On Fermat's Last Theorem

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Abstract. A solution of Fermat's Last Theorem is given, using elementary function arithmetic and inference from worked examples.

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1. Introduction

Fermat's Last Theorem was formulated in 1637 and apparently not proved successfully until 1995, when Andrew Wiles (1) did so using the latest high level number theory. The theorem states that no three positive integers a, b, c, can satisfy the equation:

$$a^p + b^p = c^p \tag{1.1}$$

if (p) is an integer greater than two.

Over the years, enthusiasts have been encouraged by the simplicity of the conjecture to try and prove it using elementary function arithmetic (2). This is one such attempt using inference from worked examples.

2. The power three case

We prove that when (b) is a positive integer in the equation:

$$a^3 + b^3 = c^3, \tag{2.0}$$

then (a) cannot be an integer, and vice-versa.

Let [c = a + e], where (e) is a positive integer, then set up two equal expressions which serve to limit the choice of (b), followed by (a) and (c):

$$\left[f(a,e) = \frac{c^3 - a^3 - e^3}{3e} \right] = \left[\frac{b^3 - e^3}{3e} = f(b,e) \right]. \tag{2.1}$$

Here, f(a,e) will be an integer when (a) is any integer, and also for some non-integer (a) values whenever f(b,e) is an integer. Substitute (s = a/e), then evaluate f(a,e) as:

$$f(a,e) = s(ks+1) \times e^2, \tag{2.1a}$$

where in this case,

$$k = 1 \tag{2.1b}$$

has been introduced for later use in this general format.

Only selected values of integer (b) can make f(b,e) an integer, to compare with a possible integer f(a,e) in Eq.(2.1). All other values of (b) make f(b,e) non-integer, then (a) would have to be non-integer. For a chosen (e) value, typical selected (b) values are given by:

$$b_n = (3n-2)e, \quad \text{or} \qquad b_n = ne, \quad \text{or} \qquad b_n = (n+3)e/4, \tag{2.2}$$

where integer ($n \geq 1$). Further selected (b_n) values are given by different multiples of (e) which depend upon the value of (e). When using (b_n) values in Eq.(2.1), while substituting [b_n = (q +1)e], then f(b,e) will be an integer given by:

$$f(b,e) = q(uq+1) \times e^2, \tag{2.3a}$$

where (q) is a function of (n) as determined by Eq.(2.2) or similar, and

$$u = (q/3+1) \tag{2.3b}$$

has also been introduced for later use.

To test whether an integer (a) can make f(a,e) equal to the selected f(b,e), equate Eq.(2.1a) and Eq.(2.3a):

$$[\ f(a,e) = (1/k)\{(ka) \times (ka+e)\}\ ] = [(1/u)\{(uqe) \times (uqe+e)\} = f(b,e)]\ . \tag{2.4a}$$

Make this more symmetrical by substituting:

$$x = (2ka+e), \tag{2.4b}$$

which is an integer if (a) is an integer; and

$$y = (2\,u\,q + 1)\,e = z\,e\,, \tag{2.4c}$$

which may be an integer; then Eq.(2.4a) can be simplified to:

$$(1/k)\{(x-e)\times(x+e)\} \;=\; (1/u)\{(y-e)\times(y+e)\}\;. \tag{2.4d}$$

The simplicity of this equation suggests the probable existence of numerical examples which possess an integer (x), independent of the cubic problem. Arbitrarily choose

$$2\times 12 = 24 = 4\times 6\;, \tag{2.4e}$$

then expand to get an expression superficially like Eq.(2.4d):

$$\{(7-5)\times(7+5)\} = \left(1/5^2\right)\{(25-5)\times(25+5)\} = 24\,. \tag{2.4f}$$

Here, the presence of integers (x ≡ 7) and (e ≡ 5) plus ($u \equiv e^2$), implies that the absence of integer (x) values in Eq.(2.4d) must be due to the (y)(u) relationship of Eq.(2.4c).

