Exercises in Error-Mitigated Lattice Gauge Theory Quantum Simulation

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The following two-day interactive workshop has been presented at the final days of the QIS bootcamp hosted by Jefferson National Lab (https://www.jlab.org/conference/QCBC23). The exercises are designed to promote collaborative exploration—beginning in the basics of lattice gauge theories, through physical descriptions of non-unitary quantum dynamics arising from interactions with an external environment, and culminating with a final project in which students perform an error-mitigated simulation of real-time dynamics in a small lattice of non-Abelian gauge theory on one of IBM's superconducting quantum devices.

Some materials of sections II and III have been extended from Ref. [1]. Some Materials of section IV have been extended from Ref. [2]. Though written in a self-guiding format, a solution manual is available to instructors upon request.

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I. SYSTEMATIC REPRESENTATION OF GAUGE THEORIES ONTO QUBITS

A. The need for lattice gauge theory

Quantum chromodynamics (QCD) is often presented as the theory of quarks and gluons interacting through the color force, with the Lagrangian written as

$$\mathcal{L}_{\rm QCD} = \bar{\psi}(x)\Big(i\gamma^\mu\partial_\mu - m - g\gamma^\mu A_\mu(x)\Big)\psi(x) - \frac{1}{2}{\rm Tr}\Big(F_{\mu\nu}(x)F^{\mu\nu}(x)\Big)\,, \tag{1}$$

where $\psi(x)$ is the quark field (a color vector) and $A_\mu(x)$ is the gluon field (a color matrix). The four-vector x is a point in spacetime. Direct interactions of one gluon with other gluons happen within the field strength tensor,

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig\left[A_\mu(x), A_\nu(x)\right]\,. \tag{2}$$

The Lagrangian $\mathcal{L}_{\rm QCD}$ is an excellent starting point for perturbative calculations at high energy colliders when the coupling g is small, but it is not very useful for quarks and gluons confined inside hadrons. When describing hadrons, we should not be trying to use individual gluons.

To study hadrons rigorously from QCD, we can use lattice gauge theory. This approach expresses QCD in terms of the gauge field, $U_\mu(x)$, rather than the gluon field. The relationship is

$$U_\mu(x) = e^{iagA_\mu(x)} \tag{3}$$

where a is the lattice spacing, i.e. the distance between neighboring sites on a hypercubic spacetime lattice.

EXERCISE I.1: On a lattice, each segment of the gauge field is called a gauge link. Consider this product of four gauge links:

$$P_{\mu\nu}(x) = U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^\dagger(x+a\hat{\nu})U_\nu^\dagger(x)\,. \tag{4}$$

- Make a simple sketch, drawing a small vector to represent each gauge link, and confirm that $P_{\mu\nu}(x)$ corresponds to a closed path in spacetime. What is the shape of that closed path?
- Taylor expand $P_{\mu\nu}(x)$ in powers of a, neglecting terms beyond $O(a^2)$. How does your answer relate to $F_{\mu\nu}(x)$?

Your expression for $F_{\mu\nu}(x)$ in terms of $P_{\mu\nu}(x)$ is a first step toward the translation of QCD from gluons to gauge links. The Wilson definition of the complete Lagrangian is

$$\begin{aligned}\mathcal{L}_W \;=\; & \bar{\psi}(x)\left(m+\frac{4}{a}\right)\psi(x) - \frac{1}{2a}\sum_\mu \bar{\psi}(x)(1-\gamma_\mu^E)U_\mu(x)\psi(x+a\hat{\mu}) \\ & -\frac{1}{2a}\sum_\mu \bar{\psi}(x)(1+\gamma_\mu^E)U_\mu^\dagger(x-a\hat{\mu})\psi(x-a\hat{\mu}) + \frac{N_c}{g^2a^4}\sum_{\mu,\nu}\left(1-\frac{1}{N_c}{\rm ReTr}P_{\mu\nu}(x)\right)\end{aligned} \tag{5}$$

where N_c is the number of colors. For QCD, $N_c = 3$.

Did you notice that the mass term seems to have gained an extra minus sign when going from $\mathcal{L}_{\rm QCD}$ to $\mathcal{L}_W$? This is actually an overall sign change for the entire Lagrangian that occurs because $\mathcal{L}_{\rm QCD}$ was written in Minkowski spacetime but $\mathcal{L}_W$ is written in Euclidean spacetime. This detail reminds us that traditional lattice gauge theory uses imaginary time and is not designed for studies of processes evolving through real time. Access to real-time lattice gauge theory (LGT) is one motivation for using quantum computers.

