Trace formulae for actions of finite unitary groups on cohomology of Artin–Schreier varieties

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Abstract

Associated to a certain additive polynomial, we introduce an Artin–Schreier variety admitting an action of a finite unitary group. We calculate the character of the cohomology as a representation of a finite unitary group. One of our main ingredients is explicit character formulae for Weil representations of unitary groups due to Gérardin. We give another trace formula for a projective hypersurface admitting an action of a finite unitary group.

1 Introduction

Let p be a prime number and q a power of it. Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$. Let $R(x) := \sum_{i=0}^{e} a_i x^{q^i} \in \mathbb{F}[x]$ with a positive integer e and $a_e \neq 0$. Assume that $a_i = 0$ if i is even. Let n be a positive integer. We consider the n-dimensional Artin–Schreier variety $X_{R,n}$ defined by

$$a^q - a = \sum_{i=1}^{n} x_i R(x_i) = \sum_{i=0}^{e} a_i \left(x_1^{q^i+1} + \cdots + x_n^{q^i+1} \right) = \sum_{i=0}^{e} a_i {}^t\boldsymbol{x}^{q^i} \boldsymbol{x}$$

in $\mathbb{A}^{n+1} = \operatorname{Spec} \mathbb{F}[a, x_1, \ldots, x_n]$, where $\boldsymbol{x} = (x_i)_{1 \leq i \leq n} \in \mathbb{A}^n$ is a column vector and ${}^t\boldsymbol{x}^{q^i}$ denotes the transpose of $\boldsymbol{x}^{q^i} = (x_j^{q^i})_{1 \leq j \leq n}$. For an integer $s \geq 1$, let $\mathbb{F}_{q^s} \subset \mathbb{F}$ be the subfield of cardinality q^s. Let

$$\mathrm{U}_n(q) := \left\{ g \in \mathrm{GL}_n(\mathbb{F}_{q^2}) \mid g^\dagger g = \mathrm{I}_n \right\},$$

where $g^\dagger := (a_{j,i}^q)$ for $g = (a_{i,j}) \in \mathrm{GL}_n(\mathbb{F}_{q^2})$. Then $X_{R,n}$ admits a natural (right) action of $\mathrm{U}_n(q)$ given by $(a, \boldsymbol{x}) \mapsto (a, g^{-1}\boldsymbol{x})$ for $g \in \mathrm{U}_n(q)$, which is well-defined, because i is odd if $a_i \neq 0$.

Let $\ell \neq p$ be a prime number. For a variety X over $\mathbb{F}$ and an integer $i \geq 0$, let $H_c^i(X, \overline{\mathbb{Q}}_\ell)$ denote the i-th ℓ-adic cohomology group of X with compact support (cf. [2] and [3, (1.1)]). We regard $H_c^i(X_{R,n}, \overline{\mathbb{Q}}_\ell)$ as a $\mathrm{U}_n(q)$-representation, where $g \in \mathrm{U}_n(q)$ acts as g^*. In this paper, we give an explicit formula of the following virtual character of $\mathrm{U}_n(q)$:

$$H_c^*(X_{R,n}, \overline{\mathbb{Q}}_\ell) = \sum_{i=0}^{2n} (-1)^i H_c^i(X_{R,n}, \overline{\mathbb{Q}}_\ell).$$

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Let $N_r(g) := \dim_{\mathbb{F}_{q^{2r}}} \mathrm{Ker}(g - \mathrm{id}_{\mathbb{F}^n_{q^{2r}}} : \mathbb{F}^n_{q^{2r}} \to \mathbb{F}^n_{q^{2r}})$ for an integer $r \geq 1$ and $g \in \mathrm{U}_n(q^r)$. We define

$$\chi_n(g) := (-1)^n(-q)^{N_1(g)} \quad \text{for } g \in \mathrm{U}_n(q),$$

which is a character. We show the following trace formula.

Theorem 1.1. *We have the equality*

$$H^*_{\mathrm{c}}(X_{R,n}, \overline{\mathbb{Q}}_\ell) = (-1)^n(q-1)\chi^e_n + 1$$

as virtual characters of $\mathrm{U}_n(q)$.

The curve $X_{R,1}$ has been studied in [4] mainly in the case $q = p = 2$ and studied in [1] in the case $q = p > 2$. The smooth compactification $\overline{X}_{R,1}$ of $X_{R,1}$ has interesting arithmetic and group-theoretic properties and has been studied in many aspects. For example, $\overline{X}_{R,1}$ is a supersingular curve, has a large automorphism group and can have many rational points.

For a skew-hermitian space (V, h) over $\mathbb{F}_{q^2}$, a Heisenberg group $\mathrm{H}(V, h)$ is introduced in [5]. The unitary group

$$\mathrm{U}(V, h) := \{g \in \mathrm{Aut}_{\mathbb{F}_{q^2}}(V) \mid h(gv, gv') = h(v, v') \ (v, v' \in V)\}$$

acts on $\mathrm{H}(V, h)$ naturally. Let $\psi \colon \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$ be a non-trivial character. Associated to ψ, an irreducible representation $\widetilde{\rho}_{\psi,V,h}$ of $\mathrm{H}(V, h) \rtimes \mathrm{U}(V, h)$ is constructed in [5]. We call $\widetilde{\rho}_{\psi,V,h}$ the Heisenberg–Weil representation (shortly HW representation) of $\mathrm{H}(V, h) \rtimes \mathrm{U}(V, h)$ associated to ψ. The restriction of $\widetilde{\rho}_{\psi,V,h}$ to $\mathrm{U}(V, h)$ is called the Weil representation of $\mathrm{U}(V, h)$, whose character is known explicitly in [5] (cf. §2.4).

We state our strategy to show Theorem 1.1. The variety $X_{R,n}$ admits the action of $\mathbb{F}_q$ given by $(a, \boldsymbol{x}) \mapsto (a + \zeta, \boldsymbol{x})$ for $\zeta \in \mathbb{F}_q$. To show Theorem 1.1, it suffices to study the ψ-isotypic part $H^i_{\mathrm{c}}(X_{R,n}, \overline{\mathbb{Q}}_\ell)[\psi] \subset H^i_{\mathrm{c}}(X_{R,n}, \overline{\mathbb{Q}}_\ell)$ for each $i \geq 0$. We have $H^i_{\mathrm{c}}(X_{R,n}, \overline{\mathbb{Q}}_\ell)[\psi] = 0$ if $i \neq n$. Hence our task is to study

$$H^n_\psi := H^n_{\mathrm{c}}(X_{R,n}, \overline{\mathbb{Q}}_\ell)[\psi].$$

The automorphism group of $X_{R,1}$ contains a certain Heisenberg group H_R as a subgroup (cf. Definition 2.4). By this fact, $X_{R,n}$ inherits an action of a certain Heisenberg group $\mathrm{H}_{R,n}$ (cf. Definition 2.28). The unitary group $\mathrm{U}_n(q)$ acts on $\mathrm{H}_{R,n}$ as group automorphisms. Then $X_{R,n}$ admits an action of $\mathrm{H}_{R,n} \rtimes \mathrm{U}_n(q)$. We can regard H^n_ψ as an $\mathrm{H}_{R,n} \rtimes \mathrm{U}_n(q)$-representation, because $\mathbb{F}_q$ is naturally regarded as a subgroup of the center of $\mathrm{H}_{R,n} \rtimes \mathrm{U}_n(q)$.

It seems highly non-trivial to compute the character of H^n_ψ geometrically. Hence, we consider to relate H^n_ψ to the HW representation above. The HW representations are defined only for the special Heisenberg groups associated to skew-hermitian spaces. Hence we need to relate $\mathrm{H}_{R,n}$ to such a Heisenberg group. Actually, we will find a skew-hermitian space $(V_{R,n}, h_{R,n})$ satisfying the following two properties:

(i) there exist a natural injective homomorphism $\mathrm{U}_n(q) \hookrightarrow \mathrm{U}(V_{R,n}, h_{R,n})$ and an isomorphism

$$\mathrm{H}_{R,n} \rtimes \mathrm{U}_n(q) \simeq \mathrm{H}(V_{R,n}, h_{R,n}) \rtimes \mathrm{U}_n(q),$$

(ii) an isomorphism $H^n_\psi \simeq \widetilde{\rho}_{\psi,V_{R,n},h_{R,n}}$ as $\mathrm{H}_{R,n} \rtimes \mathrm{U}_n(q)$-representations holds.

The first property is so important for our strategy to show Theorem 1.1 and is established in Corollary 2.33.

The second property is proved in Theorem 2.38. By this, we know that H^n_ψ is isomorphic to the e-th tensor power of the Weil representation of $\mathrm{U}_n(q)$. The character of the Weil representation of $\mathrm{U}_n(q)$ equals χ_n as shown in [5, Theorem 4.9.2]. As a result, we obtain Theorem 1.1.

By [8, Case (iii) in Theorem], the $\mathrm{U}_n(q)$-representation H^n_ψ can be expressed as a sum of Deligne–Lusztig characters if q is large enough.

If $e = 1$, the variety $X_{R,n}$ has been studied in [7] and [11]. In this case, we show that H^n_ψ realizes the Weil representation of $\mathrm{U}_n(q)$ in [7]. It is a natural and interesting generalization of this result to study $X_{R,n}$ as above.

As an application of Theorem 1.1, we give other trace formulas for projective hypersurfaces. Let $\overline{Y}_{R,n}$ be the smooth projective hypersurface defined by the homogeneous equation

$$Z^{q^e}W - ZW^{q^e} = \sum_{i=0}^{e} a_i W^{q^e-q^i}\left(X_1^{q^i+1} + \cdots + X_n^{q^i+1}\right)$$

in $\mathbb{P}^{n+1} = \operatorname{Proj} \mathbb{F}[Z, W, X_1, \ldots, X_n]$. This hypersurface has a natural $\mathrm{U}_n(q)$-action given by $[Z : W : (X_i)_{1\le i\le n}] \mapsto [Z : W : g^{-1}(X_i)_{1\le i\le n}]$ for $g \in \mathrm{U}_n(q)$.

Let $\mu_{q^e+1} := \{x \in \mathbb{F}^\times \mid x^{q^e+1} = 1\}$. We define

$$\phi_n(g) := \frac{(-1)^n}{q^e+1} \sum_{\xi\in\mu_{q^e+1}} (-q^e)^{N_e(\xi g)} \quad \text{for } g \in \mathrm{U}_n(q^e).$$

We show the following trace formula.

