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WP6100: PRS Solution (Definition)

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1. Principles

The Primary Reference Stars Solution (Step 2) shall determine the zero point corrections (more generally: the orientation parameters) of the different sets. Adjusting the abscissae resulting from a set solution according to these parameters puts them on a common celestial coordinate system with all other, similarly adjusted, abscissae.

It is possible to solve, at the same time, for any global parameters expressing some (possibly time-varying) distortion of the celestial coordinates, e.g. due to thermal instrumental effects or general relativity. Up to a few hundred global parameters can probably be handled; their number is limited by the possible weakening of the solution rather than computational considerations.

The set orientations and global parameters are calculated by imposing a certain reduction model on the abscissae of selected Primary Reference Stars (PRS), whereby each of the latter is completely characterized by five astrometric parameters. At least 1000 PRS are needed in order to have a sufficient number in every set (on the average 20 PRS/set) for a good determination of its orientation. The main criterion for selecting PRS from the programme stars is precisely the validity of that reduction model; this generally excludes all resolved and many unresolved multiple stars. No (or very few) slit errors are allowed among the abscissae used for the PRS solution, which further limits the choice. Other criteria such as the distribution of PRS with respect to position, magnitude and colour need to be discussed. Several different PRS selections can be tried, and should ideally give the same set orientations and global parameters.

The PRS Solution is formally a least-squares problem with the (relative) abscissae from the Set Solutions as 'observations' (for the right-hand sides) and the following unknowns:

(i) the astrometric parameters of the PRS: NASPAR = 5 parameters per star (NASPAR*NPRS unknowns);

(ii) the set orientation parameters (one of which is the zero point correction): NSETOR = 1 or 3 parameters per set (NSETOR*NSET unknowns);

(iii) an arbitrary number of global parameters (NGLOB $\geq$ 0 unknowns).

At present it is not clear whether a single orientation parameter (the zero point correction) is sufficient in the final solution (NSETOR = 1); possibly the coordinates of the RGC pole (α_r, δ_r) need to be included among the unknowns (NSETOR = 3). For a number of preliminary solutions, NSETOR = 1 is certainly adequate, so the software should be flexible in this respect.

In order to reduce the size of the normal equations system, either the astrometric or orientation parameters can be successively eliminated while forming the normals for the remaining unknowns. The two alternatives are discussed in NDAC/LO/020. Presently we assume that the astrometric parameters are eliminated, which appears the most natural course.

The rank deficiency due to the undefined system orientation and rotation is removed with the pseudosolution approach (NDAC/LO/018), which is preferable to the methods discussed in Annex A, pp. 53 - 55. The pseudosolution effectively adjusts the system of positions and proper motions to that used for the IC data of PRS (e.g. the FK5), without taking over any of its distortion or local errors ('soft postulation'). We also believe that this method is numerically very stable, easily adapted to the Cholesky method, and therefore also computationally efficient.

2. Summary of I/O Data

This does not include the information needed to set up PRS selection criteria and the functional influence of the global parameters on the abscissae.

The relevant Data Interface Descriptions (DID's) are:

DID#	SOURCE	DATA INPUT TO PRS SOLUTION
33	Abscissa Catalogue	• mean abscissae etc from set solutions
34	Ephemerides	• barycentric positions of the observer and the sun
35	Star Catalogue	• astrometric parameters, multiplicity, etc

DID#	DESTINATION	DATA OUTPUT FROM PRS SOLUTION
36	Zero Point Catalogue	• set orientations (zero point corrections etc)

3. Method and Basic Equations

3.1. Observation Model

For each set (j = ISET) and each PRS star (i = ID) included in it, the following observation model must be satisfied:

$$\bar{\bar{\upsilon}}(\bar{\bar{t}}_{ij},\ \underline{a}_i,\ v_{Ri},\ \underline{\varrho}_0,\ \underline{r}_j) - c_j +$$

$$+\ \Delta\upsilon_g(\bar{\bar{t}}_{ij},\ \bar{\underline{u}}_{ij},\ \underline{r}_j,\ \underline{\varrho}_S \mid \underline{g}) + \nu_{ij} = \bar{\bar{\upsilon}}_{ij} \qquad (1)$$

Here $\bar{\bar{\upsilon}}$ is the satellitocentric coordinate abscissa calculated for the mean time of observation ($\bar{\bar{t}}_{ij}$ = TOBS), astrometric parameters $\underline{a}_i = (\alpha_{oi},\ \delta_{oi},\ \mu_{\alpha i} \cos \delta_{oi},\ \mu_{\delta i},\ \Pi_i)'$ and radial velocity v_{Ri}, for satellitocentric barycentre $\underline{\varrho}_0$, and RGC pole $\underline{r}_j$. c_j is the set zero point correction and $\Delta\upsilon_g$ the global distortion function depending on time ($\bar{\bar{t}}_{ij}$), star direction ($\bar{\bar{\underline{u}}}_{ij}$), the RGC pole ($\underline{r}_j$) and satellitocentric position of the Sun ($\underline{\varrho}_S$); it is parametrized by the NGLOB-vector $\underline{g}$. $\bar{\bar{\upsilon}}_{ij}$ is the 'observed' mean abscissa from the set solution, with estimated standard error $\sigma_{\upsilon ij}$. ν_{ij} is observation noise ideally belonging to N(0, $\sigma_{\upsilon ij}$).

The PRS Solution will find a pseudosolution $\{\underline{a}_i, c_j, \underline{r}_j, \underline{g}\}$ minimizing $\Sigma_{ij}(\nu_{ij}/\sigma_{\upsilon ij})^2$.

