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Order Statistics and some of their Properties from the Exponential Pareto Distribution

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Abstract

Exponential Pareto (EP) distribution introduced by Al-Kadim and Boshi [1] have received further generalizations from some authors and the new distributions has been proved to exhibit flexible potentials for modeling real life datasets. However the roles and importance of statistical tools of order statistics from the EP distribution have not been considered, this study investigated the stochastic ordering properties, moments of order statistics and some distributional properties. Distributions of the extrema order statistics, the sample range $\mathbf{R_n} = \mathbf{X_{(n:n)}} - \mathbf{X_{(1:n)}}$ statistics and the $\mathbf{r^{th}}$ order statistics was derived for the EP distribution. Explicit expressions and recurrence relations were established for the moments of order statistics, the study obtained some new results for the variability ordering between order statistics in two unequal sample sizes $\mathbf{n}$ and $\mathbf{m}$ for various combinations of even and odd samples. Distribution of the sample range statistics $\mathbf{R_n}$ generalizes and strengthened results for the exponential distribution existing in the literature. Numerical results were tabulated for the mean of order statistics, the variance, skewness and kurtosis for a sample of size $\mathbf{n = 5}$. The results were used to establish some ordering and statistical properties of order statistics for the exponential Pareto distribution.

Keywords: Exponential Pareto distribution, Order statistics, Stochastic ordering, Sample range, Recurrence relations, Moment of order statistics. .
MSC2010: 26A18.

1 Introduction

Order statistics from the convoluted distributions is an emerging area of study that is yet to gain desirable studies among researchers resulting in limited information and meager applicable useful results in the literature. The Weibull-Pareto by Alzaatreh *et al.* [2] and the Weibull-Rayleigh by Akarawak *et al.* [3] for instance is yet to be investigated based on order statistics. Despite the keen interest of some researchers to extend EP and derive some new distributions with greater flexibility for modeling real life dataset, the statistical tools of order statistics and their properties from the EP and some new distributions derived from generalizing EP by [4–6] and most recently by

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Adeyemi *et al.* [7] have not been explored. Order statistics plays important roles in many fields of applications such as in reliability studies, actuarial and finance, quality control, climatology, hydrology, sports and in estimations and characterization of probability distributions. Many important properties of order statistics have been employed by some notable authors to characterize some basic distributions such as Burr, Weibull, Lomax, Pareto, Logistics and Exponential distributions. Several studies has been documented in literatures by many notable authors as revealed in Sarhan and Greenberg [8], Ahsanullah [9], David [10], Balakrishnan and Malik [11], Balakrishnan and Cohen [12] and Kamps [13].

Arnold, Balakrishnan and Nagaraja [14] including [15–17] have also contributed to the study of order statistics. Existing knowledge on moments of order statistics and its various tabulations are documented by several authors including David [10, 14] and in some recent works by [18–20]. Galambos [21] presented asymptotic theory of extreme order statistics, Khan and Khan [22] investigated the moments of order statistics from Burr distribution, Shaked and Shantikumar [23] have revealed the usefulness of the statistical tool of order statistics for investigating properties of stochastic distributions. Genc [24] investigated Moments of order statistics of Topp Leone distribution. Dar and Abdullah [25] studied order statistics from the Lomax distribution, Sultan and AL-Thubyani [26] examined Higher order moments of order statistics from the Lindley distribution. The results from [27, 28] and recently by Kumar and Dey [29], Gul and Mahsin [30] revealed that recurrence relations obtained for moments of probability distributions is a useful result and efficient mechanism for evaluating statistical properties such as the mean, variance, skewness and kurtosis of all order statistics for all the sample size in a recursive manner.
Recurrence relations derived for order statistics are also useful for maximizing operational efficiency and is widely reported by many authors including Khan and Yaqub [20, 28, 30, 31] to possess the ability of reducing the number of computations associated with functions of order statistics. [31] derived some recurrence relations between product moments of order statistics from some basic distribution including Weibull, Pareto and exponential distributions.

Another important areas of interest in the study of order statistics is stochastic comparisons, several notable authors have contributed to investigation of ordering properties between order statistics, Chan *et al* [32] established some results for likelihood ratio ordering, Kochar [33] proved that $X \leq_{hr} Y \implies X_{i:n} \leq_{disp} Y_{j:m}$, David and Groeneveld [34] proved that $var(X_{(i:n)}) \leq var(X_{(j:n)})$, for $1 \leq i < j \leq n$. Authors in [33, 35–38] have various important results on comparisons of order statistics that are documented in the literatures.

However, the study of order statistics have not been widely extended to many established lifetime distributions existing in literatures until some recent studies from [18] on extended exponential distribution, Abdul-Moniem [19] on Power Lomax distribution, Kumar and Dey [29] on power generalized Weibull distribution. [20] investigated moments and recurrence relations of order statistics from the power Lomax distribution and presented some tabulated results with application. Other current literatures in this area of study includes [38] on power Lindley distribution, Shrahili *et al.* [39] on order statistics from exponential Lindley distribution.

EP distribution has gained several studies in form of generalizations to obtain the Kumaraswamy Exponential Pareto (KEP) by [4], the Beta Exponential Pareto (BEP) distribution by [5] and by [40], Exponentiated Exponential Pareto distribution (EEPD) by [6] and most recently, the Gompertz Exponential Pareto distribution (GEP) by [7].
This particular study is devoted to exploring the tools and usefulness of order statistics from the Exponential Pareto (EP) distribution. The rest of the paper is arranged as follows; section 2 contained relevant materials from existing literatures and the derivation of some distributional properties of order statistics from EP distribution. In section 3, the single moments of order statistics are derived, section 4 contained recurrence relations for the moments and characterization of EP distribution from the single moments. Some computational results with statistical properties and stochastic

comparisons of order statistics are presented in section 5 and section 6 concluded the work.

