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Some Properties of Nilpotent Lattice Matrices

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Abstract: In this paper, the nilpotent matrices over distributive lattices are discussed by applying the combinatorial speculation ([9]). Some necessary and sufficient conditions for a lattice matrix A to be a nilpotent matrix are given. Also, a necessary and sufficient condition for an $n \times n$ nilpotent matrix with an arbitrary nilpotent index is obtained.

Key Words: distributive lattice, nilpotent matrix; nilpotent index; direct path.

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§1. Introduction

Since the concept of nilpotent lattice matrix was introduced by Give'on in [2], a number of researchers have studied the topic of nilpotent lattice matrices(see e.g. [2-8]). In [7], Li gave some sufficient and necessary conditions for a fuzzy matrix to be nilpotent and proved that an $n \times n$ fuzzy matrix A is nilpotent if and only if the elements on the main diagonal of the kth power A^k of A are 0 for each k in $\{1, 2, \cdots, n\}$. Ren et al.(see [8]) obtained some characterizations of nilpotent fuzzy matrices, and revealed that a fuzzy matrix A is nilpotent if and only if every principal minor of A is 0. This result was generalized to the class of distributive lattices by Tan(see [3,5]) and Zhang(see [4]). In particular, Tan gave a necessary and sufficient condition for an $n \times n$ nilpotent matrix to have the nilpotent index n in [3].

In this paper, we discuss the topic of nilpotent lattices matrices. In Section 3, we will give some characterizations of the nilpotent lattice matrices by applying the combinatorial speculation ([9]). In Section 4, a necessary and sufficient condition for an $n \times n$ nilpotent matrix with an arbitrary nilpotent index will be obtained, this result provide an answer to the open problem posed by Tan in [3].

§2. Definitions and Lemmas

For convenience, we shall use N to denote the set $\{1, 2, \cdots, n\}$ and use $|S|$ to denote the cardinality of a set S.

Let $(L, \leq, \vee, \wedge)$ be a distributive lattice with a bottom element 0 and a top element 1 and $M_n(L)$ be the set of all $n \times n$ matrices over L.

For $A \in M_n(L)$, the powers of A are defined as follows: $A^0 = I_n, A^r = A^{r-1}A, r = 1, 2, \cdots$. The (i, j)-entry of A^r is denoted by a_{ij}^r.

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A is called the zero matrix if for all $i, j \in N, a_{ij} = 0$ and denoted by 0. Let $A \in M_n(L)$. If there exists $k \geq 1, A^k = 0$, then A is called a nilpotent matrix. The least integer k satisfying $A^k = 0$ is called the nilpotent index of A and denoted by $h(A)$.

For $A \in M_n(L)$, the permanent per$A$ of A is defined as follows:

$$\text{per}A = \bigvee_{\sigma \in P_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

where P_n denotes the symmetric group of the set N.

For a matrix $A \in M_n(L)$, we denote by $A[i_1, i_2, \cdots, i_r | j_1, j_2, \cdots, j_r]$ the $r \times r$ submatrix of A whose (u, v)-entry is equal to $a_{i_u j_v} (u, v \in R)$. The matrix $A[i_1, i_2, \cdots, i_r | i_1, i_2, \cdots, i_r]$ is called a principal submatrix of order r of A, and per$A[i_1, i_2, \cdots, i_r | i_1, i_2, \cdots, i_r]$ is called a principal minor of order r of A.

For a given matrix $A \in M_n(L)$, the associated graph $G(A) : G(A) = (V, H)$ of A is the strongly complete directed weighted graph with the node set $V = N$, the arc set $H = \{(i, j) \in N \times N | a_{ij} \neq 0\}$.

