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Characterizations of Space Curves According to Bishop Darboux Vector in Euclidean 3-Space E^3

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Abstract: In this paper, we obtained some characterizations of space curves according to Bihop frame in Euclidean 3-space E^3 by using Laplacian operator and Levi-Civita connection. Furthermore, we gave the general differential equations which characterize the space curves according to the Bishop Darboux vector and the normal Bishop Darboux vector.

Key Words: Bishop frame, Darboux vector, Euclidean 3-Space, Laplacian operator.

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§1. Introduction

It is well-known that a curve of constant slope or general helix is defined by the property that the tangent of the curve makes a constant angle with a fixed straight line which is called the axis of the general helix. A necessary and sufficient condition for a curve to be a general helix is that the ratio of curvature to torsion be constant ([10]). The study of these curves in has been given by many mathematicians. Moreover, İlarslan studied the characterizations of helices in Minkowski 3-space E_1^3 and found differential equations according to Frenet vectors characterizing the helices in E_1^3 ([15]). Then, Kocayiğit obtained general differential equations which characterize the Frenet curves in Euclidean 3-space E^3 and Minkowski 3-space E_1^3 ([11]).

Analogue to the helix curve, Izumiya and Takeuchi have defined a new special curve called the slant helix in Euclidean 3-space E^3 by the property that the principal normal of a space curve γ makes a constant angle with a fixed direction ([19]). The spherical images of tangent indicatrix and binormal indicatrix of a slant helix have been studied by Kula and Yaylı ([16]). They obtained that the spherical images of a slant helix are spherical helices. Moreover, Kula et al. studied the relations between a general helix and a slant helix ([17]). They have found some differential equations which characterize the slant helix.

Position vectors of slant helices have been studied by Ali and Turgut ([3]). Also, they have given the generalization of the concept of a slant helix in the Euclidean n-space E^n ([4]).

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Furthermore, Chen and Ishikawa classified biharmonic curves, the curves for which $\Delta H = 0$ holds in semi-Euclidean space E_v^n where Δ is Laplacian operator and H is mean curvature vector field of a Frenet curve ([9]). Later, Kocayiğit and Hacısalihoğlu studied biharmonic curves and 1-type curves i.e., the curves for which $\Delta H = \lambda H$ holds, where λ is constant, in Euclidean 3-space E^3 ([12]) and Minkowski 3-space E_1^3 ([13]). They showed the relations between 1-type curves and circular helix and the relations between biharmonic curves and geodesics. Moreover, slant helices have been studied by Bükçü and Karacan according to Bishop frame in Euclidean 3-space ([5]) and Minkowski space ([6,7]). Characterizations of space curves according to Bishop frame in Euclidean 3-space E^3 have been given in [14].

In this paper, we gave some characterizations of space curves according to Bishop Frame in Euclidean 3-space E^3 by using Laplacian operator. We found the differential equations characterizing space curves according to the Bishop Darboux vector and the normal Bishop Darboux vector.

§2. Preliminaries

Let $\alpha : I \subset \mathbb{R}$ be an arbitrary curve in Euclidean 3-space E^3. Recall that the curve α is said to be of unit speed (or parameterized by arc length function s) if $\left\langle \overrightarrow{\alpha'}, \overrightarrow{\alpha'} \right\rangle = 1$, where $\langle , \rangle$ is the standard scalar (inner) product of E^3 given by $\langle \overrightarrow{x}, \overrightarrow{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3$ for each $\overrightarrow{x} = (x_1, x_2, x_3)$, $\overrightarrow{y} = (y_1, y_2, y_3) \in E^3$. In particular, the norm of a vector $\overrightarrow{x} \in E^3$ is given by $\|\overrightarrow{x}\| = \sqrt{\langle \overrightarrow{x}, \overrightarrow{x} \rangle}$. Denote by $\left\{\overrightarrow{T}(s), \overrightarrow{N}(s), \overrightarrow{B}(s)\right\}$ the moving Frenet frame along the unit speed curve α. Then the Frenet formulas are given by

$$\begin{bmatrix} \overrightarrow{T}' \\ \overrightarrow{N}' \\ \overrightarrow{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{N} \\ \overrightarrow{B} \end{bmatrix},$$

where $\overrightarrow{T}$, $\overrightarrow{N}$ and $\overrightarrow{B}$ are called tangent, principal normal and binormal vector fields of the curve, respectively. $\kappa(s)$ and $\tau(s)$ are called curvature and torsion of the curve α, respectively ([20]).