Theorem 1.2. *We have the equality*

$$H^*(\overline{Y}_{R,n}, \overline{\mathbb{Q}}_\ell) = (-1)^n(q^e - 1)\chi_n^e + n + 1 + (-1)^n\phi_n$$

as virtual characters of $\mathrm{U}_n(q)$.

We give a rough sketch of a proof of Theorem 1.2. Let $Y_{R,n}$ be the open subscheme of $\overline{Y}_{R,n}$ defined by $W \neq 0$. Let $S := \overline{Y}_{R,n} \setminus Y_{R,n}$. Then S is stable under the action of $\mathrm{U}_n(q)$. Obviously, we have $H^*(\overline{Y}_{R,n}, \overline{\mathbb{Q}}_\ell) = H^*_{\mathrm{c}}(Y_{R,n}, \overline{\mathbb{Q}}_\ell) + H^*(S, \overline{\mathbb{Q}}_\ell)$. By $x_i := X_i/W$ for $1 \le i \le n$ and $a := Z/W$, the affine variety $Y_{R,n}$ is defined by $a^{q^e} - a = \sum_{i=0}^{e} a_i {}^t\boldsymbol{x}^{q^i}\boldsymbol{x}$ in $\mathbb{A}^{n+1}$ with $\boldsymbol{x} = (x_i)_{1\le i\le n}$. Thus we have the finite Galois étale morphism $Y_{R,n} \to X_{R,n};\ (a, \boldsymbol{x}) \mapsto (\sum_{i=0}^{e-1} a^{q^i}, \boldsymbol{x})$, which is $\mathrm{U}_n(q)$-equivariant. Using this, we can relate $H^*_{\mathrm{c}}(Y_{R,n}, \overline{\mathbb{Q}}_\ell)$ to $H^*_{\mathrm{c}}(X_{R,n}, \overline{\mathbb{Q}}_\ell)$. On the other hand, S equals

$$\left\{[Z : W : X_1 : \cdots : X_n] \in \mathbb{P}^{n+1} \;\middle|\; W = X_1^{q^e+1} + \cdots + X_n^{q^e+1} = 0\right\}.$$

It should be noted that the boundary S depends only on the degree of R. We have the equality $H^*(S, \overline{\mathbb{Q}}_\ell) = n + (-1)^n\phi_n$, which is shown in [11]. Under this analysis, Theorem 1.1 implies Theorem 1.2. If $e = 1$, the formula in Theorem 1.2 is a special case of a trace formula in [6, Theorem 3].

In a subsequent paper [13], we investigate a variant of this paper.

Notation

For a finite field extension $\mathbb{F}_{q^r}/\mathbb{F}_{q^s}$, let Tr_{q^r/q^s} denote the trace map from $\mathbb{F}_{q^r}$ to $\mathbb{F}_{q^s}$.

For a finite-dimensional vector space V over a field k and a k-endomorphism $f\colon V \to V$, let $\mathrm{Tr}(f; V) \in k$ denote the trace of f.

We suppose that every closed subscheme of a variety is equipped with the reduced scheme structure.

For a finite group G, let $Z(G)$ denote its center.

2 Trace formulae for Artin–Schreier varieties

Our aim in this section is to show Theorem 2.38. Lemma 2.11 and Corollary 2.33 are very important for us to relate the cohomology of $X_{R,n}$ to the Heisenberg–Weil representation.

2.1 Heisenberg groups and affine curves

In this subsection, we construct fundamental facts on the curve $X_{R,1}$. In the case $q = p$, similar things are found in [12].

Definition 2.1. We say that $f(x) \in \mathbb{F}[x]$ is q-additive if $f(x + y) = f(x) + f(y)$ and $f(\alpha x) = \alpha f(x)$ for $\alpha \in \mathbb{F}_q$ and $x, y \in \mathbb{F}$.

Let e be a non-negative integer and let $R(x) := \sum_{i=0}^{e} a_i x^{q^i} \in \mathbb{F}[x]$ with $a_e \neq 0$, which is q-additive. We define

$$E_R(x) := R(x)^{q^e} + \sum_{i=0}^{e} (a_i x)^{q^{e-i}} \in \mathbb{F}[x], \tag{2.1}$$

$$f_R(x, y) := -\sum_{i=0}^{e-1} \left(\sum_{j=0}^{e-i-1} \left(a_i x^{q^i} y \right)^{q^j} + (xR(y))^{q^i} \right) \in \mathbb{F}[x, y]. \tag{2.2}$$

Then $E_R(x)$ is q-additive and $f_R(x, y)$ is $\mathbb{F}_q$-bilinear in a natural sense.

Lemma 2.2. *We have $f_R(x, y)^q - f_R(x, y) = -x^{q^e} E_R(y) + xR(y) + yR(x)$. In particular, $f_R(x, y)^q - f_R(x, y) = xR(y) + yR(x)$ for $y \in \mathbb{F}$ satisfying $E_R(y) = 0$.*

Proof. The former equality follows from

$$\begin{aligned} f_R(x, y)^q - f_R(x, y) &= xR(y) - (xR(y))^{q^e} + \sum_{i=0}^{e-1} \left(a_i x^{q^i} y - \left(a_i x^{q^i} y \right)^{q^{e-i}} \right) \\ &= -x^{q^e} E_R(y) + xR(y) + yR(x). \end{aligned}$$

The latter claim follows from the former one. □

From now, we assume $(p, e) \neq (2, 0)$.

Lemma 2.3. *Let $V_R := \{x \in \mathbb{F} \mid E_R(x) = 0\}$. Then V_R is an $\mathbb{F}_q$-vector space of dimension $2e$.*

Proof. Since $E_R(x)$ is q-additive, V_R is an $\mathbb{F}_q$-vector space. By definition, the derivative of $E_R(x)$ is a non-zero constant by $(p,e) \neq (2,0)$. Hence $E_R(x)$ is a separable polynomial of degree q^{2e}. This implies that $\dim_{\mathbb{F}_q} V_R = 2e$. □

Definition 2.4. Let $\mathrm{H}_R := \{(\beta,\gamma) \in V_R \times \mathbb{F} \mid \gamma^q - \gamma = \beta R(\beta)\}$ be the group defined by

$$(\beta,\gamma)\cdot(\beta',\gamma') := (\beta+\beta', \gamma+\gamma'+f_R(\beta,\beta')).$$

This is well-defined by Lemma 2.2.

For a group G and elements $g, g' \in G$, let $[g,g'] := gg'g^{-1}g'^{-1}$.

Lemma 2.5. *For $g = (\beta,\gamma)$, $g' = (\beta',\gamma') \in \mathrm{H}_R$, we have $[g,g'] = (0, f_R(\beta,\beta') - f_R(\beta',\beta))$. In particular, $f_R(\beta,\beta') - f_R(\beta',\beta) \in \mathbb{F}_q$.*

Proof. We have $g^{-1} = (-\beta, -\gamma + f_R(\beta,\beta))$. Using this, we directly check the claims. □

Definition 2.6. A group H is called a Heisenberg group if $H/Z(H)$ is abelian.

Lemma 2.7. (1) *We have $Z(\mathrm{H}_R) = \{(0,\gamma) \mid \gamma \in \mathbb{F}_q\}$. The quotient $\mathrm{H}_R/Z(\mathrm{H}_R)$ is isomorphic to V_R as groups. The group H_R is a Heisenberg group.*

(2) *The pairing $\omega_R \colon V_R \times V_R \to \mathbb{F}_q;\ (\beta,\beta') \mapsto f_R(\beta,\beta') - f_R(\beta',\beta)$ is a non-degenerate symplectic form.*

Proof. We show (1). If $e = 0$, we have $\mathrm{H}_R = \{0\} \times \mathbb{F}_q$. Hence the claims are clear. We may assume $e \geq 1$. Let $Z := \{(0,\gamma) \mid \gamma \in \mathbb{F}_q\}$. Clearly $Z \subset Z(\mathrm{H}_R)$. It suffices to show $Z(\mathrm{H}_R) \subset Z$. Let $(\beta,\gamma) \in Z(\mathrm{H}_R)$. We consider $X_\beta := \{x \in \mathbb{F} \mid f_R(\beta,x) = f_R(x,\beta)\}$ which is an $\mathbb{F}_q$-vector space since $f_R(\beta,x) - f_R(x,\beta)$ is q-additive. If $\beta \neq 0$, we have $\deg(f_R(\beta,x) - f_R(x,\beta)) = q^{2e-1}$ and hence $\dim_{\mathbb{F}_q} X_\beta \leq 2e-1$. It results that $V_R \subset X_\beta$ from Lemma 2.5 and $(\beta,\gamma) \in Z(\mathrm{H}_R)$. We obtain $\beta = 0$ from $\dim_{\mathbb{F}_q} V_R = 2e$. Hence $Z(\mathrm{H}_R) \subset Z$. The second claim is clear. The last claim follows from the first two claims.

We show (2). Assume that $\omega_R(\beta,\beta') = 0$ for every $\beta' \in V_R$. We take $(\beta,\gamma) \in \mathrm{H}_R$. Lemma 2.5 implies that $(\beta,\gamma) \in Z(\mathrm{H}_R)$. From (1), it follows that $\beta = 0$. □

Definition 2.8. (1) Let $C_R \subset \mathbb{A}^2 = \operatorname{Spec} \mathbb{F}[a,x]$ be the smooth affine curve defined by $a^q - a = xR(x)$.

(2) Let H_R act on C_R by $(a,x)\cdot(\beta,\gamma) = (a + f_R(x,\beta) + \gamma, x+\beta)$ for $(a,x) \in C_R$ and $(\beta,\gamma) \in \mathrm{H}_R$, which is well-defined by Lemma 2.2.

(3) Let $\mathbb{F}_q$ act on C_R by $(a,x) \mapsto (a+\zeta, x)$ for $\zeta \in \mathbb{F}_q$.

We take a prime number $\ell \neq p$. Let G be a finite group. Let $G^\vee$ denote its character group. For a finite-dimensional G-representation M over $\overline{\mathbb{Q}}_\ell$ and $\psi \in G^\vee$, let $M[\psi]$ denote the ψ-isotypic part of M.