For a given matrix $A \in M_n(L)$, a sequence of nodes $p = (i_0, i_1, \cdots, i_r)$ of the graph $G(A) = (V, H)$ is called a path if $(i_{k-1}, i_k) \in H$ for all $k = 1, 2, \cdots, r - 1$. These arcs, together with the nodes i_k for $k = 0, 1, \cdots, r$, are said to be on the path p. The length of a path, denoted by $l(p)$, is the number of arcs on it, in the former case, $l(p) = r$. If all nodes on a path p are pairwise distinct, then p is called a chain. A path $p = (i_0, i_1, \cdots, i_{r-1}, i_0)$ with $i_0, i_1, \cdots, i_{r-1}$ are pairwise distinct is called a cycle. For a given matrix $A \in M_n(L)$, we define:

$$C(A) = \{p | p \text{ is a cycle of } G(A)\}.$$

And for any $r \leq n$, we define:

$$S_r(A) = \{p | p \text{ is a chain of } G(A) \text{ and } l(p) = r\}.$$

For any path $p = (i_0, i_1, \cdots, i_r)$ of $G(A)$, the weight of p with respect to A, will be denoted by $W_A(p)$, is defined as

$$W_A(p) = a_{i_0 i_1} \wedge a_{i_1 i_2} \wedge \cdots \wedge a_{i_{r-1} i_r} = a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{r-1} i_r}.$$

The following lemmas are used.

Lemma 2.1([2]) *Let $A \in M_n(L)$. Then A is nilpotent if and only if $A^n = 0$.*

Lemma 2.2([4]) *Let $A = (a_{ij}) \in M_n(L)$, $A^m = (a_{ij}^m)$. Then*

$$a_{ij}^m = \bigvee_{1 \leq i_1, i_2, \cdots, i_{m-1} \leq n} a_{i i_1} a_{i_1 i_2} \cdots a_{i_{m-1} j}.$$

Lemma 2.3([4],[5]) *Let $A \in M_n(L)$. Then A is a nilpotent matrix if and only if*

$$perA[i_1, i_2, \cdots, i_k | i_1, i_2, \cdots, i_k] = 0,$$

for all $\{i_1, i_2, \cdots, i_k\} \subset N$, $k \in N$.

Lemma 2.4([4]) *Let A be a nilpotent matrix over L. Then*

$$a_{r_1r_2}a_{r_2r_3}\cdots a_{r_{m-1}r_m}a_{r_mr_1} = 0,$$

for all $\{r_1, r_2, \cdots, r_m\} \subseteq N$.

§3. Characterizations of the Nilpotent Lattice Matrices

In this section, we will give some new necessary and sufficient conditions for a lattice matrix to be a nilpotent matrix.

Theorem 3.1 *Let $A \in M_n(L)$. Then A is a nilpotent matrix if and only if for all $p \in C(A)$, $W_A(p) = 0$.*

Proof ($\Longrightarrow$) By Lemma 2.4, if A is a nilpotent matrix, then for all $p = (i_0, i_1, \cdots, i_{r-1}, i_0) \in C(A)$, $W_A(p) = a_{i_0i_1}a_{i_1i_2}\cdots a_{i_{r-1}i_0} = 0$.

($\Longleftarrow$) If for all $p \in C(A)$, $W_A(p) = 0$, we prove $A^n = 0$. By Lemma 2.2, for any typical term $a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j}$ of a^n_{ij}, there must be repetitions amongst the $n+1$ suffixes $i = i_0, i_1, \cdots, i_{n-1}, j = i_n$. Suppose that $i_s (1 \le s \le n)$ is the first one which $i_s \in \{i_0, i_1, \cdots, i_{s-1}\}$, then there exists $i_t (0 \le t \le s-1)$, such that $i_t = i_s$, so, $(i_t, i_{t+1}, \cdots, i_s) \in C(A)$. Hence

$$a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j} \le a_{i_ti_{t+1}}\cdots a_{i_{s-1}i_s} = 0,$$

and

$$a^n_{ij} = \bigvee_{1 \le i_1, i_2, \cdots, i_{n-1} \le n} a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j} = 0, \quad \forall i, j \in N.$$

That is to say, $A^n = 0$. By Lemma 2.1, A is nilpotent. □

Example 3.1 Consider the lattice L whose diagram is displayed in Fig.1.

