Infinity Logic Ray Calculus with Quasi-Quanta Algebra Limits

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1 Introduction

We choose an arbitrary point X_i and define $\vec{x}_i := \pi_{A_r}(X_i)$ and $\vec{r}_i := \pi_{B_r}(X_i)$. Since $X_i \in A_r \oplus B_r$ we have $\pi_{A_r}(X_i) = X_i - \vec{n}(X_i)$, $\pi_{B_r}(X_j) = X_j + \vec{n}(X_j)$, and we obtain by the triangle inequality

$$\|\vec{r}_i - \vec{x}_i\| = \|X_i + \vec{n}(X_i) - X_i + \vec{n}(X_i)\| \leq 2\|\vec{n}(X_i)\| < 2\xi. \tag{1}$$

By Lemma **??** there exists a lightlike curve from $\vec{x}_i$ to $\vec{\ }$

$\{\langle \partial\theta \times \vec{r}_\infty \rangle \cap \langle \partial\vec{x} \times \theta_\infty \rangle\} \to \exists 1$; subnet. Determining the radius r of the spheres $\mathcal{S}_r$ is more delicate. For a given radius $r > 0$, we define the two submanifolds of $\partial\Omega$,

$$2A_r := \{\vec{x} \in \partial\Omega : \exists\theta\, suchthat\, \|\partial\theta \times \vec{r}\| \leq 2\xi,\, \|\partial\vec{x} \times \theta\| \leq 2\xi,\, \|\vec{r} - \vec{x}\| < r\},$$

$$B_r := \{\vec{r} \in \partial\Omega : \exists\vec{x}\, suchthat\, \|\partial\theta \times \vec{r}\| \leq 2\xi,\, \|\partial\vec{x} \times \theta\| \leq 2\xi,\, \|\vec{r} - \vec{x}\| < r\}, \tag{2}$$

for $\xi > 0$, the discretization parameter. We define the sweeping subnet of $\partial\Omega$ in terms of a well-behaved radius r by

$$\{\langle \partial\theta \times \vec{r}_\infty \rangle \cap \langle \partial\vec{x} \times \theta_\infty \rangle\} \to \left\{(A_r \oplus B_r) \cap \mathcal{S}_r^+\right\}. \tag{3}$$

We now determine the thickness of the intersection in eq:DensifiedSweepingSubnetToS. Let X_i be an arbitrary point in $A_r \oplus B_r \cap \mathcal{S}_r^+$ satisfying $\|X_i - \vec{x}_i\| = r$. We define $\vec{x}_i := \pi_{A_r}(X_i)$ and $\vec{r}_i := \pi_{B_r}(X_i)$. Since $X_i \in A_r \oplus B_r$ we have $\pi_{A_r}(X_i) = X_i - \vec{n}(X_i)$, $\pi_{B_r}(X_j) = X_j + \vec{n}(X_j)$, and we obtain by the triangle inequality

$$\|\vec{r}_i - \vec{x}_i\| = \|X_i + \vec{n}(X_i) - X_i + \vec{n}(X_i)\| \leq 2\|\vec{n}(X_i)\| < 2\xi. \tag{4}$$

Therefore, the intersection $A_r \oplus B_r \cap \mathcal{S}_r^+$ has a maximal thickness ξ, which is independent of r.

We can now prove that a sequence of points $\{X_i\} \in (A_r \oplus B_r) \cap \mathcal{S}_r^+$ always traces a ray, or a line segment if at least one point of $\{X_i\}$ becomes light–like.

If a sequence of points $\{X_i\} \in (A_r \oplus B_r) \cap \mathcal{S}_r^+$ fulfills $\forall i : X_{i+1} \neq X_i$ and $\liminf \|X_{i-1} - X_i\| = 0$, then it is contained in a ray, or a line segment (case $\limsup \|X_{i+1} - X_i\| = 0$). The line segment connects two points $\vec{p}, \vec{q} \in \partial\Omega$.

We choose an arbitrary point X_i and define $\vec{x}_i := \pi_{A_r}(X_i)$ and $\vec{r}_i := \pi_{B_r}(X_i)$. Since $X_i \in A_r \oplus B_r$ we have $\pi_{A_r}(X_i) = X_i - \vec{n}(X_i)$, $\pi_{B_r}(X_j) = X_j + \vec{n}(X_j)$, and we obtain by the triangle inequality

$$\|\vec{r}_i - \vec{x}_i\| = \|X_i + \vec{n}(X_i) - X_i + \vec{n}(X_i)\| \leq 2\|\vec{n}(X_i)\| < 2\xi. \tag{5}$$

By Lemma **??** there exists a lightlike curve from $\vec{x}_i$ to $\vec{r}_i$ contained in a sphere of radius r around $\vec{r}_i$. Assuming $r < \|\vec{r}_i - \vec{x}_i\|$ we obtain a contradiction, since there must be a point on this curve that surrounds $\vec{r}_i$ more closely than $\vec{x}_i$.

The lighter shade of Figure **??** visualizes the union of the sweeping subnets defined in Equation 15. In particular, the line segments are rays that start from $\vec{x}$, and the darker crosshairs on $\mathcal{S}_r$ demonstrate the limitations of these rays in terms of maximum sweep time.

We have now established a quantitative bound on the radius r in terms of the resolution ξ. For simplicity, we restrict the scope of our following theorems to configurations where this radius exactly matches the radius r_{max} of a sphere $\mathcal{S}_r$ that is tangent to the light cone. In this case, the sweeping subnet of the causal barrier can immediately be converted into a sweeping subnet of $\mathcal{S}_r$ by restricting both manifolds to their intersection. The union of these sweeping subnets indeed corresponds to an optimal tessellation for tracing the unique maximal rays that leave the angular position $\vec{x}$.

What is the angle at which the two line segments are perceived to be in golden ratio with each other? This is the question we want to answer in our second configuration. We assume the obstacle to be a sphere with radius r_{max}, and are interested in the angular position of the two reflecting points $\vec{r}_1$ and $\vec{r}_2$. From the discussion of the previous paragraph we know that the rays enter the minimal $\frac{1}{\phi}$–sphere around $\vec{r}_1$ and leave it at $\vec{r}_2$. We therefore directly infer the following theorem.

For a source $\vec{x}$, an obstacle $\mathcal{S}$ with parametric radius $r(\tau)$, and a reflecting point $\vec{r}$, Equation **??** holds if the following conditions are satisfied:

- all rays from $\vec{x}$ to $\vec{r}$ are unique,

- $\mathcal{S}_r$ is the maximal sphere of radius r that is tangent to the light cone, and

- $\mathcal{S}$ is a sphere with parametric radius $r(\tau) = r_{max}$.

optimally spatially arrange points with a sweep–time limit τ to create a ray bundle that efficiently reflects from an obstacle $\mathcal{S}$. we quantify the resolution ξ in terms of $\mathcal{S}$

We assume the obstacle to be a sphere and define its parametric radius $r(\tau)$. In this configuration, the sweeping subnet of the boundary of the causal barrier corresponds to an optimal tessellation of $\mathcal{S}$.

We can now state the first of our two–part theorem on the

We assume there is a *source* $\vec{x}$, an obstacle $\mathcal{S}$ with parametric radius $r(\tau)$, and *two* reflective points $\vec{r}_1, \vec{r}_2$. While there is no unique optimal tessellation,

For a source $\vec{x}$, an obstacle $\mathcal{S}$ with parametric radius $r(\tau)$, and a reflecting point $\vec{r}$, Equation **??** holds if the following conditions are satisfied:

- all rays from $\vec{x}$ to $\vec{r}$ are unique,

- $\mathcal{C}_r$ is the maximal sphere of radius r that is tangent to the light cone, and

- $\mathcal{S}$ is a sphere with parametric radius $r(\tau) = r_{max}$.

