

Explicit Kernel Extraction and Proof of Symmetries of SSM Coefficients - Multi-Index version

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The Kernel of the coefficient matrix can be explicitly predicted by analysing the resonances between eigenvalues of the system-matrices. The first part of this document does this for the multi-index invariance equation.

Due to the structure of spectral subspaces, which are spanned by complex conjugate pairs of eigenvectors in the case of complex eigenvalues the coefficients resulting from the invariance equation fulfill a set of symmetries. Given some assumptions and by introducing the concept of conjugate multi-indices we prove those symmetries in the second part of the document, again using the multi-index formulation of the invariance equation.

1 Explicit calculation of inner resonances

In this section we will explain how to analytically obtain the kernel of the coefficient matrix of the invariance equation, and in fact predict when inner resonances will lead to singularities in the coefficient matrix.

Let \mathbf{R}_1 be the matrix that contains the linear reduced dynamics coefficients in its columns. It is assumed, that this matrix is diagonal. We then define

$$\mathbf{D}_k := \text{diag}\left(\sum_{j=1}^l (\mathbf{k}_1)_j (\mathbf{R}_1)_{jj}, \dots, \sum_{j=1}^l (\mathbf{k}_{z_k})_j (\mathbf{R}_1)_{jj}\right) \quad (1.1)$$

The coefficient matrix for the order k SSM coefficients is then given by

$$\mathbf{C}_k = \mathbf{D}_k \otimes \mathbf{B} - \mathbf{I}_{z_k} \otimes \mathbf{A} \quad (1.2)$$

with z_k the number of multi-indices that exist at order k . The following result is presented by Shobhit Jain and modified for our purposes [1].

Proposition 1. *If $\lambda \in \mathbb{C}$ is a generalised Eigenvalue of the matrix Pair \mathbf{A}, \mathbf{B} with left and right eigenvectors \mathbf{w}, \mathbf{v} , i.e.*

$$\begin{aligned}\mathbf{A}\mathbf{v} &= \lambda\mathbf{B}\mathbf{v} \\ \mathbf{w}^*\mathbf{A} &= \lambda\mathbf{w}^*\mathbf{B}\end{aligned}$$

and $\mu \in \mathbb{C}$ is a generalised Eigenvalue of the matrix Pair \mathbf{C}, \mathbf{D} with left and right eigenvectors \mathbf{e}, \mathbf{f} , i.e.

$$\begin{aligned}\mathbf{C}\mathbf{f} &= \mu\mathbf{D}\mathbf{f} \\ \mathbf{e}^*\mathbf{C} &= \mu\mathbf{e}^*\mathbf{D}\end{aligned}$$

then $\lambda\mu$ is a generalised eigenvalue of the matrix pair $\mathbf{C} \otimes \mathbf{A}, \mathbf{D} \otimes \mathbf{B}$ corresponding to left and right eigenvectors $\mathbf{e} \otimes \mathbf{w}, \mathbf{f} \otimes \mathbf{v}$. Clearly, if $\lambda\mu \approx 1$, then the corresponding eigenvectors correspond to the left and right near kernels of $\mathbf{D} \otimes \mathbf{B} - \mathbf{C} \otimes \mathbf{A}$

We use this proposition for the coefficient matrix as given in eq. (1.2). Associating $\mathbf{C} = \mathbf{I}$ and $\mathbf{D} = \mathbf{D}_k$ we now have to find the generalised eigenvalues of the matrix pair \mathbf{D}_k, \mathbf{I} . This is an easy task, as this merely requires the knowledge of the eigenvalues of \mathbf{D}_k which are given explicitly by its diagonal entries. The generalised eigenvalues of the matrix pair are then the inverse of the eigenvalues of \mathbf{D}_k .

Now, we recall that the diagonal entries of \mathbf{R}_1 are the generalised eigenvalues of the matrices \mathbf{A}, \mathbf{B} with largest real part. The condition for an eigenvalue of the coefficient matrix to be in the near kernel then is given by

$$\lambda_i \mu_f \approx 1 \leftrightarrow \lambda_i - \left(\sum_{j=1}^l (\mathbf{k}_f)_j (\mathbf{R}_1)_{jj} \right) \approx 0 \quad (1.3)$$

for some generalised eigenvalue λ_i of the matrix pair \mathbf{A}, \mathbf{B} and some $0 < f \leq z_k$. Therefore the coefficient matrix at order k will become singular if for some generalised eigenvalue λ_i and for some f the following resonance condition is fulfilled:

$$\lambda_i = \langle \mathbf{k}_f, \mathbf{\Lambda} \rangle, \quad \mathbf{\Lambda} = (\lambda_1, \dots, \lambda_l) \quad (1.4)$$

where $\mathbf{\Lambda}$ is the vector containing all master mode eigenvalues. These conditions can be checked a priori by knowledge of the spectrum and thus we can predict where resonances will occur.

Remark *This resonance condition now beautifully displays some key features about the existence of spectral submanifolds. If the condition is met for an eigenvalue that is not a master mode eigenvalue, a so called outer resonance, then this leads to a singular coefficient matrix and therefore there is no solution for the SSM coefficients. If the resonance is an inner resonance which occurs within the set of master mode eigenvalues however, the reduced dynamics can be chosen*

such that projection of the invariance equation onto the kernel of the coefficient matrix is zero. The SSM coefficients projected onto this subspace are then in the kernel of the coefficient matrix.

By checking this resonance condition for all master mode eigenvalues we can determine the near left kernel of the coefficient matrix explicitly. It is given by

$$\text{Ker}(\mathbf{C}_k) = \text{span}(\hat{\mathbf{e}}_f \otimes \mathbf{w}_i \in \mathbb{C}^{lN} | \lambda_i \mu_f \approx 1) \quad (1.5)$$

On a side note, one can see that for any pair (f, i) that leads to resonances, also the index pair corresponding to the complex conjugate eigenvalues will lead to resonance. Assume there are r index pairs (f, i) that fulfill this condition. Then, using the Kathri Rao product which is simply a columnwise tensor product the kernel can be written as

$$\text{Ker}(\mathbf{C}_k) = \text{span}(\mathbf{E}_{\mathcal{F}} \odot \mathbf{W}_{\mathcal{I}}) \quad (1.6)$$

with $\mathbf{W}_{\mathcal{I}} = [\mathbf{w}_{i_1}, \dots, \mathbf{w}_{i_r}]$, $\mathbf{E}_{\mathcal{F}} = [\hat{\mathbf{e}}_{f_1}, \dots, \hat{\mathbf{e}}_{f_r}]$. We define $\mathbf{K}_{\mathcal{F}, \mathcal{I}} = \mathbf{E}_{\mathcal{F}} \odot \mathbf{W}_{\mathcal{I}}$.

1.1 Reduced Dynamics

Now that the kernel has been explicitly computed we give an expression for the reduced dynamics. For this we initially norm the left and right eigenvectors such that $\mathbf{W}^* \mathbf{B} \mathbf{V} = \mathbf{1}$ where \mathbf{W}, \mathbf{V} are the matrices containing the generalised left and right eigenvectors in their columns.