Dropping the subscripts i, j, the complete formulae for $\bar{\bar{\upsilon}}(t, \underline{a}, v_R, \varrho_0, \underline{r}_j)$ are as follows [cf. (A.2.1) - (A.2.6), (A.11.1), (016.2), (016.3)]:

$$\underline{u}_o = \begin{pmatrix} \cos\delta_o \cos\alpha_o \\ \cos\delta_o \sin\alpha_o \\ \sin\delta_o \end{pmatrix} \tag{2}$$

$$\dot{\underline{u}}_o = \begin{pmatrix} -\sin\alpha_o & -\sin\delta_o\cos\alpha_o \\ \cos\alpha_o & -\sin\delta_o\sin\alpha_o \\ 0 & \cos\delta_o \end{pmatrix} \begin{pmatrix} \mu_\alpha \cos\delta_o \\ \mu_\delta \end{pmatrix} \tag{3}$$

$$\tau = \text{TOBS} - \text{TEPOCH} \tag{4}$$

$$\bar{\underline{u}} = \underline{u}_o + \tau\dot{\underline{u}}_o - \tau^2\left[R^{-1}\Pi v_R\dot{\underline{u}}_o + \tfrac{1}{2}|\dot{\underline{u}}_o|^2\underline{u}_o\right] \tag{5}$$

$$\bar{\bar{\underline{u}}} \equiv \begin{pmatrix} x_E \\ y_E \\ z_E \end{pmatrix} = \bar{\underline{u}} + \varrho_0\Pi - \bar{\underline{u}}(\bar{\underline{u}}'\varrho_0\Pi) \tag{6}$$

$$\begin{pmatrix} x_R \\ y_R \\ z_R \end{pmatrix} = \begin{pmatrix} -\sin\alpha_r & \cos\alpha_r & 0 \\ -\sin\delta_r\cos\alpha_r & -\sin\delta_r\sin\alpha_r & \cos\delta_r \\ \cos\delta_r\cos\alpha_r & \cos\delta_r\sin\alpha_r & \sin\delta_r \end{pmatrix} \begin{pmatrix} x_E \\ y_E \\ z_E \end{pmatrix} \tag{7}$$

$$\left.\begin{aligned} \bar{\bar{\alpha}} &= \mathrm{ATAN2}(y_E,\ x_E) \\ \bar{\bar{\delta}} &= \mathrm{ATAN2}(z_E,\ [x_E^2 + y_E^2]^{\frac{1}{2}}) \end{aligned}\right\} \qquad (8)$$

$$\left.\begin{aligned} \bar{\bar{\upsilon}} &= \mathrm{ATAN2}(y_R,\ x_R) \\ \bar{\bar{\rho}} &= \mathrm{ATAN2}(z_R,\ [x_R^2 + y_R^2]^{\frac{1}{2}}) \end{aligned}\right\} \qquad (9)$$

In (5), $R = 1.49597870\ 10^{11}$ m is the astronomical unit.

We cannot specify here the analytical form of the global distortion function $\Delta\upsilon_g$. Typically it would contain terms similar to spherical harmonics oriented with respect to the sun and the ecliptical or heliographic pole. A large number of such general terms could be prepared for in advance, so that they are easily switched on and off in various solutions, but the program structure must also permit incorporating arbitrary terms at a later stage.

3.2. Observation Equation

To obtain a linearized observation equation we need the partial derivatives of the left member of (1) with respect to all the unknowns (parameters). We gladly neglect the variation of $\Delta\upsilon_g$ with the astrometric parameters (through $\bar{\bar{u}}$) and the RGC pole; also the second-order terms in (5) and a few more approximations can be permitted. After division by σ_υ we have for observation ij, i.e. of star i in set j,

$$A_{ij}^{(1)}\left[\Delta\alpha_{oi}\cos\delta_{oi}\right] + A_{ij}^{(2)}\left[\Delta\delta_{oi}\right] + A_{ij}^{(3)}\left[\Delta\mu_{\alpha i}\cos\delta_{oi}\right] + A_{ij}^{(4)}\left[\Delta\mu_{\delta i}\right] +$$

$$+ A_{ij}^{(5)}\left[\Delta\Pi_i\right] + B_{ij}^{(1)}\left[\Delta c_j\right] + B_{ij}^{(2)}\left[\Delta\alpha_{rj}\cos\delta_{rj}\right] + B_{ij}^{(3)}\left[\Delta\delta_{rj}\right] +$$

$$+ \Sigma_\ell\ G_{ij}^{(\ell)}\left[\Delta g_\ell\right] + \sigma_{\upsilon ij}^{-1}\nu_{ij} = \sigma_{\upsilon ij}^{-1}(\upsilon_{ij} + c_j - \bar{\bar{\upsilon}}_{ij} - \Delta\upsilon_{gij}) \qquad (10)$$