EXERCISE I.2:
• The Dirac Lagrangian is well known in Minkowski spacetime, $\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)$. Make a change of variables from real time t to imaginary time $\tau = it$. Also make the change from the Minkowski gamma matrices to Euclidean gamma matrices by using these definitions: $\gamma^x = -i\gamma_x^E$, $\gamma^y = -i\gamma_y^E$, $\gamma^z = -i\gamma_z^E$, $\gamma^t = \gamma_\tau^E$. How does the new form of the Dirac Lagrangian compare to the original form?
• Taylor expand $\mathcal{L}_W$ in powers of a. (Neglect the term without the quark field. You can try that one later when you have more time.) Does your result confirm that $\mathcal{L}_W$ becomes equivalent to $\mathcal{L}_{\rm QCD}$ as $a \to 0$, up to the expected overall minus sign? The leading lattice artifacts are $O(a^n)$ with what power, $n = ?$.

Several different lattice discretizations are used in practice. Some approach the continuum limit more quickly than $\mathcal{L}_W$, but all must ultimately agree with $\mathcal{L}_{\rm QCD}$ as $a \to 0$.

B. Can the gauge field be integrated out?

The SU(N_c) gauge symmetry is a local symmetry. This means an arbitrary SU(N_c) transformation applied at any lattice site will produce no change in physical observables. The transformation will affect all fields at that lattice site, but these effects cancel each other within observables. Specifically, the effect of an SU(N_c) transformation $M(x)$ at the lattice site x will be

$$\psi(x) \;\to\; M(x)\psi(x)\,, \tag{6}$$
$$U_\mu(x) \;\to\; M(x)U_\mu(x)\,, \tag{7}$$
$$U_\mu(x-a\hat{\mu}) \;\to\; U_\mu(x-a\hat{\mu})M^\dagger(x)\,. \tag{8}$$

Notice that $\psi(x)$ is a vector with N_c components, while $U_\mu(x)$ and $M(x)$ are $N_c \times N_c$ matrices.

EXERCISE I.3:
• We know that each gauge link is an element of SU(N_c). Is the identity matrix an element of SU(N_c)? Does this mean a clever choice for $M(x)$ can replace a gauge link $U_\mu(x)$ by the identity matrix? If so, what is that clever choice?
• Consider a one-dimensional lattice with four sites. Are there gauge transformations that can be performed, site by site across the lattice, that will replace all gauge links by the identity? Does your answer depend on the boundary conditions (i.e. a periodic lattice versus open boundaries)? What happens to the quark field at each site because of these transformations?
• Consider a lattice in two (or more) dimensions. Will your conclusions be different?

C. From Lagrangian to Hamiltonian

The Hamiltonian of any system describes evolution through real time via e^{-iHt} acting on the state vector. For QCD, this state vector provides a complete description of the quark and gauge fields. The state vector of modern lattice QCD computations would be prohibitively enormous to store in a classical computer, but a quantum computer has the potential to overcome this problem.

EXERCISE I.4:
• Consider a collection of N qubits. To store the complete state vector classically, how much traditional RAM is needed?

• How do you expect the required number of qubits to scale with the number of sites on your lattice?

Because quarks are spin-half particles, the mapping of the quark field onto a qubit register is quite natural, but how can link variables be described by qubits? To focus on this, consider a pure gauge theory without quarks. The interesting world of glueballs and their dynamics should emerge from future quantum computations of this theory.

EXERCISE I.5:
• Write down $\mathcal{L}_W$ with $\psi(x) = 0$. Use the notation $\mathcal{L}_W = \mathcal{L}_E + \mathcal{L}_B$ where the "color electric" terms $\mathcal{L}_E$ contain time derivatives and the "color magnetic" terms $\mathcal{L}_B$ do not. Also, define your spacetime lattice to have a fundamental length a_s in the spatial directions and a different fundamental length a_t in the time direction. (Detail: a^4 in the denominator of $\mathcal{L}_W$ becomes $a_s^2 a_t^2$ within $\mathcal{L}_E$, but it becomes a_s^4 in $\mathcal{L}_B$. Can you imagine why?)
• Write $\mathcal{L}_E$ in the temporal gauge, defined by $A_\mu(x) = 0$ for $\mu = t$ at every site x, and then Taylor expand $\mathcal{L}_E$ in powers of a_s to find the leading term. Finally, take the $a_t \to 0$ limit of that term. Does your result remind you of introductory electricity and magnetism, where the energy density stored in an electric field is $\frac{1}{2}|\vec{E}|^2$ and the field itself is $\vec{E} = -\partial_t \vec{A}$ in temporal gauge?