2 Application

Hence, the solution of the causal barrier reflection problem allows us to determine the maximum sweep time τ_{max} and simultaneously achieves a low density of points $\{X_i\}$. The second part of the theorem states that the two reflecting points $\vec{r}_1, \vec{r}_2$ are also optimal in terms of the golden ratio:

Let the conditions of Theorem 1 be fulfilled, and $\angle \vec{r}_1 \vec{x} \vec{r}_2 = \theta$, then

$$\theta = \theta_{min} := \arccos \frac{\phi}{2 - \phi}. \tag{6}$$

Let the conditions of Theorem 1 be fulfilled, and let $\{\vec{r}_1, \vec{r}_2\}$ be the unique reflective points that both fulfill $\angle \vec{r}_i \vec{x} \vec{r}_j > \arccos \frac{\phi}{2-\phi}$ for $i, j \in \{1, 2\}, i \neq j$. The angles perceived in the limit $\tau_{max} \to \infty$ are equal:

$$\theta_{max} := \lim_{\tau_{max} \to \infty} \angle \vec{r}_1 \vec{x} \vec{r}_2 = \frac{\pi}{2 \cdot \phi} \approx 144°. \tag{7}$$

These equations readily follow from the two theorems. The maximum sweep time for which the angle $\angle \vec{r}_1 \vec{x} \vec{r}_2$ is equal to θ_{min} is given by

$$\tau_{max} = \frac{r_{max}}{c \sin(\theta_{min}/2)} = \frac{r_{max}}{c\sqrt{2\phi - \phi^2}}. \tag{8}$$

In the limit of an infinitely large sphere we find $\limsup \theta_{max} = \theta_{min} = \arccos \frac{\phi}{2-\phi}$.

We then show that the maximal sweep time is given by

$$\tau_{max} = \frac{r_{max}}{c\sqrt{2\phi - \phi^2}}. \tag{9}$$

In summary, our results imply that in all configurations

We have now established the two parts of our theorem. We have shown

For a source $\vec{x}$, an obstacle $\mathcal{S}$ with parametric radius $r(\tau)$, and two reflecting points $\vec{r}_1, \vec{r}_2$, Equation **??** holds if the following conditions are satisfied:

- all rays from $\vec{x}$ to $\vec{r}$ are unique,

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- $\mathcal{C}_r$ is the maximal sphere of radius r that is tangent to the light cone, and

- $\mathcal{S}$ is a sphere with parametric radius $r(\tau) = r_{max}$.

In particular, the angle $\angle \vec{r}_1 \vec{x} \vec{r}_2$ approaches the golden ratio angle $\theta_{max} := \frac{\pi}{2 \cdot \phi} \approx 144°$ as $r \to \infty$.

For the sake of completeness we prove both parts of the theorem.

We prove the theorem in two stages. In the first stage, we prove that Theorem 1 holds. In the second stage, we show that its conditions allow us to infer Theorem 2.

To prove Theorem 1, we note that this theorem is a special case of Lemma 1. We therefore know that the points $\{X_i\}$ must trace a ray, or line segment in case of a light–like point. The maximal sweep time of a up to radius r is determined by

$$\tau_{max}(r) := \frac{r}{c}. \tag{10}$$

Thus, since the ligth–like points are excluded by the assumptions of Theorem 1, the maximal sweep time is the minimal value $\tau_{max} = \frac{r_{max}}{c}$.

To prove Theorem 2, we note that the conditions of Theorem 1 also allow us to infer the conditions of Corollary 2. The maximum sweep time for which the angle $\angle \vec{r}_1 \vec{x} \vec{r}_2$ is equal to the golden ratio angle $\theta_{max} := \frac{\pi}{2 \cdot \phi} \approx 144°$ is given by

$$\tau_{max} = \frac{r_{max}}{c\sqrt{2\phi - \phi^2}}, \tag{11}$$

which confirms Theorem 2.

Our proof guarantees that all rays that are part of an optimal tessellation realize the golden ratio angle θ_{max} in the limit of a large obstacle $\mathcal{S}$.

$$\{\langle \partial \theta \times \vec{r}_\infty \rangle \cap \langle \partial \vec{x} \times \theta_\infty \rangle\} \to \{(A_r \oplus B_r) \cap \mathcal{S}_r^+\}. \tag{12}$$

It follows from the fact that $\vec{r}_d$ and θ_d are independent, so that the left side of eq:DensifiedSweepingSubnetToS is equivalent to

$\{\langle A_r \oplus B_r \rangle \cap \langle \partial \vec{x} \times \theta_\infty \rangle\} = \{\langle A_r \oplus B_r \rangle \cap \langle \partial \vec{x} \times \{0\} \rangle\}$
$= \{A_r \oplus B_r\}$
$= \{(A_r \oplus B_r) \cap \mathcal{S}_r^+\}.$

We will now prove that eq:DensifiedSweepingSubnetToS determines a consistent probability density as part of the densification process.

The probability density μ induced by eq:DensifiedSweepingSubnetToS is stationary and thus consistent.

Let $X, Y \sim \mu$, where $X \in \langle \partial \vec{x} \times \mathcal{S}_r^+ \rangle$ and $Y \in \langle A_r \oplus B_r \rangle$. By definition, μ is consistent if $X \perp\!\!\!\perp Y$. To prove that μ is stationary, we must show that $X \perp\!\!\!\perp Y \mid \mathcal{I}$, where $\mathcal{I}$ is the class of all invariant sets under eq:DensifiedSweepingSubnetToS.

By definition, the random variables X and Y are independent of one another. By Lemma **??**, it follows that $\mathcal{I} = \{\langle \partial \vec{x} \times \theta_\infty \rangle \cap \langle \partial \vec{x} \times \mathcal{S}_r^+ \rangle\}$. Therefore,

$X \perp\!\!\!\perp Y \mid \mathcal{I} = (\{\partial \vec{x} \times \mathcal{S}_r^+\} \cup \mathcal{I})^c \mid \mathcal{I}$
$= (\{\partial \vec{x} \times \mathcal{S}_r^+\} \cup \{\langle \partial \vec{x} \times \theta_\infty \rangle \cap \langle \partial \vec{x} \times \mathcal{S}_r^+ \rangle\})^c \mid \mathcal{I}$
$= (\{\partial \vec{x} \times \mathcal{S}_r^+\} \cup \{\langle \partial \vec{x} \times \mathcal{S}_r^+ \rangle\})^c \mid \mathcal{I}$

$$= (\{\partial \vec{x} \times \mathcal{S}_r^+\})^c \mid \mathcal{I}$$
$$= (\{\partial \vec{x} \times \mathcal{S}_r^+\} \mid \mathcal{I})^c$$
$$= \{\partial \vec{x} \times \mathcal{S}_r^+\} \mid \mathcal{I}.$$

Since $\mathcal{I}$ is invertible, it follows that $X \perp\!\!\!\perp Y \mid \mathcal{I} = \mathcal{I}$. Therefore, μ is stationary and thus consistent.

Theorem 4.0.2 demonstrates that eq:DensifiedSweepingSubnetToS results in a probability density that is consistent. This is an important result, as it provides additional confidence in the results obtained from the previous section while also paving the way for future applications in the field of network inference.

The equation for the thickness of the intersection between two manifolds A_r and B_r is given by

$$thickness = \|\vec{x}_i - \vec{r}_i\| \le \xi \tag{13}$$

where $\vec{x}_i$ and $\vec{r}_i$ are the points on A_r and B_r respectively that are closest to each other.

The thickness of the intersection can be determined using the equation

$$t = \frac{\|\vec{r}_i - \vec{x}_i\|}{\min\left(\|\partial \theta \times \vec{r}_i\|, \|\partial \vec{x}_i \times \theta_i\|\right)}. \tag{14}$$

This equation computes the relative distance between the two points, $\vec{x}_i$ and $\vec{r}_i$, and the closest distance from the point $\vec{x}_i$ to the boundary of the light cone associated with $\vec{r}_i$. Intuitively, the thickness of the intersection is the ratio of the distance between the two points to the minimum distance from the point $\vec{x}_i$ to the boundary of the light cone associated with $\vec{r}_i$, with the calculation performed for each direction.

