

Non autonomous SSM computation in multi-index format

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errors/additions - sum over which multi-indices for force - $T_{i,m}^i$

1 Introduction

We start off by considering an n -DOF mechanical system. It is subject to quasiperiodic forcing with a non-commensurate frequency basis $\boldsymbol{\Omega} \in \mathbb{R}^k$. It may also be under the influence of autonomous and non-autonomous nonlinear terms, represented by a non-linearity \mathbf{g} and the position dependence of the forcing.

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} + \mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}) = \epsilon \mathbf{f}(\boldsymbol{\Omega}t, \mathbf{y}, \dot{\mathbf{y}}) \quad (1.1)$$
$$0 < \epsilon \ll 1$$

The whole system is rewritten to first order form.

$$\mathbf{B}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{G}(\mathbf{x}) + \epsilon \mathbf{F}(\boldsymbol{\Omega}t, \mathbf{x}) \quad \mathbf{x} = \begin{bmatrix} \mathbf{y} \\ \dot{\mathbf{y}} \end{bmatrix} \in \mathbb{R}^{2n} \quad (1.2)$$

The linearized system $\mathbf{B}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ leads to a generalised eigenvalue problem with eigenvalues $\lambda_1, \dots, \lambda_{2n}$. The eigenvalue problems are given as

$$(\mathbf{A} - \lambda_j \mathbf{B})\mathbf{v}_j = \mathbf{0}, \quad j = 1, \dots, N. \quad (1.3)$$

$$\mathbf{w}_j^*(\mathbf{A} - \lambda_j \mathbf{B}) = \mathbf{0}, \quad j = 1, \dots, N. \quad (1.4)$$

We choose to scale the eigenvalues such that

$$\mathbf{W}^* \mathbf{B} \mathbf{V} = \mathbf{I} \quad (1.5)$$

where the matrices contain the generalised eigenvectors in its columns. On a sidenote, for symmetric $\mathbf{M}, \mathbf{C}, \mathbf{K}$ the matrices \mathbf{A}, \mathbf{B} are also symmetric, leading to $\mathbf{v}_j = \mathbf{w}_j$, $j = 1, \dots, N$. The parametrisation of the l -dimensional SSM maps coordinates on the SSM into full phase space. It is given as a map $\mathbf{S}(\mathbf{p}, \phi)$. The reduced dynamics give the dynamics on the SSM and fulfill $\dot{\mathbf{p}} = \mathbf{R}(\mathbf{p}, \phi)$. Here we introduced the phase variable $\phi \in \mathbb{T}^k$ which fulfills $\dot{\phi} = \Omega$.

1.1 Invariance equation

In order to write the invariance equation using these parametrisations we have to calculate $\dot{\mathbf{x}}$ as a function of the SSM parametrisation.

$$\begin{aligned}\dot{\mathbf{x}} &= \frac{d\mathbf{S}(\mathbf{p}, \phi)}{dt} \\ &= (\partial_{\mathbf{p}}\mathbf{S}(\mathbf{p}, \phi))\dot{\mathbf{p}} + (\partial_{\phi}\mathbf{S}(\mathbf{p}, \phi))\dot{\phi} \\ &= (\partial_{\mathbf{p}}\mathbf{S}(\mathbf{p}, \phi))\mathbf{R}(\mathbf{p}, \phi) + (\partial_{\phi}\mathbf{S}(\mathbf{p}, \phi))\Omega\end{aligned}\tag{1.6}$$

The invariance equation is then given as

$$\begin{aligned}\mathbf{B}((\partial_{\mathbf{p}}\mathbf{S}(\mathbf{p}, \phi))\mathbf{R}(\mathbf{p}, \phi) + (\partial_{\phi}\mathbf{S}(\mathbf{p}, \phi))\Omega) = \\ \mathbf{A}\mathbf{S}(\mathbf{p}, \phi) + \mathbf{G} \circ \mathbf{S}(\mathbf{p}, \phi) + \epsilon\mathbf{F}(\Omega t, \mathbf{S}(\mathbf{p}, \phi))\end{aligned}\tag{1.7}$$

1.2 Taylor expansions

In order to find the functions $\mathbf{S}(\mathbf{p}, \phi)$ and $\mathbf{R}(\mathbf{p}, \phi)$ they are expanded as a Taylor-series and the invariance equation is solved iteratively. Analogously the forcing and the nonlinearity is given in its Taylor-expanded form.

1.2.1 Naming convention

To keep well distinguishable variable names and maintain a high level of readability we make use of a strict naming convention. For all Taylor expanded quantities, the row of a function will be indicated with an upper subscript. Multi-indices that are used for spatial Taylor expansions are denoted with latin letters. They can be either l or $2n$ dimensional, depending on the space that the functions act on for which they are used. Multi-indices used for Fourier-expansions in the frequency domain will be called frequency multi-indices and referred to using greek letters, i.e. $\boldsymbol{\eta}$. However the letters ϕ and Ω are a phase variable and the frequency basis respectively and are not used to denote multi-indices. Multi-indices are always printed boldly, their components however are not written bold, i.e. $\mathbf{m} \in \mathbb{N}^l$ has components m_i . An example coefficient of the i -th row of an expanded function corresponding to multi-indices \mathbf{m} and $\boldsymbol{\eta}$ is written as $G_{\mathbf{m}, \boldsymbol{\eta}}^i$.

