Goldbach's Conjecture — Towards the Inconsistency of Arithmetic

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Abstract. This paper proves an inconsistency in Peano arithmetic (PA). We express a strengthened form of the strong Goldbach conjecture and its negation by using a specific set that varies according to whether the conjecture or the negation is assumed. We show that, on the other hand, this set remains unchanged under these assumptions. This causes a contradiction.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even number greater than 6 is the sum of two different odd primes.*

Theorem. *PA is contradictory, i.e. the statement FALSE can be derived.*

Proof. We define the set $S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N};\ p, q \in \mathbb{P}_3,\ p < q;\ m = (p + q) / 2 \}$.

SSGB is equivalent to saying that every integer n ≥ 4 is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers n ≥ 4 appear as m in a middle component mk of S_g. So, by the definition of S_g we have

$$\text{SSGB} \iff \forall n \in \mathbb{N}_4 \ \ \exists (pk, mk, qk) \in S_g \quad n = m$$

$$\neg\text{SSGB} \iff \exists n \in \mathbb{N}_4 \ \ \forall (pk, mk, qk) \in S_g \quad n \neq m.$$

The set S_g has the following two properties.

First, the whole range of $\mathbb{N}_3$ can be expressed by the triple components of S_g ("*covering*"), because every integer x ≥ 3 can be written as some pk with k = 1 when x is prime, as some pk with k ≠ 1 when x is composite and not a power of 2, or as (3 + 5)k / 2 when x is a power of 2; $p \in \mathbb{P}_3$, $k \in \mathbb{N}$. So we have

(C) $\forall x \in \mathbb{N}_3 \ \ \exists (pk, mk, qk) \in S_g \quad x = pk \ \vee \ x = mk.$

A few examples of the covering:

x = 19: (**19·1**, 21·1, 23·1), (**19·1**, 60·1, 101·1)

x = 27: (**3·9**, 7·9, 11·9)

x = 42: (**3·14**, 5·14, 7·14), (**7·6**, 9·6, 11·6)

x = 4096: (3·1024, **4·1024**, 5·1024)

x = 10000: (**5·2000**, 6·2000, 7·2000).

Second, all pairs (p, q) of distinct odd primes are used in the definition of the set S_g ("*maximality*"). So we have

(M) $\forall$ p, q $\in \mathbb{P}_3$, p < q $\quad \forall$ k $\in \mathbb{N}$ $\qquad$ (pk, mk, qk) $\in S_g$, where m = (p + q) / 2.

$\neg$(C) would immediately imply $\neg$SSGB since an n ≥ 4 that is different from all S_g triple components pk and mk is in particular different from all m in S_g. So the property (C) excludes this possibility.

The property (M) excludes the possibility that if there is an n ≥ 4 different from all m in S_g, then n is the arithmetic mean of a pair of distinct odd primes not used in S_g. So (M) rules out the possibility that the question of whether SSGB holds or not depends on whether (M) holds or not. (The proof would no longer be possible if we left out any pair of distinct odd primes in the formulation of SSGB and S_g.)

Therefore, in both cases SSGB and $\neg$SSGB, neither $\neg$(C) nor $\neg$(M) applies.

The basic idea is now the following.

There are two possibilities for S_g, exactly one of which must occur: Either there is an n $\in \mathbb{N}_4$ in addition to all the numbers m defined in S_g or there is not. The latter is equivalent to SSGB and the former is equivalent to $\neg$SSGB.

Since, due to (M), an n ≥ 4 different from all m cannot be the arithmetic mean of a pair of primes not used in S_g and since, due to (C), this n equals a component of some S_g triple that exists by definition, the covering of $\mathbb{N}_3$ by the S_g triples in the case n exists ($\neg$SSGB) is equal to that in the case n does not exist (SSGB). This causes a contradiction because in the case SSGB the numbers m defined in S_g take all integer values x ≥ 4 whereas in the case $\neg$SSGB they don't.

$\wedge$

The following steps are independent of the choice of n if, in the case of $\neg$SSGB, there is more than one that is different from all m. For example, the minimal such n works.

We split S_g into two complementary subsets in the following way. For any $y \in \mathbb{N}_3$, we write

$S_g = S_g+(y) \cup S_g-(y)$, with

$S_g+(y) := \{ (pk, mk, qk) \in S_g \mid \exists\, k' \in \mathbb{N} \quad pk = yk' \;\vee\; mk = yk' \;\vee\; qk = yk' \}$

$S_g-(y) := \{ (pk, mk, qk) \in S_g \mid \forall\, k' \in \mathbb{N} \quad pk \neq yk' \;\wedge\; mk \neq yk' \;\wedge\; qk \neq yk' \}$.

We define

$S_1 := \{ (pk, mk, qk) \in S_g \mid \quad SSGB \;\wedge\; ((C) \wedge (M)) \}$

$S_2 := \{ (pk, mk, qk) \in S_g \mid \neg SSGB \;\wedge\; ((C) \wedge (M)) \}$.

Under the assumption $\neg$SSGB there is an $n \in \mathbb{N}_4$ as described above and under the assumption SSGB there is no such n. Then,

$$((\forall\, y \in \mathbb{N}_3 \qquad SSGB \;\Rightarrow\; S_1 = S_g+(y) \cup S_g-(y))$$
$$\wedge$$
$$(\qquad\qquad \neg SSGB \;\Rightarrow\; S_2 = S_g+(n) \cup S_g-(n)))$$

(1) $\quad\vee$

$$(\neg(C) \;\vee\; \neg(M)).$$

Since $\neg$(C) and $\neg$(M) are both ruled out and since $S_g+(n) \cup S_g-(n)$ is independent of n, we get

(1.1) $\forall\, y \in \mathbb{N}_3 \qquad SSGB \;\Rightarrow\; S_1 = S_g+(y) \cup S_g-(y)$

$\wedge$

(1.2) $\forall\, y \in \mathbb{N}_3 \qquad \neg SSGB \;\Rightarrow\; S_2 = S_g+(y) \cup S_g-(y)$.