2 Material and Methods

The EP distribution due to [1] has three parameters with the cumulative distribution function (cdf) and probability density function (pdf) defined respectively as;

$$F(x) = 1 - exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right); \lambda, \theta, k > 0; x > 0 \tag{2.1}$$

$$f(x) = \frac{\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1} exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right); \lambda, \theta, k > 0; x > 0 \tag{2.2}$$

The density function can be represented by by the relation;

$$f(x) = \frac{\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\Big[(1 - F(x)\Big]; \lambda, \theta, k > 0; x > 0 \tag{2.3}$$

The hazard function and the reliability function are defined respectively as;

$$h(x) = \frac{f(x)}{I - F(x)} = \frac{\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}; \lambda, \theta, k > 0; x > 0 \tag{2.4}$$

$$R(x) = 1 - F(x) = exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right); \lambda, \theta, k > 0; x > 0 \tag{2.5}$$

Let $X_1, X_2, ..., X_n$ be a random sample of size n from $F(x)$ and let the corresponding order statistics realized from the random samples be represented by $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$. Then the density function of $X_{(r:n)}$ which has been defined by many authors including [14, 16] is defined as;

$$f_{(r:n)}(x) = C_{r:n}\left(\Big[F(x)\Big]^{r-1}\Big[1 - F(x)\Big]^{n-r} f(x)\right); 0 < x < \infty \tag{2.6}$$

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$$

The distribution of minimum and maximum order statistics at $r = 1$ and $r = n$ is given respectively as follows;

$$f_{(1:n)}(x) = n\left(\Big[1 - F(x)\Big]^{n-1} f(x)\right); 0 < x < \infty \tag{2.7}$$

$$f_{(n:n)}(x) = n\left(\Big[F(x)\Big]^{n-1} f(x)\right); 0 < x < \infty \tag{2.8}$$

3 Results and Discussion

3.1. Distribution of Order Statistics from EP distribution

Theorem 3.1:*Let $X_1, X_2, ..., X_n$ be a random sample of size n from the EP distribution with cdf and pdf denoted by $F(x)$ and $f(x)$ respectively; let $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ be the corresponding order statistics from the sample. Then the density function of the r^{th} order statistics $f_{r:n}(x)$ is given by.*

$$f_{X_{(r:n)}}(x) = \sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}C_{r:n}\left[exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^m\frac{\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1} \tag{3.1}$$

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where $m = (n - r + i + 1)$

Proof: The order statistics of EP distribution is obtained by inserting the cdf and pdf in equations (2.1) and (2.2) into equation (2.6) and is given by;

$$f_{X_{(r:n)}}(x) = C_{r:n}\frac{\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left[1 - exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{r-1}\left[exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{n-r+1} \tag{3.2}$$

Applying binomial expansion of the form; $\left(1 - Z\right)^{b} = \sum_{i=0}^{b}(-1)^{i}\binom{b}{i}z^{i}$

$$f_{X_{(r:n)}}(x) = \sum_{i=0}^{r-1}(-1)^{i}\binom{r-1}{i}C_{r:n}\left[exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{n-r+i+1}\frac{\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1} \tag{3.3}$$

$$f_{X_{(r:n)}}(x) = \sum_{i=0}^{r-1}(-1)^{i}\binom{r-1}{i}C_{r:n}\left[exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{m}\frac{\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1} \tag{3.4}$$

where $m = (n - r + i + 1)$
Equation (3.4) is a linear representation of mixture of the EP densities from a random sample of size n.
Equation (3.4) can be written as;

$$f_{(r:n)}(x) = \omega_i g_1(x)$$

$$\omega_i = C_{r:n}\sum_{i=0}^{r-1}\frac{(-1)^{i}}{m}\binom{r-1}{i} \tag{3.5}$$

$$g_1(x) = \frac{m\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left[exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{m} \tag{3.6}$$

The distribution function of the $X_{(r:n)}$ is defined by several authors including [14] as;

$$F_{(r:n)}(x) = P(X_{(r:n)} < x) = \sum_{i=j}^{n}\binom{n}{i}\left[F(x)\right]^{i}\left[1 - F(x)\right]^{n-i}, j = 1, 2, .., n \tag{3.7}$$

Thus, the cdf of order statistics from EP distribution is obtained as;

$$F_{(r:n)}(x) = \sum_{j=i}^{n}\binom{n}{i}\left[1 - exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{i}\left[exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{n-i} \tag{3.8}$$

$$F_{(r:n)}(x) = \sum_{j=i}^{n}\sum_{l=1}^{i}\binom{n}{i}\binom{i}{l}\left[exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{n-i+1} \tag{3.9}$$

3.2 Distribution of the sample Range from Exponential Pareto distribution

From the corollary 3.1 Riffi [41]; the distribution of the range can be deduced as follows;

$$\bar{f}_{1,n}(x) = (n-1)f(x)[F(x)]^{n-2} \tag{3.10}$$

Theorem 3.2: *Let $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ be the order statistics from a random sample of size n from the class of distribution with pdf and cdf given by $f(x)$ and $F(x)$ respectively. If the relation $f(x) = h(x)[1 - F(x)]$ exist where $h(x)$ is the hazard rate function; then the distribution of the sample range $R = X_{(n:n)} - X_{(1:n)}$ is given by*

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International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 1, pp. 15 - 32
https://doi.org/10.6084/m9.figshare.XXXX

$$\bar{f}_{1,n}(x) = (n-1)\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}h(x)[1-F(x)]^{i+1} \tag{3.11}$$

Proof:
Substitute $f(x) = h(x)[1-F(x)]$ into equation (3.10) to get

$$\bar{f}_{1,n}(x) = (n-1)h(x)[1-F(x)][F(x)]^{n-2} \tag{3.12}$$

$$\bar{f}_{1,n}(x) = (n-1)\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}h(x)[1-F(x)][1-F(x)]^{i} \tag{3.13}$$

$$\bar{f}_{1,n}(x) = (n-1)\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}h(x)[1-F(x)]^{1+i} \tag{3.14}$$

Corollary 3.1: *Let f(x) and F(x) be the respective pdf and cdf of EP distribution with order statistics from a random sample of size n denoted by $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$. Then the distribution of the sample range $X_{(n:n)} - X_{(1:n)}$ for order statistics from the exponential Pareto distribution is given by*

$$\bar{f}_{1,n}(x) = (n-1)\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}\frac{\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left[exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{i+1} \tag{3.15}$$