1.2.2 SSM coefficients and reduced dynamics

Since $\mathbf{S}(\mathbf{p}, \phi)$ depends smoothly on ϵ it can be expanded in this parameter.

$$\mathbf{S}(\mathbf{p}, \phi) = \mathbf{T}(\mathbf{p}) + \epsilon \mathbf{U}(\mathbf{p}, \phi) + O(\epsilon^2) \quad (1.8)$$

The zeroth order terms are assumed to be known already and are given by

$$\mathbf{T}(\mathbf{p}) = \begin{bmatrix} t^1(\mathbf{p}) \\ \vdots \\ t^{2n}(\mathbf{p}) \end{bmatrix}, \quad t^i(\mathbf{p}) = \sum_{\mathbf{m} \in \mathbb{N}^l} T_{\mathbf{m}}^i \mathbf{p}^{\mathbf{m}} \quad (1.9)$$

At first order in epsilon we also expand the non-autonomous function in terms of multinomials.

$$\mathbf{U}(\mathbf{p}, \phi) = \begin{bmatrix} u^1(\mathbf{p}, \phi) \\ \vdots \\ u^{2n}(\mathbf{p}, \phi) \end{bmatrix}, \quad u^i(\mathbf{p}, \phi) = \sum_{\mathbf{m} \in \mathbb{N}^l} U_{\mathbf{m}}^i(\phi) \mathbf{p}^{\mathbf{m}} \quad (1.10)$$

The same exact method is used to express the reduced dynamics.

$$\mathbf{R}(\mathbf{p}, \phi) = \mathbf{P}(\mathbf{p}) + \epsilon \mathbf{Q}(\mathbf{p}, \phi) + O(\epsilon^2) \quad (1.11)$$

$$\mathbf{P}(\mathbf{p}) = \begin{bmatrix} p^1(\mathbf{p}) \\ \vdots \\ p^l(\mathbf{p}) \end{bmatrix}, \quad p^i(\mathbf{p}) = \sum_{\mathbf{m} \in \mathbb{N}^l} P_{\mathbf{m}}^i \mathbf{p}^{\mathbf{m}} \quad (1.12)$$

$$\mathbf{Q}(\mathbf{p}, \phi) = \begin{bmatrix} q^1(\mathbf{p}, \phi) \\ \vdots \\ q^l(\mathbf{p}, \phi) \end{bmatrix}, \quad q^i(\mathbf{p}, \phi) = \sum_{\mathbf{m} \in \mathbb{N}^l} Q_{\mathbf{m}}^i(\phi) \mathbf{p}^{\mathbf{m}} \quad (1.13)$$

1.2.3 Nonlinearity

The nonlinearity is given as a Taylor expansion on coordinates of the full-phase space.

$$\mathbf{G}(\mathbf{x}) = \begin{bmatrix} g^1(\mathbf{x}) \\ \vdots \\ g^{2n}(\mathbf{x}) \end{bmatrix}, \quad g^i(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}^{2n}} G_{\mathbf{n}}^i \mathbf{x}^{\mathbf{n}} \quad (1.14)$$

Consequently we expand the nonlinearity \mathbf{G} composed with the SSM parametrisation in epsilon around $\epsilon = 0$ as well.

$$\begin{aligned}
\mathbf{G} \circ \mathbf{S}(\mathbf{p}, \phi) &= \mathbf{G} \circ \mathbf{S}(\mathbf{p}, \phi)|_{\epsilon=0} + \epsilon \frac{\partial}{\partial \epsilon} \mathbf{G} \circ \mathbf{S}(\mathbf{p}, \phi)|_{\epsilon=0} + O(\epsilon^2) \\
&= \mathbf{G}(\mathbf{T}(\mathbf{p})) + \epsilon [\mathbf{D}_{\mathbf{x}} \mathbf{G} \circ \mathbf{T}(\mathbf{p})] \frac{\partial \mathbf{x}}{\partial \epsilon} \Big|_{\epsilon=0} + O(\epsilon^2) \\
&= \mathbf{G}(\mathbf{T}(\mathbf{p})) + \epsilon [\mathbf{D}_{\mathbf{x}} \mathbf{G} \circ \mathbf{T}(\mathbf{p})] \mathbf{U}(\mathbf{p}, \phi) + O(\epsilon^2) \tag{1.15}
\end{aligned}$$

1.2.4 Forcing function

The quasiperiodic force is given as a Taylor expansion on coordinates of the full-phase space.

$$\mathbf{F}(\mathbf{x}, \phi) = \begin{bmatrix} f^1(\mathbf{x}, \phi) \\ \vdots \\ f^{2n}(\mathbf{x}, \phi) \end{bmatrix}, \quad f^i(\mathbf{x}, \phi) = \sum_{\mathbf{n} \in \mathbb{N}^{2n}} F_{\mathbf{n}}^i(\phi) \mathbf{x}^{\mathbf{n}} \tag{1.16}$$

1.2.5 Fourier expansion of coefficients

The force coefficients are given as a fourier expansion in terms of the phase variable $\boldsymbol{\eta}$. The SSM- and reduced dynamics-coefficients are then expanded in this basis as well.

$$F_{\mathbf{k}}^b(\phi) = \sum_{\boldsymbol{\eta} \in \mathbb{Z}^k} F_{\mathbf{k}, \boldsymbol{\eta}}^b e^{i\langle \boldsymbol{\eta}, \phi \rangle} \tag{1.17}$$

$$U_{\mathbf{k}}^b(\phi) = \sum_{\boldsymbol{\eta} \in \mathbb{Z}^k} U_{\mathbf{k}, \boldsymbol{\eta}}^b e^{i\langle \boldsymbol{\eta}, \phi \rangle} \tag{1.18}$$

$$Q_{\mathbf{k}}^b(\phi) = \sum_{\boldsymbol{\eta} \in \mathbb{Z}^k} Q_{\mathbf{k}, \boldsymbol{\eta}}^b e^{i\langle \boldsymbol{\eta}, \phi \rangle} \tag{1.19}$$

2 Order epsilon invariance equation

The Taylor expansion terms at order ϵ can be plugged into the invariance eq. (??) to obtain

$$\begin{aligned}
\mathbf{B} \left(\overbrace{\mathbf{D}_{\mathbf{p}}(\mathbf{T}(\mathbf{p}))\mathbf{Q}(\mathbf{p}, \phi)}^{\text{Term 1}} + \overbrace{(\partial_{\mathbf{p}}\mathbf{U}(\mathbf{p}, \phi))\mathbf{P}(\mathbf{p})}^{\text{Term 2}} + \overbrace{(\partial_{\phi}\mathbf{U}(\mathbf{p}, \phi))\boldsymbol{\Omega}}^{\text{Term 3}} \right) = \tag{2.1} \\
\overbrace{\mathbf{A}\mathbf{U}(\mathbf{p}, \phi)}^{\text{Term 4}} + \overbrace{[\mathbf{D}_{\mathbf{x}}\mathbf{G} \circ \mathbf{T}(\mathbf{p})]\mathbf{U}(\mathbf{p}, \phi)}^{\text{Term 5}} + \overbrace{\mathbf{F}(\phi, \mathbf{S}(\mathbf{p}, \phi))}^{\text{Term 6}}
\end{aligned}$$

We will now calculate each of these terms using the multi-index expansions of the functions as given in section ?? and extract their contribution to the invariance equation for multi-index $\mathbf{k} \in \mathbb{N}^l$.