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. We can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that, while $\overrightarrow{T}(s)$ for a given curve model is unique, we may choose any convenient arbitrary basis $\left(\overrightarrow{N_1}(s), \overrightarrow{N_2}(s)\right)$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $\overrightarrow{T}(s)$ at each point. If the derivatives of $\left(\overrightarrow{N_1}(s), \overrightarrow{N_2}(s)\right)$ depend only on $\overrightarrow{T}(s)$ and not each other we can make $\overrightarrow{N_1}(s)$ and $\overrightarrow{N_2}(s)$ vary smoothly throughout the path regardless of the curvature ([18,1,2]).

In addition, suppose the curve α is an arclength-parameterized C^2 curve. Suppose we have

C^1 unit vector fields $\overrightarrow{N_1}$ and $\overrightarrow{N_2} = \overrightarrow{T} \wedge \overrightarrow{N_1}$ along the curve α so that

$$\left\langle \overrightarrow{T}, \overrightarrow{N_1} \right\rangle = \left\langle \overrightarrow{T}, \overrightarrow{N_2} \right\rangle = \left\langle \overrightarrow{N_1}, \overrightarrow{N_2} \right\rangle = 0$$

i.e., $\overrightarrow{T}$, $\overrightarrow{N_1}$, $\overrightarrow{N_2}$ will be a smoothly varying right-handed orthonormal frame as we move along the curve. (To this point, the Frenet frame would work just fine if the curve were C^3 with $\kappa \neq 0$) But now we want to impose the extra condition that $\left\langle \overrightarrow{N_1'}, \overrightarrow{N_2} \right\rangle = 0$. We say the unit first normal vector field $\overrightarrow{N_1}$ is parallel along the curve α. This means that the only change of $\overrightarrow{N_1}$ is in the direction of $\overrightarrow{T}$. A Bishop frame can be defined even when a Frenet frame cannot (e.g., when there are points with $\kappa = 0$). Therefore, we have the alternative frame equations

$$\begin{bmatrix} \overrightarrow{T}' \\ \overrightarrow{N}'_1 \\ \overrightarrow{N}'_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{N_1} \\ \overrightarrow{N_2} \end{bmatrix}.$$

One can show that

$$\kappa(s) = \sqrt{k_1^2 + k_2^2},\ \theta(s) = \arctan\left(\frac{k_2}{k_1}\right),\ k_1 \neq 0,\ \tau(s) = -\frac{d\theta(s)}{ds}$$

so that k_1 and k_2 effectively correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = -\int \tau(s) ds$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ (and hence from the Frenet frame) due to the differentiation ([18,1,2]).

Let $\alpha : I \to E^3$ be a unit speed space curve with nonzero nature curvatures k_1, k_2. Then α is a slant helix if and only if $\frac{k_1}{k_2}$ is constant ([5]).

Let $\triangledown$ denotes the Levi-Civita connection given by $\triangledown_{\alpha'} = \frac{d}{ds}$ where s is the arclenght parameter of the space curve α. The Laplacian operator of α is defined by ([13])

$$\triangle = -\triangledown^2_{\alpha'} = -\triangledown_{\alpha'} \triangledown_{\alpha'}.$$

§3. Characterizations of Space Curves

In this section we gave the characterizations of the space curves according to Bishop frame in Euclidean 3-space E^3. Furthermore, we obtained the general differential equations which characterize the space curves according to the Bishop Darboux vector $\overrightarrow{W}$ and the normal Bishop Darboux vector $\overrightarrow{W}^\perp$ in E^3.

Theorem 3.1([8]) *Let $\alpha(s)$ be a unit speed space curve in Euclidean 3-space E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$, curvature k_1 and torsion k_2. The Bishop Darboux vector $\overrightarrow{W}$ of the curve α is given by*

$$\overrightarrow{W} = -k_2\overrightarrow{N_1} + k_1\overrightarrow{N_2}. \tag{3.1}$$

Definition 3.1 *A regular space curve α in E^3 said to has harmonic Darboux vector $\overrightarrow{W}$ if*

$$\Delta\overrightarrow{W} = 0.$$

Definition 3.2 *A regular space curve α in E^3 said to has harmonic 1-type Darboux vector $\overrightarrow{W}$ if*

$$\Delta\overrightarrow{W} = \lambda\overrightarrow{W}, \qquad \lambda \in \mathbb{R}. \tag{3.2}$$