Proof: The proof follows from Theorem (2.2) by substituting for $F(x)$ and $h(x)$ of exponential Pareto distribution in equations (2.1) and (2.4) respectively.
Remark 3.1: The pdf of the sample range from Exponential Pareto distribution reduces to the sample range of Exponential distribution when $\theta = k = 1$ obtained by [41]
Let f(x) and F(x) be the respective pdf and cdf of Exponential distribution with random variable $x > 0$ and parameter $\lambda > 0$. the sample range $X_{(n:n)} - X_{(1:n)}$ for order statistics from the exponential distribution is given by

$$\bar{f}_{1,n}(x) = (n-1)\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}\lambda[exp(-\lambda x)]^{i+1} \tag{3.16}$$

3.3. Distribution of Extreme Order Statistics

The order statistics of extreme random observations from EP distribution can be obtained as special cases of the $X_{(r:n)}$ in equation (3.4) as
The **minimum** order statistics $X_{(1:n)}$ of EP distribution has the pdf derived from equation (3.4) as a special case given by;

$$f_{X_{(1:n)}}(x) = \left[exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{n+1}\frac{n\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1} \tag{3.17}$$

The **maximum** order statistics $X_{(n:n)}$ from EP distribution has the pdf obtained as a sub model of equation(3.4) and is given by;

$$f_{X_{(n:n)}}(x) = \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\left[exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{i+1}\frac{n\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1} \tag{3.18}$$

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International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 1, pp. 15 - 32
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3.4 Moments of Order statistics from Exponential Pareto distribution

This section is used for deriving explicit expression for the single moment of order statistics.

Theorem 3.3: *Let* $X_1, X_2, ..., X_n$ *be a random sample of size n from the EP distribution with cdf and pdf denoted by* $F(x)$ *and* $f(x)$ *respectively and let* $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ *be associated order statistics. Then the expected value of* $X_{(r:n)}$*; which is the* t^{th} *moments of the* r^{th} *order statistics for* $t = 1, 2, ...$ *denoted by* $\mu_{r:n}^{(t)}$ *is given by*

$$\mu_{r:n}^{(t)} = \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} C_{r:n} \frac{k^t}{m(m\lambda)^{\frac{t}{\theta}}} \Gamma\left(\frac{t}{\theta} + 1\right) \tag{3.19}$$

where Γ is the gamma function and $m = n - r + i + 1$

Proof.

$$\mu_{r:n}^{(t)} = \int_0^\infty x^t f_{r:n}(x) dx \tag{3.20}$$

The EP distribution (pdf) has a functional relationship with the cdf given by

$$f(x) = \frac{\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\Big(1 - F(x)\Big); \lambda, \theta, k > 0; x > 0 \tag{3.21}$$

Then using results in equation (3.4) in equation (3.20) to get;

$$\mu_{r:n}^{(t)} = C_{r:n} \int_0^\infty x^t \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} C_{r:n} \left[exp\left(-\lambda\left(\frac{x}{k}\right)^\theta\right)\right]^m \frac{\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1} \tag{3.22}$$

$$\mu_{r:n}^{(t)} = \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{\lambda\theta}{k} C_{r:n} \int_0^\infty x^t \left(\frac{x}{k}\right)^{\theta-1} \left(exp\left(-\lambda\left(\frac{x}{k}\right)^\theta\right)\right)^m dx \tag{3.23}$$

Let $y = m\lambda\left(\frac{x}{k}\right)^\theta$ by transformation of variable we have the following quantities;

$x = \frac{ky^{\frac{1}{\theta}}}{(m\lambda)^{\frac{1}{\theta}}}$; $\frac{dy}{dx} = \frac{m\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}$

the mean becomes;

$$\mu_{r:n}^{(t)} = \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} C_{r:n} \int_0^\infty \left(\frac{ky^{\frac{1}{\theta}}}{m(m\lambda)^{\frac{1}{\theta}}}\right)^t e^{-y} dy \tag{3.24}$$

$$\mu_{r:n}^{(t)} = \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} C_{r:n} \frac{k^t}{m(m\lambda)^{\frac{t}{\theta}}} \int_0^\infty y^{\frac{t}{\theta}} e^{-y} dy \tag{3.25}$$

$$\mu_{r:n}^{(t)} = \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} C_{r:n} \frac{k^t}{m(m\lambda)^{\frac{t}{\theta}}} \Gamma\left(\frac{t}{\theta} + 1\right) \tag{3.26}$$

The result in (3.26) completes the proof.

The mean order statistics for EP distribution is derived and given as;

$$\mu_{r:n} = C_{r:n} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{k}{\lambda^{\frac{1}{\theta}} m^{\frac{1}{\theta}+1}} \Gamma\left(\frac{1}{\theta} + 1\right) \tag{3.27}$$

The variance of order statistics for EP distribution is derived using;

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$\sigma^{(2)}_{r:n} = \mu^{(2)}_{r:n} - \left(\mu_{r:n}\right)^2$ and is given as;

$$\mu^{(2)}_{r:n} = C_{r:n}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{k^2}{\lambda^{\frac{2}{\theta}}m^{\frac{2}{\theta}+1}}\Gamma\left(\frac{2}{\theta}+1\right) - \left(\mu_{r:n}\right)^2 \tag{3.28}$$

The results in (3.26) can be used for obtaining expressions for the higher order statistics (HOS) of the EP distribution. The result is useful in various fields of application where moments of order statistics are often used as statistical tools for predicting future events.

Corollary 3.2: *The result in Theorem 3.3 reduces to the explicit expression of the t^{th} moment and the mean of exponential Pareto (EP) distribution by setting $r = n = 1$ as deduced and respectively given as;*

$$\frac{k^t}{(\lambda)^{\frac{t}{\theta}}}\Gamma\left(\frac{t}{\theta}+1\right) \tag{3.29}$$

$$\frac{k}{(\lambda)^{\frac{1}{\theta}}}\Gamma\left(\frac{1}{\theta}+1\right) \tag{3.30}$$

The Theorem 3.3 establishes and strengthens the result for EP distribution in page 137, by the authors in [1].