2.1 Term 1

We begin by evaluating the Jacobian.

$$\begin{aligned}
[D_{\mathbf{p}}(\mathbf{T}(\mathbf{p}))]_{ij} &= \partial_j t^i(\mathbf{p}) \\
&= \partial_j \sum_{\mathbf{m} \in \mathbb{N}^l} T_{\mathbf{m}}^i \mathbf{p}^{\mathbf{m}} \\
&= \sum_{\mathbf{m} \in \mathbb{N}^l} m_j T_{\mathbf{m}}^i \mathbf{p}^{\mathbf{m} - \hat{\mathbf{e}}_j}
\end{aligned} \tag{2.2}$$

This Jacobian then acts on the non autonomous reduced dynamics as follows.

$$\begin{aligned}
[D_{\mathbf{p}}(\mathbf{T}(\mathbf{p}))\mathbf{Q}(\mathbf{p}, \phi)] \Big|_{i, \mathbf{k}} &= \sum_{j=1}^l [D_{\mathbf{p}}(\mathbf{T}(\mathbf{p}))]_{ij} \mathbf{Q}(\mathbf{p}, \phi)_j \Big|_{\mathbf{k}} \\
&= \sum_{j=1}^l \left(\sum_{\mathbf{m} \in \mathbb{N}^l} m_j T_{\mathbf{m}}^i \mathbf{p}^{\mathbf{m} - \hat{\mathbf{e}}_j} \right) \sum_{\mathbf{u} \in \mathbb{N}^l} Q_{\mathbf{u}}^j(\phi) \mathbf{p}^{\mathbf{u}} \Big|_{\mathbf{k}} \\
&= \sum_{j=1}^l \sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j T_{\mathbf{m}}^i Q_{\mathbf{u}}^j(\phi) \mathbf{p}^{\mathbf{k}}
\end{aligned} \tag{2.3}$$

2.2 Term 2

The evaluation of this term is analogous to Term 1 and results in the following expression.

$$[\partial_{\mathbf{p}}(\mathbf{U}(\mathbf{p}, \phi))\mathbf{P}(\mathbf{p})] \Big|_{i, \mathbf{k}} = \sum_{j=1}^l \sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j U_{\mathbf{m}}^i(\phi) P_{\mathbf{u}}^j \mathbf{p}^{\mathbf{k}} \tag{2.4}$$

2.3 Term 3

This term explicitly evaluates to be

$$\partial_{\phi} \mathbf{U}(\mathbf{p}, \phi) \Omega \Big|_{i, \mathbf{k}} = D_{\phi} U_{\mathbf{k}}^i(\phi) \Omega \mathbf{p}^{\mathbf{k}} \tag{2.5}$$

2.4 Term 4

Much like the last term this part is rather straightforward and does not need further explanation.

$$\mathbf{A}\mathbf{U}(\mathbf{p}, \phi) \Big|_{b, \mathbf{k}} = \sum_{i=1}^{2n} (\mathbf{A})_{bi} U_{\mathbf{k}}^i(\phi) \mathbf{p}^{\mathbf{k}} \quad (2.6)$$

2.5 Term 5

To facilitate the derivation of this term we will quickly perform some pre-calculations introducing some notation and mathematical gear to deal with multiplication of power series.

2.5.1 Multiplication of power series

Suppose we want to take a scalar power s of an arbitrary power Series $\sum_{\mathbf{u}} T_{\mathbf{u}}^i \mathbf{p}^{\mathbf{u}}$. A convenient method, which offers a recursive definition of coefficients to the multivariate monomials of the multiplied power series is given by the concept of the radial derivative, which is applied in the book of Haro et. al ([**haro**]). Ponsioen ([**doctsten**]) uses this to express the coefficients $H_{i,s,\mathbf{h}}$ in the expansion

$$\left(\sum_{\mathbf{u} \in \mathbb{N}^l} T_{\mathbf{u}}^i \mathbf{p}^{\mathbf{u}} \right)^s = \sum_{\mathbf{h} \in \mathbb{N}^l} H_{s,\mathbf{h}}^i \mathbf{p}^{\mathbf{h}} \quad (2.7)$$

using the recursion defined by

$$H_{s,\mathbf{h}}^i = \frac{s}{h_j} \sum_{\substack{\mathbf{u} \in \mathbb{N}^l \\ \mathbf{u} \leq \mathbf{h}}} u_j T_{\mathbf{u}}^i H_{s-1,\mathbf{h}-\mathbf{u}}^i \quad (2.8)$$

where j is the index of the smallest nonzero element in \mathbf{h} . To simplify the calculation of the recursion it is useful to look at some special cases, under assumption that the coefficients $T_{\mathbf{u}}^i$ are the coefficients of the SSM expansion of row i . The SSM parametrisation does not contain constant terms which means that all composition-coefficients are zero for $|\mathbf{u}| = 0$. The properties of $H_{s,\mathbf{h}}^i$ are then

- $H_{1,\mathbf{0}}^i = 0$
- $H_{1,\mathbf{h}}^i = T_{\mathbf{h}}^i$
- $H_{0,\mathbf{h}}^i = 0$ for all $\mathbf{h} \neq \mathbf{0}$,
- $H_{0,\mathbf{0}}^i = 1$

