# A VARIANT OF NIM PLAYED ON BOOLEAN MATRICES

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**Abstract**

We introduce a version of Nim played on a Boolean matrix. Each player, in turn, removes a nonzero row or column. The last player to remove a row or column wins. We investigate the Boolean matrices that represent the Ferrers diagram of an integer partition. An integer partition in which each summand is greater than the number of terms in the partition is said to be strong. The Grundy numbers of Boolean matrices that represent the Ferrers diagram of any integer partition consisting of three or fewer terms are determined. This allows us to classify the $\mathcal{P}$-positions and $\mathcal{N}$-positions of Boolean matrices that represent the Ferrers diagram of any strong integer partition.

## 1. Introduction

Combinatorial game theory (CGT) developed in the context of recreational mathematics. In their seminal work and with a spirit of playfulness, Berlekamp, Conway and Guy [3, 6] established the mathematical framework from which games of complete information could be studied. The power of this theory would soon become apparent and was utilized by many researchers (see Fraenkel's bibliography [8]). Along with its natural appeal, combinatorial game theory has applications

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in complexity theory, logic and biology. Literature on the subject continues to increase and the interested reader can find comprehensive introductions to CGT in [2, 3, 6, 14]. Additional research articles with a theoretical flavor can be found in [1, 10, 11, 12, 13].

We first recall some basic concepts from CGT which are used in this paper. Terms which are not explicitly defined can be found in [14]. A *combinatorial game* is one of complete information and no element of chance is involved in gameplay. Each player is aware of the game position at any point in the game. Under *normal play*, two players (Player 1 and Player 2) alternate taking turns and a player loses when he cannot make a move. An *impartial* combinatorial game is one where both players have the same options from any position. A *finite* game eventually terminates (with a winner and a loser, no draws allowed). It is understood that Player 1 makes the first move in any combinatorial game.

For any finite impartial combinatorial game $\Gamma$, there is an associated nonnegative integer value $\mathrm{Gr}(\Gamma)$. This value immediately tells us if $\Gamma$ is a $\mathcal{P}$*-position* (previous player win) or an $\mathcal{N}$*-position* (next player win). In particular, $\mathrm{Gr}(\Gamma) = 0$ if and only if $\Gamma$ is a $\mathcal{P}$-position. To compute $\mathrm{Gr}(\Gamma)$, we need the following definitions.

The *minimum excluded value* (or *mex*) of a multiset of nonnegative integers is the smallest nonnegative integer which does not appear in the multiset. This is denoted by $\mathrm{mex}\{t_1, t_2, t_3, \ldots, t_k\}$. Let $\Gamma$ be a finite impartial game. Then, the *Grundy number* (or *Grundy-value*) of $\Gamma$ is defined to be

$$\mathrm{Gr}(\Gamma) = \mathrm{mex}\{\mathrm{Gr}(\Delta) : \Delta \text{ is an option of } \Gamma\}.$$

The *sum* of finite impartial games is the game obtained by placing the individual games, side by side. On a player's turn, a move is made in a single summand. Under normal play, the last person to make a move wins. For any finite impartial game $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_k$, the Grundy number of $\Gamma$ is computed in the following way. First, convert $\mathrm{Gr}(\gamma_i)$ into binary. Then, compute $\bigoplus \mathrm{Gr}(\gamma_i)$, where the sum is BitXor (Nim-addition). Finally, convert this value back into a nonnegative integer.

Nim is a short impartial combinatorial game, which is played in the following manner:

- There are $n$ heaps, each containing a finite number of stones. Two players alternate turns, each time choosing a heap and removing any number ($\geqslant 1$) of stones in that heap. The player who cannot make a move loses the game.

In 1902, Bouton [5] gave a beautiful mathematical analysis and complete solution for Nim. Since then, the game of Nim and its variants have been the subject of many CGT research papers. Within the literature, studies on Nim variants with modified rule sets, Nim played on different configurations (circular, triangular and rectangular), and Nim played on graphs can be found. Variants of Nim played on Boolean matrices were analyzed in [4, 7, 9].

## 2. A Matrix Game

In this paper, we introduce and analyze yet another variant of Nim played on a Boolean matrix. A matrix $M$ of zeros and ones is chosen. A *line* of the matrix is either a row or a column. A line is *nonzero* if some entry in the line is not zero. Two players alternate turns. At each turn, a player selects a nonzero line and removes that line from the matrix. The player who makes the last move wins.

Is there a general strategy that can force a win for the first player for a reasonable collection of matrices?

**Remark 1.** We note that a line which has all zeros has no effect on the game and can be deleted. Whenever a line consists entirely of zeros after a player's move, we will remove all lines consisting entirely of zeros without affecting the outcome of the game.

**Remark 2.** Given a matrix game $A$, any permutation of rows of $A$ and any permutation of columns of $A$ yield games that are equivalent to $A$. Also, the transpose $A^T$ is equivalent to $A$.

Our first result applies to permutation matrices.

**Proposition 1.** *Let $P_n$ be any $n \times n$ permutation matrix. Then,*

- $\mathrm{Gr}(P_n) = 0$ *whenever $n$ is even and*
- $\mathrm{Gr}(P_n) = 1$ *whenever $n$ is odd.*

*Proof.* By Remark 2, any permutation matrix $P_n$ is equivalent to the the $n \times n$ identity matrix $I_n$. We observe that $\mathrm{Gr}(I_0) = 0$. We apply induction on $n$. Since every row and column of $I_n$ has exactly one entry with a value of 1, removing any column or row will leave a corresponding column or row consisting entirely of zeros. By Remark 1, we may remove this column or row consisting entirely of zeros. Thus, every allowable move results in the $(n-1) \times (n-1)$ identity matrix $I_{n-1}$. If $n$ is odd, then $\mathrm{Gr}(I_n) = \mathrm{mex}\{\mathrm{Gr}(I_{n-1})\} = \mathrm{mex}\{0\} = 1$. If $n$ is even, then $\mathrm{Gr}(I_n) = \mathrm{mex}\{\mathrm{Gr}(I_{n-1})\} = \mathrm{mex}\{1\} = 0$. $\square$

Let $J_{m,n}$ denote the $m \times n$ matrix with every entry 1. Observe that there are at most two moves from $J_{m,n}$, up to isomorphism, these being $J_{m,n} \to J_{m-1,n}$ or $J_{m,n} \to J_{m,n-1}$. Thus, $\mathrm{Gr}(J_{m,n}) \leqslant 2$. We take $J_{0,k}$ and $J_{k,0}$ to denote the game with the $0 \times 0$ matrix. Since there are no moves possible from this game, $\mathrm{Gr}(J_{0,k}) = \mathrm{Gr}(J_{k,0}) = 0$.

**Theorem 1.** *Suppose that $m, n \geqslant 1$.*

- *If $1 \in \{m, n\}$ and $m + n$ is odd, then* $\mathrm{Gr}(J_{m,n}) = 2$.

- *If* $1 \in \{m, n\}$ *and* $m+n$ *is even, then* $\mathrm{Gr}(J_{m,n}) = 1$.
- *If* $2 \in \{m, n\}$ *and* $m+n$ *is odd, then* $\mathrm{Gr}(J_{m,n}) = 2$.
- *If* $m, n \geqslant 2$ *and* $m+n$ *is even, then* $\mathrm{Gr}(J_{m,n}) = 0$.
- *If* $m, n \geqslant 3$ *and* $m+n$ *is odd, then* $\mathrm{Gr}(J_{m,n}) = 1$.

*Proof.* By Remark 2, the games $J_{m,n}$ and $J_{n,m}$ are equivalent. Thus, we may assume that $m \leqslant n$.

For $m = 1$, the only possible moves are $J_{1,n} \to J_{0,0}$ and $J_{1,n} \to J_{1,n-1}$. We observe that $\mathrm{Gr}(J_{1,1}) = \mathrm{mex}\{0\} = 1$ and $\mathrm{Gr}(J_{1,2}) = \mathrm{mex}\{1, 0\} = 2$. By induction on $k \geqslant 1$,

$$\begin{aligned}
&\mathrm{Gr}(J_{1,2k+1}) = \mathrm{mex}\{\mathrm{Gr}(J_{1,2k}), 0\} = \mathrm{mex}\{2, 0\} = 1, \text{ and}\\
&\mathrm{Gr}(J_{1,2k+2}) = \mathrm{mex}\{\mathrm{Gr}(J_{1,2k+1}), 0\} = \mathrm{mex}\{1, 0\} = 2.
\end{aligned}$$

For $m = 2$ and $n \geqslant 1$, the possible moves are $J_{2,n} \to J_{1,n}$ and $J_{2,n} \to J_{1,n-1}$. Since $\mathrm{Gr}(J_{2,1}) = 2$ and $\mathrm{Gr}(J_{2,2}) = \mathrm{mex}\{\mathrm{Gr}(J_{2,1}), \mathrm{Gr}(J_{1,2})\} = \mathrm{mex}\{2, 2\} = 0$, we have, by induction on $k \geqslant 1$,

$$\begin{aligned}
&\mathrm{Gr}(J_{2,2k+1}) = \mathrm{mex}\{\mathrm{Gr}(J_{2,2k}), \mathrm{Gr}(J_{1,2k+1})\} = \mathrm{mex}\{0, 1\} = 2, \text{ and}\\
&\mathrm{Gr}(J_{2,2k+2}) = \mathrm{mex}\{\mathrm{Gr}(J_{2,2k+1}), \mathrm{Gr}(J_{1,2k+2})\} = \mathrm{mex}\{2, 2\} = 0.
\end{aligned}$$

For $n \geqslant m \geqslant 3$, we proceed by induction on $k = m + n$. If $k$ is odd, then $\mathrm{Gr}(J_{m-1,n}) = 0$, and $\mathrm{Gr}(J_{m,n-1}) = 0$. Thus, $\mathrm{Gr}(J_{m,n}) = 1$. If $k$ is even, then $\mathrm{Gr}(J_{m-1,n}) \in \{2, 1\}$, and $\mathrm{Gr}(J_{m,n-1}) \in \{2, 1\}$. Thus, $\mathrm{Gr}(J_{m,n}) = 0$. This completes the proof. □

Let $U_n$ denote the upper triangular $n \times n$ matrix whose $(i, j)$ entry is 1 if $i \leqslant j$ and 0 if $i > j$. A computer program yields the following calculations: $\mathrm{Gr}(U_n) = 1$ for $n = 1, 3, 5$, $\mathrm{Gr}(U_n) = 2$ for $n = 7, 9$, and $\mathrm{Gr}(U_n) = 0$ for $n = 2, 4, 6, 8, 10$.

**Conjecture 1.** For any even integer $n \geq 2$, $\mathrm{Gr}(U_n) = 0$.

We now introduce some notation for games that arise from $U_n$. After deleting some lines, the resulting matrix will have at least as many ones in row $i$ as in row $i + 1$, and at least as many ones in column $i + 1$ as in column $i$. Also, each row has its ones in the last positions of the row, and each column has its ones in the first positions of the column. Thus, the position can be described by a vector $[a_1, a_2, \ldots, a_k]$, where row $i$ has $a_i$ ones, in the last $a_i$ positions of row $i$. We use $\lambda$ for the null string, and $[\lambda]$ for the lone partition of zero (the list with no elements). If $\pi = [a_1, \ldots, a_k]$ and $\rho = [a_1^{'}, \ldots, a_j^{'}]$ is a position that can be reached from $\pi$ in a single move, we write $\pi \to \rho$.