A plan is now becoming clear. In the limit of $a_t \to 0$, the lattice is purely in spatial directions. Eigenstates of the color electric terms will form a computational basis, and the color magnetic terms will provide transitions between those basis states.

The case of $N_c = 2$ is especially familiar if you recall that SU(2) is also the mathematical group that describes angular momentum. A gauge link has color electric eigenvalues $j(j+1)\frac{g^2}{2}$, where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. The corresponding eigenvectors are the basis states for the lattice. Application of a gauge link operator (arising from the color magnetic terms $\mathcal{L}_B$, for example) always makes one minimum quantum of change, so the local value of j will become $j \pm \frac{1}{2}$.

D. Two explicit examples

EXERCISE I.6: Consider SU(2) gauge theory on a lattice with just four gauge links forming a square. The Hamiltonian matrix is

$$H_1 = \begin{pmatrix} 0 & -\frac{2}{g^2} & 0 & 0 & \cdots \\ -\frac{2}{g^2} & \frac{3}{2}g^2 & -\frac{2}{g^2} & 0 & \cdots \\ 0 & -\frac{2}{g^2} & 4g^2 & -\frac{2}{g^2} & \cdots \\ 0 & 0 & -\frac{2}{g^2} & \frac{15}{2}g^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} . \tag{9}$$

• H_1 can act on a state represented by a column vector. Explain the meaning of each entry in that column vector.
• Verify that the diagonal entries of H_1 are correct. What is the fifth diagonal entry?
• Truncate H_1 to a 2×2 case valid for studying the ground state when the coupling g is large. Express the truncated H_1 as a combination of Pauli gates and the identity gate.

Did you notice that all four gauge links have the same value of j? This is required by the non-Abelian version of Gauss's law which says that, in the absence of quarks, the color flowing into any lattice site must sum to zero. Gauss's law is an important feature of SU(N_c) gauge theories like QCD, and we have

to ensure that our lattice Hamiltonians preserve it. This was quite simple for the four-site square lattice, but a larger lattice will have more than two gauge links touching each site and then Gauss's law becomes a bigger chore to manage.

EXERCISE I.7: Consider SU(2) gauge theory on a lattice with seven gauge links forming two side-by-side squares with one shared side. The Hamiltonian matrix, truncated to $j \leq \frac{1}{2}$, is

$$H_2 = \begin{pmatrix} 0 & -\frac{2}{g^2} & -\frac{2}{g^2} & 0 \\ -\frac{2}{g^2} & \frac{3}{2}g^2 & 0 & -\frac{1}{g^2} \\ -\frac{2}{g^2} & 0 & \frac{3}{2}g^2 & -\frac{1}{g^2} \\ 0 & -\frac{1}{g^2} & -\frac{1}{g^2} & \frac{9}{4}g^2 \end{pmatrix} . \tag{10}$$

- For each of the four basis states, sketch the lattice and label each gauge link by its value of j.
- In SU(2) gauge theory, checking Gauss's law uses the same mathematics as conservation of angular momentum. Verify that the four states included in H_2 respect Gauss's law. Give an example of an unacceptable basis state that is forbidden by Gauss's law.
- For large coupling g, this 4×4 truncation of H_2 is good approximation. From the infinite list of all missing basis states, which one is the most important? Extend H_2 to have a fifth row and a fifth column, representing this new basis state. What new value gets placed on the diagonal? Which new off-diagonal entries are zero and which are nonzero?
- Return to the 4×4 version of H_2 and express it as a combination of Pauli and identity gates acting on a pair of qubits.

You might be wondering why we need to worry about Gauss's law at all. It is built into QCD already, so won't it be preserved automatically? In principle, yes. For example, on the side-by-side squares we could simply assign one qubit to each gauge link. That lattice would have $2^7 = 128$ basis states of which only 4 satisfy Gauss's law, so it feels a bit inefficient. There is also a danger that the practical algorithms used to compute Hamiltonian dynamics will not maintain Gauss's law perfectly. One more worry is that today's quantum computers are noisy, so the true signal will often leak from the 4 physical basis states into the 124 others.

Mitigating errors on noisy hardware is a major concern for all quantum computations, regardless of how we choose to handle Gauss's law, so we will focus on error mitigation next.

II. ORIGINS AND DESCRIPTIONS OF QUANTUM ERRORS

When errors arise in quantum simulations, uncertainty is introduced into a device's quantum state. Ensemble measurements in the resulting system can be described by expectation values in a weighted mixture of pure quantum states, $\rho : \{p_i, |\psi_i\rangle\}$, where the *classical* probabilities p_i satisfy $\sum_i p_i = 1$. Measurement of operator expectation values, $\langle\hat{\mathcal{O}}\rangle$, in such noisy environments yields the statistical average of expectation values among the pure states present, $\langle\hat{\mathcal{O}}\rangle = \sum_i p_i \langle\psi_i|\hat{\mathcal{O}}|\psi_i\rangle$. As an extension of the wavefunction to classical mixtures of quantum states, the density matrix, $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, provides a succinct object containing all information about measurement results characterizing a mixed quantum state.