By evaluating real examples, it can be shown that (u) has to be restricted for (x) to be an integer. In Eq.(2.4d), try (e = 3, q = 24, u = 9, b_9 = 75, k = 1) so (y = 1299, z = 433), then:

$$\begin{aligned}(x-3)\times(x+3) = (1/9)\{(1299-3)\times(1299+3)\} \;&=\\ \{(433-1)\times(433+1)\} \;&= 187488\end{aligned}\;. \tag{2.5a}$$

Here (x = 433.0092), so propose changes which would produce an integer (x). First, factorise this product to get an alternative product, for use on the left hand side.

$$187488 = 378\times 496\,, \tag{2.5b}$$

then introduce the arithmetic mean [(378 + 496)/2 = 437] to represent *integer* (x = 437) in an expression similar to Eq.(2.4f):

$$\{(437-59)\times(437+59)\} = (1/59)^2\{(25547-59)\times(25547+59)\} = 187488\,. \tag{2.5c}$$

This has the same value as Eq.(2.5a), and approximate form, but it has been necessary to change factors (e = 59) and ($u/k = 59^2$), plus [y = 25547 = (59/3) x 1299] involving the original values of (y, e).

Alternatives to this choice of (x) also exist, (438, 447, and 498), whereby:

$$\{(438-66)\times(438+66)\} = (1/66)^2\{(28578-66)\times(28578+66)\} = 187488\,, \tag{2.5d}$$

$$\{(447-111)(447+111)\} = (1/111)^2\{(48063-111)(48063+111)\} = 187488\,, \tag{2.5e}$$

$$\{(498-246)(498+246)\} = (1/246)^2\{(106518-246)(106518+246)\} = 187488\,. \tag{2.5f}$$

Inspection of these four particular examples suggests that the restriction ($u/k = e^2$) would be the main requirement for solving Eq.(2.4d) with (x) integers. However, the new (y = 25547) in Eq.(2.5c) is not correctly related to the new ($u/k = 59^2$) in the way that the original (y = 1299) in Eq.(2.5a) was derived using original (u = 9) in Eq.(2.4c). That is, introduction of new ($u = 59^2 = e^2$) into Eq.(2.4c) would produce (y = 4288313579) rather than the (y = 25547) which employed the original (u = 9). This situation will exist in all numerical examples, so it appears that genuine values of (y) calculated from Eq.(2.4c) for use in Eq.(2.4d) *can never occur in expressions of the form Eq.(2.5c),* which is the unique form necessary for getting an integer (x). Thus, the derivation of Eq.(2.4d) excludes the possibility of integers (x) and (a).

To confirm this result, let Eq.(2.4d) be written to include ($u = e^2$):

$$(1/k)(x^2 - e^2) = (1/u)\{(z-1)\times(z+1)\}e^2 = (z^2 - 1) \quad . \qquad (2.6a)$$

where integer (z) in terms of (u) is given by Eq.(2.4c) with Eq.(2.3b):

$$z = (y/e) = (2uq+1) = 2u(u-1)3+1 = (6u^2 - 6u + 1) . \qquad (2.6b)$$

Now test for this restricted (x) being an integer greater than (z), if (e^2) is great enough and (u > 1) from Eq.(2.3b). That is, substitute ($x/\sqrt{k} = z + w$) for any integer ($w \geq 1$), then Eq.(2.6a) produces an impossible equality:

$$u/k + 12wu - 12wu^2 = (w+1)^2 \quad . \qquad (2.6c)$$

Thus, genuine values of (z = y/e) given by Eq.(2.6b) cannot produce an integer (x).

For simplicity in the above set of examples, it was given that ($u = 9 = e^2$) in Eq.(2.5a); but even if ($u \neq e^2$) the final outcome still stands, as follows. Consider another arbitrary expression unrelated to the cubic problem:

$$\{(17-7)\times(17+7)\} = \{(16-4)\times(16+4)\} = (4/7)^2\{(28-7)\times(28+7)\} = 240 \quad . \qquad (2.7)$$

Here, this equation has derived integers ($x \equiv 17$) and ($e \equiv 7$), plus [$u \equiv (7/4)^2$]. However, we could define ($u \equiv 7^2 \equiv e^2$) and ($k \equiv 4^2$) for consistency in (u).