**Definition 1.** Let $M$ be an $m \times n$ matrix whose entries are 0's and 1's. The *game weight* of $M$, denoted by $\mathrm{wt}(M)$, is the number of entries of $M$ that have value 1.

**Definition 2.** Given $A = [a_1, a_2, \ldots, a_k]$, the *length* of $A$, denoted by $\mathrm{len}(A)$, is the number of terms in the sequence $(a_1, a_2, \ldots, a_k)$. Thus, $\mathrm{len}(A) = k$.

Observe that $[a_1]$ is the $1 \times a_1$ matrix of all ones.

**Corollary 1.** *Suppose* $a_1 \geqslant 1$.

- *If* $a_1$ *is odd, then* $\mathrm{Gr}([a_1]) = 1$.
- *If* $a_1$ *is even, then* $\mathrm{Gr}([a_1]) = 2$.

*Proof.* By Theorem 1, $\mathrm{Gr}([a_1]) = \mathrm{Gr}(J_{1,a_1}) = 1$ if $a_1$ is odd, and $\mathrm{Gr}([a_1]) = \mathrm{Gr}(J_{1,a_1}) = 2$ if $a_1$ is even. □

## 3. Grundy Numbers of $[a_1, a_2]$

We determine the Grundy number of $[a_1, a_2]$ for all $a_1 \geqslant a_2 \geqslant 1$.

**Theorem 2.** *Suppose* $a_1 \geqslant a_2 \geqslant 1$.

- *If* $a_1$ *is even, then* $\mathrm{Gr}([a_1, a_2]) = 0$.
- *If either* $a_1$ *and* $a_2$ *are both odd with* $a_2 > 1$*, or* $a_1 = a_2 = 1$*, then* $\mathrm{Gr}([a_1, a_2]) = 2$.
- *If* $a_1 > 1$ *is odd and either* $a_2$ *is even or* 1*, then* $\mathrm{Gr}([a_1, a_2]) = 3$.

*Proof.* By Theorem 1, $\mathrm{Gr}([1, 1]) = \mathrm{Gr}(J_{2,1}) = 2$. The proof will be by induction on the game weight $\mathrm{wt}([a_1, a_2]) = a_1 + a_2$ of $[a_1, a_2]$.

`Case 1`. Assume $a_1$ is even and $a_2 = 1$. If a column of $[a_1, 1]$ is removed, the allowable positions are $[a_1 - 1, 1]$ and $[a_1 - 1]$. By the inductive hypothesis, $\mathrm{Gr}([a_1 - 1, 1]) = 2$ if $a_1 = 2$ and $\mathrm{Gr}([a_1 - 1, 1]) = 3$ if $a_1 \geqslant 4$. Also, by Corollary 1, $\mathrm{Gr}([a_1 - 1]) = 1$.

If a row of $[a_1, 1]$ is removed, the allowable positions are $[a_1]$ and $[1]$. By Corollary 1, $\mathrm{Gr}([a_1]) = 2$ and $\mathrm{Gr}([1]) = 1$. Thus, if $a_1 = 2$,

$$\mathrm{Gr}([a_1, 1]) = \mathrm{mex}\{1, 2\} = 0,$$

and if $a_1 \geqslant 4$,

$$\mathrm{Gr}([a_1, 1]) = \mathrm{mex}\{1, 2, 3\} = 0.$$

`Case 2`. Assume $a_1 > 1$ is odd and $a_2 = 1$. If a column of $[a_1, 1]$ is removed, the allowable positions are $[a_1 - 1, 1]$ and $[a_1 - 1]$. By the inductive hypothesis, $\mathrm{Gr}([a_1 - 1, 1]) = 0$, and by Corollary 1, $\mathrm{Gr}([a_1 - 1]) = 2$.

If a row of $[a_1, 1]$ is removed, the allowable positions are $[a_1]$ and $[1]$. By Corollary 1, $\mathrm{Gr}([a_1]) = 1$ and $\mathrm{Gr}([1]) = 1$. Thus,

$$\mathrm{Gr}([a_1, 1]) = \mathrm{mex}\{0, 1, 2\} = 3.$$

`Case 3`. Assume $a_1$ is even and $a_2 > 1$ is odd. If a column of $[a_1, a_2]$ is removed, the allowable positions are $[a_1 - 1, a_2]$ and $[a_1 - 1, a_2 - 1]$ which have Grundy numbers 2 and 3, respectively, by the inductive hypothesis. If a row of $[a_1, a_2]$ is removed, the allowable positions are $[a_1]$ and $[a_2]$ which have Grundy numbers 2 and 1, respectively, by Corollary 1. Thus,

$$\mathrm{Gr}([a_1, a_2]) = \mathrm{mex}\{1, 2, 3\} = 0.$$

`Case 4`. Assume both $a_1$ and $a_2$ are even. If a column of $[a_1, a_2]$ is removed, the allowable positions are $[a_1 - 1, a_2]$ and $[a_1 - 1, a_2 - 1]$. We observe that $\mathrm{Gr}([a_1 - 1, a_2]) = 3$ by the inductive hypothesis. Also, if either $a_1 = a_2 = 2$ or $a_2 \geqslant 4$, then $\mathrm{Gr}([a_1 - 1, a_2 - 1]) = 2$; otherwise, if $a_1 \geqslant 4$ and $a_2 = 2$, then $\mathrm{Gr}([a_1 - 1, a_2 - 1]) = 3$. If a row of $[a_1, a_2]$ is removed, the allowable positions are $[a_1]$ and $[a_2]$ which both have Grundy number 2 by Corollary 1. Thus,

$$\mathrm{Gr}([a_1, a_2]) = \mathrm{mex}\{2, 3\} = 0.$$

`Case 5`. Assume both $a_1$ and $a_2 > 1$ are odd. If a column of $[a_1, a_2]$ is removed, the allowable positions are $[a_1 - 1, a_2]$ and $[a_1 - 1, a_2 - 1]$ which both have Grundy number 0 by the inductive hypothesis. If a row of $[a_1, a_2]$ is removed, the allowable positions are $[a_1]$ and $[a_2]$ which both have Grundy number 1 by Corollary 1. Thus,

$$\mathrm{Gr}([a_1, a_2]) = \mathrm{mex}\{0, 1\} = 2.$$

`Case 6`. Assume $a_1$ is odd and $a_2$ is even. If a column of $[a_1, a_2]$ is removed, the allowable positions are $[a_1 - 1, a_2]$ and $[a_1 - 1, a_2 - 1]$ which both have Grundy number 0 by the inductive hypothesis. If a row of $[a_1, a_2]$ is removed, the allowable positions are $[a_1]$ and $[a_2]$ which have Grundy numbers 1 and 2, respectively, by Corollary 1. Thus,

$$\mathrm{Gr}([a_1, a_2]) = \mathrm{mex}\{0, 1, 2\} = 3.$$

$\square$

## 4. Grundy Numbers of $[a_1, a_2, a_3]$

For the game $[a_1, a_2, a_3]$, there are effectively only six possible moves: delete one of the three rows, or delete a column with one, two or three ones in it. Hence, the Grundy number of $[a_1, a_2, a_3]$ is at most 6.

**Theorem 3.** *Suppose $a_1 \geqslant 1$. Then,*

- $\mathrm{Gr}([1, 1, 1]) = 1$,
- $\mathrm{Gr}([a_1, 1, 1]) = 0$ *if $a_1 > 1$ is odd, and*
- $\mathrm{Gr}([a_1, 1, 1]) = 3$ *if $a_1$ is even.*

*Proof.* Since $[1, 1, 1]$ is isomorphic to $[3]$, we have $\mathrm{Gr}([1, 1, 1]) = \mathrm{Gr}([3]) = 1$ by Corollary 1. We induct on $a_1$.

`Case 1`. Assume $a_1$ is even. If a column of $[a_1, 1, 1]$ is removed, the allowable positions are $[a_1 - 1, 1]$ and $[a_1 - 1]$. We have $\mathrm{Gr}([a_1 - 1, 1, 1]) = 1$ for $a_2 = 2$, and $\mathrm{Gr}([a_1 - 1, 1, 1]) = 0$ for $a_2 > 2$ by the inductive hypothesis. By Corollary 1, $\mathrm{Gr}([a_1 - 1]) = 1$.

If a row of $[a_1, 1, 1]$ is removed, the allowable positions are $[a_1, 1]$ and $[1, 1]$. By Theorem 2, $\mathrm{Gr}([a_1, 1]) = 0$ and $\mathrm{Gr}([1, 1]) = 2$. Thus,

$$\mathrm{Gr}([a_1, a_2]) = \mathrm{mex}\{0, 1, 2\} = 3.$$

`Case 2`. Assume $a_1 > 1$ is odd. If a column of $[a_1, 1, 1]$ is removed, the allowable positions are $[a_1 - 1, 1]$ and $[a_1 - 1]$. We have $\mathrm{Gr}([a_1 - 1, 1, 1]) = 3$ by the inductive hypothesis, and $\mathrm{Gr}([a_1 - 1]) = 2$ by Corollary 1.

If a row of $[a_1, 1, 1]$ is removed, the allowable positions are $[a_1, 1]$ and $[1, 1]$. By Theorem 2, $\mathrm{Gr}([a_1, 1]) = 3$ and $\mathrm{Gr}([1, 1]) = 2$. Thus,

$$\mathrm{Gr}([a_1, a_2]) = \mathrm{mex}\{2, 3\} = 0.$$

□

**Theorem 4.** *Suppose $a_1$ and $a_2$ are integers such that $a_1 \geqslant a_2 > 1$.*

- *If $a_1$ is odd, then* $\mathrm{Gr}([a_1, a_2, 1]) = 1$.
- *If $a_1$ is even with $a_1 \geqslant 4$ and either $a_2$ is odd or $a_2 = 2$, then* $\mathrm{Gr}([a_1, a_2, 1]) = 2$.
- *If $a_1$ and $a_2$ are both even with $a_2 \geqslant 4$, or $a_1 = a_2 = 2$, then* $\mathrm{Gr}([a_1, a_2, 1]) = 3$.

*Proof.* The proof is by induction on the game weight of $[a_1, a_2, 1]$. We prove the result assuming $a_1 > a_2$. We leave the proof of the case $a_1 = a_2$ to the reader.