Lemma 2.9. *For $\psi \in \mathbb{F}_q^\vee \setminus \{1\}$, we have*

$$H^i_{\mathrm{c}}(C_R, \overline{\mathbb{Q}}_\ell)[\psi] = 0 \quad \textit{for } i \neq 1 \textit{ and } \dim H^1_{\mathrm{c}}(C_R, \overline{\mathbb{Q}}_\ell)[\psi] = q^e.$$

Proof. The former claim and the latter one follow from [2, Remarque 1.18(b), (c) in Sommes trig.] and [10, Remark 3.29], respectively. □

For a quasi-projective variety X over $\mathbb{F}$ with an action of a finite group G, let X/G denote the quotient of X by G. If G acts on a vector space V, let V^G denote its G-fixed part. We will use the following standard lemma several times through the paper.

Lemma 2.10. *Let $f\colon Y \to X$ be a finite Galois étale morphism between quasi-projective varieties over $\mathbb{F}$ with Galois group G.*

(1) *We have an isomorphism $f^*\colon H^i_{\mathrm{c}}(X, \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} H^i_{\mathrm{c}}(Y, \overline{\mathbb{Q}}_\ell)^G$ for any i.*

(2) *Let $K \subset G$ be a normal subgroup. Let $f'\colon Y \to Y/K$ be the quotient morphism and $\pi\colon G \to G/K$ the natural homomorphism. For any $i \in \mathbb{Z}$ and $\psi \in (G/K)^\vee$, we have an isomorphism*

$$f'^*\colon H^i_{\mathrm{c}}(Y/K, \overline{\mathbb{Q}}_\ell)[\psi] \xrightarrow{\sim} H^i_{\mathrm{c}}(Y, \overline{\mathbb{Q}}_\ell)[\psi \circ \pi].$$

Proof. The claim (1) is well-known (cf. [9, (5.10)]). We show (2). Applying (1) to $f'\colon Y \to Y/K$ deduces an isomorphism $f'^*\colon H^i_{\mathrm{c}}(Y/K, \overline{\mathbb{Q}}_\ell) \simeq H^i_{\mathrm{c}}(Y, \overline{\mathbb{Q}}_\ell)^K$. Taking the ψ-isotypic part and using $H^i_{\mathrm{c}}(Y, \overline{\mathbb{Q}}_\ell)^K[\psi] = H^i_{\mathrm{c}}(Y, \overline{\mathbb{Q}}_\ell)[\psi \circ \pi]$, we obtain the claim. □

Let $\mu_r := \{x \in \mathbb{F} \mid x^r = 1\}$ for $r \in \mathbb{Z}_{>0}$. Let $d_R := \gcd\{q^i + 1 \mid a_i \neq 0\}$. For $\xi \in \mu_{d_R}$, let $\xi\colon C_R \xrightarrow{\sim} C_R;\ (a, x) \mapsto (a, \xi x)$, which commutes with the action of $\mathbb{F}_q$ in Definition 2.8(3). The following lemma will be used in the proof of Theorem 2.38.

Lemma 2.11. *Let $\psi \in \mathbb{F}_q^\vee \setminus \{1\}$ and $\xi \in \mu_{d_R} \setminus \{1\}$. Then we have* $\mathrm{Tr}(\xi; H^1_{\mathrm{c}}(C_R, \overline{\mathbb{Q}}_\ell)[\psi]) = -1$.

Proof. We simply write C for C_R. Let D be the affine curve defined by $a^{q^e} - a = xR(x)$ over $\mathbb{F}$. Let $\mathrm{tr} := \mathrm{Tr}_{q^e/q}$ and $K := \mathrm{Ker}\,\mathrm{tr}$. We have the finite Galois étale morphism $\pi\colon D \to C;\ (a, x) \mapsto (\sum_{i=0}^{e-1} a^{q^i}, x)$ with Galois group K. Applying Lemma 2.10(2) with $(Y, G, K) = (D, \mathbb{F}_{q^e}, K)$, we obtain $\pi^*\colon H^1_{\mathrm{c}}(C, \overline{\mathbb{Q}}_\ell)[\psi] \xrightarrow{\sim} H^1_{\mathrm{c}}(D, \overline{\mathbb{Q}}_\ell)[\psi \circ \mathrm{tr}]$. Hence it suffices to show

$$\mathrm{Tr}(\xi; H^1_{\mathrm{c}}(D, \overline{\mathbb{Q}}_\ell)[\psi \circ \mathrm{tr}]) = -1.$$

Let $\overline{D}$ denote the smooth projective curve defined by the homogeneous equation $Y^{q^e} Z - Y Z^{q^e} = X Z^{q^e} R(X/Z)$ in $\mathbb{P}^2$. We have the open immersion $D \hookrightarrow \overline{D};\ (a, x) \mapsto [x : a : 1]$. Let ∞ be the closed point of $\overline{D}$ defined by $[X : Y : Z] = [0 : 1 : 0]$. Then $\overline{D} \setminus D = \{\infty\}$. Considering the projection to the $(\psi \circ \mathrm{tr})$-isotypic part, we obtain

$$\mathrm{Tr}(\xi; H^1_{\mathrm{c}}(D, \overline{\mathbb{Q}}_\ell)[\psi \circ \mathrm{tr}]) = \frac{1}{q^e} \sum_{\zeta \in \mathbb{F}_{q^e}} \psi^{-1}(\mathrm{tr}(\zeta))\, \mathrm{Tr}(\xi \circ \zeta; H^1_{\mathrm{c}}(D, \overline{\mathbb{Q}}_\ell)). \tag{2.3}$$

Let $\zeta \in \mathbb{F}_{q^e}$. The automorphism $\xi \circ \zeta$ on D extends to $\overline{D}$ and the multiplicity of it at the fixed point ∞ is one. Let $D^{\xi \circ \zeta}$ denote the set of the fixed points of $\xi \circ \zeta$ on D with multiplicities. From [2, Corollaire 5.4 in Rapport], it follows that

$$-\mathrm{Tr}(\xi \circ \zeta; H^1_{\mathrm{c}}(D, \overline{\mathbb{Q}}_\ell)) + 1 = \mathrm{Tr}(\xi \circ \zeta; H^*_{\mathrm{c}}(D, \overline{\mathbb{Q}}_\ell)) = |D^{\xi \circ \zeta}| = \begin{cases} 0 & \text{if } \zeta \neq 0, \\ q^e & \text{otherwise.} \end{cases}$$

Hence the claim follows from (2.3). □

2.2 Skew-hermitian forms

Let $R(x) = \sum_{i=0}^{e} a_i x^{q^i}$ with $e \in \mathbb{Z}_{>0}$ and $a_e \neq 0$. We assume that

$$a_i = 0 \text{ if } i \text{ is even.} \tag{2.4}$$

This implies that e is odd. By definition, V_R is regarded as an $\mathbb{F}_{q^2}$-vector space. Let ω_R be as in Lemma 2.7(2). Our aim in this subsection is to introduce a non-degenerate skew-hermitian form

$$h_R \colon V_R \times V_R \to \mathbb{F}_{q^2}$$

satisfying $\mathrm{Tr}_{q^2/q} \circ h_R = \omega_R$ (cf. Corollary 2.18). Furthermore, we can interpret H_R as a Heisenberg group associated to the skew-hermitian space (V_R, h_R) in the sense of [5]. Analysis in this subsection is important for us to show Corollary 2.33.

We define

$$\begin{aligned} e' &:= (e-1)/2 \in \mathbb{Z}_{\geq 0}, \\ d_{e-2i}(x) &:= -\sum_{j=0}^{i} \left(a_{e-2j} x^{q^{e-2j}} \right)^{q^{2i}}, \\ d_{e-2i-1}(x) &:= -d_{e-2i}(x)^q \quad \text{for } 1 \leq e-2i-1, e-2i \leq e, \\ \delta(x,y) &:= \sum_{i=1}^{e} d_i(y) x^{q^{e-i}}, \quad r(x,y) := yR(x) - xR(y) \quad \text{for } x, y \in \mathbb{F}. \end{aligned}$$

Lemma 2.12. *We have $\delta(x,y)^q + \delta(x,y) = r(x,y)^{q^e}$ for $x \in V_R$ and $y \in \mathbb{F}$.*

Proof. As $x \in V_R$, we have

$$r(x,y)^{q^e} = -y^{q^e} \sum_{i=0}^{e} (a_i x)^{q^{e-i}} - (xR(y))^{q^e}.$$

We note that $d_1(y) = -R(y)^{q^{e-1}}$. We need to show

$$d_e(y) = -a_e y^{q^e}, \quad d_i(y) + d_{i+1}(y)^q = -\left(a_i y^{q^i} \right)^{q^{e-i}} \quad \text{for } 1 \leq i \leq e-1.$$

The above equalities follow from (2.4) and a direct computation. □

For $x, y \in \mathbb{F}$, let

$$\begin{aligned} \delta_i(x,y) &:= a_i \left(x y^{q^i} - y x^{q^i} \right) \quad \text{for } i \in \mathbb{Z}_{\geq 0}, \\ h_0(x,y) &:= \sum_{i=0}^{e'-1} \left(\sum_{j=0}^{e'-i-1} \delta_{2i+1}(x,y)^{q^{2j}} + r(x,y)^{q^{2i}} \right) + \delta(x,y)^{q^{-1}}. \end{aligned}$$

Let $f_R(x,y)$ be as in (2.2).

Lemma 2.13. *We have $h_0(x,y)^q + h_0(x,y) = f_R(x,y) - f_R(y,x)$ for $x \in V_R$ and $y \in \mathbb{F}$.*

Proof. From the definition of $f_R(x,y)$, it follows that

$$f_R(x,y)-f_R(y,x)=\sum_{i=0}^{e-1}\left(\sum_{j=0}^{e-i-1}\delta_i(x,y)^{q^j}+r(x,y)^{q^i}\right).$$

Let $\mathscr{L}_+(z):=z^q+z$. Using (2.4), we easily compute

$$\begin{aligned}\mathscr{L}_+\left(\sum_{i=0}^{e'-1}\left(\sum_{j=0}^{e'-i-1}\delta_{2i+1}(x,y)^{q^{2j}}+r(x,y)^{q^{2i}}\right)\right)&=\sum_{i=0}^{e-1}\sum_{j=0}^{e-i-1}\delta_i(x,y)^{q^j}+\sum_{i=0}^{e-2}r(x,y)^{q^i}\\&=f_R(x,y)-f_R(y,x)-r(x,y)^{q^{e-1}}\\&=f_R(x,y)-f_R(y,x)-\mathscr{L}_+(\delta(x,y)^{q^{-1}}),\end{aligned}$$

where the last equality follows from Lemma 2.12. Thus we obtain the claim. □

Corollary 2.14. *We have $h_0(x,y)\in\mathbb{F}_{q^2}$ for $x,y\in V_R$.*

Proof. Lemma 2.5 implies that $f_R(x,y)-f_R(y,x)\in\mathbb{F}_q$. Thus we obtain $h_0(x,y)^q+h_0(x,y)=0$ by Lemma 2.13 and hence $h_0(x,y)\in\mathbb{F}_{q^2}$. □

Definition 2.15. We define $h_R\colon V_R\times V_R\to\mathbb{F}_{q^2}$ by

$$h_R(x,y):=h_0(x,y)^q\in\mathbb{F}_{q^2}\quad\text{for } x,y\in V_R.$$

Corollary 2.16. (1) *Let ω_R be as in Lemma 2.7(2). We have $\mathrm{Tr}_{q^2/q}\circ h_R=\omega_R$.*

(2) *The pairing $h_R\colon V_R\times V_R\to\mathbb{F}_{q^2}$ is a non-degenerate $\mathbb{F}_q$-bilinear form.*

Proof. The claim (1) follows from Lemma 2.13.