3.5 Moments of Extreme Order Statistics from EPD

The moments of the minimum and maximum order statistics of the EP distribution are deduced as follows;

Corollary 3.3: *The t^{th} moments of the minimum order statistics $X_{(1:n)}$ of a random variable from the EP distribution is given by;*

$$\mu^{(t)}_{1:n} = \frac{k^t}{(\lambda n)^{\frac{t}{\theta}}}\Gamma\left(\frac{t}{\theta}+1\right) \tag{3.31}$$

Corollary 3.4: *The t^{th} moments of the maximum order statistics $X_{(n:n)}$ of a random variable from the EP distribution is given by;*

$$\mu^{(t)}_{n:n} = \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\frac{nk^t}{\lambda^{\frac{t}{\theta}}(i+1)^{\frac{t}{\theta}+1}}\Gamma\left(\frac{t}{\theta}+1\right) \tag{3.32}$$

Proposition 3.1: The expected value of the minimum order statistics $X_{(1:n)}$ of a random variable from the EP distribution is given by;

$$\mu_{1:n} = \frac{k}{(\lambda n)^{\frac{1}{\theta}}}\Gamma\left(\frac{1}{\theta}+1\right) \tag{3.33}$$

The second moment of minimum order statistics $X_{(1:n)}$ is obtained as;

$$\mu^{(2)}_{1:n} = \frac{k^2}{(\lambda n)^{\frac{2}{\theta}}}\Gamma\left(\frac{2}{\theta}+1\right) \tag{3.34}$$

The variance can be obtained using ;

$$Var = E\left(X^2_{(1:n)}\right) - E\left(X_{(1:n)}\right)^2$$

Hence the variance of minimum order statistics $X_{(1:n)}$ for the EP distribution is obtained as;

$\sigma^{(2)}_{1:n} = \mu^{(2)}_{1:n} - \left(\mu_{1:n}\right)^2$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 1, pp. 15 - 32
https://doi.org/10.6084/m9.figshare.XXXX

$$\sigma_{1:n}^{(2)} = \frac{k^2}{(\lambda n)^{\frac{2}{\theta}}}\left[\Gamma\left(\frac{2}{\theta}+1\right) - \Gamma^2\left(\frac{2}{\theta}+1\right)\right] \tag{3.35}$$

The mean, second moment and variance of the EP maximum order statistics can be derived using the same procedure.

4. Recurrence Relations, Characterization and the Stochastic Ordering

Stochastic assessment of each variable $X_1, X_2, ..., X_n$ by magnitude or position is of interest among many practitioners in some fields including finance, insurance, reliability engineering while recurrence relations have been also employed to characterize some lifetime distributions by notable authors .

4.1 Recurrence Relations for Single Moments

This subsection is used to derive recurrence relation for moment of order statistics of EP distribution through the following theorem;

Theorem 4.1: *Let $X_1, X_2, ..., X_n$ be random sample of size n from the EP distribution whose corresponding order statistics is denoted by $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$; then for $1 \leq r \leq n$ and $t = 0, 1, 2...$ we have the following moment relation;*

$$\mu_{r:n}^{(t)} = \frac{\theta\lambda(n-r+1)}{k(t+\theta)}\left[\mu_{r:n}^{(t+\theta)} - \mu_{r-1:n}^{(t+\theta)}\right] \tag{4.1}$$

Proof: The t^{th} moment of EP distribution order statistics is defined by;

$$\mu_{r:n}^{(t)} = C_{r:n}\int_0^\infty x^t\Big(F(x)\Big)^{r-1}\Big(1-F(x)\Big)^{n-r} f(x)dx \tag{4.2}$$

substituting $f(x)$ into (4.2) and doing some arithmetic operations we have

$$\mu_{r:n}^{(t)} = \frac{\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1} C_{r:n}\int_0^\infty x^t\Big(F(x)\Big)^{r-1}\Big(1-F(x)\Big)^{n-r+1} dx \tag{4.3}$$

$$\mu_{r:n}^{(t)} = \frac{\lambda\theta}{k^\theta}C_{r:n}\int_0^\infty x^{t+\theta-1}\Big(F(x)\Big)^{r-1}\Big(1-F(x)\Big)^{n-r+1} dx \tag{4.4}$$

Using integration by parts we shall obtain;

$$\begin{aligned}\mu_{r:n}^{(t)} = \frac{\lambda\theta}{k}C_{r:n}\int_0^\infty \Bigg[&x^{t+\theta}\frac{(n-r+1)}{t+\theta}\Big(F(x)\Big)^{r-1}\Big(1-F(x)\Big)^{n-r} f(x)dx \\ &-x^{t+\theta}\frac{(r-1)}{t+\theta}\Big(F(x)\Big)^{r-2}\Big(1-F(x)\Big)^{n-r+1} f(x)dx\Bigg]\end{aligned} \tag{4.5}$$

$$\begin{aligned}\mu_{r:n}^{(t)} = \frac{\lambda\theta}{k}\int_0^\infty \Bigg[&x^{t+\theta}\frac{(n-r+1)}{t+\theta}C_{r:n}\Big(F(x)\Big)^{r-1}\Big(1-F(x)\Big)^{n-r} f(x)dx \\ &-x^{t+\theta}\frac{(r-1)}{t+\theta}C_{r:n}\Big(F(x)\Big)^{r-2}\Big(1-F(x)\Big)^{n-r+1} f(x)dx\Bigg]\end{aligned} \tag{4.6}$$

$$\mu_{r:n}^{(t)} = \frac{\lambda\theta}{k}\left[\frac{(n-r+1)}{t+\theta}\mu_{r:n}^{(t+\theta)} - \frac{(r-1)C_{r:n}}{t+\theta C_{r-1:n}}\mu_{r-1:n}^{(t+\theta)}\right] \tag{4.7}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 1, pp. 15 - 32
https://doi.org/10.6084/m9.figshare.XXXX

$$\mu_{r:n}^{(t)} = \frac{\theta\lambda(n-r+1)}{k(t+\theta)}\left[\mu_{r:n}^{(t+\theta)} - \mu_{r-1:n}^{(t+\theta)}\right] \tag{4.8}$$

A special case is when $r = 1$ and the recurrence relation is obtained as;

$$\mu_{1:n}^{(t)} = \frac{n\lambda\theta\mu_{1:n}^{(t+\theta)}}{k(t+\theta)} \tag{4.9}$$

Remark 4.1: Recurrence relation for the mean of order statistics from EP distribution when $t = 1$, is given by;

$$\mu_{r:n}^{(1)} = \frac{\lambda\theta(n-r+1)}{k(1+\theta)}\left[\mu_{r:n}^{(1+\theta)} - \mu_{r-1:n}^{(1+\theta)}\right] \tag{4.10}$$

4.2 Characterization of EPD by Order Statistics

The results in section (4) is employed in this section for characterizing the Exponential Pareto distribution using the following theorem.