`Case 1`. Assume $a_1$ is even with $a_1 \geqslant 4$ and $a_2 = 2$. If a column of $[a_1, 2, 1]$ is removed, the allowable positions are $[a_1 - 1, 2, 1]$, $[a_1 - 1, 1, 1]$ and $[a_1 - 1, 1]$. We have $\mathrm{Gr}([a_1 - 1, 2, 1]) = 1$ by the inductive hypothesis, $\mathrm{Gr}([a_1 - 1, 1, 1]) = 0$ by Theorem 3, and $\mathrm{Gr}([a_1 - 1, 1]) = 3$ by Theorem 2. If a row of $[a_1, 2, 1]$ is removed, the allowable positions are $[a_1, 2]$, $[a_1, 1]$, and $[2, 1]$ which all have Grundy number 0 by Theorem 2. Thus,

$$\mathrm{Gr}([a_1, a_2, 1]) = \mathrm{mex}\{0, 1, 3\} = 2.$$

`Case 2`. Assume $a_1$ and $a_2$ are odd with $a_2 \geqslant 3$. If a column of $[a_1, a_2, 1]$ is removed, the allowable positions are $[a_1 - 1, a_2, 1]$, $[a_1 - 1, a_2 - 1, 1]$ and $[a_1 - 1, a_2 - 1]$ which have Grundy numbers 2 by the inductive hypothesis, 2 if $a_2 = 3$ or 3 if $a_2 > 3$ by the inductive hypothesis, and 0 by Theorem 2, respectively. If a row of $[a_1, a_2, 1]$ is removed, the allowable positions are $[a_1, a_2]$, $[a_1, 1]$, and $[a_2, 1]$ which have Grundy numbers 2, 3, and 3, respectively, by Theorem 2. Thus,

$$\mathrm{Gr}([a_1, a_2, 1]) = \mathrm{mex}\{0, 2, 3\} = 1.$$

`Case 3`. Assume $a_1$ is odd and $a_2$ is even. If a column of $[a_1, a_2, 1]$ is removed, the allowable positions are $[a_1 - 1, a_2, 1]$, $[a_1 - 1, a_2 - 1, 1]$ and $[a_1 - 1, a_2 - 1]$. The Grundy number of $[a_1 - 1, a_2, 1]$ is 2 if $a_1 \geqslant 5$ and $a_2 = 2$ by the inductive hypothesis, or 3 if $a_2 \geqslant 4$ by the inductive hypothesis. Also, the Grundy number of $[a_1 - 1, a_2 - 1, 1]$ is 3 if $a_1 \geqslant 5$ and $a_2 = 2$ by Theorem 3, or 2 if $a_2 \geqslant 4$ by the inductive hypothesis. In addition, the Grundy number of $[a_1 - 1, a_2 - 1]$ is 0 by Theorem 2. If a row of $[a_1, a_2, 1]$ is removed, the allowable positions are $[a_1, a_2]$, $[a_1, 1]$, and $[a_2, 1]$ which have Grundy numbers 3, 3, and 0, respectively, by Theorem 2. Thus,

$$\mathrm{Gr}([a_1, a_2, 1]) = \mathrm{mex}\{0, 2, 3\} = 1.$$

`Case 4`. Assume $a_1$ is even and $a_2$ is odd. If a column of $[a_1, a_2, 1]$ is removed, the allowable positions are $[a_1 - 1, a_2, 1]$, $[a_1 - 1, a_2 - 1, 1]$ and $[a_1 - 1, a_2 - 1]$ which have Grundy numbers 1, 1, and 3, respectively, by the inductive hypothesis and Theorem 2. If a row of $[a_1, a_2, 1]$ is removed, the allowable positions are $[a_1, a_2]$, $[a_1, 1]$, and $[a_2, 1]$ which have Grundy numbers 0, 0, and 3, respectively, by Theorem 2. Thus,

$$\mathrm{Gr}([a_1, a_2, 1]) = \mathrm{mex}\{0, 1, 3\} = 2.$$

`Case 5`. Assume $a_1$ and $a_2$ are both even with $a_2 \geqslant 4$. If a column of $[a_1, a_2, 1]$ is removed, the allowable positions are $[a_1 - 1, a_2, 1]$, $[a_1 - 1, a_2 - 1, 1]$ and $[a_1 - 1, a_2 - 1]$ which have Grundy numbers 1, 1, and 2, respectively, by the inductive hypothesis and Theorem 2. If a row of $[a_1, a_2, 1]$ is removed, the allowable positions are $[a_1, a_2]$, $[a_1, 1]$, and $[a_2, 1]$ which all have Grundy number 0 by Theorem 2. Thus,

$$\mathrm{Gr}([a_1, a_2, 1]) = \mathrm{mex}\{0, 1, 2\} = 3.$$

□

**Theorem 5.** *Suppose $a_1$ and $a_2$ are integers such that $a_1 \geqslant a_2 \geqslant 2$.*

- *If $a_1$ and $a_2$ are both odd, then* $\mathrm{Gr}([a_1, a_2, 2]) = 0$.
- *If $a_1$ is odd and $a_2$ is even, then* $\mathrm{Gr}([a_1, a_2, 2]) = 1$.
- *If $a_1$ is even, then* $\mathrm{Gr}([a_1, a_2, 2]) = 2$.

*Proof.* The proof is by induction on the game weight of $[a_1, a_2, 2]$. We prove the result assuming $a_1 > a_2$. We leave the proof of the case $a_1 = a_2$ to the reader.

`Case 1`. Assume both $a_1$ and $a_2$ are odd. If a column of $[a_1, a_2, 2]$ is removed, the allowable positions are $[a_1 - 1, a_2, 2]$, $[a_1 - 1, a_2 - 1, 2]$, and $[a_1 - 1, a_2 - 1, 1]$. The Grundy numbers of $[a_1 - 1, a_2, 2]$ and $[a_1 - 1, a_2 - 1, 2]$ are each 2 by the inductive hypothesis. Also, the Grundy number of $[a_1 - 1, a_2 - 1, 1]$ is 2 if $a_1 \geqslant 5$ and $a_2 = 3$, or 3 if $a_2 \geqslant 5$ by Theorem 4. If a row of $[a_1, a_2, 2]$ is removed, the allowable positions are $[a_1, a_2]$, $[a_1, 2]$, and $[a_2, 2]$ which have Grundy numbers 2, 3, and 3, respectively, by Theorem 2. Thus,

$$\mathrm{Gr}([a_1, a_2, 2]) = \mathrm{mex}\{2, 3\} = 0.$$

`Case 2`. Assume $a_1$ is odd and $a_2$ is even. If a column of $[a_1, a_2, 2]$ is removed, the allowable positions are $[a_1 - 1, a_2, 2]$, $[a_1 - 1, a_2 - 1, 2]$, and $[a_1 - 1, a_2 - 1, 1]$. The Grundy numbers of $[a_1 - 1, a_2, 2]$ and $[a_1 - 1, a_2 - 1, 2]$ are each 2 by the inductive hypothesis. Also, the Grundy number of $[a_1 - 1, a_2 - 1, 1]$ is 3 if $a_2 = 2$ by Theorem 3, or 2 if $a_2 \geqslant 4$ by Theorem 4. If a row of $[a_1, a_2, 2]$ is removed, the allowable positions are $[a_1, a_2]$, $[a_1, 2]$, and $[a_2, 2]$ which have Grundy numbers 3, 3, and 0, respectively, by Theorem 2. Thus,

$$\mathrm{Gr}([a_1, a_2, 2]) = \mathrm{mex}\{0, 2, 3\} = 1.$$

`Case 3`. Assume $a_1$ is even and $a_2$ is odd. If a column of $[a_1, a_2, 2]$ is removed, the allowable positions are $[a_1 - 1, a_2, 2]$, $[a_1 - 1, a_2 - 1, 2]$, and $[a_1 - 1, a_2 - 1, 1]$ which have Grundy numbers 0, 1, and 1, respectively, by the inductive hypothesis and Theorem 4. If a row of $[a_1, a_2, 2]$ is removed, the allowable positions are $[a_1, a_2]$, $[a_1, 2]$, and $[a_2, 2]$ which have Grundy numbers 0, 0, and 3, respectively, by Theorem 2. Thus,

$$\mathrm{Gr}([a_1, a_2, 2]) = \mathrm{mex}\{0, 1, 3\} = 2.$$

`Case 4`. Assume $a_1$ and $a_2$ are both even. If a column of $[a_1, a_2, 2]$ is removed, the allowable positions are $[a_1 - 1, a_2, 2]$, $[a_1 - 1, a_2 - 1, 2]$, and $[a_1 - 1, a_2 - 1, 1]$. The Grundy numbers of $[a_1 - 1, a_2, 2]$ and $[a_1 - 1, a_2 - 1, 2]$ are 1 and 0, respectively, by the inductive hypothesis. Also, the Grundy number of $[a_1 - 1, a_2 - 1, 1]$ is 0 if $a_2 = 2$ by Theorem 3, or 1 if $a_2 \geqslant 4$ by Theorem 4. If a row of $[a_1, a_2, 2]$ is removed, the allowable positions are $[a_1, a_2]$, $[a_1, 2]$, and $[a_2, 2]$ which each have Grundy number 0 by Theorem 2. Thus,

$$\mathrm{Gr}([a_1, a_2, 2]) = \mathrm{mex}\{0, 1\} = 2.$$

□

**Theorem 6.** *Suppose $a_1 \geqslant a_2 \geqslant a_3 \geqslant 3$.*

- *If both $a_1$ and $a_2$ are odd, then $\mathrm{Gr}([a_1, a_2, a_3]) = 0$.*
- *If both $a_1$ and $a_2$ are even, then $\mathrm{Gr}([a_1, a_2, 3]) = 1$.*
- *If $a_1$ is even and $a_3 > 3$, then $\mathrm{Gr}([a_1, a_2, a_3]) = 1$.*
- *If $a_1$ is odd, $a_2$ is even, and $a_3$ is even, then $\mathrm{Gr}([a_1, a_2, a_3]) = 2$.*
- *If $a_1$ is even and $a_2$ is odd, then $\mathrm{Gr}([a_1, a_2, 3]) = 3$.*
- *If $a_1$ is odd, $a_2$ is even, and $a_3$ is odd, then $\mathrm{Gr}([a_1, a_2, a_3]) = 4$.*

*Proof.* The proof is by induction on the game weight of $[a_1, a_2, a_3]$. We prove the result assuming $a_1 > a_2 > a_3$. We leave the proof of the cases when either $a_1 = a_2$ or $a_2 = a_3$ to the reader.

`Case 1`. Assume both $a_1$ and $a_2$ are odd. If a column of $[a_1, a_2, a_3]$ is removed, the allowable positions are $[a_1 - 1, a_2, a_3]$, $[a_1 - 1, a_2 - 1, a_3]$ and $[a_1 - 1, a_2 - 1, a_3 - 1]$. The Grundy number of $[a_1 - 1, a_2, a_3]$ is 3 if $a_3 = 3$ or 1 if $a_3 > 3$ by the inductive hypothesis. Similarly, the Grundy number of $[a_1 - 1, a_2 - 1, a_3]$ is 1 by the inductive hypothesis, and the Grundy number of $[a_1-1, a_2-1, a_3-1]$ is 2 if $a_3 = 3$ by Theorem 5 or 1 if $a_3 > 3$ by the inductive hypothesis. If a row of $[a_1, a_2, a_3]$ is removed, the allowable positions are $[a_1, a_2]$, $[a_1, a_3]$, and $[a_2, a_3]$. We have $\mathrm{Gr}([a_1, a_2]) = 2$, and $\mathrm{Gr}([a_1, a_3]) = \mathrm{Gr}([a_2, a_3]) = 2$ if $a_3$ is odd or $\mathrm{Gr}([a_1, a_3]) = \mathrm{Gr}([a_2, a_3]) = 3$ if $a_3$ is even by Theorem 2. When $a_3 = 3$ or $a_3$ is even,

$$\mathrm{Gr}([a_1, a_2, a_3]) = \mathrm{mex}\{1, 2, 3\} = 0,$$

and when $a_3$ is odd,

$$\mathrm{Gr}([a_1, a_2, a_3]) = \mathrm{mex}\{1, 2\} = 0.$$

`Case 2`. Assume $a_1$ and $a_2$ are both even. If a column of $[a_1, a_2, a_3]$ is removed, the allowable positions are $[a_1 - 1, a_2, a_3]$, $[a_1 - 1, a_2 - 1, a_3]$ and $[a_1 - 1, a_2 - 1, a_3 - 1]$