Plugging this into the expression for choosing the reduced dynamics we obtain

$$\mathbf{K}_{\mathcal{F}, \mathcal{I}}^* (\mathbf{L}_k - (\mathbf{I}_{z_k} \otimes \mathbf{B} \mathbf{S}_1) \text{vec}(\mathbf{R}_k)) = 0 \quad (1.7)$$

This expression can be simplified as follows. Take an index pair (f_α, i_α) . We note that the first order SSM coefficients are given by the generalised right eigenvalues of \mathbf{A}, \mathbf{B} . Therefore due to the normation we have chosen for the generalised eigenvectors $\mathbf{w}_{i_\alpha}^* \mathbf{B} \mathbf{S}_1 = \hat{\mathbf{e}}_{i_\alpha}$, where $\mathbf{w}_{i_\alpha} \in \mathbb{C}^{2n}$, $\mathbf{B} \mathbf{S}_1 \in \mathbb{R}^{2n \times l}$, $\hat{\mathbf{e}}_{i_\alpha} \in \mathbb{R}^l$. Note that here the indices i_α are used for $2n$ dimensional vectors \mathbf{w}_{i_α} as well as l dimensional unit vectors $\hat{\mathbf{e}}_{i_\alpha}$, while f_α are used for z_k dimensional vectors. $\mathbf{I} \otimes \mathbf{B} \mathbf{S}_1 \in \mathbb{R}^{z_k \cdot 2n \times z_k \cdot l}$ is simply a rectangular blockdiagonal matrix with $\mathbf{B} \mathbf{S}_1$ on its diagonal. It follows, that $(\hat{\mathbf{e}}_{f_\alpha} \otimes \mathbf{w}_{i_\alpha})^* (\mathbf{I}_{z_k} \otimes \mathbf{B} \mathbf{S}_1) = (\hat{\mathbf{e}}_{f_\alpha} \otimes \hat{\mathbf{e}}_{i_\alpha})^T$. We define $\mathbf{G}_{\mathcal{F}, \mathcal{I}} = \mathbf{E}_{\mathcal{F}} \odot \mathbf{E}_{\mathcal{I}}$ where $\mathbf{E}_{\mathcal{I}} = [\hat{\mathbf{e}}_{i_1}, \dots, \hat{\mathbf{e}}_{i_r}]$. This implies that we can rewrite eq. (1.7) as

$$\mathbf{K}_{\mathcal{F}, \mathcal{I}}^* \mathbf{L}_k - \mathbf{G}_{\mathcal{F}, \mathcal{I}}^T \text{vec}(\mathbf{R}_k) = 0 \quad (1.8)$$

We solve this equation for the reduced dynamics by making use of the fact that $\mathbf{G}_{\mathcal{F}, \mathcal{I}}$ is a boolean matrix .

$$\text{vec}(\mathbf{R}_k) = (\mathbf{G}_{\mathcal{F}, \mathcal{I}}^T)^{-1} \mathbf{K}_{\mathcal{F}, \mathcal{I}}^* \mathbf{L}_k \quad (1.9)$$

$$= \mathbf{G}_{\mathcal{F}, \mathcal{I}} \mathbf{K}_{\mathcal{F}, \mathcal{I}}^* \mathbf{L}_k \quad (1.10)$$

2 Ordering of Multi-indices

Let a $\mathbf{u} \in \mathbb{N}^l$ be a multi-index of order u , l the dimension of the master modal subspace and l_i the number of complex conjugate eigenvalue pairs in this subspace, l_r the number of real eigenvalues of the system in this subspace. Therefore $l = 2l_i + l_r$. The coordinate directions in the master modal subspace are ordered such that the first $2l_i$ directions correspond to the complex conjugate eigenvalues, appearing in pairs. The last l_r directions are the ones corresponding to the real spectral subspaces.

2.1 Reverse Lexicographical Ordering

This is the standard ordering that is used for storing the coefficients of the SSM parametrisation and also the reduced dynamics Taylor expansion coefficients.

2.2 Conjugate Multi-Indices

Let \mathbf{u} be a multi-index. We define its conjugate multi-index \mathbf{u}_c as the multi-index, that contains the entries of \mathbf{u} corresponding to the directions of each complex conjugate eigenvalue pair in reverse order, and the ones corresponding to real eigenvalues in the same position.

$$\mathbf{u} = (u_1, u_2, u_3, \dots, u_{2l_i-1}, u_{2l_i}, u_{2l_i+1}, \dots, u_{2l_i+l_r})^T \quad (2.1)$$

$$\mathbf{u}_c = (u_2, u_1, u_4, \dots, u_{2l_i}, u_{2l_i-1}, u_{2l_i+1}, \dots, u_{2l_i+l_r})^T \quad (2.2)$$

The function $\mathbf{CFlip}(\mathbf{a}, l_i)$ (short for conjugate-flip) acting on a vector \mathbf{a} flips the coordinate directions of the first l_i coordinate pairs.

$$\mathbf{CFlip} \left(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2l_i-1} \\ a_{2l_i} \\ b_1 \\ b_2 \\ \vdots \\ b_{l_r} \end{pmatrix}, l_i \right) := \begin{pmatrix} a_2 \\ a_1 \\ \vdots \\ a_{2l_i} \\ a_{2l_i-1} \\ b_1 \\ b_2 \\ \vdots \\ b_{l_r} \end{pmatrix} \quad (2.3)$$

Therefore $\mathbf{u}_c = \mathbf{CFlip}(\mathbf{u}, l_i)$. If in an ordered set \mathbf{u} has the index f the index of the conjugate multi-index \mathbf{u}_c will be called f_c . There also exist multi-indices that are their own conjugate multi-index, which implies that for them $f = f_c$. Those multi-indices will be called symmetric multi-indices.

2.2.1 Properties

In this section we list some properties of conjugate pairs of multi-indices that will be useful for later calculations.

1. The function CFlip is linear in its first argument: $\text{CFlip}(\mathbf{u} + \lambda \mathbf{v}, l_i) = \text{CFlip}(\mathbf{u}, l_i) + \lambda \text{CFlip}(\mathbf{v}, l_i)$
2. The function CFlip is a second root of the identity, i.e. $\text{CFlip}^2 = \text{id}$
3. If for two multi-indices \mathbf{v} and \mathbf{u} the elementwise inequality $\mathbf{v} < \mathbf{u}$ is true then $\mathbf{v}_c < \mathbf{u}_c$ also holds. In particular the set $\{\mathbf{v}_c | \mathbf{v}_c < \mathbf{u}_c\}$ contains the conjugate multi-index of every multi-index in the set $\{\mathbf{v} | \mathbf{v} < \mathbf{u}\}$ and vice versa. This gives a canonical isomorphism between the sets that maps multi-indices onto their conjugate multi-index.
4. The coordinate directions of the parametrisation space of the SSM are in 1 to 1 correspondence with the order 1 multi-indices $\hat{\mathbf{e}}_j$. We therefore introduce the notion of a conjugate coordinate direction j_c , which is the coordinate direction that the conjugate multi-index $\hat{\mathbf{e}}_{j_c}$ corresponds to.
5. For any multi index \mathbf{v} that has position f in some ordered set $(\mathbf{v}_{f_c}) = (\mathbf{v}_c)$ and $(\mathbf{v}_f)_j = (\mathbf{v}_{f_c})_{j_c}$. The first equality simply states that the conjugate multi-index is in the position of the conjugate subindex f_c of the same ordered set. The second equality relates the components of two conjugate multi-indices, heuristically speaking it merely reflects the fact, that the first $2l_i$ components of a conjugate multi-index pair are flipped with their neighbour as specified above.