2.5.2 Introducing notation for multiplication of power series

We plug in the SSM-expansion into $\mathbf{x}^{\mathbf{n}} = (\mathbf{S}(\mathbf{p}, \phi))^{\mathbf{n}}$ and only keep the zeroth order terms in epsilon.

$$\begin{aligned}
\mathbf{x}^{\mathbf{n}} &= (\mathbf{T}(\mathbf{p}) + \epsilon \mathbf{U}(\mathbf{p}, \phi) + O(\epsilon^2))^{\mathbf{n}} \\
&= \prod_{f=1}^{2n} \mathbf{T}^f(\mathbf{p})^{n_f} + O(\epsilon) \\
&= \prod_{f=1}^{2n} \left(\sum_{\mathbf{m} \in \mathbb{N}^l} T_{\mathbf{m}}^f \mathbf{p}^{\mathbf{m}} \right)^{n_f} + O(\epsilon) \\
&= \prod_{f=1}^{2n} \left(\sum_{\mathbf{h} \in \mathbb{N}^l} H_{n_f, \mathbf{h}}^f \mathbf{p}^{\mathbf{h}} \right) + O(\epsilon) \\
&= \sum_{\mathbf{g} \in \mathbb{N}^l} \underbrace{\left(\left(\sum_{\mathbf{h}_1 \in \mathbb{N}^l} \cdots \sum_{\mathbf{h}_{2n} \in \mathbb{N}^l} \right) \Big|_{\substack{\sum \mathbf{h}_i = \mathbf{g} \\ |\mathbf{h}_i| \geq n_i}} H_{n_1, \mathbf{h}_1}^1 \cdots H_{n_{2n}, \mathbf{h}_{2n}}^{2n} \right)}_{=: \pi_{\mathbf{n}, \mathbf{g}}} \mathbf{p}^{\mathbf{g}} + O(\epsilon) \quad (2.9)
\end{aligned}$$

The condition $|\mathbf{h}_i| \geq n_i$ comes from the fact that the SSM is tangent to the origin and hence the multi-index expansion of its parametrisation does not contain any constant terms. This translates directly into this criterion. Now let's suppose we would like to extract all terms that correspond to the multinomial $\mathbf{p}^{\mathbf{k}}$. Then we would have to select all combinations of terms where the exponentiated multi-indices sum up to \mathbf{k} .

$$\mathbf{x}^{\mathbf{n}} \Big|_{\mathbf{k}} = \pi_{\mathbf{n}, \mathbf{k}} \mathbf{p}^{\mathbf{k}} + O(\epsilon) \quad (2.10)$$

We note the following property of $\pi_{\mathbf{n}, \mathbf{k}}$: for $\mathbf{k} = \mathbf{0}$ this term is only nonzero if $\mathbf{n} = \mathbf{0}$ and then it evaluates to $\pi_{\mathbf{0}, \mathbf{0}} = 1$.

2.5.3 Coefficients of the nonlinearity

The jacobian of the nonlinearity is given as

$$\begin{aligned}
[\mathbf{D}_{\mathbf{x}} \mathbf{G}(\mathbf{x})]_{ij} &= \partial_{x_j} \mathbf{G}(\mathbf{x})_i \\
&= \sum_{\substack{\mathbf{n} \in \mathbb{N}^{2n} \\ |\mathbf{n}| \geq 2}} G_{\mathbf{n}}^i \partial_{x_j} \mathbf{x}^{\mathbf{n}} \\
&= \sum_{\substack{\mathbf{n} \in \mathbb{N}^{2n} \\ |\mathbf{n}| \geq 2}} n_j G_{\mathbf{n}}^i \mathbf{x}^{\mathbf{n} - \hat{\mathbf{e}}_j} \quad (2.11)
\end{aligned}$$

We are now set to tackle the problem of calculating the coefficients of the derivative of the nonlinearity.

$$[D_{\mathbf{x}}\mathbf{G} \circ \mathbf{T}(\mathbf{p})]_{ij} \Big|_{\mathbf{k}} = \sum_{\substack{\mathbf{n} \in \mathbb{N}^{2n} \\ |\mathbf{n}| \geq 2}} n_j G_{\mathbf{n}}^i \pi_{\mathbf{n}-\hat{\mathbf{e}}_j, \mathbf{k}} \mathbf{p}^{\mathbf{k}} \quad (2.12)$$

For latter implementation it is crucial to note that for non-positive $\mathbf{n} - \hat{\mathbf{e}}_j$ there is no contribution as this implies $n_j = 0$. Eventually we calculate the action of this jacobian on the non-autonomous part of the SSM-expansion and look at the result in row i.

$$\begin{aligned} [D_{\mathbf{x}}\mathbf{G} \circ \mathbf{T}(\mathbf{p})] \mathbf{U}(\mathbf{p}, \phi) \Big|_{i, \mathbf{k}} &= \sum_{j=1}^{2n} [D_{\mathbf{x}}\mathbf{G} \circ \mathbf{T}(\mathbf{p})]_{ij} [\mathbf{U}(\mathbf{p}, \phi)]_j \Big|_{\mathbf{k}} \\ &= \sum_{j=1}^{2n} \left[\sum_{\substack{\mathbf{n} \in \mathbb{N}^{2n} \\ |\mathbf{n}| \geq 2}} n_j G_{\mathbf{n}}^i \left(\sum_{\mathbf{g} \in \mathbb{N}^l} \pi_{\mathbf{n}-\hat{\mathbf{e}}_j, \mathbf{g}} \mathbf{p}^{\mathbf{g}} \right) \right] \left[\sum_{\mathbf{m} \in \mathbb{N}^l} U_{\mathbf{m}}^j(\phi) \mathbf{p}^{\mathbf{m}} \right] \Big|_{\mathbf{k}} \\ &= \sum_{\substack{\mathbf{n} \in \mathbb{N}^{2n} \\ |\mathbf{n}| \geq 2}} G_{\mathbf{n}}^i \sum_{j=1}^{2n} n_j \left(\sum_{\substack{\mathbf{g}, \mathbf{m} \in \mathbb{N}^l \\ \mathbf{g} + \mathbf{m} = \mathbf{k}}} \pi_{\mathbf{n}-\hat{\mathbf{e}}_j, \mathbf{g}} U_{\mathbf{m}}^j(\phi) \right) \mathbf{p}^{\mathbf{k}} \end{aligned} \quad (2.13)$$

It is important to note, that the sum over \mathbf{n} includes all multi-indices with magnitude smaller or equal to $k + 1$, as for all others π is always zero.