which have Grundy numbers 2 if $a_3$ is even or 4 if $a_3$ is odd, 0, and 0, respectively, by the inductive hypothesis and Theorem 5. If a row of $[a_1, a_2, a_3]$ is removed, the allowable positions are $[a_1, a_2]$, $[a_1, a_3]$, and $[a_2, a_3]$ which each have Grundy number 0 by Theorem 2. When $a_3$ is even,

$$\mathrm{Gr}([a_1, a_2, a_3]) = \mathrm{mex}\{0, 2\} = 1,$$

and when $a_3$ is odd,

$$\mathrm{Gr}([a_1, a_2, a_3]) = \mathrm{mex}\{0, 4\} = 1.$$

`Case 3`. Assume $a_1$ is even, $a_2$ is odd, and $a_3 > 3$. If a column of $[a_1, a_2, a_3]$ is removed, the allowable positions are $[a_1-1, a_2, a_3]$, $[a_1-1, a_2-1, a_3]$ and $[a_1-1, a_2-1, a_3-1]$. The Grundy number of $[a_1-1, a_2, a_3]$ is 0 by the inductive hypothesis. Also, the Grundy number of $[a_1-1, a_2-1, a_3]$ is 2 if $a_3$ is even or 4 if $a_3$ is odd by the inductive hypothesis. Similarly, the Grundy number of $[a_1-1, a_2-1, a_3-1]$ is 4 if $a_3$ is even or 2 if $a_3$ is odd by the inductive hypothesis. If a row of $[a_1, a_2, a_3]$ is removed, the allowable positions are $[a_1, a_2]$, $[a_1, a_3]$, and $[a_2, a_3]$ which have Grundy numbers 0, 0, and 3 if $a_3$ is even or 2 if $a_3$ is odd, respectively, by Theorem 2. When $a_3$ is even,

$$\mathrm{Gr}([a_1, a_2, a_3]) = \mathrm{mex}\{0, 2, 3, 4\} = 1,$$

and when $a_3 > 3$ is odd,

$$\mathrm{Gr}([a_1, a_2, a_3]) = \mathrm{mex}\{0, 2, 4\} = 1.$$

`Case 4`. Assume $a_1$ is odd, $a_2$ is even, and $a_3$ is even. If a column of $[a_1, a_2, a_3]$ is removed, the allowable positions are $[a_1-1, a_2, a_3]$, $[a_1-1, a_2-1, a_3]$ and $[a_1-1, a_2-1, a_3-1]$. The Grundy numbers of $[a_1-1, a_2, a_3]$ and $[a_1-1, a_2-1, a_3]$ are each 1 by the inductive hypothesis. Also, the Grundy number of $[a_1-1, a_2-1, a_3-1]$ is 3 if $a_3 = 4$ or 1 if $a_3 > 4$ by the inductive hypothesis. If a row of $[a_1, a_2, a_3]$ is removed, the allowable positions are $[a_1, a_2]$, $[a_1, a_3]$, and $[a_2, a_3]$ which have Grundy numbers 3, 3, and 0, respectively, by Theorem 2. Thus,

$$\mathrm{Gr}([a_1, a_2, a_3]) = \mathrm{mex}\{0, 1, 3\} = 2.$$

`Case 5`. Assume $a_1$ is even, $a_2$ is odd, and $a_3 = 3$. If a column of $[a_1, a_2, 3]$ is removed, the allowable positions are $[a_1-1, a_2, 3]$, $[a_1-1, a_2-1, 3]$ and $[a_1-1, a_2-1, 2]$ which have Grundy numbers 0, 4, and 1, respectively, by the inductive hypothesis and Theorem 5. If a row of $[a_1, a_2, 3]$ is removed, the allowable positions are $[a_1, a_2]$, $[a_1, 3]$, and $[a_2, 3]$ which have Grundy numbers 0, 0, and 2, respectively, by Theorem 2. Thus,

$$\mathrm{Gr}([a_1, a_2, 3]) = \mathrm{mex}\{0, 1, 2, 4\} = 3.$$

`Case 6`. Assume $a_1$ is odd, $a_2$ is even, and $a_3$ is odd. If a column of $[a_1, a_2, a_3]$ is removed, the allowable positions are $[a_1 - 1, a_2, a_3]$, $[a_1 - 1, a_2 - 1, a_3]$ and $[a_1 - 1, a_2 - 1, a_3 - 1]$. The Grundy numbers of $[a_1 - 1, a_2, a_3]$ and $[a_1 - 1, a_2 - 1, a_3]$ are 1 and 3 if $a_3 = 3$ or 1 if $a_3 > 3$, respectively, by the inductive hypothesis. Also, the Grundy number of $[a_1 - 1, a_2 - 1, a_3 - 1]$ is 2 if $a_3 = 3$ by Theorem 5 or 1 if $a_3 > 3$ by the inductive hypothesis. If a row of $[a_1, a_2, a_3]$ is removed, the allowable positions are $[a_1, a_2]$, $[a_1, a_3]$, and $[a_2, a_3]$ which have Grundy numbers 3, 2, and 0, respectively, by Theorem 2. Thus,

$$\mathrm{Gr}([a_1, a_2, a_3]) = \mathrm{mex}\{0, 1, 2, 3\} = 4.$$

□

## 5. $\mathcal{P}$-positions and $\mathcal{N}$-positions of $[a_1, a_2, \ldots, a_k]$

In this section we determine whether $[a_1, a_2, \ldots, a_k]$ is a $\mathcal{P}$-position or an $\mathcal{N}$-position when $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k > k$. First, we introduce some definitions.

**Definition 3.** We say that a sequence of positive integers $S = (a_1, a_2, \ldots, a_k)$ is a *partition sequence* if $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k$. We say that the partition sequence $S$ is *strong* if $a_k > k$.

**Remark 3.** The sequence $S = (a_1, a_2, \ldots, a_k)$ is a partition sequence in the sense that the $a_i$'s represent the descending terms in the partition of $N = a_1 + a_2 + \cdots + a_k$. The partition sequence $S$ is strong in the sense that each term $a_i$ in the partition of $N = a_1 + a_2 + \cdots + a_k$ is greater than the number of terms $k$ in this partition.

**Definition 4.** Let $S = (a_1, a_2, \ldots, a_k)$ be a partition sequence. The *matrix associated with* $S$ is the $k \times a_1$ matrix $B = [b_{i,j}]$ where $b_{i,j} = 0$ if $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant a_1 - a_i$, and $b_{i,j} = 1$ if $1 \leqslant i \leqslant k$ and $a_1 - a_i < i \leqslant a_1$. We write $B = \mathtt{mat}(S)$ and refer to $B$ as a *partition matrix*. Let $B$ be a partition matrix. The *partition sequence associated with* $B$ is the sequence $S = (a_1, a_2, \ldots, a_k)$ where $a_i$ is the number of 1's in row $i$ of $B$, and we write $S = \mathrm{seq}(B)$. We say that $B$ is a *strong partition matrix* if $\mathrm{seq}(B)$ is a strong partition sequence. See Figure 1 for an example of the strong partition matrix $[10, 8, 7, 5]$.

**Remark 4.** We make use of the following two observations in order to show that a game position is either a $\mathcal{P}$-position or an $\mathcal{N}$-position.

- Suppose $G$ is a game position such that for every move $G'$ that Player 1 can make, Player 2 can make a countermove $G''$ such that $G''$ is a $\mathcal{P}$-position. Then, $G$ is a $\mathcal{P}$-position.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Figure 1: The strong partition matrix $[10, 8, 7, 5]$.

- Suppose $G$ is a game position such that Player 1 can make a move $G'$ that is a $\mathcal{P}$-position. Then, $G$ is an $\mathcal{N}$-position.

We first establish the conditions on a strong partition matrix $[a_1, a_2, \ldots, a_k]$, where $k \in \{4, 5\}$, that determines whether it is either a $\mathcal{P}$-position or an $\mathcal{N}$-position. This serves as the base case of our argument. Then we show that, for $k \geqslant 6$, the strong partition matrix $[a_1, a_2, \ldots, a_k]$ is a $\mathcal{P}$-position if $a_1 \equiv a_2 \equiv k \pmod 2$, and we show $[a_1, a_2, \ldots, a_k]$ is an $\mathcal{N}$-position if $a_1 \equiv k + 1 \pmod 2$. From these two fundamental cases, we are able to determine whether the strong partition matrix $[a_1, a_2, \ldots, a_k]$ is a $\mathcal{P}$-position or an $\mathcal{N}$-position depending on the parity of $a_i$ for all $1 \leqslant i \leqslant k - 1$.

The following lemma provides a condition on a strong partition matrix $A$ that allows Player 2 to remove a column with $\ell$ 1's after Player 1 has removed a column of $\ell$ 1's.

**Lemma 1.** *Suppose $k \geqslant 3$, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k > k$, and $a_\ell \equiv a_{\ell+1} \pmod 2$ for some $1 \leqslant \ell \leqslant k - 1$. If Player 1 can remove a column with $\ell$ 1's from the strong partition matrix $A = [a_1, a_2, \ldots, a_k]$, then so can Player 2. The resulting game position after these two moves is $B = [b_1, b_2, \ldots, b_k]$ where $b_i = a_i - 2$ for all $1 \leqslant i \leqslant \ell$ and $b_i = a_i$ for all $\ell < i \leqslant k$. Furthermore, $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_k > k$, $\operatorname{len}(B) = \operatorname{len}(A)$, $b_i < a_i$ for all $1 \leqslant i \leqslant \ell$, $b_i \leqslant a_i$ for all $\ell < i \leqslant k$, and $b_i \equiv a_i \pmod 2$ for all $1 \leqslant i \leqslant k$.*

*Proof.* Since Player 1 can remove a column with $\ell$ 1's, we have $a_\ell > a_{\ell+1}$. Also, since $a_\ell \equiv a_{\ell+1} \pmod 2$, we have $a_\ell \geqslant a_{\ell+1} + 2$. So, Player 2 can also remove a column with $\ell$ 1's. The resulting game positions is $B = [b_1, b_2, \ldots, b_k]$ where $b_i = a_i - 2$ for all $1 \leqslant i \leqslant \ell$ and $b_i = a_i$ for all $\ell < i \leqslant k$. Thus, $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_k > k$, $\operatorname{len}(B) = \operatorname{len}(A)$, $b_i < a_i$ for all $1 \leqslant i \leqslant \ell$, and $b_i \leqslant a_i$ for all $\ell < i \leqslant k$. Also, we have $b_i = a_i - 2 \equiv a_i \pmod 2$ for all $1 \leqslant i \leqslant \ell$, and $b_i \equiv a_i \pmod 2$ for all $\ell < i \leqslant k$. $\square$

### 5.1. $\mathcal{P}$-positions and $\mathcal{N}$-positions of $[a_1, a_2, a_3, a_4]$

We establish the conditions on $(a_1, a_2, a_3, a_4)$ that determine whether $[a_1, a_2, a_3, a_4]$ is a $\mathcal{P}$-position or an $\mathcal{N}$-position.

**Theorem 7.** *Let $a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 > 4$, and let $A = [a_1, a_2, a_3, a_4]$. Suppose either*

- $(a_1, a_2) \equiv (0, 0) \pmod 2$ *or*
- $(a_1, a_2, a_3) \equiv (0, 1, 0) \pmod 2$.

*Then,* $\mathrm{Gr}(A) = 0$.

*Proof.* The proof is by double induction on $\mathrm{len}(A)$ and $\mathrm{wt}(A)$.