For a real example, try (e = 28, q = 45, u = 16, b_{16} = 1288, k = 1) so (y = 40348, z = 1441), then Eq.(2.4d) becomes:

$$(x-28)\times(x+28) = (1/16)\{(40348-28)\times(40348+28)\} = \{(10087-7)(10087+7)\} = (7/28)^2\{(40348-28)(40348+28)\} = 101747520 \quad . \qquad (2.8a)$$

Here (x = 10087.036), so propose changes to make (x) an integer. First, factorise this product to get an alternative product:

$$101747520 = 9888 \times 10290, \tag{2.8b}$$

then introduce the arithmetic mean [(9888+10290)/2 = 10089] to represent *integer* (x = 10089) in the expression:

$$\begin{aligned}&\{(10089-201)\times(10089+201)\} = \\ &\quad (7/201)^2\{(289641-201)\times(289641+201)\} = 101747520\end{aligned} \quad . \tag{2.8c}$$

This has the same value as Eq.(2.8a), and approximate form, but it has been necessary to change factors (e = 201) and [u/k = $(201/7)^2$], plus [y = 289641 = (201/28) x 40348] employing original values of (y, e).

Alternatives to this choice of (x) also exist, such as (10178, 10294, and 10661), whereby:

$$\begin{aligned}&\{(10178-1358)\times(10178+1358)\} = \\ &\quad (7/1358)^2\{(1956878-1358)\times(1956878+1358)\} = 101747520\end{aligned} \quad , \tag{2.8d}$$

$$\begin{aligned}&\{(10294-2054)\times(10294+2054)\} = \\ &\quad (7/2054)^2\{(2959814-2054)\times(2959814+2054)\} = 101747520\end{aligned} \quad , \tag{2.8e}$$

$$\begin{aligned}&\{(10661-3451)\times(10661+3451)\} = \\ &\quad (7/3451)^2\{(4972891-3451)\times(4972891+3451)\} = 101747520\end{aligned} \quad . \tag{2.8f}$$

Inspection of these four latest examples suggests that the condition [u/k = e^2/d^2, where (d) is an integer, like 7 in Eq.(2.8a)], would be the main necessity for solving Eq.(2.4d) with (x) integer. However, the new (y = 289641) in Eq.(2.8c) is not correctly related to the new [u/k = $(201/7)^2$] in the way that (y = 40348) in Eq.(2.8a) was derived using Eq.(2.4c). That is, introduction of new [u = $(201/7)^2$, e = 201] into Eq.(2.4c) would produce (y = 818865236) rather than (y = 289641), which employed the original (u = 16). This situation will exist for all examples, so genuine values of (y) *can never occur in expressions of the form Eq.(2.8c)*, necessary for getting an integer (x) and (a). Finally, to confirm the result Eq.(2.6c) for these latest examples, let (u = e^2, and k = d^2 = 7^2) in Eq.(2.6a).

Clearly, these examples prove that (a) in Eq.(2.1) can never be an integer if (b) is an integer; and conversely any imposed value of integer (a) into Eq.(2.1) would make (b) a non-integer. This is equivalent to the proof of Eq.(1.1) for (p = 3).

3. The power four case

We prove that when (b) is a positive integer in the equation:

$$a^4 + b^4 = c^4, \qquad (3.0)$$

then (a) cannot be an integer, and vice-versa.

Let [c = a + e], where (e) is a positive integer, then set up two equal expressions which serve to limit the choice of (b), followed by (a) and (c):

$$\left[f(a,e) = \frac{c^4 - a^4 - e^4}{2e} \right] = \left[\frac{b^4 - e^4}{2e} = f(b,e) \right]. \qquad (3.1)$$

Here, f(a,e) will be an integer when (a) is any integer, and also for some non-integer (a) values whenever f(b,e) is an integer. Substitute (s = a/e), then evaluate f(a,e) as:

$$f(a,e) = s\{Ks + 2\} \times e^3, \qquad (3.1a)$$

where,

$$K = (2s + 3) \qquad (3.1b)$$

has been introduced for later use.