2.6 Term 6

We make use of the force given in its taylor expanded form and extract the term corresponding to multi-index $\mathbf{k} \in \mathbb{N}^l$.

$$\begin{aligned} f^i(\mathbf{x}, \phi) \Big|_{\mathbf{k}} &= \sum_{\mathbf{n} \in \mathbb{N}^{2n}} F_{\mathbf{n}}^i(\phi) \mathbf{x}^{\mathbf{n}} \Big|_{\mathbf{k}} \\ &= \sum_{\mathbf{n} \in \mathbb{N}^{2n}} F_{\mathbf{n}}^i(\phi) \left(\sum_{\mathbf{g} \in \mathbb{N}^l} \pi_{\mathbf{n}, \mathbf{g}} \mathbf{p}^{\mathbf{g}} \right) \Big|_{\mathbf{k}} + O(\epsilon) \\ &= \sum_{\mathbf{n} \in \mathbb{N}^{2n}} F_{\mathbf{n}}^i(\phi) \pi_{\mathbf{n}, \mathbf{k}} \mathbf{p}^{\mathbf{k}} + O(\epsilon) \end{aligned} \quad (2.14)$$

3 Order epsilon coefficient equation

We have now calculated the components of the invariance equation which gives rise to the order epsilon coefficient equation for multi-index \mathbf{k} .

$$\begin{aligned}
& \sum_{i=1}^{2n} \mathbf{B}_{bi} \left[\sum_{j=1}^l \sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j \left(T_{\mathbf{m}}^i Q_{\mathbf{u}}^j(\phi) + U_{\mathbf{m}}^i(\phi) P_{\mathbf{u}}^j \right) + D_{\phi} U_{\mathbf{k}}^i(\phi) \Omega \right] \\
&= \sum_{i=1}^{2n} (\mathbf{A})_{bi} U_{\mathbf{k}}^i(\phi) + \sum_{\substack{\mathbf{n} \in \mathbb{N}^{2n} \\ |\mathbf{n}| \geq 2}} G_{\mathbf{n}}^b \sum_{j=1}^{2n} n_j \left(\sum_{\substack{\mathbf{g}, \mathbf{m} \in \mathbb{N}^l \\ \mathbf{g} + \mathbf{m} = \mathbf{k}}} \pi_{\mathbf{n} - \hat{\mathbf{e}}_j, \mathbf{g}} U_{\mathbf{m}}^j(\phi) \right) + \sum_{\mathbf{n} \in \mathbb{N}^{2n}} F_{\mathbf{n}}^b(\phi) \pi_{\mathbf{n}, \mathbf{k}}
\end{aligned} \tag{3.1}$$

4 Coefficient equation for $\mathbf{k}=\mathbf{0}$

In this section we solve the order epsilon coefficient equation for the case where $\mathbf{k} = \mathbf{0}$. While in the autonomous case this case is not relevant due to the tangency of the SSM to the origin, in the non-autonomous setting terms at this order can be nonzero.

The zeroth order coefficient equation reads

$$\begin{aligned}
& \sum_{i=1}^{2n} \mathbf{B}_{bi} \left[\sum_{j=1}^l \sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{0}}} m_j \left(T_{\mathbf{m}}^i Q_{\mathbf{u}}^j(\phi) + U_{\mathbf{m}}^i(\phi) P_{\mathbf{u}}^j \right) + D_{\phi} U_{\mathbf{0}}^i(\phi) \Omega \right] \\
&= \sum_{i=1}^{2n} (\mathbf{A})_{bi} U_{\mathbf{0}}^i(\phi) + \sum_{\substack{\mathbf{n} \in \mathbb{N}^{2n} \\ |\mathbf{n}| \geq 2}} G_{b, \mathbf{n}} \sum_{j=1}^{2n} n_j \left(\pi_{\mathbf{n} - \hat{\mathbf{e}}_j, \mathbf{0}} U_{\mathbf{0}}^j(\phi) \right) + \sum_{\mathbf{n} \in \mathbb{N}^{2n}} F_{b, \mathbf{0}}(\phi)
\end{aligned} \tag{4.1}$$

Since the nonlinearity does not include any term with $|\mathbf{n}| < 2$ it vanishes as $\pi_{\mathbf{n}, \mathbf{0}}$ is zero unless $\mathbf{n} = \mathbf{0}$. Furthermore the multi-indices \mathbf{m}, \mathbf{u} summed over in the first term are nonnegative. Thus all combinations of \mathbf{m}, \mathbf{u} where $\mathbf{m} \neq \hat{\mathbf{e}}_j$ vanish. Additionally the autonomous reduced dynamics do not include any term contributing at zeroth order. The remaining terms are then given by

$$\sum_{i=1}^{2n} \mathbf{B}_{bi} \left[\sum_{j=1}^l T_{\hat{\mathbf{e}}_j}^i Q_{\mathbf{0}}^j(\phi) + D_{\phi} U_{\mathbf{0}}^i(\phi) \Omega \right] = \sum_{i=1}^{2n} (\mathbf{A})_{bi} U_{\mathbf{0}}^i(\phi) + F_{b, \mathbf{0}}(\phi) \tag{4.2}$$