`Case 1`. Assume both $a_1$ and $a_2$ are even. Suppose Player 1 removes a column with one 1. Since $a_1 \equiv a_2 \pmod 2$, by Lemma 1, Player 2 may also remove a column with one 1. The resulting game position is $[a_1 - 2, a_2, a_3, a_4]$ where $a_1 - 2$ and $a_2$ are even. By the inductive hypothesis, $\mathrm{Gr}([a_1 - 2, a_2, a_3, a_4]) = 0$.

Suppose Player 1 removes a column with $\ell$ 1's where $2 \leqslant \ell \leqslant 4$. We let Player 2 remove row 4. The resulting game position is $[b_1, b_2, b_3]$ where $b_i = a_i - 1$ for all $1 \leqslant i \leqslant \ell$ and $b_i = a_i$ for all $\ell < i \leqslant 3$. Since $b_1 = a_1 - 1$ and $b_2 = a_2 - 1$ are odd, we have $\mathrm{Gr}([b_1, b_2, b_3]) = 0$ by Theorem 6.

Suppose Player 1 removes row $\ell$. The resulting game position is $[b_1, b_2, b_3]$ where $b_i = a_i$ for all $1 \leqslant i < \ell$ and $b_i = a_{i+1}$ for all $\ell \leqslant i \leqslant 3$. We let Player 2 remove row 3 of $[b_1, b_2, b_3]$. The resulting game position is $[b_1, b_2]$ where $b_1 = a_2$ if $\ell = 1$ or $b_1 = a_1$ if $\ell > 1$. In either case, $b_1$ is even. By Theorem 2, $\mathrm{Gr}([b_1, b_2]) = 0$.

`Case 2`. Assume $a_1$ is even, $a_2$ is odd, and $a_3$ is even. First, suppose Player 1 removes a column with one 1. The new game position is $[a_1 - 1, a_2, a_3, a_4]$. Since $a_2 \geqslant a_3$ and $a_2 \not\equiv a_3 \pmod 2$, we have $a_2 > a_3$. We let Player 2 remove a column with two 1's. Next, suppose Player 1 removes a column with two 1's. The new game position is $[a_1 - 1, a_2 - 1, a_3, a_4]$. Since $a_1 - 1 \geqslant a_2 - 1$ and $a_1 - 1 \not\equiv a_2 - 1 \pmod 2$, we have $a_1 - 1 > a_2 - 1$. We let Player 2 remove a column with one 1. In either case, the resulting game position is $[a_1 - 2, a_2 - 1, a_3, a_4]$ where $a_1 - 2$ and $a_2 - 1$ are even. By the inductive hypothesis, $\mathrm{Gr}([a_1 - 2, a_2 - 1, a_3, a_4]) = 0$.

Suppose Player 1 removes a column with $\ell$ 1's where $3 \leqslant \ell \leqslant 4$. The new game position is $[b_1, b_2, b_3, b_4]$ where $b_i = a_i - 1$ for all $1 \leqslant i \leqslant \ell$ and $b_i = a_i$ for all $\ell < i \leqslant 4$. We let Player 2 remove row 2. The resulting game position is $[b_1, b_3, b_4]$ where $b_1 = a_1 - 1$ and $b_3 = a_3 - 1$ are odd, and $b_3 > 3$. By Theorem 6, $\mathrm{Gr}([b_1, b_3, b_4]) = 0$.

Suppose Player 1 removes row 1. The resulting game position is $[a_2, a_3, a_4]$. We let Player 2 remove row 1 of $[a_2, a_3, a_4]$. The resulting game position is $[a_3, a_4]$ where $a_3$ is even. By Theorem 2, $\mathrm{Gr}([a_3, a_4]) = 0$.

Suppose Player 1 removes row $\ell$ where $2 \leqslant \ell \leqslant 4$. The resulting game position is $[b_1, b_2, b_3]$ where $b_i = a_i$ for all $1 \leqslant i < \ell$ and $b_i = a_{i+1}$ for all $\ell \leqslant i \leqslant 3$. We let

Player 2 remove row 3 of $[b_1, b_2, b_3]$. The resulting game position is $[b_1, b_2]$ where $b_1 = a_1$ is even. By Theorem 2, $\mathrm{Gr}([b_1, b_2]) = 0$. $\square$

**Theorem 8.** *Let* $a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 > 4$*, and let* $A = [a_1, a_2, a_3, a_4]$*. Suppose either*

- $a_1 \equiv 1 \pmod 2$ *or*
- $(a_1, a_2, a_3) \equiv (0, 1, 1) \pmod 2$.

*Then,* $\mathrm{Gr}(A) \neq 0$.

*Proof.* `Case 1`. Assume both $a_1$ and $a_2$ are odd. We let Player 1 remove row 4. The resulting game position is $[a_1, a_2, a_3]$ where $a_1$ and $a_2$ are odd, and $a_3 > 4$. By Theorem 6, $\mathrm{Gr}([a_1, a_2, a_3]) = 0$.

`Case 2`. Assume $a_1$ is odd and $a_2$ is even. Since $a_1 \geqslant a_2$ and $a_1 \not\equiv a_2 \pmod 2$, $a_1 > a_2$. We let Player 1 remove a column with one 1. The resulting game position is $[a_1 - 1, a_2, a_3, a_4]$ where $a_1 - 1$ and $a_2$ are even. By Theorem 7, $\mathrm{Gr}([a_1 - 1, a_2, a_3, a_4]) = 0$.

`Case 3`. Assume $a_1$ is even, and both $a_2$ and $a_3$ are odd. We let Player 1 remove row 1. The resulting game position is $[a_2, a_3, a_4]$ where $a_2$ and and $a_3$ are odd, and $a_4 > 4$. By Theorem 6, $\mathrm{Gr}([a_2, a_3, a_4]) = 0$. $\square$

### 5.2. $\mathcal{P}$-positions and $\mathcal{N}$-positions of $[a_1, a_2, a_3, a_4, a_5]$

We now establish the conditions on $(a_1, a_2, a_3, a_4, a_5)$ that determine whether $[a_1, a_2, a_3, a_4, a_5]$ is a $\mathcal{P}$-position or an $\mathcal{N}$-position.

**Theorem 9.** *Let* $a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 \geqslant a_5 > 5$*, and let* $A = [a_1, a_2, a_3, a_4, a_5]$*. Suppose either*

- $(a_1, a_2) \equiv (1, 1) \pmod 2$ *or*
- $(a_1, a_2, a_3, a_4) \equiv (1, 0, 1, 1) \pmod 2$.

*Then,* $\mathrm{Gr}(A) = 0$.

*Proof.* The proof is by double induction on $\mathrm{len}(A)$ and $\mathrm{wt}(A)$.

`Case 1`. Assume both $a_1$ and $a_2$ are odd. Suppose Player 1 removes a column with one 1. Since $a_1 \equiv a_2 \pmod 2$, by Lemma 1, Player 2 may also remove a column with one 1. The resulting game position is $[a_1 - 2, a_2, a_3, a_4, a_5]$ where $a_1 - 2$ and $a_2$ are odd. By the inductive hypothesis, $\mathrm{Gr}([a_1 - 2, a_2, a_3, a_4, a_5]) = 0$.

Suppose Player 1 removes a column with $\ell$ 1's where $2 \leqslant \ell \leqslant 5$. The resulting game position is $[b_1, b_2, b_3, b_4, b_5]$ where $b_i = a_i - 1$ for all $1 \leqslant i \leqslant \ell$ and $b_i = a_i$ for all $\ell < i \leqslant 5$. We let Player 2 remove row 5 of $[b_1, b_2, b_3, b_4, b_5]$. The resulting game position is $[b_1, b_2, b_3, b_4]$ where $b_1 = a_1 - 1$ and $b_2 = a_2 - 1$ are even, and $b_1 \geqslant b_2 \geqslant b_3 \geqslant b_4 > 4$. By Theorem 7, $\mathrm{Gr}([b_1, b_2, b_3, b_4]) = 0$.

Suppose Player 1 removes row $\ell$ where $1 \leqslant \ell \leqslant 2$. The resulting game position is $[b_1, b_2, b_3, b_4]$ where $b_i = a_i$ for all $1 \leqslant i < \ell$ and $b_i = a_{i+1}$ for all $\ell \leqslant i \leqslant 4$. We observe that $b_1 = a_2$ if $\ell = 1$, $b_1 = a_1$ if $\ell = 2$, and $b_2 = a_3$. We consider the cases $b_2$ is odd or even separately. If $b_2 = a_3$ is odd, we let Player 2 remove row 4 of $[b_1, b_2, b_3, b_4]$. The resulting game position is $[b_1, b_2, b_3]$ where both $b_1$ and $b_2$ are odd. By Theorem 6, $\mathrm{Gr}([b_1, b_2, b_3]) = 0$. If $b_2 = a_3$ is even, we have $b_1 > b_2$ since $b_1 \geqslant b_2$ and $b_1 \not\equiv b_2 \pmod 2$. We let Player 2 remove a column with one 1. The resulting game position is $[b_1 - 1, b_2, b_3, b_4]$ where both $b_1 - 1$ and $b_2$ are even. By Theorem 7, $\mathrm{Gr}([b_1 - 1, b_2, b_3, b_4]) = 0$.

Suppose Player 1 removes row $\ell$ where $3 \leqslant \ell \leqslant 5$. The resulting game position is $[b_1, b_2, b_3, b_4]$ where $b_i = a_i$ for all $1 \leqslant i < \ell$ and $b_i = a_{i+1}$ for all $\ell \leqslant i \leqslant 4$. We let Player 2 remove row 4. The resulting game position is $[b_1, b_2, b_3]$ where $b_1 = a_1$ and $b_2 = a_2$ are odd. By Theorem 6, $\mathrm{Gr}([b_1, b_2, b_3]) = 0$.

`Case 2`. Assume $a_1$, $a_3$, and $a_4$ are odd, and $a_2$ is even. Suppose Player 1 removes a column with one 1. We let Player 2 remove row 5. The resulting position is $[a_1 - 1, a_2, a_3, a_4]$ where $a_1 - 1$ and $a_2$ are even, and $a_1 - 1 \geqslant a_2 \geqslant a_3 \geqslant a_4 > 4$. By Theorem 7, $\mathrm{Gr}([a_1 - 1, a_2, a_3, a_4]) = 0$.

Suppose Player 1 removes a column with two 1's The resulting game position is $[b_1, b_2, b_3, b_4, b_5]$ where $b_i = a_i - 1$ for all $1 \leqslant i \leqslant 2$ and $b_i = a_i$ for all $3 \leqslant i \leqslant 5$. Since $b_1 \geqslant b_2$ and $b_1 \not\equiv b_2 \pmod 2$, $b_1 > b_2$. So, we let Player 2 remove a column with one 1. The resulting position is $[b_1 - 1, b_2, b_3, b_4, b_5]$ where $b_1 - 1 = a_1 - 2$ and $b_2 = a_2 - 1$ are odd, and $b_1 - 1 \geqslant b_2 \geqslant b_3 \geqslant b_4 \geqslant b_5 > 5$. By the inductive hypothesis, $\mathrm{Gr}([b_1 - 1, b_2, b_3, b_4, b_5]) = 0$.

Suppose Player 1 removes a column with $\ell$ 1's where $3 \leqslant \ell \leqslant 5$. The resulting game position is $[b_1, b_2, b_3, b_4, b_5]$ where $b_i = a_i - 1$ for all $1 \leqslant i \leqslant \ell$ and $b_i = a_i$ for all $\ell < i \leqslant 5$. We let Player 2 remove row 5 of $[b_1, b_2, b_3, b_4, b_5]$. The resulting game position is $[b_1, b_2, b_3, b_4]$. We observe that $b_1 = a_1 - 1$ is odd, $b_2 = a_2 - 1$ is even, $b_3 = a_3 - 1$ are odd, and $b_1 \geqslant b_2 \geqslant b_3 \geqslant b_4 > 4$. By Theorem 7, $\mathrm{Gr}([b_1, b_2, b_3, b_4]) = 0$.