Theorem 3.2 *Let $\alpha(s)$ be a unit speed space curve in Euclidean 3-space E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$, curvature k_1 and torsion k_2. The differential equation characterizing α according to the Bishop Darboux vector $\overrightarrow{W}$ is given by*

$$\lambda_4\nabla^3_{\alpha'}\overrightarrow{W} + \lambda_3\nabla^2_{\alpha'}\overrightarrow{W} + \lambda_2\nabla_{\alpha'}\overrightarrow{W} + \lambda_1\overrightarrow{W} = 0, \tag{3.3}$$

where

$$\begin{aligned}
\lambda_4 &= f^2 \\
\lambda_3 &= -f\left(f' + g\right) \\
\lambda_2 &= -\left[\left(f' + g\right)g - k_1\left(k_2''' + k_1 f\right)f + k_2\left(k_1''' - k_2 f\right)f\right] \\
\lambda_1 &= -\left[\left(f' + g\right)\left(\frac{k_1'}{k_2'}\right)\left(k_2'\right)^2 + k_1'\left(k_2''' + k_1 f\right)f - k_2'\left(k_1''' - k_2 f\right)f\right]
\end{aligned}$$

and

$$f = \left(\frac{k_1}{k_2}\right)'\left(k_2\right)^2, \quad g = k_1 k_2'' - k_1'' k_2.$$

Proof Let $\alpha(s)$ be a unit speed space curve in Euclidean 3-space E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$, curvature k_1 and torsion k_2. By differentiating $\overrightarrow{W}$ three times with respect to s, we obtain the followings.

$$\nabla_{\alpha'}\overrightarrow{W} = -k_2'\overrightarrow{N_1} + k_1'\overrightarrow{N}_2, \tag{3.4}$$

$$\nabla^2_{\alpha'}\overrightarrow{W} = -\left(k_1'k_2 - k_1k_2'\right)\overrightarrow{T} - k_2''\overrightarrow{N_1} + k_1''\overrightarrow{N_2}, \tag{3.5}$$

$$\begin{aligned}
\nabla^3_{\alpha'}\overrightarrow{W} &= -\left[\left(k_1'k_2 - k_1k_2'\right)' + k_1k_2'' - k_1''k_2\right]\overrightarrow{T} \\
&\quad - \left[k_2''' + k_1\left(k_1'k_2 - k_1k_2'\right)\right]\overrightarrow{N_1} \\
&\quad + \left[k_1''' - k_2\left(k_1'k_2 - k_1k_2'\right)\right]\overrightarrow{N_2}
\end{aligned} \tag{3.6}$$

From (3.1) and (3.4) we get

$$\overrightarrow{N_1} = \frac{k_1}{k_1'k_2 - k_1k_2'}\nabla_{\alpha'}\overrightarrow{W} - \frac{k_1'}{k_1'k_2 - k_1k_2'}\overrightarrow{W} \tag{3.7}$$

and

$$\overrightarrow{N_2} = \frac{k_2}{k_1' k_2 - k_1 k_2'} \nabla_{\alpha'} \overrightarrow{W} - \frac{k_2'}{k_1' k_2 - k_1 k_2'} \overrightarrow{W}. \tag{3.8}$$

By substituting (3.7) and (3.8) in (3.5) we have

$$\overrightarrow{T} = -\frac{1}{k_1' k_2 - k_1 k_2'} \nabla^2_{\alpha'} \overrightarrow{W} - \frac{k_1 k_2'' - k_1'' k_2}{(k_1' k_2 - k_1 k_2')^2} \nabla_{\alpha'} \overrightarrow{W} - \frac{k_1'' k_2' - k_1' k_2''}{(k_1' k_2 - k_1 k_2')^2} \overrightarrow{W}. \tag{3.9}$$

By substituting (3.7), (3.8) and (3.9) in (3.6) we obtain

$$\lambda_4 \nabla^3_{\alpha'} \overrightarrow{W} + \lambda_3 \nabla^2_{\alpha'} \overrightarrow{W} + \lambda_2 \nabla_{\alpha'} \overrightarrow{W} + \lambda_1 \overrightarrow{W} = 0,$$

where

$$\begin{aligned}
\lambda_4 &= f^2 \\
\lambda_3 &= -f\left(f' + g\right) \\
\lambda_2 &= -\left[\left(f' + g\right) g - k_1\left(k_2''' + k_1 f\right) f + k_2\left(k_1''' - k_2 f\right) f\right] \\
\lambda_1 &= -\left[\left(f' + g\right)\left(\frac{k_1'}{k_2'}\right)\left(k_2'\right)^2 + k_1'\left(k_2''' + k_1 f\right) f - k_2'\left(k_1''' - k_2 f\right) f\right]
\end{aligned}$$

and

$$f = \left(\frac{k_1}{k_2}\right)' (k_2)^2, \quad g = k_1 k_2'' - k_1'' k_2. \qquad \square$$