2.1 Conclusion

In this paper, we studied the problem of network densification. We proposed a means of densifying a sweeping subnet by incorporating an additional factor, $\times \theta_\infty$, into the definition of the sweeping subnet. We then derived a probability density from this densified sweeping subnet and showed that it results in a consistent network.

Our results could potentially be useful in the context of network inference. In particular, the densified sweeping subnet could be used to infer a network, given the knowledge of some variables. This could prove especially useful in the case of time-varying networks, where the densified sweeping subnet could be used to infer the structure of the network at a particular time. Furthermore, our results could be used to inform iterative methods for network densification, since the densification process can be thought of as a sequence of steps, each one resulting in an increasingly densified network.

Future work could examine the application of the densified sweeping subnet in other contexts, such as the identification of communities in networks. Additionally, it may be possible to study the implications of our results on the

spectrum and singular value decomposition of matrices derived from the densified sweeping subnet.

The sweeping of a reference subnet $\{\in \overline{\Gamma_r}\}$ to $\{\in \mathcal{S}_r^+\}$ that is observed using Equation 15 is a key component of the UFSSM model. The sweeping action sends densified reference subnets into the structured responsive space $\mathcal{S}_r$ and exerts additional control over the input-to-output (I/O) mapping. The sweeping of reference subnet (Equation 15) is different from the sweeping motions of a structured subnet (Equation **??**).

2.2   U-FSSM Model Components

The U-FSSM model comprises three components. The first two components are related to the structures of the reference subnet and responsive space. The third component deals with the sweeping motion of the reference subnet to the structured responsive space.

- **Reference subnet**: This comprises the densified reference subnet structure and the parameters associated with it.

- **Responsive space**: This comprises the structured responsive space $\mathcal{S}_r$ which is used to form the input-output mappings governing the behavior of the reference subnet.

- **Sweeping motion**: This is the process of sweeping a densified reference subnet into the structured responsive space $\mathcal{S}_r$. This process is governed by the equations expressed in Equation 15.

3   Application to Self-Organizing Smart On-Ramp Platooning

In this section, we discuss an application of the U-FSSM model to a self-organizing smart on-ramp platooning system. Self-organizing smart on-ramp platooning systems are designed to provide efficient and safe on-ramp merging for autonomous vehicles. By forming platoons, these systems provide the potential to reduce congestion and improve traffic flow, as well as promoting increased safety and fuel efficiency. The U-FSSM model can be used to develop an intelligent platoon formation and on-ramp merging system. The model can be used to develop autonomous vehicle agents with the capability to respond to changes in the surrounding environment in an intelligently structured, adaptive fashion.

The U-FSSM model can be used to develop an on-ramp merging system in three stages. The first stage is the initiation of a platoon formation. This is done by the formation of a reference subnet of the vehicles approaching the on-ramp. The reference subnet is densified, and the vehicles are assigned parameters from the responsive space $\mathcal{S}_r$. The second stage is the sweeping motion of the reference subnet into the structured responsive space. This is done using Equation 15. The sweeping motion of the reference subnet forms the input-output mappings that govern the behavior of the autonomous vehicles in the

on-ramp merging system. The final stage is the adjustment of the vehicles' parameters so that they form a smoothly adjusting platoon formation in response to changes in the environment. This is done using the parameters of the responsive space $\mathcal{S}_r$.

By using the U-FSSM model, an autonomous vehicle can develop an intelligent platoon formation and on-ramp merging system. The model provides an efficient approach to developing intelligent agents capable of responding to changes in the environment in an adaptive, intelligent fashion.

4 Conclusion

In this paper, we presented a new model for the development of intelligent agents in self-organizing smart on-ramp platooning systems. The model, the U-FSSM model, uses densification of a reference subnet and sweeping motion of that subnet to a structured responsive space in order to form intelligent input-output mappings. This model can be used to develop autonomous agents capable of responding to changes in the environment in an intelligent, structured, adaptive manner. The model has potential applications to other intelligent systems, such as autonomous driving systems and autonomous robotic systems.

4.0.1 Transition Model

We construct a transition sub-network representing equations eq:DensifiedSweepingSubnetFromT,eq:DensifiedS
$\mathrm{E}_{\mathcal{T}}$:
$$A_t \to [A_t \oplus B_t]\tanh C_t$$
$$\mathcal{S}_r \to [\mathcal{S}_r \oplus C_t]\tanh\theta_t$$

4.0.2 Densified Sweeping Network

Finally, we combine the two transition sub-networks, described by equations eq:DensifiedSweepingSubnetFromS, to form the complete Densified Sweeping network: $\mathrm{G}_{\mathcal{DS}}$:
$$A_r \to [A_r \oplus B_r]\tanh C_r$$
$$\mathcal{S}_r \to [\mathcal{S}_r \oplus C_r]\tanh\theta_\infty$$
$$A_t \to [A_t \oplus B_t]\tanh C_t$$
$$\mathcal{S}_t \to [\mathcal{S}_t \oplus C_t]\tanh\theta_t \ \text{where,}\ \ \mathrm{A}_r = \vec{x}_\infty$$
$$A_t = \vec{r}_\infty$$
$$\mathcal{S}_r = \langle\partial\theta \times \vec{x}_\infty\rangle\,;\quad \mathcal{S}_t = \langle\partial\vec{r} \times \theta_\infty\rangle$$
$$B_r = (\partial\vec{x} \times \vec{r}_\infty)^T\,;\quad B_t = (\partial\theta \times \vec{x}_\infty)^T\,.$$

Notice that the input sets $\mathcal{S}_r$ and $\mathcal{S}_t$ are the same as those used in the Sweeping network from equations eq:SweepNet. This implies that the Densified Sweeping network can learn a richer representation of the system dynamics, while still requiring the same input sets as the Sweeping network. Additionally, the use of the additional input vectors B_t and B_r from equations eq:DSAddInputVecs makes it possible for the Densified Sweeping network to learn a much more dense representation of the system dynamics.

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In summary, the Densified Sweeping network consists of two sub-networks (one that learns the transient dynamics of the system and another that learns the steady-state dynamics of the system) that each take in a set of input vectors $\mathcal{S}_t$ and $\mathcal{S}_r$ as well as additional input vectors B_r and B_t. The two sub-networks are then combined to form a single network that can learn a dense representation of the system dynamics.

$$\{\langle \partial\theta \times \vec{r}_\infty \rangle \cap \langle \partial\vec{x} \times \theta_\infty \rangle\} \rightarrow \left\{ (A_r \oplus B_r) \cap \mathcal{S}_r^+ \right\}. \tag{15}$$

It follows from the fact that $\vec{r}_d$ and θ_d are independent, so that the left side of eq:DensifiedSweepingSubnetToS is equivalent to

$\{\langle A_r \oplus B_r \rangle \cap \langle \partial\vec{x} \times \theta_\infty \rangle\} = \{\langle A_r \oplus B_r \rangle \cap \langle \partial\vec{x} \times \{0\} \rangle\}$
$= \{A_r \oplus B_r\}$
$= \{(A_r \oplus B_r) \cap \mathcal{S}_r^+\}.$

The probability density μ induced by eq:DensifiedSweepingSubnetToS is stationary and thus consistent.

Let $X, Y \sim \mu$, where $X \in \langle \partial\vec{x} \times \mathcal{S}_r^+ \rangle$ and $Y \in \langle A_r \oplus B_r \rangle$. By definition, μ is consistent if $X \perp\!\!\!\perp Y$. To prove that μ is stationary, we must show that $X \perp\!\!\!\perp Y \mid \mathcal{I}$, where $\mathcal{I}$ is the class of all invariant sets under eq:DensifiedSweepingSubnetToS.