Suppose Player 1 removes row $\ell$ where $1 \leqslant \ell \leqslant 2$. Let $\ell' \in \{1, 2\} \setminus \{\ell\}$. We let Player 2 remove row $\ell'$. The resulting game position is $[a_3, a_4, a_5]$ where $a_3$ and $a_4$ are odd. By Theorem 6, $\mathrm{Gr}([a_3, a_4, a_5]) = 0$.

Suppose Player 1 removes row $\ell$ where $3 \leqslant \ell \leqslant 5$. The resulting game position is $[b_1, b_2, b_3, b_4]$ where $b_i = a_i$ for all $1 \leqslant i < \ell$ and $b_i = a_{i+1}$ for all $\ell \leqslant i \leqslant 4$. We let Player 2 remove row 2 of $[b_1, b_2, b_3, b_4]$ The resulting game position is $[b_1, b_3, b_4]$.

Since $b_3 = a_4$ if $\ell = 3$ or $b_3 = a_3$ if $\ell \in \{4, 5\}$, $b_3$ is odd. Thus, $b_1$ and $b_3$ are odd. By Theorem 6, $\mathrm{Gr}([b_1, b_3, b_4]) = 0$. □

**Theorem 10.** *Let* $a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 \geqslant a_5 > 5$*, and let* $A = [a_1, a_2, a_3, a_4, a_5]$*. Suppose either*

- $a_1 \equiv 0 \pmod 2$ *or*
- $(a_1, a_2, a_3, a_4) \equiv (1, 0, x, y) \pmod 2$ *where* $(x, y) \not\equiv (1, 1) \pmod 2$*.*

*Then,* $\mathrm{Gr}(A) \neq 0$.

*Proof.* `Case 1`. Assume both $a_1$ and $a_2$ are even. We let Player 1 remove row 5. The resulting game position is $[a_1, a_2, a_3, a_4]$ where $a_1$ and $a_2$ are odd, and $a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 > 5$. By Theorem 7, $\mathrm{Gr}([a_1, a_2, a_3, a_4]) = 0$.

`Case 2`. Assume $a_1$ is even and $a_2$ is odd. Since $a_1 \geqslant a_2$ and $a_1 \not\equiv a_2 \pmod 2$, $a_1 > a_2$. We let Player 1 remove a column with one 1. The resulting game position is $[a_1 - 1, a_2, a_3, a_4, a_5]$ where $a_1 - 1$ and $a_2$ are odd. By Theorem 9, $\mathrm{Gr}([a_1 - 1, a_2, a_3, a_4, a_5]) = 0$.

`Case 3`. Assume $(a_1, a_2, a_3, a_4) \equiv (1, 0, x, y) \pmod 2$ and $(x, y) \not\equiv (1, 1) \pmod 2$. We let Player 1 remove row 1. The resulting game position is $[a_2, a_3, a_4, a_5]$ where $a_2 \geqslant a_3 \geqslant a_4 \geqslant a_5 > 4$. If $x = 0$, then $(a_2, a_3) \equiv (0, 0) \pmod 2$. Also, if $(x, y) = (1, 0)$, then $(a_2, a_3, a_4) \equiv (0, 1, 0) \pmod 2$. By Theorem 7, $\mathrm{Gr}([a_2, a_3, a_4, a_5]) = 0$. □

### 5.3. Some Preliminary Results on $\mathcal{P}$-positions and $\mathcal{N}$-positions

We show that when $a_i \equiv k \pmod 2$ for all $i \in \{1, 2\}$, the game position $[a_1, a_2, \ldots, a_k]$, where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k > k$, is a $\mathcal{P}$-position.

**Theorem 11.** *Let* $k \geqslant 4$*. Suppose* $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k > k$*,* $a_i \equiv k \pmod 2$ *for all* $i \in \{1, 2\}$*, and let* $A = [a_1, a_2, \ldots, a_k]$*. Then,* $\mathrm{Gr}(A) = 0$*.*

*Proof.* The proof is by double induction on $\mathrm{len}(A)$ and $\mathrm{wt}(A)$ of $A$. Let

$$\begin{aligned}\mathcal{A} = \big\{[a_1, a_2, \ldots, a_k] : k \geqslant 4,\ a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k > k \text{ and}\\ a_i \equiv k \pmod 2 \text{ for all } i \in \{1, 2\}\big\}\end{aligned}$$

and consider the partial order on $\mathcal{A}$ given by

$$\begin{aligned}[b_1, b_2, \ldots, b_r] \preceq [c_1, c_2, \ldots, c_s] \quad \text{if and only if}\\ r \leqslant s \quad \text{and} \quad b_i \leqslant c_i \text{ for all } 1 \leqslant i \leqslant r.\end{aligned}$$

By Theorem 7, for all $A \in \mathcal{A}$ such that $\operatorname{len}(A) = 4$, $\operatorname{Gr}(A) = 0$. By Theorem 9, for all $A \in \mathcal{A}$ such that $\operatorname{len}(A) = 5$, $\operatorname{Gr}(A) = 0$. Thus, the theorem holds for $k = 4$ and $k = 5$. This establishes the base case.

Let $k \geqslant 6$ and $A = [a_1, a_2, \ldots, a_k] \in \mathcal{A}$. Suppose $\operatorname{Gr}(B) = 0$ for all $B \in \mathcal{A}$ such that $B \prec A$. We want to show that $\operatorname{Gr}(A) = 0$.

Suppose Player 1 removes a column from $A = [a_1, a_2, \ldots, a_k]$ that contains $\ell$ 1's. We consider the cases $\ell = 1$ and $\ell > 1$ separately. Suppose $\ell = 1$. Since $a_1 \equiv a_2 \pmod 2$, by Lemma 1, we have $a_1 \geqslant a_2 + 2$. So, Player 2 can also remove a column with one 1. The resulting game position is $B = [b_1, b_2, \ldots, b_k]$ where $b_1 = a_1 - 2$, and $b_i = a_i$ for all $2 \leqslant i \leqslant k$. By Lemma 1, $B \in \mathcal{A}$ and $B \prec A$. By the inductive hypothesis, $\operatorname{Gr}(B) = 0$.

Suppose Player 1 removes a column from $A = [a_1, a_2, \ldots, a_k]$ that contains $\ell$ 1's where $2 \leqslant \ell \leqslant k$. We let Player 2 remove row $k$. Let $\ell' = \ell$ if $\ell < k$, and $\ell' = k - 1$ if $\ell = k$. The resulting game position is $B = [b_1, b_2, \ldots, b_{k-1}]$ where $b_i = a_i - 1$ for all $1 \leqslant i \leqslant \ell'$, and $b_i = a_i$ for all $\ell' < i \leqslant k - 1$. We observe that $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_{k-1} > k - 1$ and $b_i = a_i - 1 \equiv k - 1 \pmod 2$ for all $i \in \{1, 2\}$. Thus, $B \in \mathcal{A}$. Since $\operatorname{len}(B) < \operatorname{len}(A)$ and $b_i \leqslant a_i$ for all $1 \leqslant i \leqslant k - 1$, $B \prec A$. By the inductive hypothesis, $\operatorname{Gr}(B) = 0$.

Suppose Player 1 removes row $\ell$ from $A = [a_1, a_2, \ldots, a_k]$. The resulting game position is $B = [b_1, b_2, \ldots, b_{k-1}]$ where $b_i = a_i$ for all $1 \leqslant i < \ell$ and $b_i = a_{i+1}$ for all $\ell \leqslant i \leqslant k - 1$. We consider the cases $\ell \leqslant 2$ and $\ell \geqslant 3$ separately. First, suppose $\ell \geqslant 3$. We let Player 2 remove row $k - 1$ of $B$. The resulting game position is $C = [c_1, c_2, \ldots, c_{k-2}]$ where $c_i = a_i$ for all $1 \leqslant i < \ell$ and $c_i = a_{i+1}$ for all $\ell \leqslant i \leqslant k - 2$. Then $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_{k-2} > k - 2$, and $c_i = a_i \equiv k - 2 \pmod 2$ for all $i \in \{1, 2\}$. Thus, $C \in \mathcal{A}$. Since $\operatorname{len}(C) < \operatorname{len}(A)$ and $c_i \leqslant a_i$ for all $1 \leqslant i \leqslant k - 2$, $C \prec A$. By the inductive hypothesis, $\operatorname{Gr}(C) = 0$.

Next, suppose $\ell \leqslant 2$. Let $\ell' \in \{1, 2\} \setminus \{\ell\}$. We further consider the cases $a_3 \equiv k \pmod 2$ and $a_3 \equiv k + 1 \pmod 2$ separately. On the one hand, if $a_3 \equiv k \pmod 2$, we let Player 2 remove row $k - 1$ of $B$. The resulting game position is $C = [c_1, c_2, \ldots, c_{k-2}]$ where $c_1 = a_{\ell'}$ and $c_i = a_{i+1}$ for all $2 \leqslant i \leqslant k - 2$. Since $c_i \equiv k - 2 \pmod 2$ for all $i \in \{1, 2\}$ and $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_{k-2} > k - 2$, $C \in \mathcal{A}$. Since $\operatorname{len}(C) < \operatorname{len}(A)$ and $c_i \leqslant a_i$ for all $1 \leqslant i \leqslant k - 2$, $C \prec A$. On the other hand, if $a_3 \equiv k + 1 \pmod 2$, then $b_1 \not\equiv b_2 \pmod 2$. Since $b_1 \geqslant b_2$, $b_1 \geqslant b_2 + 1$. So, we let Player 2 remove a column with one 1. The resulting game position is $C = [c_1, c_2, \ldots, c_{k-1}]$ where $c_1 = a_{\ell'} - 1$ and $c_i = a_{i+1}$ for all $2 \leqslant i \leqslant k - 1$. We observe that $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_{k-1} > k - 1$, $c_1 = a_{\ell'} - 1 \equiv k - 1 \pmod 2$, and $c_2 = a_3 \equiv k - 1 \pmod 2$. Thus, $C \in \mathcal{A}$. Since $\operatorname{len}(C) < \operatorname{len}(A)$ and $c_i \leqslant a_i$ for all $1 \leqslant i \leqslant k - 1$, $C \prec A$. In either case, we have $\operatorname{Gr}(C) = 0$ by the inductive hypothesis. This completes the proof. □

The next result provides a condition on $(a_1, a_2, \ldots, a_k)$ to ensure that the game position $A = [a_1, a_2, \ldots, a_k]$ is an $\mathcal{N}$-position.

**Theorem 12.** *Let $k \geqslant 5$. Suppose $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k > k$, $a_1 \equiv k+1 \pmod 2$, and let $A = [a_1, a_2, \ldots, a_k]$. Then,* $\mathrm{Gr}(A) \neq 0$.