(where the unknowns are in brackets) with coefficients

$$\sigma_\upsilon A_{ij}^{(1)} = \frac{\partial\bar{\bar{\upsilon}}}{\partial\alpha_o \cos\delta_o} = \sec^2\bar{\bar{\rho}}\left[\sin\delta_r \cos\bar{\bar{\delta}} - \cos\delta_r \sin\bar{\bar{\delta}}\cos(\bar{\bar{\alpha}} - \alpha_r)\right] \quad (11a)$$

$$\sigma_\upsilon A_{ij}^{(2)} = \frac{\partial\bar{\bar{\upsilon}}}{\partial\delta_o} = \sec^2\bar{\bar{\rho}}\cos\bar{\bar{\delta}}_r \sin(\bar{\bar{\alpha}} - \alpha_r) \quad (11b)$$

$$\sigma_\upsilon A_{ij}^{(3)} = \frac{\partial\bar{\bar{\upsilon}}}{\partial\mu_\alpha \cos\delta_o} = \sigma_\upsilon A_{ij}^{(1)}\tau \quad (11c)$$

$$\sigma_\upsilon A_{ij}^{(4)} = \frac{\partial\bar{\bar{\upsilon}}}{\partial\mu_\delta} = \sigma_\upsilon A_{ij}^{(2)}\tau \quad (11d)$$

$$\sigma_\upsilon A_{ij}^{(5)} = \frac{\partial\bar{\bar{\upsilon}}}{\partial\Pi} = \sec^2\bar{\bar{\rho}}\ (\underline{r}_\wedge \bar{\bar{\underline{u}}})'\underline{\rho}_0 \quad (11e)$$

$$\sigma_\upsilon B_{ij}^{(1)} = -1 \quad (12a)$$

$$\sigma_\upsilon B_{ij}^{(2)} = \frac{\partial\bar{\bar{\upsilon}}}{\partial\alpha_r \cos\delta_r} = -\sec^2\bar{\bar{\rho}}\cos\bar{\bar{\delta}}\left[\tan\delta_r \cos\bar{\bar{\delta}} - \sin\bar{\bar{\delta}}\cos(\bar{\bar{\alpha}} - \alpha_r)\right] \quad (12b)$$

$$\sigma_\upsilon B_{ij}^{(3)} = \frac{\partial\bar{\bar{\upsilon}}}{\partial\delta_r} = -\sec^2\bar{\bar{\rho}}\sin\bar{\bar{\rho}}\cos\bar{\bar{\delta}}\sin(\bar{\bar{\alpha}} - \alpha_r) \quad (12c)$$

$$\sigma_\upsilon G_{ij}^{(\ell)} = \frac{\partial\Delta\upsilon_g}{\partial g_\ell} \quad (13)$$

[cf. (A.12.6)]. Of course the terms $B_{ij}^{(2)}$ and $B_{ij}^{(3)}$ are omitted for NSETOR = 1.

Writing the unknowns in (10) as vectors $\underline{\Delta a}_i$, $\underline{\Delta b}_j$, $\underline{\Delta g}$ of length NASPAR, NSETOR, and NGLOB, and the coefficients as row-matrices

$$\left.\begin{aligned} \underline{A}_{ij} &\equiv (A_{ij}^{(1)} \quad \dots \quad A_{ij}^{(\mathrm{NASPAR})}) \\ \underline{B}_{ij} &\equiv (B_{ij}^{(1)} \quad \dots \quad B_{ij}^{(\mathrm{NSETOR})}) \\ \underline{G}_{ij} &\equiv (G_{ij}^{(1)} \quad \dots \quad G_{ij}^{(\mathrm{NGLOB})}) \end{aligned}\right\} \tag{14}$$

we can write the observation equations

$$\forall ij: \quad \underline{A}_{ij}\ \underline{\Delta a}_i + \underline{B}_{ij}\ \underline{\Delta b}_j + \underline{G}_{ij}\ \underline{\Delta g} + (\text{noise}) = h_{ij} \tag{15}$$

Here and in the following ij should be interpreted as a single subscript pointing at the unique observation of star i in set j. If such an observation does not exist, i.e. if star i is not included in set j, the corresponding terms are to be disregarded.

3.3. Normal Equations

From (15) we obtain directly the complete system of normals,

$$\forall i: \left[\sum_j \underline{A}'_{ij}\underline{A}_{ij}\right]\underline{\Delta a}_i + \sum_j\left[\underline{A}'_{ij}\underline{B}_{ij}\right]\underline{\Delta b}_j + \left[\sum_j \underline{A}'_{ij}\underline{G}_{ij}\right]\underline{\Delta g} = \left[\sum_j \underline{A}'_{ij}h_{ij}\right] \tag{16a}$$

$$\forall j: \sum_i\left[\underline{B}'_{ij}\underline{A}_{ij}\right]\underline{\Delta a}_i + \left[\sum_i \underline{B}'_{ij}\underline{B}_{ij}\right]\underline{\Delta b}_j + \left[\sum_i \underline{B}'_{ij}\underline{G}_{ij}\right]\underline{\Delta g} = \left[\sum_i \underline{B}'_{ij}h_{ij}\right] \tag{16b}$$

$$\sum_i\left[\sum_j \underline{G}'_{ij}\underline{A}_{ij}\right]\underline{\Delta a}_i + \sum_j\left[\sum_i \underline{G}'_{ij}\underline{B}_{ij}\right]\underline{\Delta b}_j + \left[\sum_{ij} \underline{G}'_{ij}\underline{G}_{ij}\right]\underline{\Delta g} = \left[\sum_{ij} \underline{G}'_{ij}h_{ij}\right] \tag{16c}$$

Under regular conditions both $\Sigma_j\underline{A}'_{ij}\underline{A}_{ij}$ [an (NASPAR,NASPAR)-matrix] and $\Sigma_i\underline{B}'_{ij}\underline{B}_{ij}$ [an (NSETOR, NSETOR)-matrix] are positive definite and either can serve as pivot element for eliminating $\underline{\Delta a}_i$ or $\underline{\Delta b}_j$. If the observations are accessed one star at a time [e.g. ij = i1, i2, ..., iNSET, i´1, i´2, ..., i´NSET, ...]