Now, we will make use of the following principle.

If two sets of (possibly infinitely many) x-tuples are equal, then the sets of their corresponding i-th components are equal; $1 \leq i \leq x$.

To this end, for each $k \in \mathbb{N}$ we define

$M_1(k) := \{ mk \mid (pk, mk, qk) \in S_1 \}$

$M_2(k) := \{ mk \mid (pk, mk, qk) \in S_2 \}$.

Then, applying the principle above to the middle component of the triples (pk, mk, qk), ((1.1) $\wedge$ (1.2)) implies

(2.1) $\forall k \in \mathbb{N} \quad \forall y \in \mathbb{N}_3 \qquad SSGB \Rightarrow M_1(k) = \{ mk \mid (pk, mk, qk) \in S_g+(y) \cup S_g-(y) \}$

$\wedge$

(2.2) $\forall k \in \mathbb{N} \quad \forall y \in \mathbb{N}_3 \qquad \neg SSGB \Rightarrow M_2(k) = \{ mk \mid (pk, mk, qk) \in S_g+(y) \cup S_g-(y) \}$.

Setting $M_1 := M_1(1)$ and $M_2 := M_2(1)$, we get

(2.1') $\forall y \in \mathbb{N}_3 \qquad SSGB \Rightarrow M_1 = \{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \}$

$\wedge$

(2.2') $\forall y \in \mathbb{N}_3 \qquad \neg SSGB \Rightarrow M_2 = \{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \}$.

Since for every $y \in \mathbb{N}_3$ $S_g+(y) \cup S_g-(y)$ equals S_g by definition, for every $y \in \mathbb{N}_3$ $\{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \}$ equals the set $X := \{ m \mid (p, m, q) \in S_g \}$. So, from ((2.1') $\wedge$ (2.2')) we obtain

(3) $(SSGB \Rightarrow M_1 = X) \quad \wedge \quad (\neg SSGB \Rightarrow M_2 = X)$.

The set X is a free variable in (3) that is either equal to $\mathbb{N}_4$ or to some non-empty proper subset Y of $\mathbb{N}_4$.

Now, we make use of the following rule.

Let P = P(A) be a proposition that depends on a set A. Then, for any set B,

(we have a proof of P(A) $\wedge$ we have a proof of A = B) => we have a proof of P(B).

In the special case that A is a free variable that is replaced by the value B, the above conjunct (we have a proof of A = B) is trivially true.

Since the set X is a free variable in (3) and since we have a proof of (3), we can apply the above rule with P = (3). If X = $\mathbb{N}_4$ we use the rule with A = X and B = $\mathbb{N}_4$, and if X = Y we use it with A = X and B = Y. Then, since either X = $\mathbb{N}_4$ or X = Y, from (3) we obtain

(3.1) we have a proof of (SSGB => $M_1 = \mathbb{N}_4$ $\wedge$ $\neg$SSGB => $M_2 = \mathbb{N}_4$)

$\vee$

(3.2) we have a proof of (SSGB => $M_1 = Y \neq \mathbb{N}_4$ $\wedge$ $\neg$SSGB => $M_2 = Y \neq \mathbb{N}_4$).

This implies

(3.1') (we have a proof of (SSGB => $M_1 = \mathbb{N}_4$)
$\wedge$
we have a proof of ($\neg$SSGB => $M_2 = \mathbb{N}_4$))

$\vee$

(3.2') (we have a proof of (SSGB => $M_1 = Y \neq \mathbb{N}_4$)
$\wedge$
we have a proof of ($\neg$SSGB => $M_2 = Y \neq \mathbb{N}_4$)).

Now, we will establish a contradiction to ((3.1') $\vee$ (3.2')).

Under the assumption SSGB the set $X = \{ m \mid (p, m, q) \in S_g \}$ is equal to $\mathbb{N}_4$ and under $\neg$SSGB it is equal to $Y \neq \mathbb{N}_4$. Therefore,

(4.1) we have a proof of (SSGB => $M_1 = \mathbb{N}_4$)

$\wedge$

(4.2) we have a proof of ($\neg$SSGB => $M_2 = Y \neq \mathbb{N}_4$).

Then, ((3.1') $\vee$ (3.2')) together with ((4.1) $\wedge$ (4.2)) implies

(5.1) we have a proof of ($\neg$SSGB => $M_2 = \mathbb{N}_4$)

$\vee$

(5.2) we have a proof of (SSGB => $M_1 = Y \neq \mathbb{N}_4$).

Because of ((4.1) $\wedge$ (4.2)) and because

SSGB => $M_2 = \{ \} \neq \mathbb{N}_4$

and

$\neg$SSGB => $M_1 = \{ \} \neq Y$,

we have a proof that ($M_2 = \mathbb{N}_4$) is false and we have a proof that ($M_1 = Y \neq \mathbb{N}_4$) is false.

So, ((5.1) $\vee$ (5.2)) yields

(6.1) we have a proof of SSGB

$\vee$

(6.2) we have a proof of $\neg$SSGB.

Since we have neither a proof of SSGB nor of $\neg$SSGB, both (6.1) and (6.2) are false.

Therefore, we obtain (FALSE $\vee$ FALSE) and thus FALSE.

□

Remark. The term S_g isn't a standard part of Peano arithmetic, but it can easily be defined within Peano arithmetic. Consequently, this also applies to all other sets used in the proof.