To be mathematically accurate, we have to mention that the multi-indices do not have the mathematical meaning of vectors on the parametrisation space. It is merely for order 1 multi-indices and the fact that we use diagonal linear reduced dynamics that their only non-zero entry is exactly in the component that would correspond to the coordinate direction the entry is in. It is thus well defined to speak of conjugate coordinate directions by relating them to the coordinate directions the order 1 multi-indices correspond to. However one has to be aware that there is only a correspondence between those multi-indices and the coordinate directions. There is no direct relation other than specified by pointwise operations of multi-indices \mathbf{m} on the parametrisation coordinates \mathbf{p} as $\mathbf{p}^{\mathbf{m}}$.

2.2.2 Conjugate Ordering

This is a very specific ordering designed to exploit the symmetries that the SSM coefficients and the reduced dynamics have. While all of the coefficients are stored in the standard reverse lexicographical ordering, the conjugate ordering will come into play when calculating the coefficients.

Take a set $\mathbf{U} = \{\mathbf{u}_f\}_f$ of all order u multi-indices, and let z_u be the total number of elements of that set. Let z_r be the number of multi-indices that are their own conjugate multi-index. Then there exist $z_i = (z_u - z_r)/2$ pairs of conjugate multi-indices in the set. Now for each pair of conjugate multi-indices we remove the one that has higher index in the reverse lexicographically ordered set and we

also remove the z_r symmetric multi-indices. All of the remaining multi-indices are sorted in reverse lexicographical ordering. Then we add the symmetric multi-indices at the end of that set in reverse lexicographic ordering. The set $\tilde{\mathbf{U}}$ that is left now has $z_{u,cci} = z_u - z_i$ multi-indices left in it. We will call $z_{u,cci}$ the conjugate center index at order u .

The full set in conjugate ordering is obtained by adding in the conjugate counterparts as follows: for any $\mathbf{u}_f \in \tilde{\mathbf{U}}$, the conjugate index is defined as $f_c = z_u - f + 1$. The conjugate multi-index is added to the set in the position of the conjugate index. Therefore a pair of conjugate multi-indices \mathbf{u}, \mathbf{u}_c is situated in positions f and $z_u - f + 1$ in the conjugate ordering. While this might seem tedious to construct, it is a very handy tool when dealing with the symmetries of SSM-coefficients. The following proofs do not make use of this ordering and rely only on the notion of conjugate multi-indices and conjugate coordinate directions. The conjugate ordering is designed to make tracking the conjugate pairs easy when implementing the invariance-equation.

3 Inherent Symmetries

This chapter gives detailed description and proof of inherent symmetries of the SSM-coefficients and reduced dynamics that can be used to reduce computational efforts when calculating them. We shall assume that the matrices \mathbf{A}, \mathbf{B} and the nonlinearity \mathbf{F} do not contain any imaginary values. These are necessary assumptions for the existence of the symmetries. Furthermore we take the first order reduced dynamics to be a diagonal matrix. This decouples the invariance-equation when solving for the SSM-coefficients and saves time by reducing one big linear system to several smaller ones. The higher order reduced dynamics will in general not decouple and therefore the calculation can generally not be parallelized entirely.

3.1 First order coefficients

Having chosen l master mode eigenvalues, the first order reduced dynamics is determined by those. Assume there are l_i pairs of complex eigenvalues ($\alpha_{2i-1}\alpha_{2i}$) and l_r real eigenvalues β_i chosen for the master mode, i.e. $l = 2l_i + l_r$. The reduced dynamics in multi-index notation are then given as

$$\begin{aligned} \mathbf{R}_1 &= \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_{2l_i-1}\alpha_{2l_i}, \beta_1, \dots, \beta_{l_r}) \\ &= (\mathbf{R}_{\hat{\mathbf{e}}_1}, \mathbf{R}_{\hat{\mathbf{e}}_2}, \dots, \mathbf{R}_{\hat{\mathbf{e}}_l}) \end{aligned} \quad (3.1)$$

The individual vectors are given as

$$\mathbf{R}_{\hat{\mathbf{e}}_i} = (R_{1,\hat{\mathbf{e}}_i}, \dots, R_{l,\hat{\mathbf{e}}_i})^T \quad (3.2)$$

In order to fulfill the invariance-equation at first order, given by

$$\sum_{i=1}^{2n} \mathbf{B}_{bi} \sum_{j=1}^l S_{i,\hat{\mathbf{e}}_j} R_{j,\hat{\mathbf{e}}_f} = \sum_{i=1}^{2n} \mathbf{A}_{bi} S_{i,\hat{\mathbf{e}}_f} \quad (3.3)$$

the first order SSM-coefficients have to be chosen as the right eigenvectors fulfilling $(\mathbf{B}\alpha_f - \mathbf{A}) \cdot \mathbf{v}_f = 0$ or analogously for the coordinate directions corresponding to real eigenvalues with β_f instead of α_f . The eigenvectors \mathbf{a}_f corresponding to α_f have the property that $\bar{\mathbf{a}}_f$ is the eigenvector corresponding to $\bar{\alpha}_f$. The SSM-coefficients in multi-index notation are thus given as

$$\begin{aligned} \mathbf{S}_1 &= (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{2l_i-1}, \mathbf{a}_{2l_i}, \mathbf{b}_1, \dots, \mathbf{b}_{l_r}) \\ &= (\mathbf{S}_{\hat{\mathbf{e}}_1}, \mathbf{S}_{\hat{\mathbf{e}}_2}, \dots, \mathbf{S}_{\hat{\mathbf{e}}_l}) \end{aligned} \quad (3.4)$$

where the complex conjugate pairs are grouped together. The individual vectors contain the SSM-coefficients and are given as

$$\mathbf{S}_{\hat{\mathbf{e}}_i} = (S_{1,\hat{\mathbf{e}}_i}, \dots, S_{2n,\hat{\mathbf{e}}_i})^T \quad (3.5)$$

The left eigenvectors fulfilling $\mathbf{w}_i^* \cdot (\mathbf{B}\alpha_i - \mathbf{A}) = 0$ (respectively with β_i) are given in

$$\mathbf{W} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{2l_i-1}, \mathbf{a}_{2l_i}, \mathbf{b}_1, \dots, \mathbf{b}_{l_r}) \quad (3.6)$$

What can be seen now is that the SSM-coefficients at first order fulfill a symmetry, namely $\mathbf{S}_{\hat{\mathbf{e}}_f} = \bar{\mathbf{S}}_{\hat{\mathbf{e}}_{f_c}}$ where $\hat{\mathbf{e}}_{f_c}$ is the conjugate multi-index of $\hat{\mathbf{e}}_f$ since $(\mathbf{u}_f)_c = \mathbf{u}_{f_c}$ for any multi-index.