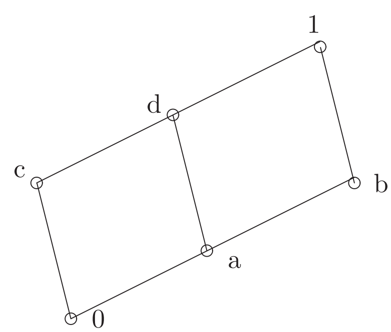

Fig.1

Obviously, L is a distributive lattice. Now let

$$A = \begin{pmatrix} 0 & a & 0 \\ c & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in M_3(L).$$

then $C(A) = \{(1,2,1),(2,1,2)\}$, and for any element p of $C(A)$, $W_A(p) = a \wedge c = 0$, hence A is a nilpotent matrix. In fact, $A^3 = 0$.

Theorem 3.2 *Let $A \in M_n(L)$. Then A is a nilpotent matrix if and only if all principal submatrices of A are nilpotent.*

Proof. ($\Longleftarrow$) Since matrix A is a principal submatrix of matrix A, A is a nilpotent matrix.

($\Longrightarrow$) Let $B = (b_{ij}) = A[i_1, i_2, \cdots, i_t | i_1, i_2, \cdots, i_t]$ is an arbitrary principal submatrix of A and let $p_1 = (k_0, k_1, \cdots, k_{r-1}, k_0) \in C(B)$. Then

$$W_B(p_1) = b_{k_0k_1} b_{k_1k_2} \cdots b_{k_{r-1}k_0} = a_{i_{k_0} i_{k_1}} a_{i_{k_1} i_{k_2}} \cdots a_{i_{k_{r-1}} i_{k_0}}.$$

Obviously, path $p = (i_{k_0}, i_{k_1}, \cdots, i_{k_{r-1}}, i_{k_0})$ is a cycle of $G(A)$, so, by Theorem 3.1, we have

$$W_A(p) = a_{i_{k_0} i_{k_1}} a_{i_{k_1} i_{k_2}} \cdots a_{i_{k_{r-1}} i_{k_0}} = 0.$$

Hence

$$W_B(p_1) = W_A(p) = 0.$$

Applying Theorem 3.1, $A[i_1, i_2, \cdots, i_t | i_1, i_2, \cdots, i_t]$ is a nilpotent matrix. This completes the proof. □

Let

$$A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix},$$

where A_1 be a $m \times m$ matrix and A_2 be a $n \times n$ matrix over distributive lattice L. Then for any $p \in C(A)$, $p \in C(A_1)$ or $p \in C(A_2)$. Hence we have the following corollary.

Corollary 3.1 *Let*

$$A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix},$$

where A_1 be a $m \times m$ matrix and A_2 be a $n \times n$ matrix over distributive lattice L. Then A is a nilpotent matrix if and only if A_1 and A_2 are nilpotent matrices.

Corollary 3.2 *Let L be a distributive lattice,*

$$A = \begin{pmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{pmatrix} \in M_n(L),$$

where $A_i \in M_{n(i)}(L), i = 1, 2, \cdots, k,$ and $n = n(1) + n(2) + \cdots + n(k)$. Then A is a nilpotent matrix if and only if $A_1, A_2, \cdots, A_k$ are all nilpotent matrices.

§4. A Characterization of Lattice Matrices with an Arbitrary Nilpotent Index

If A is a zero matrix, then $h(A)=1$; if A is a nonzero nilpotent matrix, then $h(A)\geq 2$. In the following discussion, we always suppose that A is a nonzero matrix.