$\underline{\Delta a}_i$ can be eliminated as soon as $\underline{P}_i \equiv \Sigma_j \underline{A}'_{ij}\underline{A}_{ij}$ has been completed; in this way (16a) and the first bracket in (16b) and in (16c) are not saved. Elimination of $\underline{\Delta a}_i$ is effected by substituting

$$\underline{\Delta a}_i = \underline{P}_i^{-1}\left[\sum_j \underline{A}'_{ij}h_{ij} - \sum_j \underline{A}'_{ij}\underline{B}_{ij}\underline{\Delta b}_j - \sum_j \underline{A}'_{ij}\underline{G}_{ij}\underline{\Delta g}\right] \tag{17}$$

from (16a) in (16b) and (16c); the result is

$$\forall j: \quad \sum_{i \neq i}\left(\underline{B}'_{ij}\underline{A}_{ij}\right)\underline{\Delta a}_i + \left[\left(\sum_i \underline{B}'_{ij}\underline{B}_{ij}\right)\underline{\Delta b}_j - \underline{B}'_{ij}\underline{A}_{ij}\underline{P}_i^{-1}\sum_j \underline{A}'_{ij}\underline{B}_{ij}\underline{\Delta b}_j\right] +$$

$$+ \left(\sum_i \underline{B}'_{ij}\underline{G}_{ij} - \underline{B}'_{ij}\underline{A}_{ij}\underline{P}_i^{-1}\sum_j \underline{A}'_{ij}\underline{G}_{ij}\right)\underline{\Delta g} =$$

$$= \sum_i \underline{B}'_{ij}h_{ij} - \underline{B}'_{ij}\underline{A}_{ij}\underline{P}_i^{-1}\sum_j \underline{A}'_{ij}h_{ij} \tag{18a}$$

$$\sum_{i \neq i}\left(\sum_j \underline{G}'_{ij}\underline{A}_{ij}\right)\underline{\Delta a}_i + \sum_j\left(\sum_i \underline{G}'_{ij}\underline{B}_{ij} - \left[\sum_j \underline{G}'_{ij}\underline{A}_{ij}\right]\underline{P}_i^{-1}\underline{A}'_{ij}\underline{B}_{ij}\right)\underline{\Delta b}_j +$$

$$+ \left(\sum_{ij} \underline{G}'_{ij}\underline{G}_{ij} - \left[\sum_j \underline{G}'_{ij}\underline{A}_{ij}\right]\underline{P}_i^{-1}\sum_j \underline{A}'_{ij}\underline{G}_{ij}\right)\underline{\Delta g} =$$

$$= \sum_{ij} \underline{G}'_{ij}h_{ij} - \left[\sum_j \underline{G}'_{ij}\underline{A}_{ij}\right]\underline{P}_i^{-1}\sum_j \underline{A}'_{ij}h_{ij} \tag{18b}$$

Repeating the elimination process for every PRS we get the normals in the form

$$\forall j: \quad \sum_j \underline{D}_{jj}\,\underline{\Delta b}_j + \underline{E}_j\,\underline{\Delta g} = \underline{e}_j \tag{19a}$$

$$\sum_j \underline{E}'_j\,\underline{\Delta b}_j + \underline{F}\,\underline{\Delta g} = \underline{f} \tag{19b}$$

in which the matrices of dimensions

$\underline{D}_j$ (NSETOR,NSETOR) $\quad$ $\underline{E}_j$ (NSETOR,NGLOB) $\quad$ $\underline{e}_j$ (NSETOR,1)

$\underline{F}$ (NGLOB,NGLOB) $\quad$ $\underline{f}$ (NGLOB,1)

are ($\delta_{jj'}$ is Kronecker's delta)

$$\underline{D}_{jj'} = \sum_i \left[\delta_{jj'} \underline{B}'_{ij} \underline{B}_{ij} - \underline{B}'_{ij} \underline{A}_{ij} \underline{P}_i^{-1} \underline{A}'_{ij'} \underline{B}_{ij'} \right] \tag{20a}$$

$$\underline{E}_j = \sum_i \left[\underline{B}'_{ij} \underline{G}_{ij} - \underline{B}'_{ij} \underline{A}_{ij} \underline{P}_i^{-1} \Sigma \underline{A}'_i \underline{G}_i \right] \tag{20b}$$

$$\underline{e}_j = \sum_i \left[\underline{B}'_{ij} h_{ij} - \underline{B}'_{ij} \underline{A}_{ij} \underline{P}_i^{-1} \Sigma \underline{A}'_i h_i \right] \tag{20c}$$

$$\underline{F} = \sum_i \left[\sum_j \underline{G}'_{ij} \underline{G}_{ij} - \left(\sum_j \underline{G}'_{ij} \underline{A}_{ij} \right) \underline{P}_i^{-1} \sum_j \underline{A}'_{ij} \underline{G}_{ij} \right] \tag{20d}$$

$$\underline{f} = \sum_i \left[\sum_j \underline{G}'_{ij} h_{ij} - \left(\sum_j \underline{G}'_{ij} \underline{A}_{ij} \right) \underline{P}_i^{-1} \sum_j \underline{A}'_{ij} h_{ij} \right] \tag{20e}$$

A possible algorithm for accumulating these matrices (and also the scalar $r = \Sigma_{ij} h_{ij}^2$) is given below. It is noted that the same algorithm can be used for the Set Solution, with the frame orientations (or attitude polynomials, in the case of DS) replacing $\Delta \underline{a}_i$, abscissae instead of $\Delta \underline{b}_j$, and instrument parameters instead of $\Delta \underline{g}$.