Only selected values of integer (b) can make f(b,e) an integer, to compare with a possible integer f(a,e) in Eq.(3.1). All other values of (b) make f(b,e) non-integer, then (a) would have to be non-integer. Typical selected (b) values are given by:

$$b_n = (2n-1)e, \quad \text{or} \quad b_n = ne, \quad \text{or} \quad b_n = (n+3)e/4, \qquad (3.2)$$

where integer (n ≥ 1). Further selected (b_n) values are given by different multiples of (e) which depend upon the value of (e) chosen. When using (b_n) values in Eq.(3.1), while substituting [b_n = (q +1)e], then f(b,e) will be an integer given by:

$$f(b,e) = q\{Uq + 2\} \times e^3, \qquad (3.3a)$$

where (q) is a function of (n) as determined by Eq.(3.2) or similar, and

$$U = \left(q^2/2 + 2q + 3\right) \qquad (3.3b)$$

has also been introduced for later use.

To test whether an integer (a) can make f(a,e) equal to f(b,e), equate Eq.(3.1a) and Eq.(3.3a):

$$[\, f(a,e) = (1/K)\{Ka(Ka + 2e)e\} \,] = \left[(1/U)\{Uqe(Uqe + 2e)e\} = f(b,e) \right]. \qquad (3.4a)$$

Make this more symmetrical by substituting:

$$x = (Ka + e), \qquad (3.4b)$$

which is an integer if (a) is an integer; and

$$y = (Uq + 1)e = z\,e\,, \tag{3.4c}$$

which may be an integer; then Eq.(3.4a) can be simplified to:

$$(1/K)\{(x-e)\times(x+e)\} \;=\; (1/U)\{(y-e)\times(y+e)\}\;. \tag{3.4d}$$

This equation is identical in form to Eq.(2.4d) even though (U, K, x, y) are defined differently. Consequently, the logical argument following Eq.(2.4d) will apply and lead to the same conclusion. That is, genuine values of (y) calculated from Eq.(3.4c) do not occur in expressions of the form Eq.(2.5c) and Eq.(2.8c) necessary for getting an integer (x).

Therefore (a) in Eq.(3.1) can never be an integer if (b) is an integer; and vice-versa. This is equivalent to the proof of Eq.(1.1) for (p = 4).

4. The power five case

We prove that when (b) is a positive integer in the equation:

$$a^5 + b^5 = c^5, \tag{4.0}$$

then (a) cannot be an integer, and vice-versa.

Let [c = a + e], where (e) is a positive integer, then set up two equal expressions which serve to limit the choice of (b), followed by (a) and (c):

$$\left[f(a,e) = \frac{c^5 - a^5 - e^5}{5e} \right] = \left[\frac{b^5 - e^5}{5e} = f(b,e) \right]. \tag{4.1}$$

Here, f(a,e) will be an integer when (a) is any integer, and also for some non-integer (a) values whenever f(b,e) is an integer. Substitute (s = a/e), then evaluate f(a,e) as:

$$f(a,e) = s\{k\,s + 1\}\times e^4, \tag{4.1a}$$

where,

$$k = \left(s^2 + 2s + 2\right) \tag{4.1b}$$

has been introduced for later use.

Only selected values of integer (b) can make f(b,e) an integer, to compare with a possible integer f(a,e) in Eq.(4.1). All other values of (b) make f(b,e) non-integer, then (a) would have to be non-integer. Typical selected (b) values are given by:

$$b_n = (5n-4)e, \qquad \text{or} \qquad b_n = ne, \quad \text{or} \qquad b_n = (n+3)e/4\,, \tag{4.2}$$

where integer ($n \geq 1$). Further selected (b_n) values are given by different multiples of (e) which depend upon the value of (e) chosen. When using (b_n) values in Eq.(4.1), while substituting [$b_n = (q+1)e$], then f(b,e) will be an integer given by:

$$f(b,e) = q\{uq+1\} \times e^4, \tag{4.3a}$$

where (q) is a function of (n) as determined by Eq.(4.2) or similar, and

$$u = \left(q^3/5 + q^2 + 2q + 2\right) \tag{4.3b}$$

has also been introduced for later use.