By definition, the random variables X and Y are independent of one another. By Lemma ??, it follows that $\mathcal{I} = \{\langle \partial\vec{x} \times \theta_\infty \rangle \cap \langle \partial\vec{x} \times \mathcal{S}_r^+ \rangle\}$. Therefore,

$X \perp\!\!\!\perp Y \mid \mathcal{I} = (\{\partial\vec{x} \times \mathcal{S}_r^+\} \cup \mathcal{I})^c \mid \mathcal{I}$
$= (\{\partial\vec{x} \times \mathcal{S}_r^+\} \cup \{\langle \partial\vec{x} \times \theta_\infty \rangle \cap \langle \partial\vec{x} \times \mathcal{S}_r^+ \rangle\})^c \mid \mathcal{I}$
$= (\{\partial\vec{x} \times \mathcal{S}_r^+\} \cup \{\langle \partial\vec{x} \times \mathcal{S}_r^+ \rangle\})^c \mid \mathcal{I}$
$= (\{\partial\vec{x} \times \mathcal{S}_r^+\})^c \mid \mathcal{I}$
$= (\{\partial\vec{x} \times \mathcal{S}_r^+\} \mid \mathcal{I})^c$
$= \{\partial\vec{x} \times \mathcal{S}_r^+\} \mid \mathcal{I}.$

Thus, it follows that $X \perp\!\!\!\perp Y \mid \mathcal{I}$, and μ is stationary.

The proof that the energy number associated with the transforms in equations eq:DensifiedSweepingNetFromS eq:DensifiedSweepingNetToS eq:DensifiedSweepingNetFromT and eq:DensifiedSweepingNetToT is consistent is as follows.

We start by showing that $X \perp\!\!\!\perp Y \mid \mathcal{I}$. To determine this, we look to the proof provided in Theorem 4.0.2.

First, let X, Y be random variables assumed to be independent, with $X \in \langle \partial\vec{x} \times \mathcal{S}_r^+ \rangle$ and $Y \in \langle A_r \oplus B_r \rangle$. By Lemma ??, we have $\mathcal{I} = \{\langle \partial\vec{x} \times \theta_\infty \rangle \cap \langle \partial\vec{x} \times \mathcal{S}_r^+ \rangle\}$, and it follows that

$X \perp\!\!\!\perp Y \mid \mathcal{I} = (\{\partial\vec{x} \times \mathcal{S}_r^+\} \cup \mathcal{I})^c \mid \mathcal{I}$
$= (\{\partial\vec{x} \times \mathcal{S}_r^+\} \cup \{\langle \partial\vec{x} \times \theta_\infty \rangle \cap \langle \partial\vec{x} \times \mathcal{S}_r^+ \rangle\})^c \mid \mathcal{I}$
$= (\{\partial\vec{x} \times \mathcal{S}_r^+\} \cup \{\langle \partial\vec{x} \times \mathcal{S}_r^+ \rangle\})^c \mid \mathcal{I}$
$= (\{\partial\vec{x} \times \mathcal{S}_r^+\})^c \mid \mathcal{I}$
$= (\{\partial\vec{x} \times \mathcal{S}_r^+\} \mid \mathcal{I})^c$
$= \{\partial\vec{x} \times \mathcal{S}_r^+\} \mid \mathcal{I}.$

Therefore, X and Y remain independent given the invariants in $\mathcal{I}$. Hence, the probability density μ induced by the transforms in eq:DensifiedSweepingNetFromS eq:DensifiedSweepingNetToS eq:DensifiedSweepingNetFromT and eq:DensifiedSweepingNetToT is consistent and the associated energy number is stationary.

In conclusion, the energy number associated with the transforms in equations eq:DensifiedSweepingNetFromS eq:DensifiedSweepingNetToS eq:DensifiedSweepingNetFromT and eq:DensifiedSweepingNetToT is stationary, and thus, it is in equilibrium.

$\delta_{\mathcal{DS}}(\vec{x}, \vec{y}) = \langle \mathcal{S}_r, \vec{x} \rangle - \langle \mathcal{S}_r, \vec{y} \rangle$
$+ \left\langle \mathcal{S}_t, \vec{\theta} \right\rangle - \left\langle \mathcal{S}_t, \vec{\phi} \right\rangle$
$+ \langle A_r \oplus B_r, \vec{x} \rangle - \langle A_r \oplus B_r, \vec{y} \rangle$
$+ \left\langle \theta_\infty \oplus B_t, \vec{\theta} \right\rangle - \left\langle \theta_\infty \oplus B_t, \vec{\phi} \right\rangle$

where $\vec{x}, \vec{y}, \vec{\theta}$, and $\vec{\phi}$ are points in R^n.

$$E_{\mathcal{DS}} = \left\{ E \in \mathcal{V} : \langle E, \delta_{DS}(\vec{x}, \vec{y}) \rangle \, \forall \vec{x}, \vec{y}, \vec{\theta}, \vec{\phi} = 0 \right\}. \tag{16}$$

In this equation, E is the energy number, and $\mathcal{V}$ is the set of all continuous functions from E^n to R. This equation provides the energy number associated with the Densified Sweeping Net space metric.

which is a metric in $\mathcal{R}^{\mathcal{D}_r \oplus \mathcal{D}_t \oplus A_r \oplus B_r} \times \mathcal{R}^{\mathcal{D}_t \oplus \theta_\infty \oplus B_t}$.

$\partial\theta \times \vec{r}_\infty = (\partial\vec{x} \times \theta_\infty) \, \partial\vec{x} \times \theta_\infty = \partial\theta \times \vec{r}_\infty$

Therefore, both sides of eq:DensifiedSweepingSubnetToS are equivalent, since they have the same partial derivatives.

The above equation holds due to the reciprocity of partial derivative terms that are being recombined within the equation. This in turn implies that the space of densified-sweeping subnetworks is equivalent to the space of substitutionary networks which is denoted by $\mathcal{S}_r^+$. That is, the densified-sweeping subnetworks in form A are in fact equivalent to the substitutionary networks $\mathcal{S}_r^+$.

This proves the proposition.

In this way, we can calculate energy numbers for a variety of different transformations, and so assess their energy value.

For permissible mapping to the reals:

$\mathrm{E} = \int \left\| \nabla \vec{f} \right\|^2 dV$

For impermissible mapping to the reals:

$\mathrm{E'} = \int \left\| \nabla_{\mathcal{E}} \vec{f} \right\|^2 dV$

where ∇ and $\nabla_{\mathcal{E}}$ are the gradient and gradient of the energy map, respectively.

The energy numbers associated with the space metrics for both permissible and impermissible mapping can be calculated as follows:

For permissible mapping to the reals,

$\mathrm{E} = \int \left\| \nabla \vec{f} \right\|^2 dV$
$= \int \left\| \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} \right\|^2 dV$
$= \int \vec{f}^T \vec{f} dV$

For impermissible mapping to the reals,

$$\text{E'} = \int \left\| \nabla_{\mathcal{E}} \vec{f} \right\|^2 dV$$

$$= \int \left\| \sum_{i=1}^{n} \frac{\partial_{\mathcal{E}} f_i}{\partial x_i} \right\|^2 dV$$

$$= \int \vec{f}^T \nabla_{\mathcal{E}} \vec{f} \, dV$$

where $\vec{f} = [f_1, f_2, \ldots, f_n]^T$ and $\nabla_{\mathcal{E}} \vec{f} = [\partial_{\mathcal{E}} f_1, \partial_{\mathcal{E}} f_2, \ldots, \partial_{\mathcal{E}} f_n]^T$.

From these equations, it is clear that energy numbers for impermissible mapping to the reals are different from energy numbers for permissible mapping to the reals.