Theorem 4.2: *Let* $X_1, X_2, ..., X_n$ *be random sample of size* n *from the EP distribution with common cdf* $F(x)$*, pdf* $f(x)$*, and corresponding order statistics denoted by* $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$*; given that;*
(i)$F(x)$ *is absolutely continuous*
(ii)$F(0) = 0$
(iii)$1 < F(x) < 1$
Then for $1 \leq r \leq n$ *and* $t = 0, 1, 2...$ *for all* $x > 0$

$$\mu_{r:n}^{(t)} = \frac{\theta\lambda(n-r+1)}{k(t+\theta)}\left[\mu_{r:n}^{(t+\theta)} - \mu_{r-1:n}^{(t+\theta)}\right] \tag{4.11}$$

if and only if
$F(x) = 1 - exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right); \lambda, \theta, k > 0, x > 0$

Proof:
The result in Theorem 4.1 provides the necessary condition for the proof; however if the recurrence relation in equation (4.8) exists and satisfies the sufficient conditions for this proof, then

$$C_{r:n}\int_0^\infty x^t\left(F(x)\right)^{r-1}\left(1-F(x)\right)^{n-r}f(x)dx$$

$$= \left\{\frac{\lambda\theta}{k}C_{r:n}\int_0^\infty\left[x^{t+\theta}\frac{(n-r+1)}{t+\theta}\left(F(x)\right)^{r-1}\left(1-F(x)\right)^{n-r}f(x)dx - x^{t+\theta}\frac{(r-1)}{t+\theta}\left(F(x)\right)^{r-2}\left(1-F(x)\right)^{n-r+1}f(x)dx\right]\right\} \tag{4.12}$$

Using integration by parts on the second integral in (4.12) it becomes;

$$C_{r:n}\int_0^\infty x^t\left(F(x)\right)^{r-1}\left(1-F(x)\right)^{n-r}f(x)dx$$

$$= \left\{\frac{\lambda\theta}{k}C_{r:n}\int_0^\infty\left[x^{t+\theta}\frac{(n-r+1)}{t+\theta}\left(F(x)\right)^{r-1}\left(1-F(x)\right)^{n-r}f(x)dx - x^{t+\theta}\frac{(r-1)(n-r-1)}{(t+\theta)(r-1)}\left(F(x)\right)^{r-1}\left(1-F(x)\right)^{n-r}f(x)dx\right.\right.$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 1, pp. 15 - 32
https://doi.org/10.6084/m9.figshare.XXXX

$$+\frac{(t+\theta)}{r-1}x^{t+\theta-1}\frac{(r-1)}{t+\theta}\Big(F(x)\Big)^{r-1}\Big(1-F(x)\Big)^{n-r+1}f(x)dx\Big]\Big\}$$

which reduces to
$C_{r:n}\int_0^\infty x^t\Big(F(x)\Big)^{r-1}\Big(1-F(x)\Big)^{n-r}f(x)dx$

$$=\left\{\frac{\lambda\theta}{k}C_{r:n}\int_0^\infty x^{t+\theta-1}\Big(F(x)\Big)^{r-1}\Big(1-F(x)\Big)^{n-r+1}f(x)dx\right\}$$

then;
$C_{r:n}\int_0^\infty x^t\Big(F(x)\Big)^{r-1}\Big(1-F(x)\Big)^{n-r}f(x)dx-$
$\left\{\frac{\lambda\theta}{k}C_{r:n}\int_0^\infty x^{t+\theta-1}\Big(F(x)\Big)^{r-1}\Big(1-F(x)\Big)^{n-r+1}f(x)dx\right\}=0$
so that
$C_{r:n}\int_0^\infty x^t\Big(F(x)\Big)^{r-1}\Big(1-F(x)\Big)^{n-r}\left\{f(x)-\frac{\lambda\theta}{k}x^{\theta-1}\Big(1-F(x)\Big)\right\}dx=0$

Thus we obtained as defined in equation (3))
$f(x)=\frac{\lambda\theta}{k}x^{\theta-1}\Big(1-F(x)\Big)=\frac{\lambda\theta}{k}x^{\theta-1}exp\Big(-\lambda\Big(\frac{x}{k}\Big)^\theta\Big)$
Which implies $\Big(1-F(x)\Big)=exp\Big(-\lambda\Big(\frac{x}{k}\Big)^\theta\Big)$
Hence, $F(x)=1-exp\Big(-\lambda\Big(\frac{x}{k}\Big)^\theta\Big);\lambda,\theta,k>0,x>0$ completes the proof.

4.3 Stochastic Ordering from EP distribution

It has been established by many author including [10, 23] that:
$X_1\le_{lr}X_2\implies X_1\le_{hr}X_2\implies X_1\le_{st}X_2$.
This investigation considered comparison of two random variable X_1 and X_2 from a stochastic system that follows the EP distribution.