To test whether an integer (a) can make f(a,e) equal to f(b,e), equate Eq.(4.1a) and Eq.(4.3a):

$$\left[\, f(a,e) = (1/k)\left\{k\,a\,(k\,a+e)e^2\right\}\right] = \left[\,(1/u)\left\{u\,q\,e\,(u\,q\,e+e)e^2\right\} = f(b,e)\;\right]. \tag{4.4}$$

This equation is identical in form to Eq.(2.4a) even though (u, k) are defined differently and (e^2) needs cancelling. Consequently, the logical argument following Eq.(2.4a) will apply and lead to the same conclusion that (a) is not an integer.

Therefore (a) in Eq.(4.1) can never be an integer if (b) is an integer; and vice-versa. This is equivalent to the proof of Eq.(1.1) for (p = 5).

5. The power seven case

We prove that when (b) is a positive integer in the equation:

$$a^7 + b^7 = c^7, \tag{5.0}$$

then (a) cannot be an integer, and vice-versa.

Let [c = a + e], where (e) is a positive integer, then set up two equal expressions which serve to limit the choice of (b), followed by (a) and (c):

$$\left[f(a,e) = \frac{c^7 - a^7 - e^7}{7e}\right] = \left[\frac{b^7 - e^7}{7e} = f(b,e)\right]. \tag{5.1}$$

Here, f(a,e) will be an integer when (a) is any integer, and also for some non-integer (a) values whenever f(b,e) is an integer. Substitute (s = a/e), then evaluate f(a,e) as:

$$f(a,e) = s\{ks+1\} \times e^6, \tag{5.1a}$$

where,

$$k = \left(s^4 + 3s^3 + 5s^2 + 5s + 3\right) \tag{5.1b}$$

has been introduced for later use.

Only selected values of integer (b) can make f(b,e) an integer, to compare with a possible integer f(a,e) in Eq.(5.1). All other values of (b) make f(b,e) non-integer, then (a) would have to be non-integer. Typical selected (b) values are given by:

$$b_n = (7n-6)e, \quad \text{or} \quad b_n = ne, \quad \text{or} \quad b_n = (n+3)e/4, \qquad (5.2)$$

where integer (n ≥ 1). Further selected (b_n) values are given by different multiples of (e) which depend upon the value of (e) chosen. When using (b_n) values in Eq.(5.1), while substituting [b_n = (q +1)e], then f(b,e) will be an integer given by:

$$f(b,e) = q\{uq+1\} \times e^6, \qquad (5.3a)$$

where (q) is a function of (n) as determined by Eq.(5.2) or similar, and

$$u = \left(q^5/7 + q^4 + 3q^3 + 5q^2 + 5q + 3\right) \qquad (5.3b)$$

has also been introduced for later use.

To test whether an integer (a) can make f(a,e) equal to f(b,e), equate Eq.(5.1a) and Eq.(5.3a):

$$\left[f(a,e) = (1/k)\{ka(ka+e)e^4\} \right] = \left[(1/u)\{uqe(uqe+e)e^4\} = f(b,e) \right]. \quad (5.4)$$

This equation is identical in form to Eq.(2.4a) even though (u, k) are defined differently and (e^4) needs cancelling. Consequently, the logical argument following Eq.(2.4a) will apply and lead to the same conclusion that (a) is not an integer.

Therefore (a) in Eq.(5.1) can never be an integer if (b) is an integer; and vice-versa. This is equivalent to the proof of Eq.(1.1) for (p = 7).

6. The power eleven case

We prove that when (b) is a positive integer in the equation:

$$a^{11} + b^{11} = c^{11}, \qquad (6.0)$$

then (a) cannot be an integer, and vice-versa.

Let [c = a + e], where (e) is a positive integer, then set up two equal expressions which serve to limit the choice of (b), followed by (a) and (c):

$$\left[f(a,e) = \frac{c^{11} - a^{11} - e^{11}}{11e} \right] = \left[\frac{b^{11} - e^{11}}{11e} = f(b,e) \right]. \qquad (6.1)$$

Here, f(a,e) will be an integer when (a) is any integer, and also for some non-integer (a) values whenever f(b,e) is an integer. Substitute (s = a/e), then evaluate f(a,e) as:

$$f(a,e) = s\{ks+1\}\times e^{10}, \qquad (6.1a)$$

where,

$$k = \left(s^8 + 5s^7 + 15s^6 + 30s^5 + 42s^4 + 42s^3 + 30s^2 + 15s + 5\right) \qquad (6.1b)$$

has been introduced for later use.