We show (2). Since h_0 is an $\mathbb{F}_q$-bilinear form on V_R, so is h_R. Assume that $h_R(x,y)=0$ for every $y\in V_R$. By (1), $\omega_R(x,y)=0$ for every $y\in V_R$. Since ω_R is non-degenerate as in Lemma 2.7(2), we obtain $x=0$. Hence h_R is non-degenerate. □

Lemma 2.17. *Let $x,y\in V_R$. Then*

(1) $h_R(x,y)=-\sum_{j=0}^{e'}\sum_{i=0}^{j}\left(a_{e-2i}y^{q^{e-2i}}x\right)^{q^{2j}}+\sum_{j=e'+1}^{2e'}\sum_{i=j-e'}^{e'}(a_{2i+1}y)^{q^{2(j-i)-1}}x^{q^{2j}}$,

(2) $h_R(y,x)^q+h_R(x,y)=0$.

Proof. We show (1). Let

$$\begin{aligned}A(x,y):&=-\sum_{j=0}^{e'}\sum_{i=0}^{j}\left(a_{e-2i}y^{q^{e-2i}}x\right)^{q^{2j}}=\sum_{j=0}^{e'}d_{e-2j}(y)x^{q^{2j}},\\B(x,y):&=\sum_{j=e'+1}^{2e'}\sum_{i=j-e'}^{e'}(a_{2i+1}y)^{q^{2(j-i)-1}}x^{q^{2j}}\end{aligned}$$

and $c_{2i+1}(y) := a_{2i+1}y^{q^{2i+1}}$. From the definitions of $h_R(x,y)$ and $\delta(x,y)$, it follows that

$$\begin{aligned}h_R(x,y) &= \sum_{i=0}^{e'-1}\left(\sum_{j=0}^{e'-i-1}\delta_{2i+1}(x,y)^{q^{2j+1}} + r(x,y)^{q^{2i+1}}\right) + \sum_{i=1}^{e} d_i(y)x^{q^{e-i}} \\ &= \sum_{i=0}^{e'-1}\sum_{j=0}^{e'-i-1}\left((c_{2i+1}(y)x)^{q^{2j+1}} - (a_{2i+1}y)^{q^{2j+1}}x^{q^{2(i+j+1)}}\right) \\ &+ \sum_{i=0}^{e'-1}\sum_{j=0}^{e'}(a_{2j+1}y)^{q^{2i+1}}x^{q^{2(i+j+1)}} - \sum_{i=0}^{e'-1}(R(y)x)^{q^{2i+1}} + \sum_{i=1}^{e} d_i(y)x^{q^{e-i}}.\end{aligned}$$

We will compute the coefficient of each x^{q^i} in $h_R(x,y)$. For $0 \le j \le e'-1$, the coefficient of $x^{q^{2j+1}}$ equals

$$\begin{aligned}&\sum_{i=0}^{e'-j-1} c_{2i+1}(y)^{q^{2j+1}} - R(y)^{q^{2j+1}} + d_{e-2j-1}(y) \\ &= \left(\sum_{i=0}^{e'-j-1} c_{2i+1}(y) - R(y) + \sum_{i=0}^{j} c_{e-2i}(y)\right)^{q^{2j+1}} = 0.\end{aligned}$$

The coefficient of $x^{q^{2j}}$ equals

$$\begin{cases} d_{e-2j}(y) & \text{if } 0 \le j \le e', \\ \displaystyle\sum_{i=j-e'}^{j-1} (a_{2i+1}y)^{q^{2(j-i)-1}} = \sum_{i=j-e'}^{e'} (a_{2i+1}y)^{q^{2(j-i)-1}} & \text{if } e'+1 \le j \le 2e'. \end{cases}$$

Hence

$$h_R(x,y) = A(x,y) + B(x,y). \tag{2.5}$$

We show (2). We can compute

$$A(y,x) = -\sum_{j=e'}^{2e'}\sum_{i=j-e'}^{e'}(a_{2i+1}y)^{q^{2(j-i)}}x^{q^{2j+1}}, \quad B(y,x) = \sum_{j=0}^{e'-1}\sum_{i=0}^{j}(c_{e-2i}(y)x)^{q^{2j+1}}.$$

These imply that

$$\begin{aligned}A(x,y)^q + B(y,x) &= -(R(y)x)^{q^e}, \\ B(x,y)^q + A(y,x) &= -\sum_{i=0}^{e'}(a_{2i+1}y)^{q^{e-2i-1}}x^{q^e}.\end{aligned}$$

Summing up these equalities and using (2.5) and $y \in V_R$, we obtain $h_R(y,x)^q + h_R(x,y) = -E_R(y)x^{q^e} = 0$. Thus the claim follows. □

Corollary 2.18. *The pairing $h_R\colon V_R \times V_R \to \mathbb{F}_{q^2}$ is a non-degenerate skew-hermitian form.*

Proof. By Lemma 2.17(1), $h_R(\alpha x, y) = \alpha h_R(x,y)$ for $\alpha \in \mathbb{F}_{q^2}$. Hence the claim follows from Corollary 2.16(2) and Lemma 2.17(2). □

2.3 Preparation to construct group isomorphism

Our aim in this subsection is to show Lemma 2.20 and Proposition 2.21, which induce a group isomorphism in Proposition 2.30.

Definition 2.19. We define

$$g_R(x) := \sum_{i=0}^{e'-1} \sum_{j=0}^{e-2i-2} \left(a_{2i+1} x^{q^{2i+1}+1}\right)^{q^j} + \sum_{i=1}^{e'} \sum_{j=e'-i}^{e'-1} \left(a_{2i+1} x^{q^{2i+1}+1}\right)^{q^{2j+1}} \in \mathbb{F}[x].$$

Lemma 2.20. *We have $g_R(x)^q - g_R(x) = -xR(x) - h_R(x,x)$ for $x \in V_R$.*

Proof. Let $b_{2i+1}(x) := a_{2i+1} x^{q^{2i+1}+1}$. Substituting x to y at the equality in Lemma 2.17(1), we have

$$\begin{aligned} h_R(x,x) &= -\sum_{j=0}^{e'} \sum_{i=0}^{j} b_{e-2i}(x)^{q^{2j}} + \sum_{j=e'+1}^{2e'} \sum_{i=j-e'}^{e'} b_{2i+1}(x)^{q^{2(j-i)-1}} \\ &= -\sum_{i=0}^{e'} \sum_{j=e'-i}^{e'} b_{2i+1}(x)^{q^{2j}} + \sum_{i=1}^{e'} \sum_{j=e'-i}^{e'-1} b_{2i+1}(x)^{q^{2j+1}}. \end{aligned}$$

Let $\mathscr{L}_-(z) := z^q - z$. Then we compute

$$\begin{aligned} \mathscr{L}_-\left(\sum_{i=0}^{e'-1} \sum_{j=0}^{e-2i-2} b_{2i+1}(x)^{q^j}\right) &= \sum_{i=0}^{e'-1} \left(b_{2i+1}(x)^{q^{e-2i-1}} - b_{2i+1}(x)\right) \\ &= \sum_{i=0}^{e'} b_{2i+1}(x)^{q^{e-2i-1}} - xR(x), \\ \mathscr{L}_-\left(\sum_{i=1}^{e'} \sum_{j=e'-i}^{e'-1} b_{2i+1}(x)^{q^{2j+1}}\right) &= \sum_{i=1}^{e'} \sum_{j=e'-i+1}^{e'} b_{2i+1}(x)^{q^{2j}} - \sum_{i=1}^{e'} \sum_{j=e'-i}^{e'-1} b_{2i+1}(x)^{q^{2j+1}} \\ &= -\sum_{i=0}^{e'} b_{2i+1}(x)^{q^{e-2i-1}} - h_R(x,x). \end{aligned}$$

Summing up these equalities, we obtain the claim. □

Proposition 2.21. *We have $g_R(x+y) - g_R(x) - g_R(y) = h_R(x,y) - f_R(x,y)$ for $x, y \in V_R$.*

Proof. We rewrite (2.2) as follows:

$$-f_R(x,y) = \sum_{i=0}^{e-1} \left(R(y)^{q^i} + \sum_{j=0}^{i} (a_j y)^{q^{i-j}}\right) x^{q^i}. \tag{2.6}$$

We define $\{h_i(y)\}_{0 \le i \le 4e'}$ by

$$h_R(x,y) - f_R(x,y) = \sum_{i=0}^{4e'} h_i(y) x^{q^i}. \tag{2.7}$$

Let $c_{2i+1}(y) := a_{2i+1}y^{q^{2i+1}}$. Clearly $R(y) = \sum_{i=0}^{e'} c_{2i+1}(y)$. From Lemma 2.17(1) and (2.6), it results that

$$h_{2j}(y) = \sum_{i=0}^{e'-j-1} c_{2i+1}(y)^{q^{2j}} + \sum_{i=0}^{2j}(a_i y)^{q^{2j-i}}, \quad h_{2j+1}(y) = R(y)^{q^{2j+1}} + \sum_{i=0}^{2j+1}(a_i y)^{q^{2j-i+1}} \tag{2.8}$$

for $0 \leq 2j,\ 2j+1 \leq 2e'$ and

$$h_{2j}(y) = \sum_{i=j-e'}^{e'} (a_{2i+1}y)^{q^{2(j-i)-1}}, \quad h_{2j+1}(y) = 0 \quad \text{for } 2e'+1 \leq 2j,\ 2j+1 \leq 4e'. \tag{2.9}$$