TABLE 1. Accumulation of normals with successive elimination of astrometric parameters.

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1. zero all $\underline{D}_{jj'}$, $\underline{E}_j$, $\underline{e}_j$, and $\underline{F}$, $\underline{f}$, r
2. loop through the stars i:

   2.1. zero $(\underline{P}\ \underline{Q}\ \underline{R}\ \underline{s})$ of dimension (NASPAR, NASPAR+NSETOR+NGLOB+1)

   2.2. loop through the sets j:

      2.2.1. input/compute $(\underline{A}_{ij}\ \underline{B}_{ij}\ \underline{G}_{ij}\ h_{ij})$

      2.2.2. $\underline{H}_j := \underline{A}'_{ij}\underline{B}_{ij}$

      2.2.3. $(\underline{P}\ \underline{Q}\ \underline{R}\ \underline{s}) := (\underline{P}\ \underline{Q}\ \underline{R}\ \underline{s}) + \underline{A}'_{ij}(\underline{A}_{ij}\ \underline{B}_{ij}\ \underline{G}_{ij}\ h_{ij})$

   2.3. next j

   2.4. if $\underline{P}$ is positive definite, factorize and reduce,

   $$(\underline{P}\ \underline{Q}\ \underline{R}\ \underline{s}) := \underline{P}'^{-\frac{1}{2}}(\underline{P}\ \underline{Q}\ \underline{R}\ \underline{s})$$

   otherwise, go to 3.

   2.5. loop through the sets j:

      2.5.1. $\underline{H}_j := \underline{P}'^{-\frac{1}{2}}\underline{H}_j$

      2.5.2. accumulation of normals (except off-diagonal $\underline{D}_{jj'}$),

   $$\begin{pmatrix} \underline{D}_{jj} & \underline{E}_j & \underline{e}_j \\ \cdot & \underline{F} & \underline{f} \\ \cdot & \cdot & r \end{pmatrix} := \begin{pmatrix} \underline{D}_{jj} & \underline{E}_j & \underline{e}_j \\ \cdot & \underline{F} & \underline{f} \\ \cdot & \cdot & r \end{pmatrix} +$$

   $$+ \begin{pmatrix} \underline{B}'_{ij} \\ \underline{G}'_{ij} \\ h_{ij} \end{pmatrix} \begin{pmatrix} \underline{B}_{ij} & \underline{G}_{ij} & h_{ij} \end{pmatrix} - \begin{pmatrix} \underline{H}'_j \\ \underline{R}' \\ \underline{s}' \end{pmatrix} \begin{pmatrix} \underline{H}'_j & \underline{R} & \underline{s} \end{pmatrix}$$

      2.5.3. loop through the previous sets $j' < j$:

         2.5.3.1. $\underline{D}_{j'j} := \underline{D}_{j'j} - \underline{H}'_{j'}\underline{H}_j$

      2.5.4. next j'

   2.6. next j

3. next i
4. end

More concisely, Eqns (15) - (20) can be written in matrix form as shown below, which however effectively conceals the advantageous structure of the equations. Let M be the total number of observations and $\underline{A}$, $\underline{B}$, $\underline{G}$, $\underline{\nu}$, $\underline{h}$, $\Delta\underline{a}$, $\Delta\underline{b}$, $\Delta\underline{g}$ matrices of dimensions

$\underline{A}$(M, NPRS*NASPAR)	$\Delta\underline{a}$(NPRS*NASPAR,1)	$\underline{\nu}$(M,1)
$\underline{B}$(M, NSET*NSETOR)	$\Delta\underline{b}$(NSET*NSETOR,1)	$\underline{h}$(M,1)
$\underline{G}$(M,NGLOB)	$\Delta\underline{g}$(NGLOB,1)	

The observation equations are

$$(\underline{A}\quad \underline{B}\quad \underline{G})\begin{pmatrix}\Delta\underline{a}\\ \Delta\underline{b}\\ \Delta\underline{g}\end{pmatrix} + \underline{\nu} = \underline{h} \tag{15'}$$

and the normals,

$$\begin{pmatrix}\underline{A}'\underline{A} & \underline{A}'\underline{B} & \underline{A}'\underline{G}\\ \underline{B}'\underline{A} & \underline{B}'\underline{B} & \underline{B}'\underline{G}\\ \underline{G}'\underline{A} & \underline{G}'\underline{B} & \underline{G}'\underline{G}\end{pmatrix}\begin{pmatrix}\Delta\underline{a}\\ \Delta\underline{b}\\ \Delta\underline{g}\end{pmatrix} = \begin{pmatrix}\underline{A}'\underline{h}\\ \underline{B}'\underline{h}\\ \underline{G}'\underline{h}\end{pmatrix} \tag{16'}$$