Only selected values of integer (b) can make f(b,e) an integer, to compare with a possible integer f(a,e) in Eq.(6.1). All other values of (b) make f(b,e) non-integer, then (a) would have to be non-integer. Typical selected (b) values are given by:

$$b_n = (11n-10)e, \quad \text{or} \quad b_n = n\,e, \quad \text{or} \quad b_n = (n+3)e/4, \quad (6.2)$$

where integer (n ≥ 1). Further selected (b_n) values are given by different multiples of (e) which depend upon the value of (e) chosen. When using (b_n) values in Eq.(6.1), while substituting [b_n = (q +1)e], then f(b,e) will be an integer given by:

$$f(b,e) = q\{u\,q+1\}\times e^{10}, \qquad (6.3a)$$

where (q) is a function of (n) as determined by Eq.(6.2) or similar, and

$$u = \left(q^9/11 + q^8 + 5q^7 + 15q^6 + 30q^5 + 42q^4 + 42q^3 + 30q^2 + 15q + 5\right) \qquad (6.3b)$$

has also been introduced for later use.

To test whether an integer (a) can make f(a,e) equal to f(b,e), equate Eq.(6.1a) and Eq.(6.3a):

$$\left[\,f(a,e) = (1/k)\left\{k\,a\,(k\,a+e)e^8\right\}\right] = \left[(1/u)\left\{u\,q\,e\,(u\,q\,e+e)e^8\right\} = f(b,e)\;\right]. \quad (6.4)$$

This equation is identical in form to Eq.(2.4a) even though (u, k) are defined differently and (e^8) needs cancelling. Consequently, the logical argument following Eq.(2.4a) will apply and lead to the same conclusion that (a) is not an integer.

Therefore (a) in Eq.(6.1) can never be an integer if (b) is an integer; and vice-versa. This is equivalent to the proof of Eq.(1.1) for (p = 11).

7. The power thirteen case

We prove that when (b) is a positive integer in the equation:

$$a^{13} + b^{13} = c^{13}, \qquad (7.0)$$

then (a) cannot be an integer, and vice-versa.

Let [c = a + e], where (e) is a positive integer, then set up two equal expressions which serve to limit the choice of (b), followed by (a) and (c):

$$\left[f(a,e) = \frac{c^{13} - a^{13} - e^{13}}{13e} \right] = \left[\frac{b^{13} - e^{13}}{13e} = f(b,e) \right]. \qquad (7.1)$$

Here, f(a,e) will be an integer when (a) is any integer, and also for some non-integer (a) values whenever f(b,e) is an integer. Substitute (s = a/e), then evaluate f(a,e) as:

$$f(a,e) = s\{ks + 1\} \times e^{12}, \qquad (7.1a)$$

where,

$$k = \left(s^{10} + 6s^9 + 22s^8 + 55s^7 + 99s^6 + 132s^5 + 132s^4 + 99s^3 + 55s^2 + 22s + 6 \right) \qquad (7.1b)$$

has been introduced for later use.

Only selected values of integer (b) can make f(b,e) an integer, to compare with a possible integer f(a,e) in Eq.(7.1). All other values of (b) make f(b,e) non-integer, then (a) would have to be non-integer. Typical selected (b) values are given by:

$$b_n = (13n - 12)e, \quad \text{or} \quad b_n = ne, \quad \text{or} \quad b_n = (n+3)e/4, \qquad (7.2)$$

where integer (n ≥ 1). Further selected (b_n) values are given by different multiples of (e) which depend upon the value of (e) chosen. When using (b_n) values in Eq.(7.1), while substituting [b_n = (q +1)e], then f(b,e) will be an integer given by:

$$f(b,e) = q\{uq + 1\} \times e^{12}, \qquad (7.3a)$$

where (q) is a function of (n) as determined by Eq.(7.2) or similar, and

$$u = \left(q^{11}/13 + q^{10} + 6q^9 + 22q^8 + 55q^7 + 99q^6 + 132q^5 + 132q^4 + 99q^3 + 55q^2 + 22q + 6 \right) \qquad (7.3b)$$

has also been introduced for later use.