EXERCISE II.1:
- Given a density matrix only (i.e., without access to a pure-state convex decomposition), determine the operation used to calculate operator expectation values, $\langle\hat{\mathcal{O}}\rangle(\rho)$.
- Consider the following family of pure-state convex decompositions,

$$\rho(\alpha) = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2| \quad , \tag{11}$$

with $|\psi_1\rangle = \begin{pmatrix} \cos\frac{\alpha}{2} \\ 0 \\ 0 \\ \sin\frac{\alpha}{2} \end{pmatrix}$ and $|\psi_2\rangle = \begin{pmatrix} -\sin\frac{\alpha}{2} \\ 0 \\ 0 \\ \cos\frac{\alpha}{2} \end{pmatrix}$. Note that members of this family include: $\rho(0) = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$ and $\rho(\pi/2) = \frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\Phi^-\rangle\langle\Phi^-|$, where $|\Phi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$ are entangled Bell states. Discuss the use of measurements to observe the value of α.

• Utilizing the connection between partial tracing and losing access to a qubit or to the results of a partial projective measurement, find the single-qubit measurement bases that connect $\rho(\alpha)$ to the state $|GHZ\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}$. In other words, in what basis could you measure a qubit in $|GHZ\rangle$ such that a combination of the measurement ensembles would produce the convex decomposition $\rho(0)$? $\rho(\pi/2)$? $\rho(\alpha)$?

The non-unitary evolution of a density matrix in a noisy environment (error channel, $\mathcal{E}(\rho)$) can be modeled physically as *unitary* operators acting upon an enlarged Hilbert space including a quantum environment that becomes inaccessible,

$$\mathcal{E}(\rho) = \sum_k \langle e_k|U\left(\rho \otimes |0\rangle\langle 0|\right)U^\dagger|e_k\rangle = \sum_k E_k \rho E_k^\dagger \quad . \tag{12}$$

Let's walk through this expression. Beginning in the ground state of the environment, the full-space unitary may produce entanglement between the system and its environment. The loss of the environment from experimental access is expressed through the partial tracing operation, i.e., projectively measuring in any orthonormal basis of the environment Hilbert space, $\{|e_k\rangle\}$, without retention of the measurement result. From the perspective of the system alone, such interactions may be expressed by a set of non-unitary Kraus operators, $E_k = \langle e_k|U|0\rangle$, where the kets provide projection in the environment Hilbert space and thus E_k is an operator on the system.

Consider an example of the amplitude damping quantum channel, which physically models decay of a qubit's excited state due to, for example, spontaneous emission. For a decay rate, Γ, leading to a probability of decay in time δ_t as $p = \Gamma\delta_t$, the interaction unitary between system and environment may be characterized by the following transitions:

$$U|0\rangle_s|0\rangle_e = |0\rangle_s|0\rangle_e \qquad \text{and} \qquad U|1\rangle_s|0\rangle_e = \sqrt{1-p}|1\rangle_s|0\rangle_e + \sqrt{p}|0\rangle_s|1\rangle_e \tag{13}$$

As such, the error operators from the perspective of the system alone are,

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \qquad \text{and} \qquad E_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \quad . \tag{14}$$

Through sequential application of this error channel, you should convince yourself that the persistence probability in the qubit excited state is $(1-\Gamma\delta_t)^{t/\delta_t}$, which has a form $\sim e^{-\Gamma t}$ in the limit of continuous time, $\delta_t \to 0$. This exponential decay thus characterizes the timescale for maintaining the classical state $|1\rangle$ and relates to the T_1, or longitudinal coherence time of the quantum device.