Additionally the reduced dynamics fulfill $\mathbf{R}_{\hat{\mathbf{e}}_f} = \text{CFlip}(\bar{\mathbf{R}}_{\hat{\mathbf{e}}_{f_c}}, l_i)$.

In the next two sections we will show that given the restrictions named above, all higher order coefficients will fulfill these symmetries as well. This effectively reduces the number of multi-indices for which the coefficients have to be computed by l_i . Since the coefficient matrix decouples for diagonal linear reduced dynamics this also reduces the number of equations.

3.2 Symmetries in the invariance-equation

We consider the autonomous invariance-equation for the SSM-coefficients.

$$\begin{aligned}
& \sum_{i=1}^{2n} \left(\mathbf{B}_{bi} \sum_{j=1}^l \sum_{a=1}^{z_k} (\mathbf{k}_a)_j S_{i, \mathbf{k}_a} R_{j, (\mathbf{k}_f - \mathbf{k}_a + \hat{\mathbf{e}}_j)} - \mathbf{A}_{bi} S_{i, \mathbf{k}_f} \right) \\
&= \underbrace{\sum_{\substack{\mathbf{m} \\ |\mathbf{m}| \leq k}} \mathbf{F}_{b, \mathbf{m}} \pi_{\mathbf{m}, \mathbf{k}_f} - \sum_{i=1}^{2n} \mathbf{B}_{bi} \sum_{\substack{j=1 \\ u_j > 0}}^l \left(\sum_{\substack{\mathbf{g}, \mathbf{u} \\ \mathbf{g} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}_f \\ |\mathbf{u}| \neq 1, k}} u_j S_{i, \mathbf{u}} R_{j, \mathbf{g}} \right)}_{:= (\mathbf{L}_k)_{(2n(f-1)+b)}} \\
&+ \underbrace{\sum_{i=1}^{2n} \mathbf{B}_{bi} \sum_{\substack{j=1 \\ u_j > 0}}^l \left(\sum_{a=1}^l \delta_{ja} S_{i, \hat{\mathbf{e}}_a} R_{j, (\mathbf{k}_f - \hat{\mathbf{e}}_a + \hat{\mathbf{e}}_j)} \right)}_{(\mathbf{b}_k)_{2n(f-1)+b}} \tag{3.7}
\end{aligned}$$

We show by induction that $(\mathbf{L}_k)_{(2n(f-1)+b)} = \overline{(\mathbf{L}_k)_{(2n(f_c-1)+b)}}$. Assume that for all orders $m < k$ the reduced dynamcis, the SSM-coefficients and the composition coefficients fulfill

1. $\mathbf{R}_{\mathbf{m}} = \text{CFlip}(\overline{\mathbf{R}_{\mathbf{m}_c}}, l_i)$
2. $\mathbf{S}_{\mathbf{m}} = \overline{\mathbf{S}_{\mathbf{m}_c}}$
3. $H_{i, s, \mathbf{m}} = \overline{H_{i, s, \mathbf{m}_c}}$

This in particular means that for symmetric multi-indices with $\mathbf{m} = \mathbf{m}_c$ the corresponding SSM- and composition coefficients are real. Initially we have a look at the composition coefficients of power series. We look at all composition coefficients that contribute in the invariance-equation, i.e. $s > 1$. Therefore the sum in the following equation only includes terms where $\mathbf{u} < \mathbf{k}_f$.

$$H_{i, s, \mathbf{k}_f} = \frac{s}{(\mathbf{k}_f)_j} \sum_{\mathbf{u} < \mathbf{k}_f} (\mathbf{u})_j S_{i, \mathbf{u}} H_{i, s-1, \mathbf{k}_f - \mathbf{u}} \tag{3.8}$$

$$= \frac{s}{(\mathbf{k}_f)_j} \sum_{\mathbf{u} < \mathbf{k}_f} (\mathbf{u})_j \overline{S_{i, \mathbf{u}_c} H_{i, s-1, (\mathbf{k}_f - \mathbf{u})_c}} \tag{3.9}$$

$$= \frac{s}{(\mathbf{k}_f)_j} \sum_{\mathbf{u} < \mathbf{k}_f} (\mathbf{u})_j \overline{S_{i, \mathbf{u}_c} H_{i, s-1, \mathbf{k}_{f_c} - \mathbf{u}_c}} \tag{3.10}$$

$$= \frac{s}{(\mathbf{k}_f)_j} \sum_{\mathbf{u}_c < \mathbf{k}_{f_c}} (\mathbf{u})_j \overline{S_{i, \mathbf{u}_c} H_{i, s-1, \mathbf{k}_{f_c} - \mathbf{u}_c}} \tag{3.11}$$

$$= \frac{s}{(\mathbf{k}_{f_c})_{j_c}} \sum_{\mathbf{u}_c < \mathbf{k}_{f_c}} (\mathbf{u}_c)_{j_c} \overline{S_{i, \mathbf{u}_c} H_{i, s-1, \mathbf{k}_{f_c} - \mathbf{u}_c}} \tag{3.12}$$

We get from eq. (3.8) to eq. (3.9) by using the induction assumption. The next step uses the linearity of CFlip . In the third step we make use of the

fact, that instead of summing over \mathbf{u} we can also directly sum over \mathbf{u}_c . And as shown in section 2.2.1 the set of all \mathbf{u}_c is given by all multi-indices that fulfill $\mathbf{u}_c < \mathbf{k}_{f_c}$. In the last step we realise, that $(\mathbf{k}_f)_j = (\mathbf{k}_{f_c})_{j_c}$ and also $(\mathbf{u})_j = (\mathbf{u}_c)_{j_c}$. The conjugate coordinate direction j_c is given by the coordinate direction of the conjugate multi-index of $\hat{\mathbf{e}}_j$. If \mathbf{k}_f has its minimal nonzero entry in entry j , then \mathbf{k}_{f_c} has its minimal nonzero entry in entry j_c . It follows that $H_{i,s,\mathbf{k}_f} = \overline{H_{i,s,\mathbf{k}_{f_c}}}$. For the first contribution in $(\mathbf{L}_k)_{(2n(f-1)+b)}$ we then obtain

$$\pi_{\mathbf{m},\mathbf{k}_f} = \left(\sum_{\substack{\mathbf{h}_1 \\ |\mathbf{h}_1| \geq m_1}} \cdots \sum_{\substack{\mathbf{h}_{2n} \\ |\mathbf{h}_{2n}| \geq m_{2n}}} \right)_{\sum \mathbf{h}_i \stackrel{!}{=} \mathbf{k}_f} H_{1,m_1,\mathbf{h}_1} \cdots H_{2n,m_{2n},\mathbf{h}_{2n}} \quad (3.13)$$

$$= \left(\sum_{\substack{\mathbf{h}_1 \\ |\mathbf{h}_1| \geq m_1}} \cdots \sum_{\substack{\mathbf{h}_{2n} \\ |\mathbf{h}_{2n}| \geq m_{2n}}} \right)_{\sum \mathbf{h}_i \stackrel{!}{=} \mathbf{k}_f} \overline{H_{1,m_1,(\mathbf{h}_1)_c} \cdots H_{2n,m_{2n},(\mathbf{h}_{2n})_c}} \quad (3.14)$$