Both $\underline{A}'\underline{A}$ and $\underline{B}'\underline{B}$ are block-diagonal and positive definite; using $\underline{A}'\underline{A}$ as pivot to eliminate $\Delta\underline{a}$ yields

$$\begin{pmatrix}\underline{D} & \underline{E}\\ \underline{E}' & \underline{F}\end{pmatrix}\begin{pmatrix}\Delta\underline{b}\\ \Delta\underline{g}\end{pmatrix} = \begin{pmatrix}\underline{e}\\ \underline{f}\end{pmatrix} \tag{19'}$$

with

$$\underline{D} = \underline{B}'\underline{B} - \underline{B}'\underline{A}(\underline{A}'\underline{A})^{-1}\underline{A}'\underline{B} \tag{20a'}$$

$$\underline{E} = \underline{B}'\underline{G} - \underline{B}'\underline{A}(\underline{A}'\underline{A})^{-1}\underline{A}'\underline{G} \tag{20b'}$$

$$\underline{e} = \underline{B}'\underline{h} - \underline{B}'\underline{A}(\underline{A}'\underline{A})^{-1}\underline{A}'\underline{h} \tag{20c'}$$

$$\underline{F} = \underline{G}'\underline{G} - \underline{G}'\underline{A}(\underline{A}'\underline{A})^{-1}\underline{A}'\underline{G} \tag{20d'}$$

$$\underline{f} = \underline{G}'\underline{h} - \underline{G}'\underline{A}(\underline{A}'\underline{A})^{-1}\underline{A}'\underline{h} \tag{20e'}$$

3.4. Elimination of Rank Deficiency

The PRS normals (16′) or (19′) should have a rank defect of NULLSP = 6 for the undefined coordinate system orientation and rotation. In order to apply the psedosolution method described in NDAC/LO/018 we need to specify

(i) exactly NULLSP fictitious 'observations' suitable to fix the coordinate orientation and rotation;

(ii) exactly NULLSP linearly independent vectors from the null space of the normal equations matrix.

Fictitious observations. If we disregard at first the system rotation, it is clear that the system of celestial coordinates resulting from a PRS solution would be completely fixed by only specifying the abscissa origins (zero points) on three mutually orthogonal RGC's. That advantageous configuration is almost achievable with a revolving angle close to 45° (Fig. 1). If three sets j_1, j_2, j_3 are thus selected early in the mission, and a similar triplet j_4, j_5, j_6 a few years later, we remove the indeterminacy by postulating

$$c_{j_1} = c_{j_2} = \ldots = c_{j_{\mathrm{NULLSP}}} = 0 \tag{21}$$

As observation equations, these conditions can be given arbitrarily large or even infinite weight (equivalent to deleting the corresponding six unknowns from the normals); in practice a large positive number is simply added to the relevant NULLSP diagonal elements. The normal equations are then solved as usual.

Null space vectors. Let $\underline{\varepsilon}_o$ be the misorientation (at mid-epoch TEPOCH, $\tau = 0$) and $\underline{\omega}$ the uniform rotation of the PRS celestial coordinates with respect to some preferred frame. Null vectors can be generated as derivatives of the unknowns with respect to the components of $\underline{\varepsilon}_o$ and $\underline{\omega}$:

$$\underline{W} = \begin{pmatrix} \dfrac{\partial \Delta \underline{b}}{\partial \varepsilon_{ox}} & \dfrac{\partial \Delta \underline{b}}{\partial \varepsilon_{oy}} & \cdots & \dfrac{\partial \Delta \underline{b}}{\partial \omega_z} \\[2ex] \dfrac{\partial \Delta \underline{g}}{\partial \varepsilon_{ox}} & \dfrac{\partial \Delta \underline{g}}{\partial \varepsilon_{oy}} & \cdots & \dfrac{\partial \Delta \underline{g}}{\partial \omega_z} \end{pmatrix} \tag{22}$$

