_N} \right) \right]^2 + \left[ \left( \frac{1}{k_N} \right)' \right]^2 + \left[ \left( \frac{\cosh \theta}{k_C} \right)' - \frac{\tau}{k_N} \right]^2}{(\theta')^2 + \|W\|^2}}, \\ k_{B^*} = \sqrt{1 + \left( \frac{\theta'}{\|W\|} \right)^2}, \\ k_{C^*} = \sqrt{\left| 1 + \left( \frac{\kappa(k_C)^3}{\lambda \tau (\sqrt{k_C^2 - 1})'} \right)^2 \right|}. \end{array} \right.$$

Let  $\gamma_{T^*}$  be  $\alpha^* : I \rightarrow \mathbb{L}^3$  spacelike binormal spacelike Mannheim partner curve's  $\alpha_{T^*}^*(s) = T^*(s)$  tangent indicator's geodesic curvature in  $S_1^2$ . Then;

$$\gamma_{T^*} = \|\bar{D}_{T_{T^*}} T_{T^*}\|$$

From Gauss equation, it can be written that;

$$D_{T_{T^*}} T_{T^*} = \bar{D}_{T_{T^*}} T_{T^*} + \varepsilon g(S(T_{T^*}), T_{T^*}) T^*$$

where

$$\varepsilon = g(T^*, T^*) = +1, \quad S(T_{T^*}) = -T_{T^*} \text{ and } g(S(T_{T^*}), T_{T^*}) = +1.$$

If those values are written in Gauss equation, and if the equation of (3.1) and (3.48) are considered, we can say that;

$$\bar{D}_{T_{T^*}} T_{T^*} = \left( \frac{\|W\|}{\theta'} \right) N \quad (3.61)$$

If we get norm of (3.61), we can write that;

$$\gamma_{T^*} = \frac{\|W\|}{\theta'}$$

$\bar{D}_{T_{T^*}} T_{T^*} = 0$  if and only if  $(\overline{T^*})$  curve geodesic spray is an integral curve. From the equation of (3.61), we can find that;  $\kappa = 0, \tau = 0$  which means  $\alpha$  is a straight line.

**Result 3.7** Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs.  $\alpha$  Mannheim curve does not have any partner curve because  $\alpha$  Mannheim curve is a straight line.

Let  $\gamma_{N^*}$  be geodesic indicator in  $H_0^2$  for  $\alpha_{N^*}^*(s) = N^*(s)$  principal normal indicator. We can write that;

$$\gamma_{N^*} = \|\bar{D}_{T_{N^*}} T_{N^*}\|$$

From Gauss equation, it can be written that;

$$D_{T_{N^*}} T_{N^*} = \bar{D}_{T_{N^*}} T_{N^*} + \varepsilon g(S(T_{N^*}), T_{N^*}) N^*$$

where

$$\varepsilon = g(N^*, N^*) = +1, S(T_{N^*}) = -T_{N^*} \text{ and } g(S(T_{N^*}), T_{N^*}) = +1.$$

If those values are written in Gauss equation, and if the equations of (3.1) and (3.52) are considered, we can say that

$$\begin{aligned} \bar{D}_{T_{N^*}} T_{N^*} &= \left[ \frac{\left(\frac{-\sinh \theta}{k_C}\right)' + \left(\frac{\kappa}{k_N}\right)}{\sqrt{(\theta')^2 + \|W\|^2}} + \cosh \theta \right] T + \\ &\quad \left[ \frac{\left(\frac{1}{k_N}\right)'}{\sqrt{(\theta')^2 + \|W\|^2}} \right] N + \left[ \frac{\left(\frac{\cosh \theta}{k_C}\right)' - \frac{\tau}{k_N}}{\sqrt{(\theta')^2 + \|W\|^2}} - \sinh \theta \right] B \end{aligned} \quad (3.62)$$

If we get norm of (??), we can write that;

$$\begin{aligned} \gamma_{N^*} &= \left( \left[ \frac{\left(\frac{-\sinh \theta}{k_C}\right)' + \left(\frac{\kappa}{k_N}\right)}{\sqrt{(\theta')^2 + \|W\|^2}} + \cosh \theta \right]^2 + \left[ \frac{\left(\frac{1}{k_N}\right)'}{\sqrt{(\theta')^2 + \|W\|^2}} \right]^2 \right. \\ &\quad \left. + \left[ \frac{\left(\frac{\cosh \theta}{k_C}\right)' - \frac{\tau}{k_N}}{\sqrt{(\theta')^2 + \|W\|^2}} - \sinh \theta \right]^2 \right)^{\frac{1}{2}} \end{aligned}$$

$\bar{D}_{T_{N^*}} T_{N^*} = 0$  if and only if  $(N^*)$  curve geodesic spray is an integral curve. In this case, we can write that;

$$\left\{ \begin{array}{l} \frac{\left(\frac{-\sinh \theta}{k_C}\right)' + \left(\frac{\kappa}{k_N}\right)}{\sqrt{(\theta')^2 + \|W\|^2}} + \cosh \theta = 0, \\ \frac{\left(\frac{1}{k_N}\right)'}{\sqrt{(\theta')^2 + \|W\|^2}} = 0, \\ \frac{\left(\frac{\cosh \theta}{k_C}\right)' - \frac{\tau}{k_N}}{\sqrt{(\theta')^2 + \|W\|^2}} - \sinh \theta = 0 \end{array} \right.$$

This value cannot be 0.