$$= \left(\sum_{\substack{\mathbf{h}_1 \\ |\mathbf{h}_1| \geq m_1}} \cdots \sum_{\substack{\mathbf{h}_{2n} \\ |\mathbf{h}_{2n}| \geq m_{2n}}} \right)_{\sum (\mathbf{h}_i)_c \stackrel{!}{=} \mathbf{k}_{f_c}} \overline{H_{1,m_1,(\mathbf{h}_1)_c} \cdots H_{2n,m_{2n},(\mathbf{h}_{2n})_c}} \quad (3.15)$$

$$= \overline{\pi_{\mathbf{m},\mathbf{k}_{f_c}}} \quad (3.16)$$

This can be done, since instead of searching for combinations $\{\mathbf{h}_i\}_i$ that sum up to \mathbf{k}_f and then taking the composition coefficients corresponding to their conjugate multi-index counterpart we can also look for those conjugate counterparts directly as we have argued in section 2.2.1. And since \mathbf{CFlip} is linear $\sum \mathbf{h}_i \stackrel{!}{=} \mathbf{k}_f \leftrightarrow \sum (\mathbf{h}_i)_c \stackrel{!}{=} \mathbf{k}_{f_c}$.

Now let us look at the second contribution to $(\mathbf{L}_k)_{(2n(f-1)+b)}$.

$$\sum_{\substack{j=1 \\ u_j > 0}}^l \left(\sum_{\substack{\mathbf{g}, \mathbf{u} \\ \mathbf{g} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}_f \\ |\mathbf{u}| \neq 1, k}} (\mathbf{u})_j S_{i,\mathbf{u}} R_{j,\mathbf{g}} \right) = \sum_{\substack{j=1 \\ (\mathbf{u})_j > 0}}^l \left(\sum_{\substack{\mathbf{g}, \mathbf{u} \\ \mathbf{g} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}_f \\ |\mathbf{u}| \neq 1, k}} (\mathbf{u})_j \overline{S_{i,\mathbf{u}_c} R_{j,\mathbf{g}}} \right) \quad (3.17)$$

$$= \sum_{\substack{j=1 \\ (\mathbf{u})_j > 0}}^l \left(\sum_{\substack{\mathbf{g}, \mathbf{u} \\ \mathbf{g} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}_f \\ |\mathbf{u}| \neq 1, k}} (\mathbf{u})_j \overline{S_{i,\mathbf{u}_c} R_{j_c,\mathbf{g}_c}} \right) \quad (3.18)$$

$$= \sum_{\substack{j=1 \\ (\mathbf{u})_j > 0}}^l \left(\sum_{\substack{\mathbf{g}, \mathbf{u} \\ \mathbf{g} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}_f \\ |\mathbf{u}| \neq 1, k}} (\mathbf{u}_c)_{j_c} \overline{S_{i,\mathbf{u}_c} R_{j_c,\mathbf{g}_c}} \right) \quad (3.19)$$

$$= \sum_{\substack{j=1 \\ (\mathbf{u}_c)_{j_c} > 0}}^l \left(\sum_{\substack{\mathbf{g}_c, \mathbf{u}_c \\ \mathbf{g}_c + \mathbf{u}_c - \hat{\mathbf{e}}_{j_c} = \mathbf{k}_{f_c} \\ |\mathbf{u}_c| \neq 1, k}} (\mathbf{u}_c)_{j_c} \overline{S_{i,\mathbf{u}_c} R_{j_c,\mathbf{g}_c}} \right) \quad (3.20)$$

The arguments that justify the individual steps are analogous to the ones used above. These results imply that

$$(\mathbf{L}_k)_{(2n(f-1)+b)} = \overline{(\mathbf{L}_k)_{(2n(f_c-1)+b)}} \quad (3.21)$$

If $f = f_c$ then

$$\sum_{\substack{j=1 \\ u_j > 0}}^l \left(\sum_{\substack{\mathbf{g}, \mathbf{u} \\ \mathbf{g} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}_f \\ |\mathbf{u}| \neq 1, k}} (\mathbf{u})_j S_{i, \mathbf{u}} R_{j, \mathbf{g}} \right) = \sum_{\substack{j=1 \\ (\mathbf{u}_c)_j > 0}}^l \left(\sum_{\substack{\mathbf{g}_c, \mathbf{u}_c \\ \mathbf{g}_c + \mathbf{u}_c - \hat{\mathbf{e}}_{j_c} = \mathbf{k}_{f_c} \\ |\mathbf{u}_c| \neq 1, k}} (\mathbf{u}_c)_{j_c} \overline{S_{i, \mathbf{u}_c} R_{j_c, \mathbf{g}_c}} \right) \quad (3.22)$$

$$= \sum_{\substack{j=1 \\ (\mathbf{u}_c)_j > 0}}^l \left(\sum_{\substack{\mathbf{g}_c, \mathbf{u}_c \\ \mathbf{g}_c + \mathbf{u}_c - \hat{\mathbf{e}}_{j_c} = \mathbf{k}_f \\ |\mathbf{u}_c| \neq 1, k}} (\mathbf{u}_c)_{j_c} S_{i, \mathbf{u}_c} R_{j_c, \mathbf{g}_c} \right) \quad (3.23)$$

This simply means that for every triple of multi-indices \mathbf{g}, \mathbf{u} and $\hat{\mathbf{e}}_j$ their conjugate counterparts are also a combination that is summed over. This implies that if $(\mathbf{u})_j S_{i, \mathbf{u}} R_{j, \mathbf{g}}$ is part of the sum, then so is $(\mathbf{u}_c)_{j_c} \overline{S_{i, \mathbf{u}_c} R_{j_c, \mathbf{g}_c}}$. Since $(\mathbf{u})_j = (\mathbf{u}_c)_{j_c}$ and we sum over j the complex conjugate of both expressions is also part of the sum and the imaginary parts cancel out. Therefore if $f = f_c$ then $(\mathbf{L}_k)_{(2n(f-1)+b)} \in \mathbb{R}$.

3.3 Symmetry of reduced dynamics

Proposition 2. *The order k reduced dynamics fulfill*

$$\mathbf{R}_{\mathbf{k}_f} = \text{CFlip}(\overline{\mathbf{R}_{\mathbf{k}_{f_c}}}, l_i) \quad (3.24)$$

for all order k multi-indices given that SSM-coefficients fulfill the symmetry $\mathbf{S}_{\mathbf{m}} = \overline{\mathbf{S}_{\mathbf{m}_c}}$ for all order m multi-indices with $m < k$ and that this symmetry holds for the reduced dynamics at all lower orders. If $f = f_c$ then the coefficients are real.

Proof. We prove the symmetry for the reduced dynamics coefficients by induction. At order 1 the statement is fulfilled since \mathbf{R}_1 contains $\mathbf{\Lambda}$ on its diagonal. Assume all order $m < k$ reduced dynamics coefficients fulfill $\mathbf{R}_{\mathbf{m}} = \text{CFlip}(\overline{\mathbf{R}_{\mathbf{m}_c}}, l_i)$. Additionally assume that the SSM-coefficients fulfill $\mathbf{S}_{\mathbf{m}} = \overline{\mathbf{S}_{\mathbf{m}_c}}$ at lower orders.