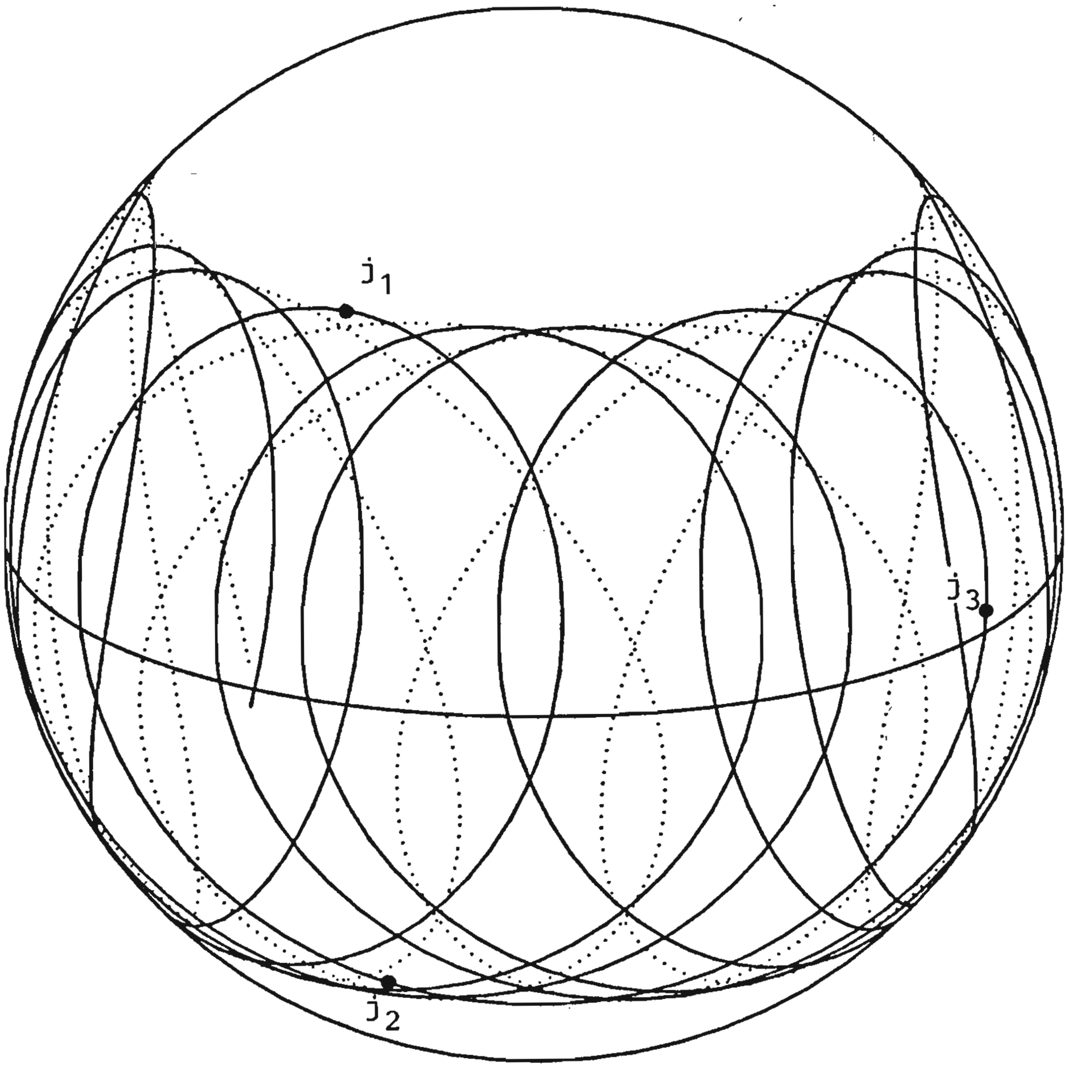

FIGURE 1. Example showing how three sets (j_1, j_2, j_3) in a relatively short time interval can be chosen for almost orthogonal RGC's. Scanning parameters: $\xi = 43^{\circ}$, $K = 6.4$. The time from j_1 to j_3 is about 40 days.

$$\underline{W} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\sec\delta_{rj}\cos\alpha_{rj} & -\sec\delta_{rj}\sin\alpha_{rj} & 0 & -\tau_j\sec\delta_{rj}\cos\alpha_{rj} & -\tau_j\sec\delta_{rj}\sin\alpha_{rj} & 0 \\ \sin\delta_{rj}\cos\alpha_{rj} & \sin\delta_{rj}\sin\alpha_{rj} & -\cos\delta_{rj} & \tau_j\sin\delta_{rj}\cos\alpha_{rj} & \tau_j\sin\delta_{rj}\sin\alpha_{rj} & -\tau_j\cos\delta_{rj} \\ -\sin\alpha_{rj} & \cos\alpha_{rj} & 0 & -\tau_j\sin\alpha_{rj} & \tau_j\cos\alpha_{rj} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ [\Delta c_j] \\ [\Delta\alpha_{rj}\cos\delta_{rj}] \\ [\Delta\delta_{rj}] \\ \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} [\Delta g_\ell] \end{matrix} \qquad \ldots \quad (23)$$

For NSETOR = 3 and NULLSP = 6, the resulting (NSET*NSETOR+NGLOB, NULLSP)-matrix is outlined on the preceding page, Eqn (23). The unknowns are indicated in brackets to the right; τ_j is the average mean observation time for set j.

For orthonormalizing $\underline{W} \equiv (\underline{w}_1 \ \ \underline{w}_2 \ \ \ldots \ \ \underline{w}_{NULLSP})$, the Modified Gram-Schmidt algorithm is recommended:

---

1. for m = 1 to NULLSP
   - 1.1. d := $\underline{w}_m'\underline{w}_m$
   - 1.2. for n = m+1 to NULLSP
     - 1.2.1. s := $\underline{w}_n'\underline{w}_m/d$
     - 1.2.2. $\underline{w}_n := \underline{w}_n - \underline{w}_m s$
   - 1.3. next n
   - 1.4. $\underline{w}_m := \underline{w}_m/d^{\frac{1}{2}}$
2. next m
3. end

---

4. Process Description

Before entering the PRS Solution proper, the Abscissa Catalogue must be re-ordered according to object identification number (ID), e.g. as follows:

```
-----------------------------------------------------------------
ID, NOBS                                 - object identifier,
                                           number of abscissae
:
ISET, TOBS, ABSC, ORD, SDABSC, GOF       } NOBS entries
:
-----------------------------------------------------------------
ID, NOBS                                 - next object
:
```

As NOBS will vary from 0 to ~60 (average ~40) for different objects, it is probably useful to accumulate NOBS(ID) in the Star Catalogue (or a separate file) during the progress of Set Solutions; in this way the re-ordered Abscissa Catalogue can be set up compactly by running through the set results tapes once only.

The re-ordered Abscissa Catalogue is input also for the Backsubstitution (Step 3) and Double Stars. In all cases only sequential access is required.

A. Initiate the PRS Solution by defining PRS and set selection criteria, global parameters (number and form), and options such as NASPAR = 2 or 5, NSETOR = 1 or 3. Surveying a set catalogue allows to determine NSET and then set up the normal equations structure.

B. Accumulate normals as indicated in Table 1. While looping through the stars, the PRS selection criteria are applied. These must in particular reject suspected double stars and stars with unresolved slit errors. Note that a Step 3 analysis of prospective PRS can be made after point 2.4 in Table 1: neglecting $\Delta \underline{b}$ and $\Delta \underline{g}$, we have $\chi^2 = \underline{s}'\underline{s}$, current $\Delta \underline{a}_i = \underline{P}^{-\frac{1}{2}}\underline{s}$, and abscissa residuals = $(h_{ij} - \underline{A}_{ij}\Delta \underline{a}_i)\sigma_{\upsilon ij}$.