To test whether an integer (a) can make f(a,e) equal to f(b,e), equate Eq.(7.1a) and Eq.(7.3a):

$$\left[f(a,e) = (1/k)\left\{ ka(ka + e)e^{10} \right\} \right] = \left[(1/u)\left\{ uqe(uqe + e)e^{10} \right\} = f(b,e) \right]. \qquad (7.4)$$

This equation is identical in form to Eq.(2.4a) even though (u,k) are defined differently and (e^{10}) needs cancelling. Consequently, the logical argument following Eq.(2.4a) will apply and lead to the same conclusion that (a) is not an integer.

Therefore (a) in Eq.(7.1) can never be an integer if (b) is an integer; and vice-versa. This is equivalent to the proof of Eq.(1.1) for (p = 13).

8. The power p (prime number) case

We prove that when prime number ($p \geq 3$), and (b) is a positive integer in the equation:

$$a^p + b^p = c^p, \qquad (8.0)$$

then (a) cannot be an integer, and vice-versa.

Let [c = a + e], where (e) is a positive integer, then set up two equal expressions which serve to limit the choice of (b), followed by (a) and (c):

$$\left[f(a,e) = \frac{c^p - a^p - e^p}{p\,e} \right] = \left[\frac{b^p - e^p}{p\,e} = f(b,e) \right]. \qquad (8.1)$$

Here, f(a,e) will be an integer when (a) is any integer, and also for some non-integer (a) values whenever f(b,e) is an integer. Substitute (s = a/e), then evaluate f(a,e) as:

$$f(a,e) = s\{ks + 1\} \times e^{p-1}, \qquad (8.1a)$$

where,

$$k = \left[(s+1)^p - \left(s^p + ps + 1\right)\right] / \left(ps^2\right) \qquad (8.1b)$$

has been introduced for later use.

Only selected values of integer (b) can make f(b,e) an integer, to compare with a possible integer f(a,e) in Eq.(8.1). All other values of (b) make f(b,e) non-integer, then (a) would have to be non-integer. Typical selected (b) values are given by:

$$b_n = [p(n-1)+1]e, \quad \text{or} \quad b_n = ne, \quad \text{or} \quad b_n = (n+3)e/4, \qquad (8.2)$$

where integer ($n \geq 1$). Further selected (b_n) values are given by different multiples of (e) which depend upon the value of (e) chosen. When using (b_n) values in Eq.(8.1), while substituting [$b_n = (q+1)e$], then f(b,e) will be an integer given by:

$$f(b,e) = q\{uq + 1\} \times e^{p-1}, \qquad (8.3a)$$

where (q) is a function of (n) as determined by Eq.(8.2) or similar, and

$$u = \left[(q+1)^p - (pq+1)\right] / \left(pq^2\right) \qquad (8.3b)$$

has also been introduced for later use.

To test whether an integer (a) can make f(a,e) equal to f(b,e), equate Eq.(8.1a) and Eq.(8.3a):

$$\left[f(a,e) = (1/k)\left\{ka(ka+e)e^{p-3}\right\} \right] = \left[(1/u)\left\{uqe(uqe+e)e^{p-3}\right\} = f(b,e) \right]. \qquad (8.4)$$

This equation is identical in form to Eq.(2.4a) even though (u, k) are defined differently and (e^{p-3}) needs cancelling. Consequently, the logical argument following Eq.(2.4a) will apply and lead to the same conclusion that (a) is not an integer.

Therefore (a) in Eq.(8.1) can never be an integer if (b) is an integer; and vice-versa. This is equivalent to the proof of Eq.(1.1) for any prime number $p \geq 3$.

References

1) Wiles, A.J. (1995) Annals of Mathematics 141, No.3, pp 443-551

2) Wikipedia. https://en.wikipedia.org/wiki/Fermat%27s_Last_Theorem