The reduced dynamics at order k are calculated in section 1.1 and given by $\text{vec}(\mathbf{R}_k) = \mathbf{G}_{\mathcal{F}, \mathcal{I}} \mathbf{K}_{\mathcal{F}, \mathcal{I}}^* \mathbf{L}_k$. Assume that for an index pair (f, i) and the corresponding generalised left eigenvectors $\hat{\mathbf{e}}_f, \mathbf{w}_i$ and eigenvalues μ_f, λ_i an inner resonance occurs. We distinguish between a complex resonance where $\lambda_i \in \mathbb{C}$ and a real resonance where $\lambda_i \in \mathbb{R}$.

Complex inner resonance Assume α_i and μ_f lead to a resonance. In this case clearly this also leads to an inner resonance for the complex conjugate counterparts $(\overline{\mu_f}, \overline{\alpha_i})$. The complex conjugate eigenvalue of α_i has index i_c by construction. For the other eigenvalue the situation is less trivial. We show, that $\overline{\mu_f} = \mu_{f_c}$. Take $\mu_f = 1/\delta_f$ and $\mathbf{\Lambda}$ as the vector containing all master mode eigenvalues.

$$\overline{\delta_f} = \sum_{j=1}^l \overline{(\mathbf{\Lambda})_j} (\mathbf{k}_f)_j \quad (3.25)$$

$$= \sum_{j=1}^l (\mathbf{\Lambda})_{j_c} (\mathbf{k}_f)_j \quad (3.26)$$

$$= \sum_{j=1}^l (\mathbf{\Lambda})_j (\mathbf{k}_f)_{j_c} \quad (3.27)$$

$$= \sum_{\bar{j}=1}^l (\mathbf{\Lambda})_{\bar{j}} (\mathbf{k}_{f_c})_{\bar{j}} \quad (3.28)$$

As we pointed out at first order there is a 1 to 1 correspondence between coordinate directions and multi-indices which allows us to use the notion of conjugate coordinate directions. A change of variable leads to the third equality. Consequently we realise that $(\mathbf{k}_f)_{j_c} = (\mathbf{k}_{f_c})_j$. This proves that the complex conjugate counterparts of the eigenvalues α_i and μ_f are α_{i_c} and μ_{f_c} . Note that if $f = f_c$ then $\delta_f = \overline{\delta_f}$. That implies that in the case of a complex eigenvalue leading to resonances the corresponding eigenvalue μ_f cannot correspond to a symmetric multi-index \mathbf{k}_f and therefore $f \neq f_c$, which proves that $f = f_c$ is not possible in this case. This leads to two contributions $\mathbf{v}_{f,i}$ and \mathbf{v}_{f_c,i_c} to the reduced dynamics:

$$\mathbf{v}_{f,i} = (\hat{\mathbf{e}}_f \otimes \hat{\mathbf{e}}_i)(\hat{\mathbf{e}}_f \otimes \mathbf{a}_i)^* \mathbf{L}_k \quad (3.29)$$

$$\mathbf{v}_{f_c,i_c} = (\hat{\mathbf{e}}_{f_c} \otimes \hat{\mathbf{e}}_{i_c})(\otimes \mathbf{a}_{i_c})^* \mathbf{L}_k \quad (3.30)$$

Here $\hat{\mathbf{e}}_f \in \mathbb{R}^{z_k}$ and $\hat{\mathbf{e}}_i \in \mathbb{R}^l$, and analogously for f_c, i_c . The first tensor product can be explicitly written as $\hat{\mathbf{e}}_f \otimes \hat{\mathbf{e}}_i = \hat{\mathbf{e}}_{(l(f-1)+i)} \in \mathbb{R}^{l * z_k}$ and similarly $\hat{\mathbf{e}}_{f_c} \otimes \hat{\mathbf{e}}_{i_c} = \hat{\mathbf{e}}_{(l(f_c-1)+i_c)}$. We calculate the term $\mathbf{v}_{f,i}$ explicitly.

$$\mathbf{v}_{f,i} = (\hat{\mathbf{e}}_f \otimes \hat{\mathbf{e}}_i)(\hat{\mathbf{e}}_f \otimes \mathbf{a}_i)^* \mathbf{L}_k \quad (3.31)$$

$$= \left(\sum_{j=1}^{2n} (\mathbf{a}_i)_j (\mathbf{L}_k)_{(2n(f-1)+j)} \right) \hat{\mathbf{e}}_{(l(f-1)+i)} \quad (3.32)$$

We see that $\mathbf{v}_{f,i}$ contributes to $\mathbf{R}_{\mathbf{k}_f}$ since $0 < i < l$. To see this recall that $\mathbf{v}_{f,i}$ is the contribution to $\mathbf{vec}(\mathbf{R}_k)$. Explicitly $(\mathbf{R}_{\mathbf{k}_f})_i = \langle \hat{\mathbf{e}}_{(l(f-1)+i)}, \mathbf{v}_{f,i} \rangle$. From

the results of the previous section for the vector \mathbf{L}_k and the fact that $\mathbf{a}_i = \overline{\mathbf{a}_{i_c}}$ it follows directly that $(\mathbf{a}_{i_c})_j(\mathbf{L}_k)_{(2n(f_c-1)+j)} = \overline{(\mathbf{a}_i)_j(\mathbf{L}_k)_{(2n(f-1)+j)}}$. Therefore we have

$$\mathbf{v}_{f_c, i_c} = \left(\sum_{j=1}^{2n} (\mathbf{a}_{i_c})_j (\mathbf{L}_k)_{(2n(f_c-1)+j)} \right) \hat{\mathbf{e}}_{(l(f_c-1)+i_c)} \quad (3.33)$$

$$= \overline{\left(\sum_{j=1}^{2n} (\mathbf{a}_i)_j (\mathbf{L}_k)_{(2n(f-1)+j)} \right) \hat{\mathbf{e}}_{(l(f-1)+i)}} \quad (3.34)$$

and $(\mathbf{R}_{\mathbf{k}_{f_c}})_{i_c} = \langle \hat{\mathbf{e}}_{(l(f_c-1)+i_c)}, \mathbf{v}_{f_c, i_c} \rangle = \overline{(\mathbf{R}_{\mathbf{k}_f})_i}$. Here once again i and i_c correspond to the directions of the order 1 multi-indices $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}_{i_c}$. To see this recall that \mathbf{R}_1 contains $\mathbf{\Lambda}$ on its diagonal. Note that every index pair (i, f) that leads to resonances has a unique reduced dynamics coefficient that is set to nonzero. This relation is one to one.