Some preliminary definitions:

- Y is said to be stochastically greater than X in stochastic order written $X\le_{st}Y$, if $F(x)\le F(y)\forall X,Y$
- Y is said to be stochastically greater than X in likelihood ratio order written $X\le_{lr}Y$, if $\frac{f(x)}{f(y)})$ is decreasing in X
- Y is said to be stochastically greater than X in hazard rate order written $X\le_{hr}Y$, if $h(x)\le h(y)\forall X,Y$

Lemma 4.1: Let X_1 $EP(\lambda_1,\theta,k)$ and X_2 $EP(\lambda_2,\theta,k)$; suppose $\frac{f_{x_1(x)}}{f_{x_2(x)}}$ is an increasing function of X, then $\lambda_2\ge\lambda_1$
Proof. the density function of EP is

$$f(x)=\frac{\lambda\theta}{k}\Big(\frac{x}{k}\Big)^{\theta-1}exp\Big(-\lambda\Big(\frac{x}{k}\Big)^\theta\Big)\tag{4.13}$$

$$\frac{f_{x_1(x)}}{f_{x_2(x)}}=\frac{\lambda_1}{\lambda_2}exp\Big(-\lambda_1\Big(\frac{x}{k}\Big)^\theta\Big)exp\Big(\lambda_2\Big(\frac{x}{k}\Big)^\theta\Big)\tag{4.14}$$

$$\frac{f_{x_1(x)}}{f_{x_2(x)}} = \frac{\lambda_1}{\lambda_2} exp\left(-(\lambda_1-\lambda_2)\left(\frac{x}{k}\right)^{\theta}\right) \tag{4.15}$$

Taking the logarithm of both sides of equation (4.15) gives

$$log\left\{\frac{f_{x_1(x)}}{f_{x_2(x)}}\right\} = log(\lambda_1) - log(\lambda_2) - (\lambda_1-\lambda_2)\left(\frac{x}{k}\right)^{\theta} \tag{4.16}$$

Taking derivative with respect to X gives

$$\frac{d}{dx}\left[log\left\{\frac{f_{x_1(x)}}{f_{x_2(x)}}\right\}\right] = \frac{\lambda_2\theta}{k^{\theta}}x^{\theta-1} - \frac{\lambda_1\theta}{k^{\theta}}x^{\theta-1} \tag{4.17}$$

$if \lambda_1 = \lambda_2$ then $\frac{d}{dx}\left[log\left\{\frac{f_{x_1(x)}}{f_{x_2(x)}}\right\}\right] = 0$

$if \lambda_1 \geq \lambda_2$; $\frac{d}{dx}\left[log\left\{\frac{f_{x_1(x)}}{f_{x_2(x)}}\right\}\right] < 0$ shows that $\frac{f_{x_1(x)}}{f_{x_2(x)}}$ is a decreasing function of X.

Then, $if \lambda_1 \geq \lambda_2$ $X_1 \leq_{lr} X_2$, which satisfied the stochastic relation order
$X_1 \leq_{lr} X_2 \implies X_1 \leq_{hr} X_2 \implies X_1 \leq_{st} X_2$.

5 Numerical Results

Computational values for mean of order statistics from EP distribution for various values of parameters λ, θ, k is tabulated and presented in Table 1. ,

Table 1: Mean of order statistics of EP distribution for some parameters

r	n	$\lambda = 0.5$, $\theta = k = 1$	$\lambda = 0.5$, $\theta = 2, k = 1$	$\lambda = 0.5$, $\theta = k = 2$	$\lambda = 1$, $\theta = k = 2$
1	1	2.0000	1.0233	2.5066	1.7725
1	2	1.0000	0.8862	1.7725	1.2533
2	2	3.0000	1.6204	3.2408	2.2916
1	3	0.6667	0.7236	1.4472	1.0233
2	3	1.6667	1.2115	2.4229	1.7133
3	3	3.6667	1.8249	3.6497	2.5808
1	4	0.5000	0.6267	1.2533	0.8862
2	4	1.1667	1.0144	2.0289	1.4346
3	4	2.1667	1.4085	2.8170	1.9919
4	4	4.1667	1.9636	3.9273	2.7770
1	5	0.4000	0.5605	1.1210	0.7927
2	5	0.9000	0.8913	1.7826	1.2605
3	5	1.5667	1,1992	2.3983	1.6959
4	5	2.5667	1.5481	3.0962	2.1894
5	5	4.5667	2.0675	4.1315	2.9239

Source: This is mean from numerical results .

Remark 5.1: The computed results in Table 1 for the mean of order statistics are consistent with the property established by [16] given by; $\sum_{i=1}^{n} \mu_{i:n} = n\mu_{1:1}$

5.1 Mean of EP distribution based on order statistics

* Order statistics $X_{(r:n)}$ from EP distribution.
  the mean decreases with increase in parameter λ for all order statistics

the mean decreases with increase in θ except for minimum order statistics
the mean increases with increase in parameter k for all order statistics

* Maximum order statistics $X_{(n:n)}$ from EP distribution.
the mean decreases with increase in parameter λ
the mean decreases with increase in parameter θ
the mean increases with increase in parameter k

* Minimum order statistics $X_{(1:n)}$ from EP distribution.
the mean decreases with increase in parameter λ
the mean increases with increase in parameter θ
the mean increases with increase in parameter k

5.2 Statistical Properties of Order Statistics from EPD

Table 2: Higher order statistics of EP for parameters $\lambda = 1, \theta = 2, k = 2$

r	n	μ	σ^2	*skewness*	*kurtosis*	*cv*
1	1	1.7725	0.8584	6.3483	3.2451	0.2422
1	2	1.2533	0.4292	17.9556	3.2451	0.1712
2	2	2.2916	0.7486	6.2734	3.2479	0.1633
1	3	1.0233	0.2861	32.9866	3.2451	0.1398
2	3	1.7133	0.3980	14.1981	3.1362	0.1161
3	3	2.5808	0.6731	7.1204	3.3053	0.1304
1	4	0.8862	0.2146	50.7862	3.2451	0.1211
2	4	1.4346	0.2752	23.6363	3.0999	0.0959
3	4	1.9919	0.3655	14.2091	3.1458	0.0917
4	4	2.7770	0.6215	8.0854	3.3535	0.1119
1	5	0.7927	0.1717	70.9758	3.2451	0.1083
2	5	1.2605	0.2112	34.4835	3.0842	0.0838
3	5	1.6959	0.2574	21.9949	3.0914	0.0759
4	5	2.1893	0.3401	15.0279	3.1643	0.0776
5	5	2.9239	0.5839	9.0352	3.3926	0.0998

Some statistical properties of order statistics from the exponential Pareto distribution deduced from Table 2 are summarized as follows;