C. Pseudosolution of normals
   - Ca. Select NULLSP (3 or 6) and fix sets $(j_1 \ldots j_{NULLSP})$
   - Cb. Calculate orthonormal $\underline{W}$
   - Cc. Modify diagonal elements for fix sets
   - Cd. Conventional solution of $\Delta \underline{b}$, $\Delta \underline{g}$
   - Ce. Modify solution according to (018.22)
   - Cf. If elements of the inverse (pseudocovariances) are required, proceed according to (018.36)

D. Output results to Abscissa Zero Points

DID 33	OUTPUT FROM : ABSCISSA CATALOGUE INPUT TO : PRS SOLUTION	PAGE : 1 OF 1 DATE : 83.05.02

DESIGNATION	ANNEX A	EXPLANATION	UNIT	RANGE MIN	RANGE MAX	RANGE DEF	NUM ACC ABS	NUM ACC REL	DIGITS
		1. <u>For each star</u>:							
ID	i (p35)	object identification number	-	0	10^8	0	1	-	8
		1.1. <u>For each set containing the object</u>:							
ISET	j (p35)	set identification number	-	1	9.10^5	-	1	-	6
TOBS	$t_o+\tau$ (p4)	mean time of observation = $\overline{\overline{t}}_{ij}$	s	0	2.10^8	-	10^{-6}	-	15
ABSC	υ_{ji} (p35)	mean abscissa = $\overline{\overline{\upsilon}}_{ij}$ (NDAC/LO/016)	rad	$-\pi$	π	-	10^{-11}	-	12
ORD	ρ_{ji} (p35)	mean ordinate = $\overline{\overline{\rho}}_{ij}$ (NDAC/LO/016)	rad	$-\frac{1}{2}\pi$	$\frac{1}{2}\pi$	-	10^{-11}	-	12
SDABSC	σ_{ji} (p37)	standard deviation of ABSC = $\sigma_{\upsilon ij}$	rad	-10^{-5}	19^{-5}	-	10^{-11}	-	7
GOF	-	goodness of fit	-	-10^{38}	10^{38}		-	10^{-6}	6

DID 34	OUTPUT FROM : EPHEMERIDES INPUT TO : PRS SOLUTION	PAGE : 1 OF 1 DATE : 83.05.02

DESIGNATION	ANNEX A	EXPLANATION	UNIT	RANGE MIN	MAX	DEF	NUM ACC ABS	REL	DIG-ITS
		1. <u>For each star:</u>							
		1.1. <u>For each set containing the object</u>							
TOBS	$t_o+\tau$ (p4)	mean time of observation	s	0	10^8	-	10^{-6}	-	15
ROBS(K), K=1,3	$-\varrho_0$ (p5)	barycentric coord. of observer	m	-2.10^{11}	2.10^{11}	-	10	-	11
RSUN(K), K=1,3	-	barycentric coord. of the Sun	m	-2.10^9	2.10^9	-	10^5	-	5

DID 35	OUTPUT FROM : STAR CATALOGUE INPUT TO : PRS SOLUTION	PAGE : 1 OF 1 DATE : 83.05.02

DESIGNATION	ANNEX A	EXPLANATION	UNIT	RANGE MIN	RANGE MAX	RANGE DEF	NUM ACC ABS	NUM ACC REL	DIG-ITS
		1. <u>For each PRS candidate</u>:							
ID	-	object identification number	-	0	9.10^8	0	1	-	8
RAØ	α_o (p4)	Right Ascension at TEPOCH	rad	$-\pi$	π	-	10^{-11}	-	12
DECØ	δ_o (p4)	Declination at TEPOCH	rad	$-\frac{1}{2}\pi$	$\frac{1}{2}\pi$	-	10^{-11}	-	12
PMRA	$\mu_\alpha \cos \delta_o$ (p4)	Proper Motion in R.A. at TEPOCH	rad/s	-10^{-11}	10^{-11}	0	10^{-19}	-	9
PMDEC	μ_δ (p4)	Proper Motion in Dec at TEPOCH	rad/s	-10^{-11}	10^{-11}	0	10^{-19}	-	9
PX	$\tilde{\omega}$ (p4)	trigonometric parallax	rad	-5.10^{-6}	5.10^{-6}	0	10^{-11}	-	6
VR	v_R (p4)	Radial Velocity	m/s	-10^6	10^6	0	1	-	6
BMAG	-	blue (B) magnitude	mag	-2	15	99	10^{-2}	-	4
BMV, SDBMV	-	colour index (B-V) and s.d.	mag	-1	3	99	10^{-2}	-	4
MULT	-	multiplicity information (TBD)	-	0	9999	0	1	-	4

DID 36	OUTPUT FROM : PRS SOLUTION INPUT TO : ZERO POINT CATALOGUE	PAGE : 1 OF 1 DATE : 83.05.02

DESIGNATION	ANNEX A	EXPLANATION	UNIT	RANGE MIN	RANGE MAX	RANGE DEF	NUM ACC ABS	NUM ACC REL	DIG-ITS
		1. <u>For each set:</u>							
ISET	j (p35)	set identification number	-	1	9.10^5	-	1	-	6
CSET	c_j (p34)	abscissa zero point correction	rad	$-\pi$	π	0	10^{-11}	-	12
RASET	α_{rj} (p31)	Right Ascension of RGC pole	rad	$-\pi$	π	-	10^{-11}	-	12
DECSET	δ_{rj} (p31)	Declination of RGC pole	rad	$-\frac{1}{2}\pi$	$\frac{1}{2}\pi$	-	10^{-11}	-	12