Clearly

$$\begin{aligned}
&g_R(x+y) - g_R(x) - g_R(y)\\
&= \sum_{i=0}^{e'-1}\sum_{j=0}^{e-2i-2}\left(a_{2i+1}x^{q^{2i+1}}y + c_{2i+1}(y)x\right)^{q^j} + \sum_{i=1}^{e'}\sum_{j=e'-i}^{e'-1}\left(a_{2i+1}x^{q^{2i+1}}y + c_{2i+1}(y)x\right)^{q^{2j+1}}\\
&= \sum_{i=0}^{e'-1}\sum_{j=0}^{e-2i-2}\left(a_{2i+1}x^{q^{2i+1}}y + c_{2i+1}(y)x\right)^{q^j} + \sum_{i=1}^{e'}\sum_{j=e'-i}^{e'-1}(c_{2i+1}(y)x)^{q^{2j+1}}\\
&\quad + \sum_{i=1}^{e'}\sum_{j=e'-i}^{e'-1}\left(a_{2i+1}x^{q^{2i+1}}y\right)^{q^{2j+1}}.
\end{aligned} \tag{2.10}$$

Let $[a]$ be the integral part of $a \in \mathbb{R}$. The equalities (2.8) and (2.9) imply that

$$\begin{aligned}
&\sum_{i=0}^{e'-1}\sum_{j=0}^{e-2i-2}\left(a_{2i+1}x^{q^{2i+1}}y + c_{2i+1}(y)x\right)^{q^j} + \sum_{i=1}^{e'}\sum_{j=e'-i}^{e'-1}(c_{2i+1}(y)x)^{q^{2j+1}}\\
&= \sum_{j=0}^{2e'}\left(\sum_{i=0}^{[\frac{e-j-2}{2}]} c_{2i+1}(y)^{q^j} + \sum_{i=0}^{[\frac{j-1}{2}]}(a_{2i+1}y)^{q^{j-(2i+1)}}\right)x^{q^j} + \sum_{j=0}^{e'-1}\sum_{i=e'-j}^{e'-1}(c_{2i+1}(y)x)^{q^{2j+1}}\\
&= \sum_{i=0}^{2e'} h_i(y)x^{q^i} \quad \text{and}
\end{aligned}$$

$$\sum_{i=1}^{e'}\sum_{j=e'-i}^{e'-1}\left(a_{2i+1}x^{q^{2i+1}}y\right)^{q^{2j+1}} = \sum_{j=e'+1}^{2e'}\sum_{i=j-e'}^{e'}(a_{2i+1}y)^{q^{2(j-i)-1}}x^{q^{2j}} = \sum_{i=2e'+1}^{4e'} h_i(y)x^{q^i}.$$

Summing up these two equalities and using (2.7) and (2.10), we obtain the claim. □

Remark 2.22. Substituting x to y at the equality in Proposition 2.21 and using $g_R(2x) = 4g_R(x)$, we have $h_R(x,x) - f_R(x,x) = 2g_R(x)$. This implies that

$$g_R(x) = \frac{h_R(x,x) - f_R(x,x)}{2} \quad \text{if } p \neq 2.$$

2.4 Weil representation of unitary group

In this subsection, we follow [5, §3.3]. Let V be a vector space of dimension n over $\mathbb{F}_{q^2}$ with a non-degenerate skew-hermitian form $h\colon V \times V \to \mathbb{F}_{q^2}$. We call a pair (V, h) a skew-hermitian space over $\mathbb{F}_{q^2}$. For skew-hermitian spaces (V, h) and (V', h') over $\mathbb{F}_{q^2}$ of the same dimension, we have an isomorphism $(V, h) \simeq (V', h')$.

Definition 2.23. Let $\mathrm{H}(V, h) := \{(\beta, \gamma) \in V \times \mathbb{F} \mid \gamma^q - \gamma = -h(\beta, \beta)\}$ be the group defined by

$$(\beta, \gamma) \cdot (\beta', \gamma') = (\beta + \beta', \gamma + \gamma' + h(\beta, \beta')).$$

Lemma 2.24. *The center $Z(\mathrm{H}(V, h))$ equals $\{(0, \gamma) \mid \gamma \in \mathbb{F}_q\}$.*

Proof. This is shown in [5, Lemma 3.1(a)]. □

We identify $Z(\mathrm{H}(V, h))$ with $\mathbb{F}_q$ by $(0, \gamma) \mapsto \gamma$.

Lemma 2.25. *For $\psi \in \mathbb{F}_q^\vee \setminus \{1\}$, there exists a unique irreducible $\mathrm{H}(V, h)$-representation $\rho_{\mathrm{H}(V,h),\psi}$ of dimension q^n containing ψ restricted to the center $Z(\mathrm{H}(V, h)) \simeq \mathbb{F}_q$.*

Proof. This is stated in [5, the first paragraph of §3.3] (cf. [5, lemma 1.2(b)]). □

Recall $\mathrm{U}(V, h) = \{g \in \mathrm{Aut}_{\mathbb{F}_{q^2}}(V) \mid h(gv, gv') = h(v, v')\ (v, v' \in V)\}$. This group acts on $\mathrm{H}(V, h)$ by the group automorphism $\mathrm{H}(V, h) \xrightarrow{\sim} \mathrm{H}(V, h)$; $(\beta, \gamma) \mapsto (g\beta, \gamma)$ for $g \in \mathrm{U}(V, h)$. We define $\mathrm{HU}(V, h) := \mathrm{H}(V, h) \rtimes \mathrm{U}(V, h)$.

Theorem 2.26. *For $\psi \in \mathbb{F}_q^\vee \setminus \{1\}$, there exists a unique representation $\widetilde{\rho}_{\psi,V,h}$ of $\mathrm{HU}(V, h)$ satisfying*

(i) *an isomorphism $\widetilde{\rho}_{\psi,V,h}|_{\mathrm{H}(V,h)} \simeq \rho_{\mathrm{H}(V,h),\psi}$ as $\mathrm{H}(V, h)$-representations and*

(ii) *the equality*

$$\mathrm{Tr}(g; \widetilde{\rho}_{\psi,V,h}) = (-1)^n(-q)^{N(g,V)} \quad \text{for } g \in \mathrm{U}(V, h), \tag{2.11}$$

where $N(g, V) := \dim_{\mathbb{F}_{q^2}} \mathrm{Ker}(g - \mathrm{id}_V \colon V \to V)$.

Proof. This follows from [5, Theorems 3.3 and 4.9.2]. □

Definition 2.27. We call $\widetilde{\rho}_{\psi,V,h}$ the Heisenberg–Weil representation of $\mathrm{HU}(V, h)$ associated to ψ. We define $\rho_{V,h} := \widetilde{\rho}_{\psi,V,h}|_{\mathrm{U}(V,h)}$ whose isomorphism class as a $\mathrm{U}(V, h)$-representation is independent of ψ by (2.11). This is called the Weil representation of $\mathrm{U}(V, h)$.

2.5 Main theorem

Let $R(x) = \sum_{i=0}^{e} a_i x^{q^i} \in \mathbb{F}[x]$ with $e \in \mathbb{Z}_{>0}$ and $a_e \neq 0$ satisfying (2.4). For $\boldsymbol{x} = (x_i)_{1\le i\le n} \in \mathbb{A}^n$, let $\boldsymbol{R}(\boldsymbol{x}) := (\sum_{i=0}^{e} a_i x_j^{q^i})_{1\le j\le n} = \sum_{i=0}^{e} a_i \boldsymbol{x}^{q^i} \in \mathbb{A}^n$. Clearly $\boldsymbol{R}(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{R}(\boldsymbol{x}) + \boldsymbol{R}(\boldsymbol{y})$ in a natural sense.

We define $f_{R,n}\colon \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^1$ by

$$f_{R,n}(\boldsymbol{x}, \boldsymbol{y}) := \sum_{i=1}^{n} f_R(x_i, y_i) = -\sum_{i=0}^{e-1} \left(\sum_{j=0}^{e-i-1} \left(a_i {}^t\boldsymbol{x}^{q^i} \boldsymbol{y} \right)^{q^j} + \left({}^t\boldsymbol{x}\boldsymbol{R}(\boldsymbol{y}) \right)^{q^i} \right)$$

for $\boldsymbol{x} = (x_i)_{1\le i\le n}, \boldsymbol{y} = (y_i)_{1\le i\le n} \in \mathbb{A}^n$.

Definition 2.28. Let $V_{R,n} := V_R^{\oplus n}$ and let

$$\mathrm{H}_{R,n} := \left\{ (\boldsymbol{\beta}, \gamma) \in V_{R,n} \times \mathbb{F} \mid \gamma^q - \gamma = {}^t\boldsymbol{\beta}\boldsymbol{R}(\boldsymbol{\beta}) \right\}$$

be the group defined by $(\boldsymbol{\beta}, \gamma) \cdot (\boldsymbol{\beta}', \gamma') = (\boldsymbol{\beta} + \boldsymbol{\beta}', \gamma + \gamma' + f_{R,n}(\boldsymbol{\beta}, \boldsymbol{\beta}'))$. This is well-defined by Lemma 2.2.

Recall that V_R is regarded as an $\mathbb{F}_{q^2}$-vector space by (2.4) and

$$\mathrm{U}_n(q) = \left\{ g \in \mathrm{GL}_n(\mathbb{F}_{q^2}) \mid g^\dagger g = \mathrm{I}_n \right\},$$

where $g^\dagger = (a_{j,i}^q)$ for $g = (a_{i,j}) \in \mathrm{GL}_n(\mathbb{F}_{q^2})$. An $\mathbb{F}_{q^2}$-linear map $g \colon \mathbb{F}_{q^2}^n \to \mathbb{F}_{q^2}^n$ induces an $\mathbb{F}_{q^2}$-linear map $g \colon V_{R,n} \to V_{R,n}$ if we identify $V_{R,n}$ with $V_R \otimes_{\mathbb{F}_{q^2}} \mathbb{F}_{q^2}^n$. This defines an action of $\mathrm{U}_n(q)$ on $V_{R,n}$. For $g \in \mathrm{U}_n(q)$, we have ${}^t\boldsymbol{x}\boldsymbol{R}(\boldsymbol{y}) = {}^t(g\boldsymbol{x})\boldsymbol{R}(g\boldsymbol{y})$ and $f_{R,n}(g\boldsymbol{x}, g\boldsymbol{y}) = f_{R,n}(\boldsymbol{x}, \boldsymbol{y})$. Thus $\mathrm{U}_n(q)$ acts on $\mathrm{H}_{R,n}$ as group automorphisms by $(\boldsymbol{\beta}, \gamma) \mapsto (g\boldsymbol{\beta}, \gamma)$ for $g \in \mathrm{U}_n(q)$.