**Result 3.8** Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs. There is no Mannheim partner curve on hyperbolic sphere to be made  $\alpha^*$  Mannheim partner curve's  $(N^*)$  principal normal indicator geodesic spray is an integral curve.

Let  $\gamma_{B^*}$  be geodesic curve in  $S_1^2$  for  $\alpha_{B^*}^*(s) = B^*(s)$  principal normal indicator. In this case, we can write that;

$$\gamma_{B^*} = \left\| \bar{\bar{D}}_{T_{B^*}} T_{B^*} \right\|$$

From Gauss Equation, it can be written that;

$$D_{T_{B^*}} T_{B^*} = \bar{\bar{D}}_{T_{B^*}} T_{B^*} + \varepsilon g(S(T_{B^*}), T_{B^*}) B^*$$

where

$$\varepsilon = g(B^*, B^*) = -1, \quad S(T_{B^*}) = -T_{B^*} \text{ and } g(S(T_{B^*}), T_{B^*}) = -1.$$

If those values are written in Gauss Equation, then,  $B^* = N$ . And if the equation of (3.55) is considered, we can say that;

$$\bar{\bar{D}}_{T_{B^*}} T_{B^*} = \frac{\theta'}{\|W\|} (-\sinh \theta T + \cosh \theta B) \quad (3.63)$$

If we get norm of (3.63), we can write that;

$$\gamma_{B^*} = \frac{\theta'}{\|W\|}$$

$\bar{\bar{D}}_{T_{B^*}} T_{B^*} = 0$  if and only if  $(\overline{B^*})$  curve geodesic spray is an integral curve. In this case, from the equation of (3.63), we can write that;

$$\begin{cases} \frac{-\theta' \sinh \theta}{\|W\|} = 0, \\ \frac{\theta' \cosh \theta}{\|W\|} = 0 \end{cases}$$

where  $\theta' = 0$ . In this case, from the equations of (3.13) and (2.18),  $\frac{\kappa}{\tau} = \text{constant}$ . This means  $\alpha$  Mannheim curve is a helix.

**Result 3.9** Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs. If  $\alpha$  Mannheim curve is a helix, Mannheim partner of  $\alpha$  is a straight line.

Let  $\gamma_{C^*}$  be geodesic curve in  $S_1^2$  for  $\alpha_{C^*}^*(s) = C^*(s)$ . In this case, we can write that;

$$\gamma_{C^*} = \left\| \bar{\bar{D}}_{T_{C^*}} T_{C^*} \right\|$$

From Gauss Equation, it can be written that;

$$D_{T_{C^*}} T_{C^*} = \bar{\bar{D}}_{T_{C^*}} T_{C^*} + \varepsilon g(S(T_{C^*}), T_{C^*}) C^*$$

where

$$\varepsilon = g(C^*, C^*) = +1, \quad S(T_{C^*}) = -T_{C^*} \text{ and } g(S(T_{C^*}), T_{C^*}) = -1.$$

If those values are written in Gauss equation, then,

$$C^* = -\sin \varphi^* T^* + \cos \varphi^* B^*$$

And if the equation of (3.57) is considered, we can say that;

$$\bar{D}_{T_{C^*}} T_{C^*} = -\frac{\|W^*\|}{(\varphi^*)'} N^* \quad (3.64)$$

If the equation of (3.60) is considered, geodesic curve is,

$$\gamma_{C^*} = \frac{\kappa(k_C)^3}{\lambda\tau \left( \sqrt{k_C^2 - 1} \right)'}$$

$\bar{D}_{T_{C^*}} T_{C^*} = 0$  if and only if  $(\bar{C}^*)$  curve geodesic spray is an integral curve. In this case, from the equation of (3.64), we can write that;  $\kappa^* = \tau^* = 0$ . And from the equation of (3.9),  $\kappa = 0$ .

**Result 3.10** Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs.  $\alpha$  Mannheim curve does not have any partner curve because  $\alpha$  Mannheim curve is a straight line.

**Result 3.11** Let  $(\alpha, \alpha^*)$  be timelike-spacelike Mannheim curve pairs. In terms of  $S_1^2$  or  $H_0^2$ ,  $\alpha^*$  curve's  $(T^*)$ ,  $(N^*)$  and  $(B^*)$  spherical indicator curves' and  $(C^*)$  fixed pol curve's geodesic curvatures are respectively;

$$\begin{aligned} \gamma_{T^*} &= \gamma_C = \frac{\|W\|}{\theta'} \\ \gamma_{N^*} &= \left( \left[ \frac{\left( \frac{-\sinh \theta}{k_C} \right)' + \left( \frac{\kappa}{k_N} \right)}{\sqrt{(\theta')^2 + \|W\|^2}} + \cosh \theta \right]^2 + \left[ \frac{\left( \frac{1}{k_N} \right)'}{\sqrt{(\theta')^2 + \|W\|^2}} \right]^2 \right. \\ &\quad \left. + \left[ \frac{\left( \frac{\cosh \theta}{k_C} \right)' - \frac{\tau}{k_N}}{\sqrt{(\theta')^2 + \|W\|^2}} - \sinh \theta \right]^2 \right)^{\frac{1}{2}} \\ \gamma_{B^*} &= \frac{\theta'}{\|W\|}, \gamma_{C^*} = \frac{\kappa(k_C)^3}{\lambda\tau k_C'}. \end{aligned}$$

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