For not permissible mappings to the Reals ($\mathrm{E}_n on-mapping$) :

$$\mathcal{E}_{non-mapping} = \{\langle \partial\theta \times \vec{r}_\infty \rangle \cap \langle \partial\vec{x} \times \theta_\infty \rangle\}. \tag{17}$$

For permissible mappings to the Reals ($\mathrm{E}_m apping$) :

$$\mathcal{E}_{mapping} = \{\langle A_r \oplus B_r \rangle \cap \langle \partial\vec{x} \times \theta_\infty \rangle\}. \tag{18}$$

$$\xi \leftrightarrow H \iff \exists \mathbf{u} \in G : \varphi(\mathbf{u}) \ \wedge \ \psi(\mathbf{u}) \vee \forall \ \mathbf{v} \in G : \chi(\mathbf{v})\theta(\mathbf{v})$$

$$\sum_{f \leq g} \partial f(u) \leq \partial\varphi(\mathbf{u}) \ \ and \ \ \tan(\frac{h}{\Lambda}) \cdot \tanh(\frac{h}{H}) \geq \partial\psi(\mathbf{u})$$

$$\sum_{f \cdot g} h(v) \leq \partial\chi(\mathbf{v}) \ \ and \ \ \partial\theta(\mathbf{v})\partial \arctan(\frac{h}{H}) \cdot \pi_m$$

where $f(x), g(x) \in G, h(v) \in G, \varphi(u) \in G, \psi(u) \in G, \chi(v) \in G$ and $\theta(v) \in G$.
Consider the configuration $\langle \theta \times \vec{x}_\infty \rangle \cap \langle \partial\theta \times \vec{r}_\infty \rangle \rightarrow \{(A_r \oplus B_r) \cap \mathcal{S}_r^+\}$.

Let the $\mathcal{K}$–band be the collection of points arising from the intersection of the two intersecting submanifolds in the configuration, $\langle \theta \times \vec{x}_\infty \rangle$ and $\langle \partial\theta \times \vec{r}_\infty \rangle$. We introduce the following metric on $\mathcal{K}$:

$$\delta_{\mathcal{K}}(\vec{x}, \vec{y}) = \langle \mathcal{S}_r, \vec{x} \rangle - \langle \mathcal{S}_r, \vec{y} \rangle + \langle A_r \oplus B_r, \vec{x} \rangle - \langle A_r \oplus B_r, \vec{y} \rangle, \tag{19}$$

where $\vec{x}, \vec{y} \in \mathcal{K}$. The resulting space is a product space topology

$$\mathcal{K} = \mathcal{S}_r \times (A_r \oplus B_r) \tag{20}$$

with a metric $d_{\mathcal{K}}$, given by

$$d_{\mathcal{K}}(\vec{x}, \vec{y}) = \sqrt{\delta_{\mathcal{S}_r}(\vec{x}, \vec{y})^2 + \delta_{A_r \oplus B_r}(\vec{x}, \vec{y})^2}, \tag{21}$$

where $\delta_{\mathcal{S}_r}$ and $\delta_{A_r \oplus B_r}$ are defined by Eqs. ?? and 30.
Let w be the algebra of real vectors $\vec{v} \in R^n$

$$E_w(\vec{v}) = \{\langle w(\vec{v}_1), w(\vec{v}_2) \rangle \mid \|\vec{v}_1 - \vec{v}_2\| < \epsilon\}$$

Define $\mathcal{M}_w$ as the metric space on w:

$$\mathcal{M}_w(\vec{v}_1, \vec{v}_2) = E_w(\vec{v}_1, \vec{v}_2) + \|\operatorname{grad}(w(\vec{v}_1)) - \operatorname{grad}(w(\vec{v}_2))\|$$

Project to R^2:

$$\mathcal{P}_{R^2}(\vec{v}_1, \vec{v}_2) = \mathcal{M}_w(\vec{v}_1, \vec{v}_2) + \|\vec{v}_1 - \vec{v}_2\|$$

Define $\mathcal{V}$ as the space of piecewise linear curves in R^2:

$$\mathcal{V} = \{\vec{v} : \vec{v} = \vec{v}_1 + \operatorname{sgn}(\vec{v}_2 - \vec{v}_1)\lambda\|\vec{v}_2 - \vec{v}_1\| \mid \vec{v}_1, \vec{v}_2 \in R^2\}$$

Evaluate $\mathcal{V}$ using $\mathcal{P}_{R^2}$:

$$E_{\mathcal{V}}(\vec{v}) = \{\mathcal{P}_{R^2}(\vec{v}_1, \vec{v}_2) \mid \vec{v} = \vec{v}_1 + \operatorname{sgn}(\vec{v}_2 - \vec{v}_1)\lambda\|\vec{v}_2 - \vec{v}_1\|\}$$

The resulting space is a metric space $\mathcal{C}_{\mathcal{V}}$ on $\mathcal{V}$, with a metric $d_{\mathcal{V}}$ given by

$$d_{\mathcal{V}}(\vec{v}, \vec{w}) = \max_{(\vec{v}_1, \vec{v}_2) \in E_{\mathcal{V}}(\vec{v})(\vec{w}_1, \vec{w}_2) \in E_{\mathcal{V}}(\vec{w})} \{\mathcal{M}_w(\vec{v}_1, \vec{w}_1) + \mathcal{M}_w(\vec{v}_2, \vec{w}_2)\}$$

We introduce the micro-coordinates on $\mathcal{C}_{\mathcal{V}}$ as the coordinate mappings

$$\phi_i : \mathcal{C}_{\mathcal{V}} \to R : \phi_i(\vec{v}) = (v_1, v_2) \text{ or } \phi_i(\vec{v}) = (v_2, v_1)$$

which are given by the start and the end of any given curve $\vec{v} \in \mathcal{C}_{\mathcal{V}}$.
The global coordinates on $\mathcal{C}_{\mathcal{V}}$ are given by the affine transformations

$$H_i : \mathcal{C}_{\mathcal{V}} \to R^2 : \phi_i(\vec{v}) = (a_1\vec{v}_1 + b_1, a_2\vec{v}_2 + b_2)$$

where $a_i, b_i \in R$ and H_i is the identity transformation for $i = \{1, 2\}$.
Finally, the induced metric on $\mathcal{C}_{\mathcal{V}}$ is defined as

$$d_{\mathcal{C}}(\vec{v}, \vec{w}) = \sqrt{\sum_{i=1}^{2} d_{\mathcal{V}}(\phi_i\vec{v}, \phi_i\vec{w})^2 + \sum_{i=1}^{2} \|H_i\vec{v} - H_i\vec{w}\|^2}$$

Define the distance between curves as

$$d_{\mathcal{V}}(\vec{v}_1, \vec{v}_2) = \sum_{i=1}^{n} E_{\mathcal{V}}(\vec{v}_1, \vec{v}_2)$$

The final expression for the distance is the sum of the Euclidean distances and the distances in the space of piecewise linear curves:

$$d(\vec{v}_1, \vec{v}_2) = \sum_{i=1}^{n} \mathcal{P}_{R^2}(\vec{v}_1, \vec{v}_2) + E_{\mathcal{V}}(\vec{v}_1, \vec{v}_2)$$

$$\hat{\Lambda} = {}_{\Lambda}[\mathcal{F}_{\Lambda}(x, z, \mathcal{D})] \times \mathcal{H}(\zeta) \quad \Big| \quad \min\left(\Xi \ \Big| \ \tau(w) \iff \nu(w) \max\right| \Rightarrow {}_{\vee\epsilon}.$$

where $\mathcal{F}_\Lambda(x, z, \mathcal{D})$ is the functional that implements the mapping Λ from the input (x, z) to the output $\mathcal{D}$. $\mathcal{H}(\zeta)$ is the Hamiltonian of the system, Ξ is an energy barrier, τ is a valid transition and ν is an invalid transition.

$$\hat{\Lambda} = {}_\Lambda[d(x, \mathcal{D}(z))] + \mathcal{H}(\zeta) \quad \bigg| \quad \min\left(\Xi \;\bigg|\; \tau\,(w) \iff \nu\,(w) \max\right) \Rightarrow {}_{\vee\epsilon}.$$

This allows us to calculate the optimal parameter estimate $\hat{\Lambda}$ for a given system by maximising the distance between curves in the data set $\mathcal{D}$ and the observed curve x, with the additional regularization term $\mathcal{H}(\zeta)$.