EXERCISE II.2: Phase-type errors may arise from the presence of scattering events that do not shift the energy. Consider a unitary system-environment interaction in which the excited state of the system yields an environment sensitive to external interactions, i.e., with transitions

$$U|0\rangle_s|0\rangle_e = |0\rangle_s|0\rangle_e \qquad \text{and} \qquad U|1\rangle_s|0\rangle_e = \sqrt{1-p}|1\rangle_s|0\rangle_e + \sqrt{p}|1\rangle_s|1\rangle_e \quad . \tag{15}$$

• Find the Kraus operators describing the quantum channel of this environment interaction from the perspective of the system alone.
• Using the mixed-state Bloch-vector representation, $\rho = \frac{\mathbb{I}+\vec{r}\cdot\vec{\sigma}}{2}$, choose a few pure states on the surface of the Bloch sphere and plot their trajectories as a function of time (or decay time steps) when subject to this type of environment interaction.
• Described visually, what does this channel do to states on the Bloch sphere? Discuss why this channel is called the "phase damping channel".
• In the limit of continuous time, write the time-dependent functional form of the diagonal and off-diagonal matrix elements of the system's density matrix beginning from an arbitrary initial state, $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = a|0\rangle + b|1\rangle$.
• In a language of quantum circuitry including a "wait" gate during which environmental interactions take place, describe an operation and measurement protocol that would allow a determination of the relevant decay constant for the phase damping channel. Plot the chosen observed probability as a function of system-environment interaction time.

Similar to the variability in convex decompositions of a mixed state—for example, associated with the projective basis used to trace an environment Hilbert space—error channels from the perspective of a quantum system alone can be generated by an array of distinct physical processes. Understanding the responsible physical process can be vital for designing effective error mitigation techniques. One method for lifting degeneracy in underlying error sources is to observe responses to targeted improvement or error mitigation protocols.

EXERCISE II.3: Consider a source of error arising from fluctuations in unitary controls or electromagnetic background fields between runs of an ensemble measurement. One model of these effects is to describe the ensemble by a stochastically sampled Z-rotation.

$$\mathcal{E}_{z,\sigma}\Big(|\psi\rangle\langle\psi|\Big) = \frac{1}{\sqrt{2\pi\sigma^2}}\int \mathrm{d}\theta\ R_z(\theta)|\psi\rangle\langle\psi|R_z^\dagger(\theta)e^{-\frac{\theta^2}{2\sigma^2}} \tag{16}$$

where $R_z(\theta) = e^{-iZ\theta/2}$.
• For an arbitrary initial state, $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = a|0\rangle + b|1\rangle$, calculate the updated density matrix.
• Compare your result to the evolution of the system's density matrix under the phase damping channel from the process in Eq. (15) from Exercise II.2. What time-dependent functional form of the dispersion, σ, would make the observed error channels equivalent?
• Find the basis transformation unitary, U, for which $U^\dagger R_z(\theta)U = R_z(-\theta)$. With $\Gamma = 2$, calculate and plot (up to $t \sim 18$) the measurement protocol from Exercise II.2 with one insertion of U applied at time $t = 2$. (*Hint:* What is the relationship between t and a factor rescaling θ in Eq. (16)? Break the rotation into two, one in each basis, and introduce a scaling factor to each rotation angle.)
• If the noise were incoherent (for example, without experimental access to individual pure states due to entanglement generated with an environment Hilbert space, see Exercise II.2), would this protocol improve the phase coherence?

In the next section, we will explore select error mitigation techniques directed toward quantum simulation applications of simple models with symmetries and the gauge theories discussed in Section I.

III. ERROR MITIGATED GAUGE THEORY QUANTUM SIMULATION

For large-scale quantum information processing to be successful, there must be a natural or artificial source of robustness to interactions with the external quantum and classical environment. Many error

correction approaches utilize a technique of periodic measurements to maintain population in a subspace. The frequency at which the periodic measurement is advisably performed depends upon both the fidelity of the measurement and the error rate experienced by the quantum system.

By measuring a quantity that discriminates between states inside/outside the desired subspace (often computed onto an auxiliary qubit to do so non-destructively), the process projects a quantum system either into or outside the desired subspace where ideally a correction procedure can be devised. Which scenario occurs, of course, is governed by the probabilistic nature of the measurement result. Basic designs of such *partial* projective measurements extract one bit of information (whether or not the state is in the subspace), leaving a coherent quantum state within the subspace upon correction or post-selection. The limiting case of this protocol for a one-dimensional subspace (not too interesting from the perspective of protected dynamics, but a useful limit to consider) can be appreciated as the Quantum Zeno Effect.

EXERCISE III.1: Consider a qubit in state $|0\rangle$ that is subject to external fields whose effects can be modeled at leading order as a bit-flip Hamiltonian, $\frac{X+Y}{\sqrt{2}}$, i.e., $U_{ext} = e^{-i\frac{X+Y}{\sqrt{2}}t}$.

• If you aimed to retain the $|0\rangle$ quantum state through repeated measurement in the computational basis, how many (non-destructive) measurements would you have to perform in order to produce a probability of at least 99% that the state has remained in the vicinity of the $|0\rangle$ state throughout one unit of time evolution? In other words, if post-selecting for the $|0\rangle$ outcome, how many measurements are needed to retain sampling statistics of at least 99%.