Definition 2.29. (1) Let h_R be as in Corollary 2.18. We define a non-degenerate skew-hermitian form $h_{R,n} \colon V_{R,n} \times V_{R,n} \to \mathbb{F}_{q^2}$ by

$$h_{R,n}(\boldsymbol{\beta}, \boldsymbol{\beta}') := \sum_{i=1}^n h_R(\beta_i, \beta_i') \quad \text{for } \boldsymbol{\beta} = (\beta_i)_{1 \le i \le n}, \boldsymbol{\beta}' = (\beta_i')_{1 \le i \le n} \in V_{R,n}.$$

(2) Let $g_R(x)$ be as in Definition 2.19. We define

$$g_{R,n}(\boldsymbol{\beta}) := \sum_{i=1}^n g_R(\beta_i) \quad \text{for } \boldsymbol{\beta} = (\beta_i)_{1 \le i \le n} \in V_{R,n}.$$

Let $\mathrm{H}(V_{R,n}, h_{R,n})$ be as in Definition 2.23.

Proposition 2.30. *We have the group isomorphism*

$$\phi \colon \mathrm{H}_{R,n} \xrightarrow{\sim} \mathrm{H}(V_{R,n}, h_{R,n});\ (\boldsymbol{\beta}, \gamma) \mapsto (\boldsymbol{\beta}, \gamma + g_{R,n}(\boldsymbol{\beta})).$$

Proof. We simply write f, g and h for $f_{R,n}$, $g_{R,n}$ and $h_{R,n}$, respectively. Then

$$\begin{aligned} (\gamma + g(\boldsymbol{\beta}))^q - (\gamma + g(\boldsymbol{\beta})) &= \gamma^q - \gamma + g(\boldsymbol{\beta})^q - g(\boldsymbol{\beta}) \\ &= {}^t\boldsymbol{\beta}\boldsymbol{R}(\boldsymbol{\beta}) - {}^t\boldsymbol{\beta}\boldsymbol{R}(\boldsymbol{\beta}) - h(\boldsymbol{\beta}, \boldsymbol{\beta}) = -h(\boldsymbol{\beta}, \boldsymbol{\beta}), \end{aligned}$$

where the second equality follows from Lemma 2.20. Thus ϕ is well-defined. One has

$$\begin{aligned} \phi((\boldsymbol{\beta}, \gamma) \cdot (\boldsymbol{\beta}', \gamma')) &= \phi(\boldsymbol{\beta} + \boldsymbol{\beta}', \gamma + \gamma' + f(\boldsymbol{\beta}, \boldsymbol{\beta}')) \\ &= (\boldsymbol{\beta} + \boldsymbol{\beta}', \gamma + \gamma' + f(\boldsymbol{\beta}, \boldsymbol{\beta}') + g(\boldsymbol{\beta} + \boldsymbol{\beta}')), \\ \phi(\boldsymbol{\beta}, \gamma) \cdot \phi(\boldsymbol{\beta}', \gamma') &= (\boldsymbol{\beta}, \gamma + g(\boldsymbol{\beta})) \cdot (\boldsymbol{\beta}', \gamma' + g(\boldsymbol{\beta}')) \\ &= (\boldsymbol{\beta} + \boldsymbol{\beta}', \gamma + \gamma' + g(\boldsymbol{\beta}) + g(\boldsymbol{\beta}') + h(\boldsymbol{\beta}, \boldsymbol{\beta}')). \end{aligned}$$

Hence we know that ϕ is a group homomorphism by Proposition 2.21. Clearly ϕ is injective. Since the source and target of ϕ have the same cardinality, ϕ is bijective. Thus the claim follows. □

Lemma 2.31. *Let $g \in \mathrm{U}_n(q)$ and $\boldsymbol{\beta}, \boldsymbol{\beta}' \in V_{R,n}$. Then*

(1) $h_{R,n}(g\boldsymbol{\beta}, g\boldsymbol{\beta}') = h_{R,n}(\boldsymbol{\beta}, \boldsymbol{\beta}')$,

(2) $g_{R,n}(g\boldsymbol{\beta}) = g_{R,n}(\boldsymbol{\beta})$.

Proof. We consider the skew-hermitian form $h_0 \colon \mathbb{F}_{q^2}^n \times \mathbb{F}_{q^2}^n \to \mathbb{F}_{q^2}$; $(\boldsymbol{x}, \boldsymbol{y}) \mapsto {}^t\boldsymbol{x}^q \boldsymbol{y}$. Under the identification $V_{R,n} \simeq V_R \otimes_{\mathbb{F}_{q^2}} \mathbb{F}_{q^2}^n$, the form $h_{R,n}$ equals $h_R \otimes h_0$. Hence $h_{R,n}(g\boldsymbol{\beta}, g\boldsymbol{\beta}') = \sum_{i=1}^n h_R(\beta_i, \beta_i')h_0(ge_i, ge_i) = h_{R,n}(\boldsymbol{\beta}, \boldsymbol{\beta}')$, where $\boldsymbol{\beta} = (\beta_i)_{1 \leq i \leq n}$, $\boldsymbol{\beta}' = (\beta_i')_{1 \leq i \leq n}$ and $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{F}_{q^2}^n$.

We show (2). Using Definition 2.19, we can write

$$g_{R,n}(\boldsymbol{\beta}) = \sum_{i=0}^{e'-1} \sum_{j=0}^{e-2i-2} (a_{2i+1} {}^t\boldsymbol{\beta}^{q^{2i+1}} \boldsymbol{\beta})^{q^j} + \sum_{i=1}^{e'} \sum_{j=e'-i}^{e'-1} (a_{2i+1} {}^t\boldsymbol{\beta}^{q^{2i+1}} \boldsymbol{\beta})^{q^{2j+1}}.$$

Terms ${}^t\boldsymbol{\beta}^{q^{2i+1}} \boldsymbol{\beta}$ in $g_{R,n}(\boldsymbol{\beta})$ are stable by g. Hence (2) follows. □

Lemma 2.31(1) implies the injective homomorphism

$$\mathrm{U}_n(q) \hookrightarrow \mathrm{U}(V_{R,n}, h_{R,n}). \tag{2.12}$$

The group $\mathrm{U}_n(q)$ acts on $\mathrm{H}(V_{R,n}, h_{R,n})$ through (2.12) and the action of $\mathrm{U}(V_{R,n}, h_{R,n})$ on $\mathrm{H}(V_{R,n}, h_{R,n})$.

Remark 2.32. We identify $V_{R,n} = V_R \otimes_{\mathbb{F}_{q^2}} \mathbb{F}_{q^2}^n$ and $h_{R,n} = h_R \otimes h_0$ as in the proof of Lemma 2.31(1). We have the injective homomorphism

$$\mathrm{U}(V_R, h_R) \times \mathrm{U}_n(q) \hookrightarrow \mathrm{U}(V_{R,n}, h_{R,n}); (f, g) \mapsto f \otimes g,$$

which appears in the Howe correspondence (cf. [8, §1]).

Corollary 2.33. *We have the group isomorphism*

$$\phi \rtimes \mathrm{id}_{\mathrm{U}_n(q)} \colon \mathrm{H}_{R,n} \rtimes \mathrm{U}_n(q) \xrightarrow{\sim} \mathrm{H}(V_{R,n}, h_{R,n}) \rtimes \mathrm{U}_n(q);\ (x, g) \mapsto (\phi(x), g).$$

Proof. It suffices to show that $\phi \rtimes \mathrm{id}_{\mathrm{U}_n(q)}$ is a group homomorphism. This follows from

$$\phi(g \cdot (\boldsymbol{\beta}, \gamma)) = \phi(g\boldsymbol{\beta}, \gamma) = (g\boldsymbol{\beta}, \gamma + g_{R,n}(g\boldsymbol{\beta})) = (g\boldsymbol{\beta}, \gamma + g_{R,n}(\boldsymbol{\beta})) = g \cdot \phi(\boldsymbol{\beta}, \gamma)$$

for $(\boldsymbol{\beta}, \gamma) \in \mathrm{H}_{R,n}$ and $g \in \mathrm{U}_n(q)$, where the third equality follows from Lemma 2.31(2). □

Definition 2.34. Let $g \in \mathrm{U}_n(q)$. Recall $N_1(g) = \dim_{\mathbb{F}_{q^2}} \mathrm{Ker}(g - \mathrm{id}_{\mathbb{F}_{q^2}^n} \colon \mathbb{F}_{q^2}^n \to \mathbb{F}_{q^2}^n)$. We define $\chi_n(g) := (-1)^n(-q)^{N_1(g)}$.

Then χ_n is the character of the Weil representation of $\mathrm{U}_n(q)$ by (2.11).

Let $\rho_{V_{R,n}, h_{R,n}}$ be the $\mathrm{U}(V_{R,n}, h_{R,n})$-representation as in Definition 2.27. We regard $\rho_{V_{R,n}, h_{R,n}}$ as a $\mathrm{U}_n(q)$-representation via (2.12).

Lemma 2.35. *We have the equality $\rho_{V_{R,n}, h_{R,n}} = \chi_n^e$ as characters of $\mathrm{U}_n(q)$.*

Proof. We note that $\dim_{\mathbb{F}_{q^2}} V_{R,n} = en$. Let $g \in \mathrm{U}_n(q)$. Applying $V_R \otimes_{\mathbb{F}_{q^2}} (-)$ to $g - \mathrm{id}_{\mathbb{F}_{q^2}^n} \colon \mathbb{F}_{q^2}^n \to \mathbb{F}_{q^2}^n$, we obtain $g - \mathrm{id}_{V_{R,n}} \colon V_{R,n} \to V_{R,n}$. Let $N(g, V_{R,n})$ be as in Theorem 2.26(ii). Then $N(g, V_{R,n}) = (\dim_{\mathbb{F}_{q^2}} V_R) \cdot N_1(g) = eN_1(g)$. Thus the claim follows from (2.11). □

Definition 2.36. (1) Let $X_{R,n}$ be the n-dimensional smooth affine variety defined by

$$a^q - a = \sum_{i=1}^{n} x_i R(x_i) = \sum_{i=0}^{e} a_i \left(x_1^{q^i+1} + \cdots + x_n^{q^i+1} \right) = {}^t\boldsymbol{x}\boldsymbol{R}(\boldsymbol{x}) \quad \text{in } \mathbb{A}^{n+1},$$

where $\boldsymbol{x} = (x_i)_{1 \le i \le n} \in \mathbb{A}^n$.