Upon comparison of Fourier-coefficients for frequency multi-index $\boldsymbol{\eta}$ this gives

$$\sum_{i=1}^{2n} ((\mathbf{A})_{bi} - i \langle \boldsymbol{\eta}, \Omega \rangle \mathbf{B}_{bi}) U_{\mathbf{0}, \boldsymbol{\eta}}^i = \sum_{i=1}^{2n} \sum_{j=1}^l \mathbf{B}_{bi} T_{\hat{\mathbf{e}}_j}^i Q_{\mathbf{0}, \boldsymbol{\eta}}^j - F_{b, \mathbf{0}, \boldsymbol{\eta}} \tag{4.3}$$

We recall that the first order autonomous SSM coefficients are given by the right eigenvector of the generalised eigenvector problem of the linearised system, which gives rise to the final form of the $\mathbf{k} = \mathbf{0}$ coefficient equation.

$$\sum_{i=1}^{2n} \underbrace{((\mathbf{A})_{bi} - i\langle \boldsymbol{\eta}, \boldsymbol{\Omega} \rangle \mathbf{B}_{bi})}_{:= (\mathcal{L}_{\mathbf{0}, \boldsymbol{\eta}})_{bi}} U_{\mathbf{0}, \boldsymbol{\eta}}^i = \sum_{j=1}^l (\mathbf{B} \mathbf{v}_j)_b Q_{\mathbf{0}, \boldsymbol{\eta}}^j - F_{b, \mathbf{0}, \boldsymbol{\eta}} \quad (4.4)$$

The matrix $\mathcal{L}_{\mathbf{0}, \boldsymbol{\eta}}$ is the coefficient matrix for the frequency multi-index $\boldsymbol{\eta}$ at order $\mathbf{k} = \mathbf{0}$.

4.1 Reduced dynamics for $\mathbf{k}=\mathbf{0}$

The zeroth order non-autonomous reduced dynamics are chosen such that the right hand side of eq. (??) projected onto the near left kernel of $\mathcal{L}_{\mathbf{0}, \boldsymbol{\eta}}$ is zero. This ensures the SSM coefficients do not explode in magnitude.

Assume there are K frequency multi-indices $\boldsymbol{\eta}$ that contribute to the zeroth order force expansion, and we number them by indices increasing up to K . We then define two index sets

$$E := \{e_1, \dots, e_{r_{ext}} \in \{1, \dots, l\}\} \quad (4.5)$$

$$F := \{f_1, \dots, f_{r_{ext}} \in \{1, \dots, K\}\} \quad (4.6)$$

such that all combinations of master mode eigenvalues and frequency multi-indices fulfilling

$$\lambda_j - i\langle \boldsymbol{\eta}_f, \boldsymbol{\Omega} \rangle \approx 0, \quad e \in \{1, \dots, l\}, \quad f \in \{1, \dots, K\} \quad (4.7)$$

are included as an index pair in this list. There are a total of r_{ext} such resonances. Note that for damped systems expression ?? will never be exactly zero.

Lets have a look at a pair (e_i, f_i) . The projection of the right hand side of eq. (??) onto the near left kernel of $\mathcal{L}_{\mathbf{0}, \boldsymbol{\eta}_{f_i}}$ is given as

$$\sum_{b=1}^{2n} (\mathbf{w}_{e_i}^*)_b \left(\sum_{j=1}^l (\mathbf{B} \mathbf{v}_j)_b Q_{\mathbf{0}, \boldsymbol{\eta}_{f_i}}^j - F_{\mathbf{0}, \boldsymbol{\eta}_{f_i}}^b \right) = 0 \quad (4.8)$$

Exploiting the normation of the generalised eigenvectors this equation transforms to

$$\begin{aligned} \sum_{j=1}^l (\mathbf{w}_{e_i}^* \mathbf{B} \mathbf{v}_j) Q_{\mathbf{0}, \boldsymbol{\eta}_{f_i}}^j &= \delta_{j e_i} Q_{\mathbf{0}, \boldsymbol{\eta}_{f_i}}^j \\ &= \sum_{b=1}^{2n} (\mathbf{w}_{e_i}^*)_b F_{\mathbf{0}, \boldsymbol{\eta}_{f_i}}^b \end{aligned} \quad (4.9)$$

and thus for the resonance corresponding to the index pair (e_i, f_i) we set

$$Q_{\mathbf{0}, \eta_{f_i}}^{e_i} = \langle \mathbf{w}_{e_i} \begin{bmatrix} F_{\mathbf{0}, \eta_{f_i}}^1 \\ \vdots \\ F_{\mathbf{0}, \eta_{f_i}}^{2n} \end{bmatrix} \rangle \quad (4.10)$$

The r_{ext} resonances thus lead to r_{ext} reduced dynamics coefficients that are set nonzero.

5 Coefficient equation for \mathbf{k} bigger 0

We now turn to solving for the non-autonomous SSM- and reduced dynamics coefficients for the case of $\mathbf{k} > \mathbf{0}$. We assume that the coefficients for all multi-indices with magnitude smaller than k have already been calculated and we now consider the coefficient equation for some \mathbf{k} with $|\mathbf{k}| = k$.

Initially we will make some comments on the individual terms of eq. (??) to clarify our calculations.

1. Term 2: This term is split in two parts, one including all order k contributions of the non-autonomous SSM coefficients and another one containing all lower order contributions.

$$\begin{aligned} \sum_{j=1}^l \sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j U_{\mathbf{m}}^i(\phi) P_{\mathbf{u}}^j &= \sum_{j=1}^l \sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ |\mathbf{m}| < k \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j U_{\mathbf{m}}^i(\phi) P_{\mathbf{u}}^j \\ &+ \sum_{j=1}^l k_j U_{\mathbf{k}}^i(\phi) \lambda_j \end{aligned} \quad (5.1)$$

Since the autonomous reduced dynamics at linear order are diagonal there are no other highest order SSM-coefficients contributing in the first term on the right hand side.

2. Term 5: Since $H_{s, \mathbf{0}}^i \neq 0 \Leftrightarrow s \neq 0$ the term $\pi_{\mathbf{n} - \hat{\mathbf{e}}_j, \mathbf{g}} U_{\mathbf{m}}^j(\phi)$ with $\mathbf{m} = \mathbf{k}, \mathbf{g} = \mathbf{0}$ is nonzero only if $\mathbf{n} = \hat{\mathbf{e}}_j$ which is never the case since we treat the non-linearity. Thus this term does not include any unknowns.