Not only can a single quantum state be maintained through the Zeno Effect with complete projective measurements, but maintaining subspaces with dimension greater than one can also be enforced with *partial* projective measurements. By measuring a binary piece of information indicating whether the quantum state is inside or outside the desired subspace, evolution departing from the subspace can be systematically slowed and removed. Common subspaces to consider are: the degenerate ground-state subspace of the toric code protected by local stabilizer measurements or the physical subspace of a gauge theory protected by local measurements of Gauss's law.

Exercise III.2: Consider the slightly more interesting scenario of two qubits beginning in the Bell state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ and subject to a quantum simulation of unitary time evolution of the form $U_{\text{noiseless}} = e^{-i(X\otimes Y)t}$. In the proposed scenario, the time evolution will experience *coherent* single-qubit errors such that the unitary becomes $U_{\text{noisy}} = e^{-i\left(X\otimes Y+\mathbb{I}\otimes\frac{X+Y}{\sqrt{2}}+\frac{X+Y}{\sqrt{2}}\otimes\mathbb{I}\right)t}$.

• Projected into the Bell basis, plot the evolution of noisy and noiseless probabilities for times up to t = 2. For how long do the noisy probabilities remain within ±1% from their noiseless values?

• Identify an operator (e.g., parity) whose expectation values distinguish the populated states from those that remain unpopulated in the noiseless evolution. Show that this operator commutes with the Hamiltonian, and thus presents a static symmetry under time evolution (Ehrenfest's theorem).

• Consider the procedure of periodically projectively measuring (non-destructively) the operator whose expectation values identify the positive-parity subspace, and post-selecting upon the results. Describe experimentally how such measurements could be performed, either abstractly in quantum circuit notation or in a quantum architecture of your choice.

• Show that, at the cost of measurement statistics, this procedure can improve the fidelity of the noisy simulation. How many such measurements would be required to maintain probabilities within ±1% from their noiseless values (not relative) to a simulation time of t = 1.5 with a loss of no more than 20% measurement statistics?

As discussed in Section II, both coherent and incoherent sources of error may be at play in a quantum simulation. However, there are techniques that introduce a randomized set of quantum gates into a

quantum circuit with the outcome of restructuring sources of error from coherent modifications of the Hamiltonian to incoherent classical noise. One such technique is *Pauli Twirling*, which introduces a randomization of Pauli bases throughout a simulation to turn coherent noise into incoherent channels, in some cases destructively interfering coherent noise entirely. The Pauli twirling that will be utilized in the following example introduces randomization at the level of the Pauli basis of the time evolution. This can be considered analogous to performing periodic gauge transformations of an LGT quantum simulation, modifying the unitary evolution operator but leaving the physical subspace evolution unchanged.

Exercise III.3: Consider the error mitigation technique of Pauli twirling.
• Viable basis transformations of the noiseless operators leave invariant the logical circuits of time evolution. Find all Pauli operators, within the set of 16 $P_j = \{\mathbb{I}_2, X, Y, Z\} \otimes \{\mathbb{I}_2, X, Y, Z\}$, that satisfy the criteria

$$U_{\text{noiseless}} = P^\dagger U_{\text{noiseless}} P \quad . \tag{17}$$

• Devise a stochastic strategy of these invariant basis changes that maintains, up to $t = 1.5$, the Bell basis measurement probabilities within $\pm 10\%$ of their noiseless values (not relative). How many layers of basis changes were used? (*Hint:* though probabilistic numerical simulation can be used, density matrix methods may capture the stochastic features of the procedure more naturally.)
• Is the utilization of all basis transformations necessary to achieve this goal?

For coherent errors, Section II discussed how additional unitary circuit elements could improve simulation fidelity by reducing sensitivity to noise terms in the Hamiltonian. For incoherent errors, data analysis informed by a relevant error channel can allow recovery of simulation results through associated post-processing.