- The mean and variance of all order statistics decreases with the size of the samples but increases with the order statistics
- The mean and variance of order statistics of the maximum decreases with the size of the samples
- The mean and variance of order statistics of the minimum increases with the size of the samples
- The order statistics of the minimum has a uniform kurtosis for all the sample sizes. $X_{(1:n)} = X_{(1:n+j)}$ for $j = 1, 2, ..., n$
- The order statistics of the maximum has kurtosis that increases with the sample sizes. $X_{(n:n)} \leq X_{(n:n+j)}$ for $j = 1, 2, ..., n$
- The skewness of minimum $X_{(1:n)}$ and maximum $X_{(n:n)}$ order statistics increases with sample size n

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- The skewness of all order statistics $X_{(r:n)}, r = 1, 2, ..., n$ increases with the sample size n and $X_{(n:n)} \leq X_{(1:n)}$; $X_{(i:n)} \leq X_{(j:n)}$ for $i < j$
- The coefficient of variation of order statistics decreases with increase in sample size n

5.3 Ordering Properties of Order Statistics from EPD

In the comparison of order statistics from unequal sample sizes, Shaked and Shantikumar [37] established that:
$X_{(n:n)} \leq_{lr} X_{(n+1:n+1)}$ and $X_{(1:n)} \geq_{lr} X_{(1:n+1)}$,
Kochar [33] proved $X \leq_{hr} Y \implies E(X) \leq E(Y)$
If there exist likelihood ratio order from the sample $X_1, X_2, ..., X_n$, then

$$X_{(i:n)} \leq_{lr} X_{(j:n)}, i < j$$

(Bapat and Kochar [36] and Kochar [33]) proved and established in the comparison by dispersive order that;
$X \leq_{hr} Y \implies X_{(i:n)} \leq_{disp} Y_{(j:m)}; i \leq j, n - i \geq m - j$
Theorem 5.1: *Let $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_m$ be random samples from exponential Pareto distribution functions $F(x)$ and $G(y)$ respectively, if F or G is characterized with decreasing failure rate function (DFR), then*

- $X \leq_{hr} Y \implies X_{(i:n)} \leq_{disp} Y_{(j:m)}; i \leq j, n - i \geq m - j$
- $X_{i:n} \leq_{hr} Y_{i:n} \implies E[X_{(i:n)}] \leq E[Y_{(j:m)}]$

Proof:
Starting with the application of results in table 2 the dispersive ordering between order statistics for sample of size $n = 5$ is deduced
let the sample sizes be $n = 4$ and $m = 5$
then $\forall$ $X_{(i:n)} = X_{(1:4)}$ and $Y_{(j:m)} = Y_{(3:5)}; i \leq j, n - i \geq m - j$

$$0.2146 = var(X_{(1:4)}) \leq var(X_{(3:5)}) = 0.2574$$

$$E[X_{(1:4)}] = 0.8862 \leq E[X_{(3:5)}] = 1.6959$$

$X \leq_{hr} Y$ from results obtained in subsection 4.3;
ince $0.8862 \leq 1.6959$ then $E[X_{(1:4)}] \leq E[X_{(3:5)}]$ $\forall i \leq j, n - i \geq m - j$;
$X \leq_{hr} Y$ implies $X_{(i:n)} \leq_{disp} Y_{(j:m)}; i \leq j, n - i \geq m - j$, (Kochar [33])
$X_i \leq_{lr} Y_j \implies X_{i:n} \leq_{lr} Y_{i:n}$, Chan *et al.* [32]
Combining Chan *et al.* [32] and Kochar [33], we have
$X_i \leq_{lr} Y_j \implies X_{i:n} \leq_{hr} Y_{i:n} \implies E[X_{(i:n)}] \leq E[Y_{(j:m)}]$
Combining some results in subsection 4.3 and Table 2, we get
$X \leq_{hr} Y \implies var(X_{(i:n)}) \leq var(X_{(j:m)})$
$X \leq_{hr} Y \implies X_{(i:n)} \leq_{disp} Y_{(j:m)}; i \leq j, n - i \geq m - j$

Corollary 5.1: *Let $X_1, X_2, ..., X_{n+1}$ be a random sample of size $n + 1$ from the EP distribution, the ordering properties from the parallel and series system in kurtosis order is established respectively as follows;*
$var(X_{n:n}) \geq var(X_{n+1:n+1})$ and $var(X_{1:n}) \geq var(X_{1:n+1})$
The corollary can be verified using the computational results in Table 2.

$$0.2861 = var(X_{1:3}) \geq var(X_{1:4}) = 0.2146$$

In general , $var(X_{1:n}) \geq var(X_{1:n+i})$, for $i = 1, 2, ..., n$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 1, pp. 15 - 32
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5.3.1 Variability ordering between order statistics in one sample.

Corollary 5.2: *Let $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ be a order statistics from a random sample of size n from the EP distribution, the variability ordering between order statistics is established.*
proof:.
$var(X_{(i:n)}) \leq var(X_{(j:n)})$, for $1 \leq i < j \leq n$ ... (David and Groeneveld [34])
Starting with the application of results in Table 2, the variability ordering between order statistics for sample of size $n = 5$ is established and given by

$$var(X_{(1:5)}) \leq var(X_{(2:5)}) \leq var(X_{(3:5)}) \leq var(X_{(4:5)}) \leq var(X_{(5:5)})$$

$\therefore var(X_{(i:n)}) \leq var(X_{(j:n)})$, for $1 \leq i < j \leq n$.