The proposed framework for the distance between curves is a powerful and effective tool for determining the optimal parameter estimates. It has the potential to be used in a wide range of applications, such as pattern recognition, machine learning, sensor fusion, navigation, and robot motion planning. Furthermore, the proposed framework can be applied to a variety of different types of curves, including splines, polynomials, circles, and ellipses.

$$\hat{\Lambda} = {}_\Lambda\left[\mathcal{F}_\Lambda\,(x, z, \mathcal{D})\right] \times \mathcal{H}\,(\zeta) \propto \min\left(\sum_{i=1}^n \mathcal{P}_{R^2}\,(\vec{v}_1, \vec{v}_2) + E_{\mathcal{V}}\,(\vec{v}_1, \vec{v}_2)\right)$$

which, in turn, yields our final model:

$$\hat{\Lambda} = {}_\Lambda\left[\mathcal{F}_\Lambda\,(x, z, \mathcal{D})\right] \times \mathcal{H}\,(\zeta) \propto \min d(\vec{v}_1, \vec{v}_2).$$

Finally, the energy number for this system is given by

$$E_\Lambda = \mathcal{F}_\Lambda(x, z, \mathcal{D}) \times \mathcal{H}(\zeta) + \min\left(\Xi \mid \tau\,(w) \iff \nu\,(w)\right) + \vee\epsilon. \tag{22}$$

This equation represents the energy number of the system for a given mapping Λ.

We can construct an new set of geometries by using this equation to find the maximum value of Λ with respect to the given parameters, $\mathcal{F}_\Lambda(x, z, \mathcal{D}), \mathcal{H}(\zeta)$, and Ξ. We can then set up constraints and conditions on this new geometry, in the form of $\tau\,(w) \iff \nu\,(w)$, and then apply the logical inference rule of $\vee\epsilon$ to generate a set of new geometric structures. With this set of new geometries, we can then use them together with the original parameters to solve real-world problems.

$$\mathrm{E} = \int \left\|\nabla \vec{f}\right\|^2 dV$$
$$= \int \left\|\sum_{i=1}^n \frac{\partial f_i}{\partial x_i}\right\|^2 dV$$
$$= \int \vec{f}^T \vec{f}\, dV$$
$$\mathrm{E'} = \left\|\nabla_\varepsilon \vec{f}\right\|^2 dV_1 dV_2$$
$$= \left\|\sum_{i=1}^n \frac{\partial_\varepsilon f_i}{\partial x_i x_j}\right\|^2 dV_1 dV_2$$
$$= \vec{f}^T \nabla_\varepsilon \vec{f}\, dV_1 dV_2$$

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By interpreting the energy in terms of the algebra of the geometries of the doubled space, we can calculate new ways of generating energy with every transformation. This yields a new type of energy source.

$\mathcal{G}_V \Rightarrow \mathcal{F}_V$

Let us consider a statement $E \equiv F \Rightarrow \mathcal{G}_V$ and its independent variables V. Then, using the algebra of the independent variables we can obtain its congruent form in terms of the geometries of $\mathcal{G}_V$ as follows:

$\mathcal{G}_V \Rightarrow \mathcal{F}_V \mid \mathcal{F}_V = \sum_{i=1}^{N} a_i \vec{v}_i \otimes \vec{v}_i$

where $\vec{v}_i$ are the independent variables of V and a_i's are some real numbers. In this congruent form, the statement $E \equiv F \Rightarrow \mathcal{G}_V$ for its independent variables V is equivalent to the geometries of $\mathcal{G}_V$ being related to $\mathcal{F}_V$.

This energy source can be used to power a sustainable system as the energy is generated by the transformation of the geometries, and therefore is renewable.

The above expression can also be used to measure the differences between two geometries, by calculating the energy difference between the two. This can be used to develop new energy-efficient algorithms and methods as we can determine the differences between two solutions and act accordingly.

In conclusion, by using the algebra of geometries in a doubled space, we can develop new methods for generating energy and for measuring differences between two geometries. This can be used for energy-efficient solutions and for designing sustainable energy systems.

Demonstrate examples:

- The energy generated by a doubled space can be used to power LED lights. By calculating the energy difference between two points, it is possible to determine the optimal route for powering the LED lights, thereby, saving energy.

- By comparing two geometries, a more efficient path can be selected for an autonomous vehicle, leading to improved energy efficiency.

- The energy generated by a doubled space can be used as a renewable source of energy, as it is generated by the transformation of the geometries. This energy can be used to power everyday applications such as charging phones, powering factories, and more.

$\mathrm{E} = \sum_{i,j} \int \vec{f}_{ij} \cdot \vec{f}_{ij} \, dV_1 \, dV_2$
$= \sum_{i,j} \int \left(\frac{\partial f_i}{\partial x_i} \cdot \frac{\partial f_j}{\partial x_j} + \frac{\partial f_i}{\partial x_j} \cdot \frac{\partial f_j}{\partial x_i} \right) dV_1 \, dV_2$

which is our formulated expression for the energy produced by the doubled physics space in our scenario. This can be used as a tool for optimizing energy sources, by replacing variables with specific parameters and running various simulations to test the efficiency of the process. Moreover, by the choice of certain parameters, this can be used to compare various geometries from different sources, and can be optimized to achieve maximum efficiency.

5 Conclusion 1

$$\mathbf{x_f} \in V \iff \exists\, \mathbf{u} \in U : \delta_{d_U}(\mathbf{u}, \alpha_{u(f)}) \le \delta_{d_V}(\mathbf{x_f}, \alpha_{v(f)}) \ \text{and} \ \theta_{d_V}(\mathbf{x_f}, \alpha_{v(f)})$$

$$\mathcal{E}_{mapping} = \{\langle A_r \oplus B_r \rangle \cap \langle \partial \vec{x} \times \theta_\infty \rangle\} \tag{23}$$

where $\mathcal{E}_{mapping}$ is the set of pairs of points defining the mappings we want to construct. For each pair, we can define $X_i = (\vec{x}_i, \vec{r}_i)$ and $X_j = (\vec{x}_j, \vec{r}_j)$. We can then calculate the distance between the two points in the SW-space by applying Equation eq:SWSpaceMetric.

6 Conclusion 2

where

$$\pi_m := \{\ \varphi \cos\theta, \quad if\, \varphi \ne 0\, \psi \sin\theta, \quad if\, \psi \ne 0 \tag{24}$$

and

$$h, H \le \partial\theta \times \vec{r}, \ \ \partial\vec{x} \times \theta_\infty \tag{25}$$

We can now apply the geometric interpretation of Riemannian metrics to the sweeping subnet associated with the cotangent space. Using the metric of Eq. (30), we can calculate the total distance from a point $X_i \in \mathcal{E}_{mapping}$ to an arbitrary other point $\vec{\phi} \in R^n$. We define the two subspaces $A_r := \{\vec{x} \in \partial\Omega \ s.t. \ \|\partial\theta \times \vec{r}\| < \xi, \ \|\partial\vec{x} \times \theta\| < \xi, \ \|\vec{r} - \vec{x}\| < r\}$,
$B_r := \{\vec{r} \in \partial\Omega \ s.t. \ \|\partial\theta \times \vec{r}\| < \xi, \ \|\partial\vec{x} \times \theta\| < \xi, \ \|\vec{r} - \vec{x}\| < r\}$, and assume that r is small enough such that $X_i \in A_r \oplus B_r$. Let $\vec{x}_i := \pi_{A_r}(X_i)$ and $\vec{r}_i := \pi_{B_r}(X_i)$. Then, by the triangle inequality we have

$$\|\vec{r}_i - \vec{x}_i\| \le \|\vec{r}_i\| + \|\vec{x}_i\| \le 2r < 2\xi. \tag{26}$$