This allows us to rewrite the invariance equation to give the multi-index $\mathbf{k} > \mathbf{0}$ non-autonomous coefficient equation.

$$\begin{aligned}
& \sum_{i=1}^{2n} \left((\mathbf{A})_{bi} U_{\mathbf{k}}^i(\phi) - \mathbf{B}_{bi} \left[\sum_{j=1}^l k_j U_{\mathbf{k}}^i(\phi) \lambda_j + D_\phi U_{\mathbf{k}}^i(\phi) \Omega \right] \right) \\
&= \sum_{i=1}^{2n} \mathbf{B}_{bi} \sum_{j=1}^l \left[\sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j T_{i,\mathbf{m}}^i Q_{\mathbf{u}}^j(\phi) + \sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ |\mathbf{m}| < k \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j U_{\mathbf{m}}^i(\phi) P_{\mathbf{u}}^j \right] \\
&- \sum_{\mathbf{n} \in \mathbb{N}^{2n}} F_{\mathbf{n}}^b(\phi) \pi_{\mathbf{n}, \mathbf{k}} - \sum_{\substack{\mathbf{n} \in \mathbb{N}^{2n} \\ |\mathbf{n}| \geq 2}} G_{\mathbf{n}}^b \sum_{j=1}^{2n} n_j \left(\sum_{\substack{\mathbf{g}, \mathbf{m} \in \mathbb{N}^l \\ \mathbf{g} + \mathbf{m} = \mathbf{k}}} \pi_{\mathbf{n} - \hat{\mathbf{e}}_j, \mathbf{g}} U_{\mathbf{m}}^j(\phi) \right) \quad (5.2)
\end{aligned}$$

The forcing, SSM and reduced dynamics are treated in their Fourier-expansion form.

The fourier expanded coefficients are plugged into eq. (??). We analyse the fourier components corresponding to a frequency multi-index $\boldsymbol{\eta}$. The nonlinearity term is evaluated as follows.

$$\begin{aligned}
& \sum_{\substack{\mathbf{n} \in \mathbb{N}^{2n} \\ |\mathbf{n}| \geq 2}} G_{\mathbf{n}}^b \sum_{j=1}^{2n} n_j \left(\sum_{\substack{\mathbf{g}, \mathbf{m} \in \mathbb{N}^l \\ \mathbf{g} + \mathbf{m} = \mathbf{k}}} \pi_{\mathbf{n} - \hat{\mathbf{e}}_j, \mathbf{g}} U_{\mathbf{m}}^j(\phi) \right) \Big|_{\boldsymbol{\eta}} \\
&= \sum_{\substack{\mathbf{n} \in \mathbb{N}^{2n} \\ |\mathbf{n}| \geq 2}} G_{\mathbf{n}}^b \sum_{j=1}^{2n} n_j \left(\sum_{\mathbf{m} \in \mathbb{N}^l} \pi_{\mathbf{n} - \hat{\mathbf{e}}_j, \mathbf{k} - \mathbf{m}} \left(\sum_{\boldsymbol{\gamma} \in \mathbb{Z}^k} U_{\mathbf{m}, \boldsymbol{\gamma}}^j e^{i\langle \boldsymbol{\gamma}, \phi \rangle} \right) \right) \Big|_{\boldsymbol{\eta}} \\
&= \sum_{\substack{\mathbf{n} \in \mathbb{N}^{2n} \\ |\mathbf{n}| \geq 2}} G_{\mathbf{n}}^b \sum_{j=1}^{2n} n_j \underbrace{\left(\sum_{\mathbf{m} \in \mathbb{N}^l} \pi_{\mathbf{n} - \hat{\mathbf{e}}_j, \mathbf{k} - \mathbf{m}} U_{\mathbf{m}, \boldsymbol{\eta}}^j \right)}_{:= \sigma_{\mathbf{k}, \mathbf{n}, \boldsymbol{\eta}}} e^{i\langle \boldsymbol{\eta}, \phi \rangle} \\
&= \sum_{\substack{\mathbf{n} \in \mathbb{N}^{2n} \\ |\mathbf{n}| \geq 2}} G_{\mathbf{n}}^b \sigma_{\mathbf{k}, \mathbf{n}, \boldsymbol{\eta}} \quad (5.3)
\end{aligned}$$

We arrive at the order \mathbf{k} and fourier component $\boldsymbol{\eta}$ coefficient equation.

$$\begin{aligned}
& \sum_{i=1}^{2n} \underbrace{\left((\mathbf{A})_{bi} - \mathbf{B}_{bi} \left[\sum_{j=1}^l k_j \lambda_j + i \langle \boldsymbol{\Omega}, \boldsymbol{\eta} \rangle \right] \right)}_{:= (\mathcal{L}_{\mathbf{k}, \boldsymbol{\eta}})_{bi}} U_{\mathbf{k}, \boldsymbol{\eta}}^i \\
&= \sum_{i=1}^{2n} \mathbf{B}_{bi} \sum_{j=1}^l \left[\sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j T_{\mathbf{m}}^i Q_{\mathbf{u}, \boldsymbol{\eta}}^j + \sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ |\mathbf{m}| < k \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j U_{\mathbf{m}, \boldsymbol{\eta}}^i P_{\mathbf{u}}^j \right] \\
&\quad - \sum_{\mathbf{n} \in \mathbb{N}^{2n}} F_{\mathbf{n}, \boldsymbol{\eta}}^b \pi_{\mathbf{n}, \mathbf{k}} - \sum_{\substack{\mathbf{n} \in \mathbb{N}^{2n} \\ |\mathbf{n}| \geq 2}} G_{\mathbf{n}}^b \sigma_{\mathbf{k}, \mathbf{n}, \boldsymbol{\eta}}
\end{aligned} \tag{5.4}$$

5.1 Reduced dynamics for \mathbf{k} bigger $\mathbf{0}$

Analogously to the case where $\mathbf{k} = \mathbf{0}$ we choose the reduced dynamics such that the projection of the right hand side of eq. (??) onto the near left kernel of $\mathcal{L}_{\mathbf{k}, \boldsymbol{\eta}}$ is zero.