Exercise III.4: Leveraging the density matrix formalism for noisy quantum evolution discussed in Section II, the following exercise will focus upon the two-plaquette system of SU(2) lattice gauge theory established in Section I, Hamiltonian of Eq. (10), experiencing depolarizing quantum noise.
• Begin in the basis state of left-plaquette excitation with coupling $g = 1$ and t in units of $2/g^2$. For time evolutions out to $t = 8$, plot the oscillatory probabilities for the left plaquette to be in the $j = 1/2$ excited state (treat the fourth basis state as both the left and right plaquettes excited). Plot the same evolution for the probability of the right plaquette to be excited in the final state.
• The depolarizing channel introduces population of the maximally-mixed state into a density matrix

$$\mathcal{E}_p(\rho) = (1-p)\rho + p\frac{I}{d} \quad , \tag{18}$$

where d is the dimensionality of the state, i.e., $d = 2^n$ for a state of n qubits. Visually describe what this channel does to states on the Bloch sphere.
• Incorporate the depolarizing error channel to your simulation. Through visual comparison of your depolarized evolution with the experimental data reported in the top panel of Fig 3 in Ref. [2], find a decay parameter, $\Gamma = p/\delta_t$, that approximately captures the reported rate of coherence loss.
• By performing the depolarizing channel on the two-qubit system without active time evolution, calculate the observable's noisy decay from the initial state. Plot the exponential envelope together with your noisy and noiseless evolutions.
• Consider the survival probability of $|0\rangle^{\otimes n}$ after one application of the channel, $P_0(p) = \mathcal{E}(|0\rangle\langle 0|)_{1,1} = (1-p) + p/2^n$. By analysing and exactly summing the recursion of this survival probability under many applications of the error channel, show that the decay follows the form

$$P_0(t) = \frac{1}{2^n}\left(1 + e^{-t\Gamma}(2^n - 1)\right) \quad , \tag{19}$$

(*Note: the final asymptotic analysis will utilize the same replacements,* $p = \Gamma\delta_t$ *and* $N_{applications} = t/\delta_t$ *in the limit* $\delta t \to 0$*, as was utilized in and discussed above Exercise II.2.*). By comparing this form to that calculated previously in the two-qubit system without active time evolution, discuss the relevant Hilbert space dimensionality for the local observable's decay.

• Eq. (8) of Ref. [2] outlines a post-processing rescaling procedure that accommodates the decaying envelope of coherent fluctuations:

$$\frac{P_{\text{true}} - \frac{1}{2}}{P_{\text{computed}} - \frac{1}{2}}\Big|_{\text{physics}} = \frac{P_{\text{true}} - \frac{1}{2}}{P_{\text{computed}} - \frac{1}{2}}\Big|_{\text{calibration}} \quad . \tag{20}$$

Taking the noisy time evolution as the physics calculation and the decay envelope as the calibration information, implement this self-mitigation/renormalization to find P_{true} at each time step. Compare this final mitigated result extracted from this noisy environment to the noiseless simulation.

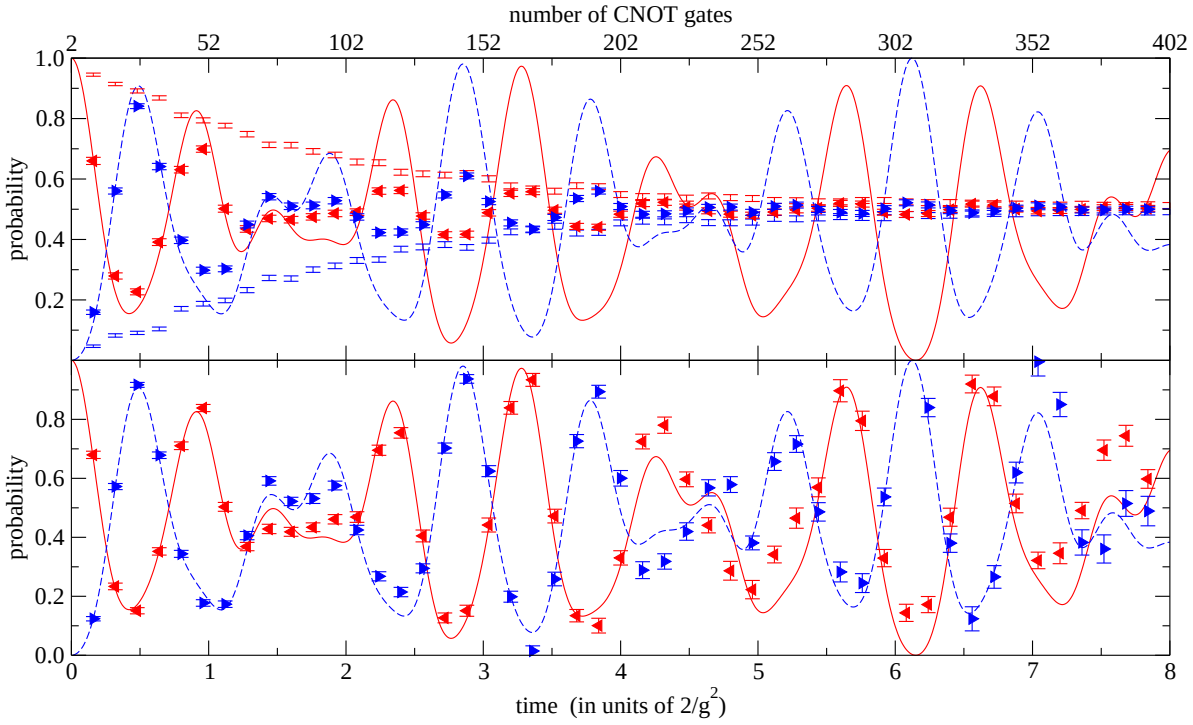
Finally, it is time to take the ideas and experiences of the past exercises and apply it to LGT quantum simulation on real quantum devices.