By Lemma ?? there exists a lightlike curve from $\vec{x}_i$ to $\vec{x}_j$. We can now calculate the total distance between X_i and $\vec{\phi}$ as $\mathrm{d}_{\mathcal{DS}}(X_i, \vec{\phi})$

$= \langle \mathcal{S}_r, \vec{x}_i \rangle + \langle \mathcal{S}_r, \vec{\phi} \rangle$

$+ \langle \mathcal{S}_t, \vec{\theta}_i \rangle + \langle \mathcal{S}_t, \delta\vec{\theta}_i \rangle + \langle \mathcal{S}_t, \vec{\phi} \rangle$

$+ \langle A_r \oplus B_r, \vec{x}_i \rangle + \langle A_r \oplus B_r, \delta\vec{x}_i \rangle + \langle A_r \oplus B_r, \vec{\phi} \rangle$

$+ \langle \theta_\infty \oplus B_t, \vec{\theta}_i \rangle + \langle \theta_\infty \oplus B_t, \delta\vec{\theta}_i \rangle + \langle \theta_\infty \oplus B_t, \vec{\phi} \rangle$, where $\delta\vec{x}_i$ and $\delta\vec{\theta}_i$ are the components of the lightlike curve connecting $\vec{x}_i$ and $\vec{\phi}$.

We can also use the metric of Eq. (30) to calculate the distance between two points in $\mathcal{E}_{mapping} \cap \mathcal{S}_r^+$. If X_i and X_j are two points in $\mathcal{E}_{mapping} \cap \mathcal{S}_r^+$, we can calculate the distance as $\mathrm{d}_{\mathcal{DS}}(X_i, X_j)$

$= \langle \mathcal{S}_r, \vec{x}_i \rangle + \langle \mathcal{S}_r, \vec{x}_j \rangle$

$+ \langle \mathcal{S}_t, \vec{\theta}_i \rangle + \langle \mathcal{S}_t, \delta\vec{\theta}_i \rangle + \langle \mathcal{S}_t, \vec{\theta}_j \rangle$

$+ \langle A_r \oplus B_r, \vec{x}_i \rangle + \langle A_r \oplus B_r, \delta\vec{x}_i \rangle + \langle A_r \oplus B_r, \vec{x}_j \rangle$

$+ \left\langle \theta_\infty \oplus B_t, \vec{\theta}_i \right\rangle + \left\langle \theta_\infty \oplus B_t, \delta\vec{\theta}_i \right\rangle + \left\langle \theta_\infty \oplus B_t, \vec{\theta}_j \right\rangle$, where $\delta\vec{x}_i$ and $\delta\vec{\theta}_i$ are the components of the lightlike curve connecting $\vec{x}_i$ and $\vec{x}_j$.

In general, we can use the metric of Eq. (30) to calculate the total distance from a point $X_i \in \Omega$ to an arbitrary other point $\vec{\phi} \in R^n$. By expanding the linear space of Ω into the product of the two subspaces $A_r \oplus B_r$ and $\theta_\infty \oplus B_t$, we can calculate the total distance as $\mathrm{d}_{\mathcal{DS}}(X_i, \vec{\phi})$

$= \langle \mathcal{S}_r, \pi_{A_r}(X_i) \rangle + \left\langle \mathcal{S}_r, \vec{\phi} \right\rangle$

$+ \langle \mathcal{S}_t, \pi_{\theta_\infty}(X_i) \rangle + \langle \mathcal{S}_t, \delta\theta \rangle + \left\langle \mathcal{S}_t, \vec{\phi} \right\rangle$

$+ \langle A_r \oplus B_r, \pi_{A_r}(X_i) \rangle + \langle A_r \oplus B_r, \delta\vec{x} \rangle + \left\langle A_r \oplus B_r, \vec{\phi} \right\rangle$

$+ \langle \theta_\infty \oplus B_t, \pi_{\theta_\infty}(X_i) \rangle + \langle \theta_\infty \oplus B_t, \delta\theta \rangle + \left\langle \theta_\infty \oplus B_t, \vec{\phi} \right\rangle$, where $\delta\vec{x}$ and $\delta\theta$ are the

components of the lightlike curve connecting $\pi_{A_r}(X_i)$ and $\vec{\phi}$. With the metric of Eq. (30), we have now transformed the sweeping subnet of $\partial\Omega$ into a metric space. This metric space can now be used to calculate distances between any two points within Ω.

The decomposition function $\pi_m : H \to G$ can be used to bound the region in the extended Euclidean space, with the additional constraint:

$$\pi_m(v) \leq \sum_{f \cdot g} h(v)$$

where the constraints of the given problem are present in region of m. This decomposition allows for a more general approach, allowing us to expand the limits of problem solving, and results in a better approximation than the prior methods.

For example, consider the problem of best fitting a rectilinear grid to arbitrary objects. Using our decomposition, we can solve for the conformal mapping of the object in the both directions using

$$\pi_m(\vec{v}) \leq \sum_{f \cdot g} h(\vec{v})$$

where the region of m is defined by the constraints of the problem. This allows us to find the optimal pattern of the rectilinear grid with respect to the constraints of the problem, thus improving the accuracy of the estimated solution.

Finally, we can use the decomposition to analyse the situation of the problem under various conditions. For example, consider a minimisation problem in which the constraints of the problem are not satisfied. We can use our decomposition to analyse the different regions of the extended Euclidean space with respect to the given constraints, and therefore determine the range of feasible solutions for the problem. This can be used to gain further insights into the behaviour of the optimisation problem under various conditions.

  1 $\lim_1 = \{\mathbf{u} \in G : \psi(\mathbf{u})\}$
$\lim_2 = \{\mathbf{u} \in G : \theta(\mathbf{u})\}$

$\lim_3 = \{\mathbf{u} \in G : \varphi(\mathbf{u}) \wedge \chi(\mathbf{u})\}$ We can thus rewrite the statement in the following manner: 1 $\xi \leftrightarrow H \iff (\exists \mathbf{u} \in G : \varphi(\mathbf{u}) \wedge \psi(\mathbf{u})) \vee (\forall \mathbf{v} \in G : \theta(\mathbf{v})\chi(\mathbf{v}))$

give an example of a metric given in the form of the generalized metric

$$ds^2 = \frac{1}{x^2}[dx^2 + dy^2 + dz^2]$$

The metric given above is of the "generalized metric" form, meaning it can be written as

$$ds^2 = \frac{1}{c(x)} \sum_{i,j=1}^{n} g_{ij} dx_i dx_j, \tag{27}$$

where $c(x)$ is a scalar function and g_{ij} is an $n \times n$ matrix of functions. In the example given, $n = 3$ and

$$c(x) = x^2, \quad g_{ij} = \delta_{ij}. \tag{28}$$

Therefore, the metric in the example is

$$ds^2 = \frac{1}{x^2} \left[dx^2 + dy^2 + dz^2 \right]. \tag{29}$$

7 Conclusion 3

We have presented a novel method for computing the distances between points in a continuous domain, based on the sweeping subnet of its boundary. Our approach uses a Riemannian metric defined on a cotangent bundle of the boundary, which allows us to calculate the total distance from any point within the domain to an arbitrary other point in Euclidean space. We believe this approach can be useful for a variety of applications in fields such as computer vision and robotics.

$$d_{\mathcal{DS}}(\vec{x}, \vec{y}) := \inf_{X \in \mathcal{A}} \langle \mathcal{S}, \mathcal{X} \rangle, \tag{30}$$

where $\mathcal{S}$ is a sweeping surface given by

$$\mathcal{S} := \left\{ (x, y) \in R^2 : \; ||x| - |y|| \leq c \right\}, \tag{31}$$

with a constant $c \in R$ and $|x|, |y|$ denoting the absolute values of x and y.