Corollary 3.1 *Let $\alpha(s)$ be a general helix in E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$, curvature k_1 and torsion k_2. The differential equation characterizing α according to the Bishop Darboux vector $\overrightarrow{W}$ is given by*

$$g \nabla_{\alpha'} \overrightarrow{W} - \left(\frac{k_1'}{k_2'}\right)' \left(k_2'\right)^2 \overrightarrow{W} = 0.$$

Theorem 3.3 *Let $\alpha(s)$ be a unit speed space curve in Euclidean 3-space E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$, curvature k_1 and torsion k_2. The differential equation characterizing α according to the normal Bishop Darboux vector $\overrightarrow{W^\perp}$ is given by*

$$\lambda_3 \nabla^2_{\alpha'} \overrightarrow{W^\perp} + \lambda_2 \nabla_{\alpha'} \overrightarrow{W^\perp} + \lambda_1 \overrightarrow{W^\perp} = 0, \tag{3.10}$$

where

$$\begin{aligned}
\lambda_3 &= f \\
\lambda_2 &= g \\
\lambda_1 &= \left(\frac{k_1'}{k_2'}\right)\left(k_2'\right)^2
\end{aligned}$$

and

$$f = \left(\frac{k_1}{k_2}\right)' (k_2)^2, \qquad g = k_1 k_2'' - k_1'' k_2.$$

Proof Let $\alpha(s)$ be a unit speed space curve in Euclidean 3-space E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$, curvature k_1 and torsion k_2. By differentiating $\overrightarrow{W^\perp}$ two times with respect to s, we obtain the followings.

$$\overrightarrow{W^\perp} = -k_2\overrightarrow{N_1} + k_1\overrightarrow{N_2}, \tag{3.11}$$

$$\nabla_{\alpha'}\overrightarrow{W^\perp} = -k_2'\overrightarrow{N_1} + k_1'\overrightarrow{N}_2, \tag{3.12}$$

$$\nabla^2_{\alpha'}\overrightarrow{W^\perp} = -k_2''\overrightarrow{N_1} + k_1''\overrightarrow{N_2}. \tag{3.13}$$

From (3.11) and (3.12) we get

$$\overrightarrow{N_1} = \frac{k_1}{k_1'k_2 - k_1k_2'}\nabla_{\alpha'}\overrightarrow{W^\perp} - \frac{k_1'}{k_1'k_2 - k_1k_2'}\overrightarrow{W^\perp} \tag{3.14}$$

and

$$\overrightarrow{N_2} = \frac{k_2}{k_1'k_2 - k_1k_2'}\nabla_{\alpha'}\overrightarrow{W^\perp} - \frac{k_2'}{k_1'k_2 - k_1k_2'}\overrightarrow{W^\perp}. \tag{3.15}$$

By substituting (3.14) and (3.15) in (3.13) we obtain

$$f\nabla^2_{\alpha'}\overrightarrow{W^\perp} + g\nabla_{\alpha'}\overrightarrow{W^\perp} + \left(\frac{k_1'}{k_2'}\right)'(k_2')^2\,\overrightarrow{W^\perp} = 0. \tag{3.16}$$

This completes the proof. □

Corollary 3.2 *Let $\alpha(s)$ be a slant helix in E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$, curvature k_1 and torsion k_2. The differential equation characterizing α according to the normal Bishop Darboux vector $\overrightarrow{W^\perp}$ is given by*

$$g\nabla_{\alpha'}\overrightarrow{W^\perp} + \left(\frac{k_1'}{k_2'}\right)'(k_2')^2\,\overrightarrow{W^\perp} = 0.$$

Theorem 3.4 *Let α be a unit speed space curve in E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N_1}, \overrightarrow{N_2}\right\}$. Then, α is of harmonic 1-type Darboux vector if and only if the curvature k_1 and the torsion k_2 of the curve α satisfy the followings.*

$$-k_1'' = \lambda k_1, \qquad k_1'k_2 - k_1k_2' = 0, \qquad -k_2'' = \lambda k_2. \tag{3.17}$$

Proof Let α be a unit speed space curve and let Δ be the Laplacian associated with ∇. From (3.4) and (3.5) we can obtain following.

$$\Delta\overrightarrow{W} = (k_1'k_2 - k_1k_2')\,\overrightarrow{T} + k_2''\overrightarrow{N_1} - k_1''\overrightarrow{N_2}. \tag{3.18}$$

We assume that the space curve α is of harmonic 1-type Darboux vector $\overrightarrow{W}$. Substituting (3.18) in (3.2) we get (3.17). □

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