*Proof.* We consider the cases $a_2 \equiv k \pmod 2$ and $a_2 \equiv k+1 \pmod 2$ separately. First, suppose $a_2 \equiv k \pmod 2$. Since $a_1 \geqslant a_2$ and $a_1 \not\equiv a_2 \pmod 2$, $a_1 \geqslant a_2 + 1$. So, we let Player 1 remove a column with one 1. The resulting game position is $B = [b_1, b_2, \ldots, b_k]$ where $b_1 = a_1 - 1$ and $b_i = a_i$ for all $2 \leqslant i \leqslant k$. Then $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_k > k$, $b_1 = a_1 - 1 \equiv k \pmod 2$, and $b_2 = a_2 \equiv k \pmod 2$. By Theorem 11, $\mathrm{Gr}(B) = 0$. Thus, $\mathrm{Gr}(A) \neq 0$.

Next, suppose $a_2 \equiv k+1 \pmod 2$. We let Player 1 remove row $k$. The resulting game position is $B = [b_1, b_2, \ldots, b_{k-1}]$ where $b_i = a_i$ for all $1 \leqslant i \leqslant k-1$. Then $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_{k-1} > k-1$, and $b_i = a_i \equiv k-1 \pmod 2$ for all $i \in \{1, 2\}$. By Theorem 11, $\mathrm{Gr}(B) = 0$. Thus, $\mathrm{Gr}(A) \neq 0$. □

### 5.4. The Main Result

We introduce the following notation in order to state when $[a_1, a_2, \ldots, a_k]$ is a $\mathcal{P}$-position or an $\mathcal{N}$-position.

**Notation 1.** Suppose $(b_1, b_2, \ldots, b_k)$ is a sequence of 0's and 1's. We let $(b_1, b_2, \ldots, b_k)^n$ represent the concatenation of $n$ copies of $(b_1, b_2, \ldots, b_k)$. For example, $(1,0)^3 = (1,0,1,0,1,0)$.

We state the conditions that are needed on the strong partition sequence $S = (a_1, a_2, \ldots, a_k)$ in order to determine whether the strong partition matrix $\mathrm{mat}(S)$ is a $\mathcal{P}$-position or an $\mathcal{N}$-position. Theorems 13 and 14 provide the conditions to ensure that $\mathrm{mat}(S)$ is a $\mathcal{P}$-position or an $\mathcal{N}$-position, respectively, when $k$ is even, and Theorems 15 and 16 provide the conditions to ensure that $\mathrm{mat}(S)$ is a $\mathcal{P}$-position or an $\mathcal{N}$-position, respectively, when $k$ is odd.

**Theorem 13.** *Let $k = 2p \geqslant 4$ be even, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k > k$, and $A = [a_1, a_2, \ldots, a_k]$. Then,* $\mathrm{Gr}(A) = 0$ *if and only if either*

- $(a_i : 1 \leqslant i \leqslant 2q+2) \equiv ((0,1)^q, 0, 0) \pmod 2$ *for some integer* $0 \leqslant q \leqslant p-3$ *or*
- $(a_i : 1 \leqslant i \leqslant k-1) \equiv ((0,1)^{p-2}, 0, x, y) \pmod 2$ *and* $(x, y) \not\equiv (1,1) \pmod 2$.

**Theorem 14.** *Let $k = 2p \geqslant 4$ be even, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k > k$, and $A = [a_1, a_2, \ldots, a_k]$. Then,* $\mathrm{Gr}(A) \neq 0$ *if and only if* $(a_i : 1 \leqslant i \leqslant 2q+1) \equiv ((0,1)^q, 1) \pmod 2$ *for some integer* $0 \leqslant q \leqslant p-1$.

**Theorem 15.** *Let $k = 2p+1$ be odd, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k > k$, and $A = [a_1, a_2, \ldots, a_k]$. Then,* $\mathrm{Gr}(A) = 0$ *if and only if* $(a_i : 1 \leqslant i \leqslant 2q+2) \equiv ((1,0)^q, 1, 1) \pmod 2$ *for some integer* $0 \leqslant q \leqslant p-1$.

**Theorem 16.** *Let $k = 2p + 1$ be odd, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k > k$, and $A = [a_1, a_2, \ldots, a_k]$. Then,* $\mathrm{Gr}(A) \neq 0$ *if and only if either*

- $(a_i : 1 \leqslant i \leqslant 2q+1) \equiv ((1,0)^q, 0) \pmod 2$ *for some integer* $0 \leqslant q \leqslant p-2$ *or*
- $(a_i : 1 \leqslant i \leqslant k-1) \equiv ((1,0)^{p-1}, x, y) \pmod 2$ *and* $(x, y) \not\equiv (1,1) \pmod 2$.

In order to establish Theorems 13, 14, 15, and 16, we first demonstrate that Theorem 13 is equivalent to Theorem 14 when $k$ is even, and Theorem 15 is equivalent to Theorem 16 when $k$ is odd. Next, we establish the necessity of Theorems 13, 14, 15, and 16. These arguments establish the claims made in Theorems 13, 14, 15, and 16. Then, we consider the special case $k = 7$ as an example.

**Lemma 2.** *When $k \geqslant 4$ is even, Theorem 13 is equivalent to Theorem 14. Similarly, when $k \geqslant 5$ is odd, Theorem 15 is equivalent to Theorem 16.*

*Proof.* We demonstrate that when $k \geqslant 4$ is even, Theorem 13 is equivalent to Theorem 14. The proof that Theorem 15 is equivalent to Theorem 16 when $k$ is odd is similar, and we leave the proof of that case to the reader.

Let $k = 2p \geqslant 4$ be even. The proof will be by backwards induction on $0 \leqslant q \leqslant p-2$. We show that, for each integer $q$ such that $0 \leqslant q \leqslant p-2$, among the $2^{2p-2q-1}$ choices of parity of $a_i$ where $2q+1 \leqslant i \leqslant 2p-1$, there are

$$(2^2 - 1) + 2^3 + 2^5 + \cdots + 2^{2p-2q-2} = \tfrac{1}{3}(2^{2p-2q-1} + 1)$$

choices of parity for which $[a_1, a_2, \ldots, a_k]$ is a $\mathcal{P}$-position and

$$1 + 2^2 + 2^4 + \cdots + 2^{2p-2q-2} = \tfrac{1}{3}(2^{2p-2q} - 1)$$

complementary choices of parity for which $[a_1, a_2, \ldots, a_k]$ is an $\mathcal{N}$-position.

We let $S = (a_i : 1 \leqslant i \leqslant k)$, $A = \mathrm{mat}(S)$, and $q = p-2$. By Theorem 13, if $(a_i : 1 \leqslant i \leqslant 2p-1) \equiv \big((0,1)^{p-2}, 0, x, y\big) \pmod 2$ and $(x,y) \not\equiv (1,1) \pmod 2$, then $\mathrm{Gr}(A) = 0$. By Theorem 14, if $(a_i : 1 \leqslant i \leqslant 2p-1) \equiv \big((0,1)^{p-2}, 1\big) \pmod 2$ or $(a_i : 1 \leqslant i \leqslant 2p-3) \equiv \big((0,1)^{p-3}, 1\big) \pmod 2$, then $\mathrm{Gr}(A) \neq 0$. Thus, among the $2^3 = 2^{2p-2q-1}$ choices of parity of $a_i$ where $2q+1 \leqslant i \leqslant 2p-1$, $2^2 - 1 = \frac{1}{3}(2^3+1)$ choices of parity yield $\mathrm{Gr}(A) = 0$ and $2^0 + 2^2 = \frac{1}{3}(2^4 - 1)$ complementary choices of parity yield $\mathrm{Gr}(A) \neq 0$. This establishes the base case.

Suppose that, for some integer $1 \leqslant q \leqslant p-2$, among the $2^{2p-2q-1}$ choices of parity of $a_i$ where $2q+1 \leqslant i \leqslant 2p-1$,

$$(2^2 - 1) + 2^3 + 2^5 + \cdots + 2^{2p-2q-3} = \tfrac{1}{3}(2^{2p-2q-1} + 1)$$

choices of parity yield $\mathrm{Gr}(A) = 0$ and

$$2^0 + 2^2 + 2^4 + \cdots + 2^{2p-2q-2} = \tfrac{1}{3}(2^{2p-2q} - 1)$$

complementary choices of parity yield $\mathrm{Gr}(A) \neq 0$. By Theorem 13, if $(a_i : 1 \leqslant i \leqslant 2q) \equiv \big((0,1)^{q-1}, 0, 0\big) \pmod 2$, then $\mathrm{Gr}(A) = 0$. By Theorem 14, if $(a_i : 1 \leqslant i \leqslant 2q-1) \equiv \big((0,1)^{q-1}, 1\big) \pmod 2$, then $\mathrm{Gr}(A) \neq 0$. Thus, among the $2^{2p-2(q-1)-1}$ choices of parity of $a_i$ where $2(q-1)+1 \leqslant i \leqslant 2p-1$,

$$(2^2 - 1) + 2^3 + 2^5 + \cdots + 2^{2p-2(q-1)-3} = \tfrac{1}{3}(2^{2p-2(q-1)-1} + 1)$$

choices of parity yield $\mathrm{Gr}(A) = 0$ and

$$2^0 + 2^2 + 2^4 + \cdots + 2^{2p-2(q-1)-2} = \tfrac{1}{3}(2^{2p-2(q-1)} - 1)$$

complementary choices of parity yield $\mathrm{Gr}(A) \neq 0$. This completes the proof. $\square$

**Remark 5.** We let $\langle x \rangle$ denote the *nearest integer function* defined by $\langle x \rangle \in \mathbb{Z}$ such that $x - \frac{1}{2} < \langle x \rangle \leqslant x + \frac{1}{2}$. Let $A = [a_1, a_2, \ldots, a_k]$ be a strong partition matrix. For $k \geqslant 4$, among the $2^{k-1}$ choices of parity of $a_i$ where $1 \leqslant i \leqslant k-1$, there are $\langle \frac{1}{3}\, 2^{k-1} \rangle$ choices of parity that yield $\mathrm{Gr}(A) = 0$ and $\langle \frac{1}{3}\, 2^{k} \rangle$ complementary choices of parity that yield $\mathrm{Gr}(A) \neq 0$.

We provide the proofs of Theorems 13, 14, 15, and 16.

*Proof of necessity of Theorems 13 and 15.* The proof is by double induction on the length of $A$ and the game weight of $A$. Let

$$\begin{aligned}
&\mathcal{E} = \big\{[a_1, a_2, \ldots, a_k] : k = 2p \geqslant 4 \text{ is even},\ a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k > k,\ \text{and either}\\
&(a_i : 1 \leqslant i \leqslant 2q+2) \equiv ((0,1)^q, 0, 0) \pmod 2 \text{ for some integer } 0 \leqslant q \leqslant p-3,\ \text{or}\\
&(a_i : 1 \leqslant i \leqslant k-1) \equiv ((0,1)^{p-2}, 0, x, y) \pmod 2 \text{ and } (x,y) \not\equiv (1,1) \pmod 2\big\},\\
&\mathcal{O} = \big\{[a_1, a_2, \ldots, a_k] : k \geqslant 5 \text{ is odd},\ a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k > k,\ \text{and}\\
&(a_i : 1 \leqslant i \leqslant 2q+2) \equiv ((1,0)^q, 1, 1) \pmod 2 \text{ for some integer } 0 \leqslant q \leqslant p-1\big\},\\
&\mathcal{C} = \mathcal{E} \cup \mathcal{O},
\end{aligned}$$

and consider the partial order on $\mathcal{C}$ given by

$$\begin{aligned}
[b_1, b_2, \ldots, b_r] \preceq [c_1, c_2, \ldots, c_s] &\quad \text{if and only if}\\
r \leqslant s \ \text{ and } \ b_i \leqslant c_i &\text{ for all } 1 \leqslant i \leqslant r.
\end{aligned}$$

By Theorem 7, for all $A \in \mathcal{C}$ such that $\texttt{len}(A) = 4$, $\mathrm{Gr}(A) = 0$. By Theorem 9, for all $A \in \mathcal{C}$ such that $\texttt{len}(A) = 5$, $\mathrm{Gr}(A) = 0$. Thus, the theorem holds for $k = 4$ and $k = 5$. This establishes the base case.