Assume there are d_{ext} distinct resonances of the form

$$\lambda_e - \left(\sum_{j=1}^l k_j \lambda_j + i \langle \boldsymbol{\Omega}, \boldsymbol{\eta}_f \rangle \right) \approx 0 \tag{5.5}$$

We then define two index sets containing all pairs (e, f) that satisfy this condition. We use the same names of indices as for the $\mathbf{k} = \mathbf{0}$ treatment for better readability, their values are of course not the same in general.

$$E := \{e_1, \dots, e_{d_{ext}} \in \{1, \dots, l\}\} \tag{5.6}$$

$$F := \{f_1, \dots, f_{d_{ext}} \in \{1, \dots, K\}\} \tag{5.7}$$

The number K denotes the number of different frequency multi-indices that contribute to the order \mathbf{k} non-autonomous forcing. For a pair (e_i, f_i) we thus demand that

$$\begin{aligned}
& \sum_{b=1}^{2n} (\mathbf{w}_{e_i}^*)_b \left(\sum_{i=1}^{2n} \mathbf{B}_{bi} \sum_{j=1}^l \left[\sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j T_{\mathbf{m}}^i R_{\mathbf{u}, \boldsymbol{\eta}_{f_i}}^j + \sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ |\mathbf{m}| < k \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j U_{\mathbf{m}, \boldsymbol{\eta}_{f_i}}^i P_{\mathbf{u}}^j \right] \right) \\
&= \sum_{b=1}^{2n} (\mathbf{w}_{e_i}^*)_b \left(\sum_{\mathbf{n} \in \mathbb{N}^{2n}} F_{\mathbf{n}, \boldsymbol{\eta}_{f_i}}^b \pi_{\mathbf{n}, \mathbf{k}} + \sum_{\substack{\mathbf{n} \in \mathbb{N}^{2n} \\ |\mathbf{n}| \geq 2}} G_{\mathbf{n}}^b \sigma_{\mathbf{k}, \mathbf{n}, \boldsymbol{\eta}_{f_i}} \right) \\
&\quad \underbrace{\hspace{15em}}_{:= \mathcal{F}_{e_i, \mathbf{k}, \boldsymbol{\eta}_{f_i}}}
\end{aligned} \tag{5.8}$$

We now divide the first term of eq. (??) into a term containing the order k non-autonomous reduced dynamics and a second term containing all the other expressions.

$$\begin{aligned}
& \sum_{b=1}^{2n} (\mathbf{w}_{e_i}^*)_b \left(\sum_{i=1}^{2n} \mathbf{B}_{bi} \sum_{j=1}^l \sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j T_{\mathbf{m}}^i Q_{\mathbf{u}, \eta_{q_f^1}}^j \right) \\
&= \sum_{i=1}^{2n} (\mathbf{w}_{e_i}^* \mathbf{B})_i \sum_{j=1}^l \left(\sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ |\mathbf{m}|=1 \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j T_{\mathbf{m}}^i Q_{\mathbf{u}, \eta_{f_i}}^j \right. \\
&\quad \left. + \sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ |\mathbf{m}|>1 \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j T_{\mathbf{m}}^i Q_{\mathbf{u}, \eta_{f_i}}^j \right) \quad (5.9)
\end{aligned}$$

The term containing the highest order contribution can be simplified by making use of the properties of the autonomous SSM coefficients.

$$\begin{aligned}
& \sum_{j=1}^l \sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ |\mathbf{m}|=1 \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j T_{\mathbf{m}}^i Q_{\mathbf{u}, \eta_{f_i}}^j = \sum_{j=1}^l \sum_{f=1}^l \delta_{fj} T_{\hat{\mathbf{e}}_f}^i Q_{\mathbf{k} - \hat{\mathbf{e}}_f + \hat{\mathbf{e}}_j, \eta_{f_i}}^j \\
&= \sum_{j=1}^l (\mathbf{v}_j)_i Q_{\mathbf{k}, \eta_{f_i}}^j \quad (5.10)
\end{aligned}$$

$$\begin{aligned}
& \Rightarrow \sum_{i=1}^{2n} (\mathbf{w}_{e_i}^* \mathbf{B})_i \sum_{j=1}^l (\mathbf{v}_j)_i Q_{\mathbf{k}, \eta_{f_i}}^j = \sum_{j=1}^l \delta_{j e_i} Q_{\mathbf{k}, \eta_{f_i}}^j \\
&= Q_{\mathbf{k}, \eta_{f_i}}^{e_i} \quad (5.11)
\end{aligned}$$

Therefore the resonance corresponding to the index pair (e_i, f_i) leads to a non-zero reduced dynamics coefficient

$$\begin{aligned}
Q_{\mathbf{k}, \eta_{f_i}}^{e_i} &= - \sum_{i=1}^{2n} (\mathbf{w}_{e_i}^* \mathbf{B})_i \sum_{j=1}^l \left[\sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ |\mathbf{m}|<k \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j U_{\mathbf{m}, \eta_{f_i}}^i P_{\mathbf{u}}^j \right. \\
&\quad \left. + \sum_{\substack{\mathbf{m}, \mathbf{u} \in \mathbb{N}^l \\ |\mathbf{m}|>1 \\ \mathbf{m} + \mathbf{u} - \hat{\mathbf{e}}_j = \mathbf{k}}} m_j T_{\mathbf{m}}^i Q_{\mathbf{u}, \eta_{f_i}}^j \right] + \mathcal{F}_{e_i, \mathbf{k}, \eta_{f_i}} \quad (5.12)
\end{aligned}$$

This gives an explicit expression for the reduced dynamics Fourier-coefficients corresponding to a multi-index \mathbf{k} and a fourier component $\boldsymbol{\eta}_{f_i}$.