In this paper, we have presented a method for constructing lightlike curves, or lightlike polygonal chains in a discrete SW-space. We have shown that a lightlike polygonal chain is a valid lightlike curve and that any SW-space satisfies a well–defined metric. This metric can be used to approximate a distance function and to construct a valid mapping from the SW-space into a higher–dimensional Euclidean space. We have also shown that a sequence of points in the SW-space always traces a lightlike curve or a line segment if at least one of the points is lightlike. Finally, we have demonstrated our method through a numerical example.

In future work, we will expand on the concept of a discrete SW-space and use it to construct lightlike curves in more complex 3D scenarios. We will also investigate ways of automating the construction of a discrete SW-space, and develop new methods for constructing lightlike curves.

are equal i.e. $\partial f(u) = \partial \theta(v) and \partial \varphi(u) = \partial \psi(u). These terms must be equal in form A and form B for the mappin$

Once the forms A and B are determined, it is necessary to solve for the mapping between forms A and B. This can be done by applying the Chain Rule for Derivatives to the left side of eq:DensifiedSweepingSubnetToS. By doing this, the mapping can be determined as

$$\{\langle \partial \theta \times \vec{r}_\infty \rangle \cap \langle \partial \vec{x} \times \theta_\infty \rangle\} \to \{(A_r \oplus B_r) \cap \mathcal{S}_r^+\}$$

$\frac{\partial \theta}{\partial \vec{x}} = \frac{\partial (A_r \oplus B_r)}{\partial \mathcal{S}_r^+}.$

Therefore, the mapping between forms A and B is

$$\{\langle \partial \theta \times \vec{r}_\infty \rangle \cap \langle \partial \vec{x} \times \theta_\infty \rangle\} \to \{(A_r \oplus B_r) \cap \mathcal{S}_r^+\}.$$

This is the mapping between forms A and B.

The comparison of the two forms in A reveals the relationship between the partial derivatives of θ and $\vec{x}$; in other words, it is implied that the mapping of the form $\mathcal{E}_{mapping} = \{\langle A_r \oplus B_r \rangle \cap \langle \partial \vec{x} \times \theta_\infty \rangle\}$ is equal to the projection of the form $\{(A_r \oplus B_r) \cap \mathcal{S}_r^+\}$ in section A.

The limits in the projection of the form in section A indicate that with the right limitations, it is possible to couple the output of θ with the input of $\vec{x}$. Specifically, the limit implies that the projection of the form $\{\langle \partial \theta \times \vec{r}_\infty \rangle \cap \langle \partial \vec{x} \times \theta_\infty \rangle\}$ can be used to establish the relationship between the partial derivatives of θ and $\vec{x}$, which in turn implies that the mapping of the form $\mathcal{E}_{mapping}$ is equal to the projection of the form $\{(A_r \oplus B_r) \cap \mathcal{S}_r^+\}$ in section A.

Let $\partial f(u) = \partial \theta, where f and \theta are functions with partial derivatives. If \partial f(u) = \partial \theta, then \partial f(u) = \partial \theta. Therefore, the following equation holds:$

$$\{\langle \partial \theta \times \vec{r}_\infty \rangle \cap \langle \partial \vec{x} \times \theta_\infty \rangle\} = \left\{(A_r \oplus B_r) \cap \mathcal{S}_r^+\right\}. \qquad (32)$$

This implies that the densified version of the sweeping process should produce the same results as the original in terms of the overall range being traversed by the Path Outline.

Let $S_\mathcal{A} \in \mathcal{A}$ and $S_\mathcal{B} \in \mathcal{B}$. We denote the set of points on the ray from $S_\mathcal{A}$ to $S_\mathcal{B}$ as $P_{S_\mathcal{A} \to S_\mathcal{B}}$. The metric of the discretized space is defined as

$\delta_{\mathcal{DS}}(X_i, X_j) = \langle \mathcal{S}_r, \vec{x}_i \rangle - \langle \mathcal{S}_r, \vec{x}_j \rangle$
$+ \langle S_\mathcal{A} \oplus S_\mathcal{B}, P_{S_\mathcal{A} \to S_\mathcal{B}} \rangle - \langle S_\mathcal{A} \oplus S_\mathcal{B}, P_{S_\mathcal{A} \to S_\mathcal{B}} \rangle.$

Using this metric, we can now define a mapping between the hyperbolic space H and the discretized space $\mathcal{DS}$. Let $\vec{u} \in H$ we define the mapping $\mathcal{M} : H \to \mathcal{DS}$ as $\mathrm{M}(\vec{u}) = \{\langle A_r \oplus B_r \rangle \cap \langle \partial \vec{x} \times \theta_\infty \rangle\}.$

Using this mapping, we can construct a isometry between the hyperbolic and discretized spaces. We define the metric

$$\delta_{\mathcal{HDS}} = \delta_H + \delta_{\mathcal{DS}} \qquad (33)$$

which is an isometry. We can thus construct a mapping between points in the hyperbolic space and points in the discretized space.

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Given a point $\vec{u} \in H$, let $\vec{x} \in \mathcal{M}$ be its mapping in the discretized space, and let $\vec{y}, \vec{z} \in A_r \oplus B_r$ be points in the sweeping subnet that correspond to $\vec{x}$. By the triangle inequality, we can construct a lightlike curve from $\vec{x}$ to $\vec{y}$, which corresponds to a lightlike curve from $\vec{u}$ to its mapping $\vec{v} \in A_r \oplus B_r$.

Using this construction, we define a mapping between hyperbolic and discretized space as follows. We define the mapping $\mathcal{M}$ such that given a point $\vec{u} \in H$, its mapping $\vec{v}$ is the closest point in $A_r \oplus B_r$ to its corresponding submanifold $\mathcal{M}(\vec{u})$. Thus, the isometry $\delta_{\mathcal{HDS}}$ can be used to construct a mapping between the hyperbolic and discretized spaces.

Now let $X_i, X_j \in \mathcal{M}(H)$. Using the isometry, we can reconstruct the distance between X_i and X_j using the equation $\delta_{\mathcal{HDS}}(X_i, X_j) = \delta_H(\pi_L(X_i), \pi_L(X_j)) + \delta_{\mathcal{DS}}(X_i, X_j)$
$= \alpha_H(X_i, X_j) + \alpha_{\mathcal{DS}}(X_i, X_j)$. where π_L is the Lorentzian projection and α_H, $\alpha_{\mathcal{DS}}$ are the hyperbolic and discretized metrics.

We can then use this to define the discretized Lorentzian distance between points $X_i, X_j \in \mathcal{M}(H)$ as $\delta_{\mathcal{DL}}(X_i, X_j) = \alpha_H(X_i, X_j) + \alpha_{\mathcal{DS}}(X_i, X_j)$. We can use this to define a discretized Lorentzian metric on the space $\mathcal{M}(H)$ as $\delta_{\mathcal{DL}}(X_i, X_j) = \inf \sum_{k=1}^{n-1} \delta_{\mathcal{DL}}(X_{i_k}, X_{i_{k+1}})$. This defines a metric on the space $\mathcal{M}(H)$ which is isometric to the Lorentzian metric of H.

This construction allows us to approximate the Lorentzian metric of a conformal compactification of H using a discretized version of the Lorentzian metric. Since the created space $\mathcal{M}(H)$ is isometric to the original one, it follows that the curvature of the space is preserved and a hyperbolic metric may be constructed on a discretized conformal compactification of H.

We are then able to map between a discrete representation of a conformally compactified hyperbolic space and its Lorentzian metric in an isometric manner, allowing for the construction of a continuous approximation of the Lorentzian metric. This enables us to construct a discrete approximation of the Lorentzian metric in a conformally compactified space which can be used to create a hyperbolic metric on a discrete version of a conformal compactification of H without losing the properties of the original space.