Real inner resonance

Assume a real eigenvalue β_i leads to resonance at order k together with $\mu_f = 1/\delta_f$. Then $\beta_i - \delta_f \approx 0$. This implies $\delta_f \in \mathbb{R}$. Explicitly the resonance condition is $\beta_i - (\sum_{j=1}^l (\mathbf{k}_f)_j (\mathbf{R}_1)_{jj}) \approx 0$. But since the complex parts in δ_f have to cancel each other out the resonance condition also is fulfilled for \mathbf{k}_{f_c} as

$$\sum_{j=1}^l (\mathbf{k}_f)_j (\mathbf{R}_1)_{jj} = \overline{\sum_{j=1}^l (\mathbf{k}_f)_j (\mathbf{R}_1)_{jj}} \quad (3.35)$$

$$= \sum_{j=1}^l (\mathbf{k}_f)_j \overline{(\mathbf{R}_1)_{jj}} \quad (3.36)$$

$$= \sum_{j=1}^l (\mathbf{k}_{f_c})_j (\mathbf{R}_1)_{jj} \quad (3.37)$$

There are now two possibilities. The first one is that $\mathbf{k}_{f_c} \neq \mathbf{k}_f$. Then

$$\mathbf{v}_{f, i} = (\hat{\mathbf{e}}_f \otimes \hat{\mathbf{e}}_i) (\hat{\mathbf{e}}_f \otimes \hat{\mathbf{e}}_i)^* \mathbf{L}_k \quad (3.38)$$

$$= \left(\sum_{j=1}^{2n} (\mathbf{f}_i)_j (\mathbf{L}_k)_{(2n(f-1)+j)} \right) \hat{\mathbf{e}}_{(l(f-1)+2l_i+i)} \quad (3.39)$$

$$\mathbf{v}_{f_c, i} = \left(\sum_{j=1}^{2n} (\mathbf{f}_i)_j (\mathbf{L}_k)_{(2n(f_c-1)+j)} \right) \hat{\mathbf{e}}_{(l(f_c-1)+2l_i+i)} \quad (3.40)$$

$$= \overline{\left(\sum_{j=1}^{2n} (\mathbf{f}_i)_j (\mathbf{L}_k)_{(2n(f-1)+j)} \right) \hat{\mathbf{e}}_{(l(f-1)+2l_i+i)}} \quad (3.41)$$

Note here that \mathbf{CFlip} has no effect on the index i since it corresponds to a coordinate direction that is associated with a real eigenvalue, i.e. $i = i_c$. We obtain $\overline{\mathbf{v}_{f,i}} = \mathbf{v}_{f_c, i_c}$. They contribute to $(\mathbf{R}_{\mathbf{k}_f})_{2l_i+i}$ and $(\mathbf{R}_{\mathbf{k}_{f_c}})_{2l_i+i}$ respectively. The second possibility is $\mathbf{k}_{f_c} = \mathbf{k}_f$. In this case

$$\mathbf{v}_{f,i} = (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_f)(\mathfrak{b}_i \otimes \hat{\mathbf{e}}_f)^* \mathbf{L}_k \quad (3.42)$$

$$= \left(\sum_{j=1}^{2n} (\mathfrak{b}_i)_j (\mathbf{L}_k)_{(2n(f-1)+j)} \right) \hat{\mathbf{e}}_{(l(f-1)+2l_i+i)} \quad (3.43)$$

$$= \left(\sum_{j=1}^{2n} (\mathfrak{b}_i)_j \overline{(\mathbf{L}_k)_{(2n(f_c-1)+j)}} \right) \hat{\mathbf{e}}_{(l(f-1)+2l_i+i)} \quad (3.44)$$

Since $f = f_c$ we know that $(\mathbf{L}_k)_{(2n(f-1)+j)} \in \mathbb{R}$. Because $i = i_c$ we obtain $\mathbf{v}_{f,i} = \mathbf{v}_{f_c, i_c} \in \mathbb{R}$.

End of Proof

We have shown that for all the reduced dynamics coefficients $(\mathbf{R}_{\mathbf{k}_{f_c}})_{i_c} = \overline{(\mathbf{R}_{\mathbf{k}_f})_i}$ since an index pair (f, i) that triggers a resonance is in one to one correspondence with a reduced dynamics coefficient. This proves that the reduced dynamics at order k fulfill

$$\mathbf{CFlip}(\mathbf{R}_{\mathbf{k}_f}, l_i) = \overline{\mathbf{R}_{\mathbf{k}_{f_c}}} \quad (3.45)$$

□

3.4 Symmetry of SSM-coefficients

We now seek to prove the symmetry of the order k SSM-coefficients.

Proposition 3. *Assume that for all order $m < k$ multi-indices the SSM-coefficients fulfill $\mathbf{S}_{\mathbf{m}} = \overline{\mathbf{S}_{\mathbf{m}_c}}$, and that for all order $m \leq k$ multi-indices $\mathbf{CFlip}(\mathbf{R}_{\mathbf{m}}, l_i) = \overline{\mathbf{R}_{\mathbf{m}_c}}$ is fulfilled. Then the following symmetry holds for the order k SSM-coefficients*

$$\forall \mathbf{k} \in \mathbb{N}^l, |\mathbf{k}| = k : \mathbf{S}_{\mathbf{k}} = \overline{\mathbf{S}_{\mathbf{k}_c}} \quad (3.46)$$

Proof. In order to prove the statement we show that the invariance-equation corresponding to two conjugate multi indices are complex conjugate to each other. Then the resulting SSM-coefficients will also be complex conjugate to each other. We have already shown that a part of the right hand side of the invariance-equation has this property in section 3.2. What is left is to show that the coefficient matrix and $(\mathbf{b}_k)_{2n(f-1)+b}$ also fulfill this property. We look at a order k multi-index \mathbf{k}_f with conjugate multi-index \mathbf{k}_{f_c} . The right hand term in the invariance-equation for $\mathbf{S}_{\mathbf{k}_f}$ is

$$\sum_{\substack{j=1 \\ u_j > 0}}^l \sum_{a=1}^l \delta_{ja} S_{i, \hat{\mathbf{e}}_a} R_{j, (\mathbf{k}_f - \hat{\mathbf{e}}_a + \hat{\mathbf{e}}_j)} = \sum_{\substack{j=1 \\ u_j > 0}}^l S_{i, \hat{\mathbf{e}}_j} R_{j, \mathbf{k}_f} \quad (3.47)$$

$$= \sum_{\substack{j=1 \\ u_j > 0}}^l \overline{S_{i, \hat{\mathbf{e}}_{j_c}} R_{j_c, \mathbf{k}_{f_c}}} \quad (3.48)$$

$$= \sum_{\substack{j=1 \\ u_j > 0}}^l \overline{S_{i, \hat{\mathbf{e}}_{j_c}} R_{j, \mathbf{k}_{f_c}}} \quad (3.49)$$

This is exactly the complex conjugate of the contribution of the same term to the invariance-equation for $\mathbf{S}_{\mathbf{k}_{f_c}}$. For the coefficient matrix we make use of the fact, that the first order reduced dynamics are diagonal and therefore $R_{j, (\mathbf{k}_f - \mathbf{k}_a + \hat{\mathbf{e}}_j)} = \delta_{af} R_{j, \hat{\mathbf{e}}_j}$. We rewrite the coefficient matrix term using this.