IV. FINAL PROJECT

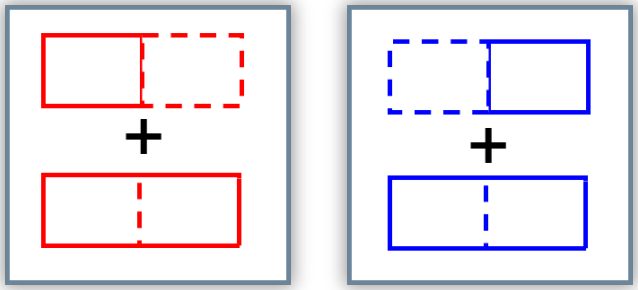
A. Activity Goal

For this final project, we will have the opportunity to combine the concepts discussed in previous sections in order to perform an error mitigated quantum computation on actual IBM hardware. We will work together to collectively produce a plot of the time evolution of a 2-plaquette SU(2) lattice.

Our goal is to recreate the following figure from Ref. [2] and see firsthand how important error mitigation is on current hardware:



This plot shows the time evolution of a 2 plaquette lattice, with coupling $g = 1.0$, from the initial state where the left plaquette is turned on. The top panel shows the raw results and the lower panel shows the results once the self-mitigation ratio has been applied. The triangles in the top half correspond to the physics data and the error bars without symbols correspond to the mitigation data. Each of these points were computed on `ibm_lagos` with 148 twirling runs with 10000 shots each. The red solid lines are the exact probability of the left plaquette being excited and the blue dashed lines are the exact probability of the right plaquette being excited.



B. Activity Instructions and Exercises

• This activity will use the jupyter notebook QCBC.ipynb.
• Once you have downloaded the notebook, log in to your IBMQ account at https://quantum-computing.ibm.com/login. From your dashboard, launch the IBM Quantum Lab and then upload the notebook.
• Modify the code inputs to compute your assigned timesteps. You can check your code against the exact results using the simulator before running on the actual hardware.
• Once your code has run on the quantum computer, record your results in the 'group results' tab of the google sheet.

As you step through the notebook, you will come across the following exercises that you can complete as you go or perhaps while you wait for your code to run on the quantum computer.

EXERCISE IV.1: For randomized compiling, each CNOT gate gets replaced by one of 16 options which leaves the function of the circuit unchanged. Fill in the blanks, in the list below, for the twirling options of the CNOT gate, CX_{jk}, where j is the control and k is the target:

1. CX_{jk}
2. $X_k CX_{jk} X_k$
3. $Y_k CX_{jk} Z_j Y_k$
4. $\square_k CX_{jk} Z_j Z_k$
5. $X_j CX_{jk} X_j X_k$
6. $X_j \square_k CX_{jk} \square_j$
7. $X_j Y_k CX_{jk} Y_j Z_k$
8. $X_j Z_k CX_{jk} Y_j Y_k$

9. $Y_j CX_{jk} Y_j X_k$
10. $Y_j X_k CX_{jk} Y_j$
11. $\square_j Y_k CX_{jk} X_j Z_k$
12. $Y_j Z_k CX_{jk} X_j Y_k$
13. $\square_j CX_{jk} \square_j$
14. $Z_j X_k CX_{jk} Z_j X_k$
15. $Z_j Y_k CX_{jk} \square_k$
16. $Z_j Z_k CX_{jk} Z_k$

EXERCISE IV.2:

- Why does the self-mitigation ratio (Eq. (20)) have factors of 1/2 in it?
- Is it possible to measure the state where both plaquettes are not excited? How would you modify the self-mitigation ratio to compute this?
- Can you come up with a general expression for the self-mitigation ratio when measuring a particular n-qubit state?

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[1] N. Klco, Exercises: Introductory course in quantum information science (2023).
[2] S. A Rahman, R. Lewis, E. Mendicelli, and S. Powell, Phys. Rev. D **106**, 074502 (2022), arXiv:2205.09247 [hep-lat].