(2) The group $\mathrm{U}_n(q)$ acts on $\mathrm{H}_{R,n}$ by $(\boldsymbol{\beta}, \gamma) \mapsto (g\boldsymbol{\beta}, \gamma)$ for $g \in \mathrm{U}_n(q)$. The semidirect group $\mathrm{H}_{R,n} \rtimes \mathrm{U}_n(q) \ni ((\boldsymbol{\beta}, \gamma), g)$ acts on $X_{R,n} \ni (a, \boldsymbol{x})$ by

$$(a, \boldsymbol{x}) \cdot ((\boldsymbol{\beta}, \gamma), g) = \left(a + \gamma + f_{R,n}(\boldsymbol{x}, \boldsymbol{\beta}), g^{-1}(\boldsymbol{x} + \boldsymbol{\beta})\right),$$

which is well-defined since $f_{R,n}(\boldsymbol{x}, \boldsymbol{\beta})^q - f_{R,n}(\boldsymbol{x}, \boldsymbol{\beta}) = {}^t\boldsymbol{x}\boldsymbol{R}(\boldsymbol{\beta}) + {}^t\boldsymbol{\beta}\boldsymbol{R}(\boldsymbol{x})$ by Lemma 2.2.

(3) Let $\mathbb{F}_q$ act on $X_{R,n}$ by $(a, \boldsymbol{x}) \mapsto (a + \zeta, \boldsymbol{x})$ for $\zeta \in \mathbb{F}_q$.

We note that the action of $Z(\mathrm{H}(V_{R,n}, h_{R,n})) \simeq \mathbb{F}_q$ (cf. Lemma 2.24) on $X_{R,n}$ equals the one in Definition 2.36(3).

Lemma 2.37. *Assume* $(n, q) = (2, 2)$. *Let* $\iota = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{U}_2(2)$. *For* $\psi \in \mathbb{F}_q^\vee \setminus \{1\}$, *we have* $\mathrm{Tr}(\iota; H_{\mathrm{c}}^2(X_{R,2}, \overline{\mathbb{Q}}_\ell)[\psi]) = -2^e$.

Proof. The Künneth formula implies the claim in the same way as in [7, Lemma 1.4]. □

For $\psi \in \mathbb{F}_q^\vee \setminus \{1\}$, $H_{\mathrm{c}}^i(X_{R,n}, \overline{\mathbb{Q}}_\ell)[\psi]$ is regarded as a representation of $\mathrm{H}(V_{R,n}, h_{R,n}) \rtimes \mathrm{U}_n(q)$ by Corollary 2.33. The homomorphism (2.12) implies an injection $\mathrm{H}(V_{R,n}, h_{R,n}) \rtimes \mathrm{U}_n(q) \hookrightarrow \mathrm{HU}(V_{R,n}, h_{R,n})$. Via this, we regard the $\mathrm{HU}(V_{R,n}, h_{R,n})$-representation $\widetilde{\rho}_{\psi, V_{R,n}, h_{R,n}}$ in Definition 2.27 as an $\mathrm{H}(V_{R,n}, h_{R,n}) \rtimes \mathrm{U}_n(q)$-representation. Now we state our main theorem in this paper.

Theorem 2.38. *Let* $\psi \in \mathbb{F}_q^\vee \setminus \{1\}$.

(1) *We have an isomorphism*

$$H_{\mathrm{c}}^n(X_{R,n}, \overline{\mathbb{Q}}_\ell)[\psi] \simeq \widetilde{\rho}_{\psi, V_{R,n}, h_{R,n}}$$

as $\mathrm{H}(V_{R,n}, h_{R,n}) \rtimes \mathrm{U}_n(q)$*-representations.*

(2) *We have the equality* $H_{\mathrm{c}}^n(X_{R,n}, \overline{\mathbb{Q}}_\ell)[\psi] = \chi_n^e$ *as characters of* $\mathrm{U}_n(q)$.

Proof. The claim (2) follows from (1) and Lemma 2.35. We show (1). We write $H_{\psi,n}^i$ and (V, h) for $H_{\mathrm{c}}^i(X_{R,n}, \overline{\mathbb{Q}}_\ell)[\psi]$ and $(V_{R,n}, h_{R,n})$, respectively. Let $\mathscr{L}_\psi(s)$ be the smooth Artin–Schreier sheaf on $\mathbb{A}^1 = \operatorname{Spec} \mathbb{F}[s]$ defined by the Galois étale covering $a^q - a = s$ and ψ in the sense of [2, Définition 1.7 in Sommes trig.]. For a morphism of varieties $f \colon X \to \mathbb{A}^1$, let $\mathscr{L}_\psi(f) := f^*\mathscr{L}_\psi(s)$. We regard a polynomial $g(x) \in \mathbb{F}[x]$ as a morphism

$\mathbb{A}^1 \to \mathbb{A}^1;\ x \mapsto g(x)$. Considering the Galois étale morphism $X_{R,n} \to \mathbb{A}^n;\ (a, \boldsymbol{x}) \mapsto \boldsymbol{x}$ with Galois group $\mathbb{F}_q$, we see that $H^i_{\psi,n} = H^i_{\mathrm{c}}(\mathbb{A}^n, \mathscr{L}_\psi(\sum_{j=1}^n x_j R(x_j)))$ for an integer i. By $X_{R,1} = C_R$ and Lemma 2.9, we have $H^i_{\psi,1} = 0$ for $i \neq 1$ and $\dim H^1_{\psi,1} = q^e$. From this, the Künneth formula in [2, (2.4.1)* in Sommes trig.] and $\mathscr{L}_\psi(\sum_{i=1}^n x_i R(x_i)) = \boxtimes_{i=1}^n \mathscr{L}_\psi(x_i R(x_i))$, it results that

$$H^n_{\psi,n} \simeq \bigotimes_{i=1}^n H^1_{\mathrm{c}}(\mathbb{A}^1, \mathscr{L}_\psi(x_i R(x_i))) \simeq (H^1_{\psi,1})^{\otimes n}. \tag{2.13}$$

Hence $\dim H^n_{\psi,n} = q^{en}$. This and Lemma 2.25 imply an isomorphism $H^n_{\psi,n} \simeq \rho_{\mathrm{H}(V,h),\psi}$ as $\mathrm{H}(V,h)$-representations. Recall that $\rho_{\mathrm{H}(V,h),\psi}$ is irreducible as in Lemma 2.25. By Schur's lemma, there exists $\chi \in \mathrm{U}_n(q)^\vee$ such that an isomorphism

$$H^n_{\psi,n} \simeq \widetilde{\rho}_{\psi,V,h} \otimes \chi$$

as $\mathrm{H}(V,h) \rtimes \mathrm{U}_n(q)$-representations holds. We need to show $\chi = 1$. For $(\xi_i)_{1 \leq i \leq n} \in \mu^n_{q+1}$, let $\mathrm{diag}(\xi_1, \ldots, \xi_n) \in \mathrm{U}_n(q)$ denote the diagonal matrix. From Lemma 2.11 and (2.13), it follows that

$$\mathrm{Tr}(\mathrm{diag}(\xi_1, \ldots, \xi_n); H^n_{\psi,n}) = (-1)^n \quad \text{for } (\xi_i)_{1 \leq i \leq n} \in (\mu_{q+1} \setminus \{1\})^n. \tag{2.14}$$

On the other hand, Lemma 2.35 implies

$$\mathrm{Tr}\, \rho_{V_{R,n}, h_{R,n}}(\mathrm{diag}(\xi_1, \ldots, \xi_n)) = (-1)^{ne} = (-1)^n$$

since e is odd. Furthermore, if $(n, q) = (2, 2)$, $\mathrm{Tr}\, \rho_{V_{R,2}, h_{R,2}}(\iota) = (-2)^e = -2^e$. Using (2.14) and Lemma 2.37 in the same manner as in the proof of [7, Theorem 1.5], we obtain the claim. □

Remark 2.39. By Remark 2.32, Theorem 2.38 and [8, Case (iii) in Theorem], the $\mathrm{U}_n(q)$-representation $H^n_{\mathrm{c}}(X_{R,n}, \overline{\mathbb{Q}}_\ell)[\psi]$ is expressed as a sum of Deligne–Lusztig characters.

For a variety X over $\mathbb{F}$ equipped with a $\mathrm{U}_n(q)$-action, let

$$H^*_{\mathrm{c}}(X, \overline{\mathbb{Q}}_\ell) := \sum_{i=0}^\infty (-1)^i H^i_{\mathrm{c}}(X, \overline{\mathbb{Q}}_\ell),$$

which is a virtual character of $\mathrm{U}_n(q)$.

Corollary 2.40. (1) *We have isomorphisms*

$$\begin{aligned} &H^i_{\mathrm{c}}(X_{R,n}, \overline{\mathbb{Q}}_\ell)[\psi] = 0 \quad \textit{for any } \psi \in \mathbb{F}_q^\vee \setminus \{1\} \textit{ and } i \neq n, \\ &H^i_{\mathrm{c}}(X_{R,n}, \overline{\mathbb{Q}}_\ell) = 0 \quad \textit{for } i \neq n, 2n \textit{ and } H^{2n}_{\mathrm{c}}(X_{R,n}, \overline{\mathbb{Q}}_\ell) \simeq \overline{\mathbb{Q}}_\ell. \end{aligned}$$

(2) *We have the equalities*

$$\begin{aligned} H^n_{\mathrm{c}}(X_{R,n}, \overline{\mathbb{Q}}_\ell) &= (q-1)\chi^e_n, \\ H^*_{\mathrm{c}}(X_{R,n}, \overline{\mathbb{Q}}_\ell) &= (-1)^n (q-1)\chi^e_n + 1 \end{aligned}$$

as virtual characters of $\mathrm{U}_n(q)$.