If $p=(i_0,i_1,\cdots,i_{r-1})\in S_{r-1}(A)$ and $W_A(p)\neq 0$, then $a_{i_0i_1}a_{i_1i_2}\cdots a_{i_{r-2}i_{r-1}}=W_A(p)$ is a term of $a^{r-1}_{i_0i_{r-1}}$, so, $A^{r-1}\neq 0$. Hence, we have

Lemma 4.1 *Let $A\in M_n(L)$ be a nilpotent matrix. If there exists $p\in S_{r-1}(A)$, such that $W_A(p)\neq 0$, then $h(A)\geq r$.*

Example 3.1(continued). Since $p=(3,2,1)\in S_2(A)$ and $W_A(p)=1\wedge c=c\neq 0$, we have $h(A)\geq 3$, i.e., $h(A)=3$.

Lemma 4.2 *Let $A\in M_n(L)$ be a nilpotent matrix. If $S_r(A)=\emptyset$ or for every $p\in S_r(A), W_A(p)=0$, then $h(A)\leq r$.*

Proof Suppose that $a_{ii_1}a_{i_1i_2}\cdots a_{i_{r-1}j}$ is a typical term of a^r_{ij}. If $|\{i,i_1,\cdots,i_{r-1},j\}|<r+1$, let $i_s(1\leq s\leq r, i=i_0, j=i_r)$ be the first one which $i_s\in\{i,i_1,\cdots,i_{s-1}\}$, then there exists $i_t(0\leq t\leq s-1)$, such that $i_s=i_t$, so, $(i_t,i_{t+1},\cdots,i_s)\in C(A)$, therefore, we have

$$a_{ii_1}a_{i_1i_2}\cdots a_{i_{r-1}j}\leq a_{i_ti_{t+1}}\cdots a_{i_{s-1}i_s}=0.$$

If $|\{i,i_1,\cdots,i_{r-1},j\}|=r+1$, then $(i,i_1,\cdots,i_{r-1},j)\in S_r(A)$, so

$$W_A(p)=a_{ii_1}a_{i_1i_2}\cdots a_{i_{r-1}j}=0.$$

Thus, for any $i,j\in N$ and in any cases, we can obtain:

$$a_{ii_1}a_{i_1i_2}\cdots a_{i_{r-1}j}=0.$$

Therefore

$$a^r_{ij}=\bigvee_{1\leq i_1,i_2,\cdots,i_{r-1}\leq n}a_{ii_1}a_{i_1i_2}\cdots a_{i_{r-1}j}=0,\quad \forall i,j\in N.$$

This means that $A^r=0$, i.e., $h(A)\leq r$. □

Now, a characterization on lattice matrices with an arbitrary nilpotent index can be given in the following.

Theorem 4.1 *Let $A\in M_n(L)$ be a nilpotent matrix. Then $h(A)=r(r\in N)$ if and only if there exists $p\in S_{r-1}(A), W_A(p)\neq 0$ and for all $p\in S_r(A), W_A(p)=0$.*

Proof ($\Longleftarrow$) If A is a nilpotent matrix, by Lemma 4.1, $h(A)\geq r$, and by Lemma 4.2, $h(A)\leq r$, thus $h(A)=r$.

($\Longrightarrow$)Since $h(A)=r$ implies $A^{r-1}\neq 0$, there exist $i_0,i_1,\cdots,i_{r-2},j_0\in N$, such that $a_{i_0i_1}a_{i_1i_2}\cdots a_{i_{r-2}j_0}\neq 0$, by Lemma 2.4, this means that $i_0,i_1,\cdots,i_{r-2},j_0$ are pairwise distinct, i.e., there exists $p=(i_0,i_1,\cdots,i_{r-2},j_0)\in S_{r-1}(A), W_A(p)=a_{i_0i_1}a_{i_1i_2}\cdots a_{i_{r-2}j_0}\neq 0$. On the other hand, if $h(A)=r$, then for all $p\in S_r(A), W_A(p)=0$(otherwise, if there exist $p\in S_r(A), W_A(p)\neq 0$, by lemma 4.1, $h(A)\geq r+1$, this is a contradiction). □

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