Let $k \geqslant 6$ and $A = [a_1, a_2, \ldots, a_k] \in \mathcal{C}$. Suppose for all $B \in \mathcal{C}$ such that $B \prec A$, $\mathrm{Gr}(B) = 0$. We want to show that $\mathrm{Gr}(A) = 0$. We consider the cases $k$ is even and $k$ is odd separately.

Suppose $k$ is even. First, suppose $(a_i : 1 \leqslant i \leqslant 2q+2) \equiv ((0,1)^q, 0, 0) \pmod 2$ for some integer $0 \leqslant q \leqslant p-3$. If $q = 0$, then $(a_1, a_2) \equiv (0,0) \pmod 2$. By Theorem 11, $\mathrm{Gr}(A) = 0$. So, we may assume $q \geqslant 1$.

Suppose Player 1 removes a column from $A = [a_1, a_2, \ldots, a_k]$ that contains one 1. We let Player 2 remove row $k$ of $A$. The resulting game position is $B = [b_1, b_2, \ldots, b_{k-1}]$ where $b_1 = a_1 - 1$ and $b_i = a_i$ for all $1 < i \leqslant k-1$. We observe that $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_{k-1} > k-1$, $\mathrm{len}(B) = k-1$ is odd, and $(b_1, b_2) \equiv (1,1) \pmod 2$. By Theorem 11, $\mathrm{Gr}(B) = 0$. Thus, $\mathrm{Gr}(A) = 0$.

Suppose Player 1 removes a column from $A = [a_1, a_2, \ldots, a_k]$ that contains two 1's. The resulting game position is $B = [b_1, b_2, \ldots, b_{k-1}]$ where $b_i = a_i - 1$ for all $1 \leqslant i \leqslant 2$, and $b_i = a_i$ for all $2 < i \leqslant k$. Since $b_1$ is odd, $b_2$ is even, and $b_1 \geqslant b_2$, we have $b_1 \geqslant b_2 + 1$. So, we let Player 2 remove a column that contains one 1. The resulting game position is $C = [c_1, c_2 \ldots, c_k]$ where $c_1 = a_1 - 2$, $c_2 = a_2 - 1$, and $c_i = a_i$ for all $3 \leqslant i \leqslant k$. We observe that $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_k > k$, $\mathrm{len}(C) = k$ is even, and $(c_1, c_2) \equiv (0,0) \pmod 2$. By Theorem 11, $\mathrm{Gr}(C) = 0$. Thus, $\mathrm{Gr}(A) = 0$.

Suppose Player 1 removes a column from $A = [a_1, a_2, \ldots, a_k]$ that contains $\ell$ 1's such that $3 \leqslant \ell \leqslant k$. We let Player 2 remove row 2 of $A$. The resulting game position is $B = [b_1, b_2, \ldots, b_{k-1}]$ where $b_1 = a_1 - 1$, $b_i = a_{i+1} - 1$ for all $2 \leqslant i < \ell$, and $b_i = a_{i+1}$ for all $\ell \leqslant i \leqslant k-1$. We observe that $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_{k-1} > k-1$, $\mathrm{len}(B) = k-1$ is odd, and $(b_1, b_2) \equiv (1,1) \pmod 2$. By Theorem 11, $\mathrm{Gr}(B) = 0$. Thus, $\mathrm{Gr}(A) = 0$.

If Player 1 removes row 1 of $A = [a_1, a_2, \ldots, a_k]$, we let Player 2 remove row 2 of $A$. If Player 1 removes row 2 of $A$, we let Player 2 remove row 1 of $A$. Then the resulting game position is $B = [b_1, b_2, \ldots, b_{k-2}]$ where $b_i = a_{i+2}$ for all $1 \leqslant i \leqslant k-2$. Since $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_{k-2} > k-2$, $\mathrm{len}(B) = k-2$ is even, and $(b_i : 1 \leqslant i \leqslant 2q) \equiv ((0,1)^{q-1}, 0, 0) \pmod 2$, we have $B \in \mathcal{C}$. Since $\mathrm{len}(B) < \mathrm{len}(A)$ and $b_i \leqslant a_i$ for all $1 \leqslant i \leqslant k-2$, we have $B \prec A$. By the inductive hypothesis, $\mathrm{Gr}(B) = 0$. Thus, $\mathrm{Gr}(A) = 0$.

Suppose Player 1 removes row $\ell$ of $A = [a_1, a_2, \ldots, a_k]$ where $3 \leqslant \ell \leqslant k$. The resulting position is $B = [b_1, b_2, \ldots, b_{k-1}]$ where $b_i = a_i$ for all $1 \leqslant i < \ell$ and $b_i = a_{i+1}$ for all $\ell \leqslant i \leqslant k-1$. Since $b_1$ is even, $b_2$ is odd, and $b_1 \geqslant b_2$, we have $b_1 \geqslant b_2 + 1$. So, we let Player 2 remove a column with one 1. The resulting position is $C = [c_1, c_2, \ldots, c_{k-1}]$ where $c_1 = a_1 - 1$, $c_i = a_i$ for all $2 \leqslant i < \ell$ and $c_i = a_{i+1}$ for all $\ell \leqslant i \leqslant k-1$. We observe that $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_{k-1} > k-1$, $\mathrm{len}(C) = k-1$ is odd, and $(c_1, c_2) \equiv ((1,1) \pmod 2$. By Theorem 11, $\mathrm{Gr}(C) = 0$. Thus, $\mathrm{Gr}(A) = 0$.

Suppose $k$ is even and $(a_i : 1 \leqslant i \leqslant k-1) \equiv ((0,1)^{p-2}, 0, x, y) \pmod 2$ where $(x, y) \not\equiv (1,1) \pmod 2$. Suppose Player 1 removes a column with $\ell$ 1's of $A = [a_1, a_2, \ldots, a_k]$ where $1 \leqslant \ell \leqslant k$. An argument similar to that in paragraphs 4, 5, and 6 of this proof shows that $\mathrm{Gr}(A) = 0$. We leave the details to the reader.

Suppose Player 1 removes row $\ell$ of $A = [a_1, a_2, \ldots, a_k]$ where $1 \leqslant \ell \leqslant k$. An argument as above shows that $\mathrm{Gr}(A) = 0$. We leave the details to the reader.

Suppose $k$ is odd and $(a_i : 1 \leqslant i \leqslant 2q+2) \equiv ((1,0)^q, 1, 1) \pmod 2$ for some integer $0 \leqslant q \leqslant p-1$. The proof is similar to the proof that $\mathrm{Gr}(A) = 0$ when $k$ is even and $(a_i : 1 \leqslant i \leqslant 2q+2) \equiv ((0,1)^q, 0, 0) \pmod 2$ for some integer $0 \leqslant q \leqslant p-3$. We leave the details to the reader. $\square$

*Proof of necessity of Theorem 14.* Let $(a_i : 1 \leqslant i \leqslant 2q+1) \equiv ((0,1)^q, 1) \pmod 2$ for some integer $0 \leqslant q \leqslant p-1$. First, suppose $q = 0$. Then, by Theorem 12, $\mathrm{Gr}(A) \neq 0$. Next, suppose $q \geqslant 1$. We let Player 1 remove row 1. The resulting game position is $B = [b_1, b_2, \ldots, b_{k-1}]$ where $b_i = a_{i+1}$ for all $1 \leqslant i \leqslant k-1$. Then $(b_i : 1 \leqslant i \leqslant 2q) \equiv ((1,0)^{q-1}, 1, 1) \pmod 2$. By Theorem 15, $\mathrm{Gr}(B) = 0$. Thus, $\mathrm{Gr}(A) \neq 0$. $\square$

*Proof of necessity of Theorem 16.* Let $(a_i : 1 \leqslant i \leqslant 2q+1) \equiv ((1,0)^q, 0) \pmod 2$ for some integer $0 \leqslant q \leqslant p-2$. First, suppose $q = 0$. Then, by Theorem 12, $\mathrm{Gr}(A) \neq 0$. Next, suppose $q \geqslant 1$. We let Player 1 remove row 1. The resulting game position is $B = [b_1, b_2, \ldots, b_{k-1}]$ where $b_i = a_{i+1}$ for all $1 \leqslant i \leqslant k-1$. Then $(b_i : 1 \leqslant i \leqslant 2q) \equiv ((0,1)^{q-1}, 0, 0) \pmod 2$. By Theorem 13, $\mathrm{Gr}(B) = 0$. Thus, $\mathrm{Gr}(A) \neq 0$.

Let $(a_i : 1 \leqslant i \leqslant k-1) \equiv ((1,0)^{p-1}, x, y) \pmod 2$ and $(x, y) \not\equiv (1,1) \pmod 2$. We let Player 1 remove row 1. The resulting game position is $B = [b_1, b_2, \ldots, b_{k-1}]$ where $b_i = a_{i+1}$ for all $1 \leqslant i \leqslant k-1$. Then, $(b_i : 1 \leqslant i \leqslant k-2) \equiv ((0,1)^{p-2}, 0, x, y) \pmod 2$ and $(x, y) \not\equiv (1,1) \pmod 2$. By Theorem 13, $\mathrm{Gr}(B) = 0$. Thus, $\mathrm{Gr}(A) \neq 0$. $\square$

**Example 1.** We consider the special case $k = 7$. Then, Theorems 15 and 16 specialize to the following two propositions.

**Proposition 2.** *Let* $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_7 > 7$ *and* $A = [a_1, a_2, \ldots, a_7]$. *Then,* $\mathrm{Gr}(A) = 0$ *if and only if either*

- $(a_1, a_2) \equiv (1,1) \pmod 2$,
- $(a_1, a_2, a_3, a_4) \equiv (1,0,1,1) \pmod 2$, *or*
- $(a_1, a_2, a_3, a_4, a_5, a_6) \equiv (1,0,1,0,1,1) \pmod 2$.

**Proposition 3.** *Let* $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_7 > 7$ *and* $A = [a_1, a_2, \ldots, a_7]$. *Then,* $\mathrm{Gr}(A) \neq 0$ *if and only if either*

- $a_1 \equiv 0 \pmod 2$,
- $(a_1, a_2, a_3) \equiv (1,0,0) \pmod 2$, *or*
- $(a_1, a_2, a_3, a_4, a_5, a_6) \equiv (1,0,1,0,x,y) \pmod 2$ *and* $(x, y) \not\equiv (1,1) \pmod 2$.

$$\begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{bmatrix}$$

Figure 2: The strong partition matrix $A = [19, 18, 15, 14, 11, 9, 8]$ is a $\mathcal{P}$-position.

**Example 2.** We consider the strong partition matrix $A = [19, 18, 15, 14, 11, 9, 8]$ depicted in Figure 2. Then, $\operatorname{seq}(A) = (a_i : 1 \leqslant i \leqslant 7) = (19, 18, 15, 14, 11, 9, 8)$. Since $(a_i : 1 \leqslant i \leqslant 6) \equiv \big((1,0)^2, 1, 1\big) \pmod 2$, $A$ is a $\mathcal{P}$-position by Proposition 2.

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