Proof. We write $H^i_{\psi,n}$ for $H^i_{\mathrm{c}}(X_{R,n}, \overline{\mathbb{Q}}_\ell)[\psi]$. Clearly $H^i_{\mathrm{c}}(X_{R,n}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\psi \in \mathbb{F}_q^\vee} H^i_{\psi,n}$ for any i. We show (1). If $\psi = 1$, we have $H^i_{\psi,n} = H^i_{\mathrm{c}}(X_{R,n}, \overline{\mathbb{Q}}_\ell)^{\mathbb{F}_q} \simeq H^i_{\mathrm{c}}(X_{R,n}/\mathbb{F}_q, \overline{\mathbb{Q}}_\ell) = H^i_{\mathrm{c}}(\mathbb{A}^n, \overline{\mathbb{Q}}_\ell)$ by $X_{R,n}/\mathbb{F}_q \simeq \mathbb{A}^n$ and Lemma 2.10(1). Thus $H^i_{\psi,n}$ is zero if $i \neq 2n$ and is isomorphic to $\overline{\mathbb{Q}}_\ell$ if $i = 2n$. Let $\psi \in \mathbb{F}_q^\vee \setminus \{1\}$. Lemma 2.9 implies that $H^i_{\psi,1} = 0$ for $i \neq 1$. Using the Künneth formula similarly as (2.13), we see that $H^i_{\psi,n} = 0$ for $i \neq n$. Therefore we have obtained (1).

We show (2). The former equality follows from Theorem 2.38. The latter equality follows from (1) and the former one. □

3 Trace formulae for projective hypersurfaces

As in §2.5, assume that $R(x) = \sum_{i=0}^{e} a_i x^{q^i} \in \mathbb{F}[x]$ with $e > 0$ and $a_e \neq 0$ satisfies (2.4). Our aim in this section is to show Theorem 3.8 as an application of Theorem 2.38.

Definition 3.1. (1) Let $Y_{R,n}$ be the smooth affine variety defined by

$$a^{q^e} - a = \sum_{i=0}^{e} a_i \left(x_1^{q^i+1} + \cdots + x_n^{q^i+1} \right) = {}^t\boldsymbol{x}\boldsymbol{R}(\boldsymbol{x}) \quad \text{in } \mathbb{A}^{n+1},$$

where $\boldsymbol{x} = (x_i)_{1 \le i \le n} \in \mathbb{A}^n$.

(2) The unitary group $\mathrm{U}_n(q)$ acts on $Y_{R,n}$ by $(a, \boldsymbol{x}) \mapsto (a, g^{-1}\boldsymbol{x})$ for $g \in \mathrm{U}_n(q)$.

(3) Let $\mathbb{F}_{q^e}$ act on $Y_{R,n}$ by $(a, \boldsymbol{x}) \mapsto (a + \zeta, \boldsymbol{x})$ for $\zeta \in \mathbb{F}_{q^e}$. For $\psi \in \mathbb{F}_{q^e}^\vee$ and an integer i, the ψ-isotypic part $H^i_{\mathrm{c}}(Y_{R,n}, \overline{\mathbb{Q}}_\ell)[\psi]$ is regarded as a $\mathrm{U}_n(q)$-representation.

For a scheme X over $\mathbb{F}$, we often write $H^i_{\mathrm{c}}(X)$ for $H^i_{\mathrm{c}}(X, \overline{\mathbb{Q}}_\ell)$. We fix ${}^0\psi \in \mathbb{F}_q^\vee \setminus \{1\}$. Let $\xi \in \mathbb{F}_{q^e}^\times$. We have the finite Galois étale morphism

$$Y_{R,n} \to X_{\xi R,n};\ (a, \boldsymbol{x}) \mapsto \left(\sum_{i=0}^{e-1} (\xi a)^{q^i}, \boldsymbol{x} \right)$$

whose Galois group is the kernel of $\mathrm{Tr}_\xi \colon \mathbb{F}_{q^e} \to \mathbb{F}_q;\ x \mapsto \mathrm{Tr}_{q^e/q}(\xi x)$. For $\xi \in \mathbb{F}_{q^e}$ and $\psi \in \mathbb{F}_{q^e}^\vee$, we define $\psi_\xi(x) := \psi(\xi x)$ for $x \in \mathbb{F}_{q^e}$.

Corollary 3.2. *Let $\psi := {}^0\psi \circ \mathrm{Tr}_{q^e/q} \in \mathbb{F}_{q^e}^\vee$ and $\xi \in \mathbb{F}_{q^e}^\times$. For any $i \geq 0$, we have an isomorphism $H^i_{\mathrm{c}}(Y_{R,n})[\psi_\xi] \simeq H^i_{\mathrm{c}}(X_{\xi R,n})[{}^0\psi]$ as $\mathrm{U}_n(q)$-representations.*

Proof. By $Y_{R,n}/\operatorname{Ker}\mathrm{Tr}_\xi \simeq X_{\xi R}$ and Lemma 2.10(2), $H^i_{\mathrm{c}}(Y_{R,n})[\psi_\xi] \simeq H^i_{\mathrm{c}}(X_{\xi R,n})[{}^0\psi]$. □

Definition 3.3. We define

$$\phi_n(g) := \frac{(-1)^n}{q^e + 1} \sum_{\xi \in \mu_{q^e+1}} (-q^e)^{N_e(\xi g)} \quad \text{for } g \in \mathrm{U}_n(q^e), \tag{3.1}$$

which is an irreducible unipotent character of $\mathrm{U}_n(q^e)$ corresponding to the partition $(n-1, 1)$ of n (cf. [6, Theorems 1 and 3]).

Definition 3.4. Let $\overline{Y}_{R,n}$ be the smooth projective hypersurface defined by

$$Z^{q^e}W - ZW^{q^e} = \sum_{i=0}^{e} a_i W^{q^e-q^i}\left(X_1^{q^i+1} + \cdots + X_n^{q^i+1}\right) \quad \text{in } \mathbb{P}^{n+1}. \tag{3.2}$$

We regard $Y_{R,n}$ as an open subscheme of $\overline{Y}_{R,n}$ by $a = Z/W$ and $x_i = X_i/W$ for $1 \leq i \leq n$. Let $\mathrm{U}_n(q)$ act on $\overline{Y}_{R,n}$ by $[Z : W : (X_i)_{1\leq i\leq n}] \mapsto [Z : W : g^{-1}(X_i)_{1\leq i\leq n}]$ for $g \in \mathrm{U}_n(q)$. This action is an extension of the one in Definition 3.1(2). The closed subscheme

$$S := \left\{[Z : W : X_1 : \cdots : X_n] \in \mathbb{P}^{n+1} \;\middle|\; W = X_1^{q^e+1} + \cdots + X_n^{q^e+1} = 0\right\} \subset \overline{Y}_{R,n} \tag{3.3}$$

is stable under the action of $\mathrm{U}_n(q)$.

Lemma 3.5. *We have* $S = \overline{Y}_{R,n} \setminus Y_{R,n}$.

Proof. Substituting $W = 0$ to (3.2), we obtain $a_e(X_1^{q^e+1} + \cdots + X_n^{q^e+1}) = 0$. Hence $\overline{Y}_{R,n} \setminus Y_{R,n}$ equals S as $a_e \neq 0$. □

Lemma 3.6. *We have the equality* $H^*(S) = n + (-1)^n\phi_n$ *as virtual characters of* $\mathrm{U}_n(q)$.

Proof. The scheme S is denoted by R and ϕ_n is denoted by ψ_n in the notation of [11]. The claim is shown in the proof of [11, Lemma 3.6]. □

Lemma 3.7. (1) *We have isomorphisms*

$$H_c^i(Y_{R,n}) = 0 \quad \textit{for } i \neq n, 2n, \quad H_c^n(Y_{R,n}) \simeq \bigoplus_{\xi\in\mathbb{F}_{q^e}^\times} H_c^n(X_{\xi R,n})[{}^0\psi], \quad H_c^{2n}(Y_{R,n}) \simeq \overline{\mathbb{Q}}_\ell.$$

(2) *We have the equality* $H_c^*(Y_{R,n}) = (-1)^n(q^e-1)\chi_n^e+1$ *as virtual characters of* $\mathrm{U}_n(q)$.

Proof. The assertion (2) follows from (1) and Theorem 2.38(2). We show (1). The claim $H_c^{2n}(Y_{R,n}) \simeq \overline{\mathbb{Q}}_\ell$ is clear. Let $\psi := {}^0\psi \circ \mathrm{Tr}_{q^e/q}$. Then the isomorphism $\mathbb{F}_{q^e} \xrightarrow{\sim} \mathbb{F}_{q^e}^\vee;\ \xi \mapsto \psi_\xi$ holds. For any $i \neq 2n$, we have isomorphisms

$$H_c^i(Y_{R,n}) = \bigoplus_{\xi\in\mathbb{F}_{q^e}^\times} H_c^i(Y_{R,n})[\psi_\xi] \simeq \bigoplus_{\xi\in\mathbb{F}_{q^e}^\times} H_c^i(X_{\xi R,n})[{}^0\psi],$$

where the first equality follows from Lemma 2.10(1) and $i \neq 2n$ and the second isomorphism follows from Corollary 3.2. Hence the claim follows from Corollary 2.40(1). □

Theorem 3.8. *We have the equality*

$$H^*(\overline{Y}_{R,n}, \overline{\mathbb{Q}}_\ell) = (-1)^n(q^e-1)\chi_n^e + n + 1 + (-1)^n\phi_n$$

as virtual characters of $\mathrm{U}_n(q)$.

Proof. Lemma 3.5 implies that $H^*(\overline{Y}_{R,n}) = H_c^*(Y_{R,n}) + H^*(S)$. Hence the claim follows from Lemmas 3.6 and 3.7(2). □

Remark 3.9. Assume $e = 1$. Then $\overline{Y}_{R,n}$ is isomorphic to the Fermat hypersurface S_{n+2} defined by $\sum_{i=1}^{n+2} x_i^{q+1} = 0$ in $\mathbb{P}^{n+1}$ as in [11, Lemma 3.2]. We regard $\mathrm{U}_n(q)$ as a subgroup of $\mathrm{U}_{n+2}(q)$ by $g \mapsto \begin{pmatrix} g & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_2 \end{pmatrix}$ for $g \in \mathrm{U}_n(q)$. Theorem 3.8 asserts

$$\mathrm{Tr}(g; H^*(S_{n+2}, \overline{\mathbb{Q}}_\ell)) = n + 1 + (-1)^{n+2}\phi_{n+2}(g) \quad \text{for } g \in \mathrm{U}_n(q).$$

This equality is shown in [6, Theorem 3].

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Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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