$$\sum_{i=1}^{2n} \left(\mathbf{B}_{bi} \sum_{j=1}^l \sum_{a=1}^{z_k} (\mathbf{k}_a)_j S_{i, \mathbf{k}_a} R_{j, (\mathbf{k}_f - \mathbf{k}_a + \hat{\mathbf{e}}_j)} - \mathbf{A}_{bi} S_{i, \mathbf{k}_f} \right) \quad (3.50)$$

$$= \sum_{i=1}^{2n} \left(\mathbf{B}_{bi} \sum_{j=1}^l (\mathbf{k}_f)_j S_{i, \mathbf{k}_f} R_{j, \hat{\mathbf{e}}_j} - \mathbf{A}_{bi} S_{i, \mathbf{k}_f} \right) \quad (3.51)$$

$$= \sum_{i=1}^{2n} \left(\mathbf{B}_{bi} \sum_{j=1}^l (\mathbf{k}_f)_j R_{j, \hat{\mathbf{e}}_j} - \mathbf{A}_{bi} \right) S_{i, \mathbf{k}_f} \quad (3.52)$$

We now make use of the symmetry of the reduced dynamics and obtain

$$\sum_{i=1}^{2n} \left(\mathbf{B}_{bi} \sum_{j=1}^l (\mathbf{k}_f)_j R_{j, \hat{\mathbf{e}}_j} - \mathbf{A}_{bi} \right) \quad (3.53)$$

$$= \sum_{i=1}^{2n} \left(\mathbf{B}_{bi} \sum_{j=1}^l (\mathbf{k}_f)_j \overline{R_{j_c, \hat{\mathbf{e}}_{j_c}}} - \mathbf{A}_{bi} \right) \quad (3.54)$$

$$= \sum_{i=1}^{2n} \left(\mathbf{B}_{bi} \sum_{j=1}^l (\mathbf{k}_f)_{j_c} \overline{R_{j, \hat{\mathbf{e}}_j}} - \mathbf{A}_{bi} \right) \quad (3.55)$$

$$= \sum_{i=1}^{2n} \left(\mathbf{B}_{bi} \sum_{j=1}^l (\mathbf{k}_{f_c})_j \overline{R_{j, \hat{\mathbf{e}}_j}} - \mathbf{A}_{bi} \right) \quad (3.56)$$

$$= \sum_{i=1}^{2n} \left(\mathbf{B}_{bi} \sum_{j=1}^l (\mathbf{k}_{f_c})_j R_{j, \hat{\mathbf{e}}_j} - \mathbf{A}_{bi} \right) \quad (3.57)$$

The first step made use of the assumption that the system matrices \mathbf{A} and \mathbf{B} are real. Consequently we use, that we sum over all j , i.e. for every j , j_c is also included in the sum and we therefore can change the variable. Then we once again make use of the fact that the j -th component of any multi-index is the j_c -th component of its conjugate counterpart. The last expression is exactly the complex conjugate of the coefficient matrix term corresponding to row b of the coefficient matrix for $\mathbf{S}_{\mathbf{k}_f}$ and we have shown that the invariance-equation of a conjugate pair of multi-indices are complex conjugate pairs of each other. We have thus proven the proposition. If a multi-index is its own conjugate multi-index, then the SSM-coefficients corresponding to it will be real. \square

This also implies that for the composition coefficients with $s = 1$ the symmetry $H_{i,s,\mathbf{k}} = \overline{H_{i,s,\mathbf{k}_c}}$ holds for order k multi-indices \mathbf{k} and \mathbf{k}_c .

Corollary 1. *Assume we construct the SSM over spectral subspaces $E_{i_1}, E_{i_2}, \dots, E_{i_l}, E_{i_{l+1}}, \dots, E_{i_{l+r}}$, which yields a l dimensional SSM. Let λ_{i_α} be an eigenvalue of one of those subspaces. For a multi-index $\mathbf{k} \in \mathbb{N}^l$ let $\tilde{\mathbf{k}}$ be the same multi-index but with the components corresponding to spectral subspace E_{i_α} set to zero. In the case of $\lambda_{i_\alpha} \in \mathbb{C}$ it is given as $\tilde{\mathbf{k}} = \mathbf{k} - (\langle \mathbf{k}, \hat{\mathbf{e}}_{2\alpha} \rangle \hat{\mathbf{e}}_{2\alpha} + \langle \mathbf{k}, \hat{\mathbf{e}}_{2\alpha+1} \rangle \hat{\mathbf{e}}_{2\alpha+1})$. If $\lambda_{i_\alpha} \in \mathbb{R}$ then $\tilde{\mathbf{k}} = \mathbf{k} - (\langle \mathbf{k}, \hat{\mathbf{e}}_{\alpha+l_i} \rangle \hat{\mathbf{e}}_{\alpha+l_i})$. Assume, that the following non resonance condition is fulfilled for all spectral subspaces $E_{i_1}, E_{i_2}, \dots, E_{i_{l+r}}$.*

$$\forall \mathbf{k} \in \mathbb{N}^l, |\tilde{\mathbf{k}}| > 0 : \langle \mathbf{k}, \mathbf{\Lambda} \rangle \neq \lambda_{i_\alpha} \quad (3.58)$$

Then and only then the reduced dynamics on the SSM decouple for all spectral subspaces and the FRC can be obtained individually for each of them.

Proof. To prove this we revisit the proof for the symmetry of the reduced dynamics coefficients. For a pair (μ_f, λ_i) to trigger a resonance $\lambda_i - \delta_f \approx 0$ has to be fulfilled where

$$\delta_f = \sum_{j=1}^l (\mathbf{\Lambda})_j (\mathbf{k}_f)_j \quad (3.59)$$

We take a look at subspace E_{i_α} . The reduced dynamics on this subspace corresponding to a multi-index \mathbf{k}_f are $\mathbf{R}_{\mathbf{k}_f}|_\alpha$. In the case of a two-dimensional subspace that would be the coefficients $R_{2\alpha, \mathbf{k}_f}$ and $R_{2\alpha+1, \mathbf{k}_f}$ and in a subspace corresponding to a real eigenvalue it would be $R_{\alpha+l_i, \mathbf{k}_f}$.

In order to decouple from all other subspaces these coefficients have to be zero for all multi-indices \mathbf{k}_f that have nonzero entries in coordinate directions other than 2α and $2\alpha+1$ for a complex subspace and $\alpha+l_i$ in the case of $\lambda_{i_\alpha} \in \mathbb{R}$. Multi-indices that fulfill this condition are exactly the ones where $|\tilde{\mathbf{k}}_f| > 0$. This means that for such a multi-index the index pair $(2\alpha, f)$ and $(2\alpha+1, f)$ respectively $(\alpha+l_i, f)$ must not lead to a resonance. This condition is given as $\lambda_{i_\alpha} - \delta_f \neq 0$ which can be rewritten as

$$\sum_{j=1}^l (\mathbf{\Lambda})_j (\mathbf{k}_f)_j \neq \lambda_{i_\alpha} \quad (3.60)$$

This expression is equivalent to the condition stated in the corollary. The opposite direction of the proof directly follows as only resonances that violate this condition lead to coupled reduced dynamics. \square

It is straightforward to see that a set of subspaces that does only have resonances as described in the corollary within itself but not with subspaces outside of the set will have reduced dynamics that are decoupled from the dynamics in spectral subspaces outside of the set.

References

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