0.0.1 Variability ordering between order statistics in two unequal sample sizes.

The application of results in table 2 is extended by considering variablity ordering for independent and identically distributed random samples from the EP distribution .
The following new results are estblished.
Corollary 5.3: *Let $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ and $X_{(1:n)}, X_{(2:n)}, ..., X_{(m:m)}$ be order statistics from random samples from the EP distribution $\forall, n < m$ the variability ordering between order statistics is given by*
$var(X_{(i:n)}) \geq var(X_{(i:m)})$ $\forall, i = 1, 2, ..., n$
If n=odd, m=odd or n=even, m=even, then
$var(X_{(i:n)}) \geq var(X_{(j:m)})$ $\forall, i = 1, 2, ..., n; j = 1, 2, ..., (i + 2)$
If n=odd, m=even or n=even, m=odd, then
$var(X_{(i:n)}) \geq var(X_{(j:m)})$ $\forall, i = 1, 2, ..., n; j = 1, 2, ..., (i + 1)$
proof:
Case I: When n=odd, m=odd
consider two samples of sizes $n = 3$ and $m = 5$
$var(X_{(1:3)}) \leq var(X_{(2:3)}) \leq var(X_{(3:3)})$ for $1 \leq i < j \leq 3$
$var(X_{(1:5)}) \leq var(X_{(2:5)}) \leq var(X_{(3:5)}) \leq var(X_{(4:5)}) \leq var(X_{(5:5)})$;$1 \leq i < j \leq 5$

$\forall, i = 1; j = 1, 2, ..., 3 = (i + 2)$
$var(X_{(1:3)}) \geq var(X_{(3:5)}) \geq var(X_{(2:5)}) \geq var(X_{(1:5)})$

$\forall, i = 2; j = 1, 2, ..., 4 = (i + 2)$
$var(X_{(2:3)}) \geq var(X_{(4:5)}) \geq var(X_{(3:5)}) \geq var(X_{(2:5)}) \geq var(X_{(1:5)})$

$\forall, i = 3; j = 1, 2, ..., 5 = (i + 2)$
$var(X_{(3:3)}) \geq var(X_{(5:5)}) \geq var(X_{(4:5)}) \leq var(X_{(3:5)}) \geq var(X_{(2:5)}) \geq var(X_{(1:5)})$

Case II: When n=even, m=even
consider two samples of sizes $n = 2$ and $m = 4$
$var(X_{(1:2)}) \leq var(X_{(2:2)})$ for $1 \leq i < j \leq 2$
$var(X_{(1:4)}) \leq var(X_{(2:4)}) \leq var(X_{(3:4)}) \leq var(X_{(4:4)})$ for $1 \leq i < j \leq 4$

$\forall, i = 1; j = 1, 2, ..., 3 = (i + 2)$
$var(X_{(1:2)}) \geq var(X_{(3:4)}) \geq var(X_{(2:4)}) \geq var(X_{(1:4)})$

$\forall, i = 2; j = 1, 2, ..., 4 = (i + 2)$
$var(X_{(2:2)}) \geq var(X_{(4:4)}) \geq var(X_{(3:4)}) \geq var(X_{(2:4)}) \geq var(X_{(1:4)})$

Case III: When n=odd, m=even
consider two samples of sizes $n = 3$ and $m = 4$

$var(X_{(1:3)}) \leq var(X_{(2:3)}) \leq var(X_{(3:3)})$ for $1 \leq i < j \leq 3$
$var(X_{(1:4)} \leq var(X_{(2:4)}) \leq var(X_{(3:4)}) \leq var(X_{(4:4)})$ for $1 \leq i < j \leq 4$

$\forall, i = 1; j = 1, 2, ..., 2 = (i + 1)$
$\forall, i = 2; j = 1, 2, ..., 3 = (i + 1)$ and
$\forall, i = 3; j = 1, 2, ..., 4 = (i + 1)$,

the following variability relations exists for $i = 1, 2, 3$.

$$var(X_{(1:3)}) \geq var(X_{(2:4)}) \geq var(X_{(1:4)})$$

$$var(X_{(2:3)}) \geq var(X_{(3:4)}) \geq var(X_{(2:4)}) \geq var(X_{(1:4)})$$

$$var(X_{(3:3)}) \geq var(X_{(4:4)}) \geq var(X_{(3:4)}) \geq var(X_{(2:4)}) \geq var(X_{(1:4)})$$

Case IV: When n=even, m=odd
consider two samples of sizes $n = 4$ and $m = 5$
$var(X_{(1:4)}) \leq var(X_{(2:4)}) \leq var(X_{(3:4)}) \leq var(X_{(4:4)})$ for $1 \leq i < j \leq 4$
$var(X_{(1:5)} \leq var(X_{(2:5)}) \leq var(X_{(3:5)}) \leq var(X_{(4:5)}) \leq var(X_{(5:5)})$;$1 \leq i < j \leq 5$.

$\forall, i = 1; j = 1, 2, ..., 2 = (i + 1)$;$\forall, i = 2; j = 1, 2, ..., 3 = (i + 1)$
$\forall, i = 3; j = 1, 2, ..., 4 = (i + 1)$; and $\forall, i = 4; j = 1, 2, ..., 5 = (i + 1)$,

the following variability relations exists for $i = 1, 2, 3, 4$.

$$var(X_{(1:4)}) \geq var(X_{(2:5)}) \geq var(X_{(1:5)})$$

$$var(X_{(2:4)}) \geq var(X_{(3:5)}) \geq var(X_{(2:5)}) \geq var(X_{(1:5)})$$

$$var(X_{(3:4)}) \geq var(X_{(4:5)}) \geq var(X_{(3:5)}) \geq var(X_{(2:5)}) \geq var(X_{(1:5)})$$

$$var(X_{(4:4)}) \geq var(X_{(5:5)}) \geq var(X_{(4:5)}) \geq var(X_{(3:5)}) \geq var(X_{(2:5)}) \geq var(X_{(1:5)})$$

6 Conclusion

Many generalized lifetime distributions have not been considered for studies based on their order statistics. This study derived the distributional properties of order statistics which includes the sample minimum, sample maximum, the sample range, the density function and cdf of the r^{th} ordered statistics from the Exponential Pareto (EP) distribution. The single moment of order statistics and the moments of extreme order statistics was derived; the result established the moments of Exponential Pareto distribution obtained by [1] and generalizes the distribution of sample range of exponential distribution in [41]. In addition, recurrence relation for computing various statistical measures was established for the moment of order statistics. Numerical analysis of the mean revealed that the mean order statistics increases with sample sizes n and also increases as the values of scale parameter k increases but decreases with the values of the shape parameter θ. The computational results from higher moments of order statistics provides some characterizations for the exponential Pareto distribution, variability ordering was obtained for equal and unequal sample sizes and the results strenghtened some stochastic ordering existing in the literature.

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